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GLOBAL STABILITY OF TRAVELING WAVES FOR A SPATIALLY DISCRETE DIFFUSION SYSTEM WITH TIME DELAY

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Abstract. This article deals with the global stability of traveling waves of a spatially discrete diffusion system with time delay and without quasi-monotonicity. Using the Fourier transform and the weighted energy method with a suitably selected weighted function, we prove that the monotone or non-monotone traveling waves are exponentially stable in $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ with the exponential convergence rate $e^{-\mu t}$ for some constant $\mu > 0$.

1. **Introduction.**

In this article, we consider the following spatially discrete diffusion system with time delay

$$\begin{cases} \partial_t v_1(x,t) = d_1 \mathcal{D}[v_1](x,t) - \alpha v_1(x,t) + h(v_2(x,t-\tau_1)), \\ \partial_t v_2(x,t) = d_2 \mathcal{D}[v_2](x,t) - \beta v_2(x,t) + g(v_1(x,t-\tau_2)) \end{cases} \tag{1}$$

with the initial data

$$v_i(x,s) = v_{i0}(x,s),\ x \in \mathbb{R},\ s \in [-\tau_i, 0],\ i = 1,2, \tag{2}$$

where $t > 0$, $x \in \mathbb{R}$, $d_i \geq 0$ and

$$\mathcal{D}[v_i](x,t) = v_i(x+1,t) - 2v_i(x,t) + v_i(x-1,t),\ i = 1,2.$$

Here $v_1(x,t)$ and $v_2(x,t)$ biologically stand for the spatial density of the bacterial population and the infective human population at point $x \in \mathbb{R}$ and time $t \geq 0$, respectively. Both bacteria and humans are assumed to diffuse, d_1 and d_2 are diffusion coefficients; the term $-\alpha v_1$ is the natural death rate of the bacterial population and the nonlinearity $h(v_2)$ is the contribution of the infective humans to the growth rate of the bacterial; $-\beta v_2$ is the natural diminishing rate of the infective population due to the finite mean duration of the infectious population and the nonlinearity $g(v_1)$ is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic, and τ_1, τ_2 are time delays. The nonlinearities g and h satisfy the following hypothesis:

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(H1): $g \in C^2([0,K_1],\mathbb{R})$, $g(0)=h(0)=0$, $K_2=g(K_1)/\beta>0$, $h\in C^2([0,K_2],\mathbb{R})$, $h(g(K_1)/\beta)=\alpha K_1$, $h(g(v)/\beta)>\alpha v$ for $v\in(0,K_1)$, where K_1 is a positive constant.

According to (H1), the spatially homogeneous system of (1) admits two constant equilibria

$$(v_{1-},v_{2-})=\mathbf{0}:=(0,0) \quad \text{and} \quad (v_{1+},v_{2+})=\mathbf{K}:=(K_1,K_2).$$

It is clear that (H1) is a basic assumption to ensure that system (1) is monostable on $[\mathbf{0},\mathbf{K}]$. When $g'(u)\ge 0$ for $u\in[0,K_1]$ and $h'(v)\ge 0$ for $v\in[0,K_2]$, system (1) is a quasi-monotone system. Otherwise, if $g'(u)\ge 0$ for $u\in[0,K_1]$ or $h'(v)\ge 0$ for $v\in[0,K_2]$ does not hold, system (1) is a non-quasi-monotone system. In this article, we are interested in the existence and stability of traveling wave solutions of (1) connecting two constant equilibria $(0,0)$ and (K_1,K_2). A *traveling wave solution* (in short, traveling wave) of (1) is a special translation invariant solution of the form $(v_1(x,t),v_2(x,t))=(\phi_1(x+ct),\phi_2(x+ct))$, where $c>0$ is the wave speed. If ϕ_1 and ϕ_2 are monotone, then (ϕ_1,ϕ_2) is called a traveling wavefront. Substituting $(\phi_1(x+ct),\phi_2(x+ct))$ into (1), we obtain the following wave profile system with the boundary conditions

$$\begin{cases} c\phi_1'(\xi)=d_1\mathcal{D}[\phi_1](\xi)-\alpha\phi_1(\xi)+h(\phi_2(\xi-c\tau_1)),\\ c\phi_2'(\xi)=d_2\mathcal{D}[\phi_2](\xi)-\beta\phi_2(\xi)+g(\phi_1(\xi-c\tau_2)),\\ (\phi_1,\phi_2)(-\infty)=(v_{1-},v_{2-}), \quad (\phi_1,\phi_2)(+\infty)=(v_{1+},v_{2+}), \end{cases} \tag{3}$$

where $\xi=x+ct$, $'=\frac{d}{d\xi}$, $\mathcal{D}[\phi_i](\xi)=\phi_i(\xi+1)-2\phi_i(\xi)+\phi_i(\xi-1)$, $i=1,2$.

System (1) is a discrete version of classical epidemic model

$$\begin{cases} \partial_t v_1(x,t)=d_1\partial_{xx}v_1(x,t)-a_1v_1(x,t)+h(v_2(x,t-\tau_1)),\\ \partial_t v_2(x,t)=d_2\partial_{xx}v_2(x,t)-a_2v_2(x,t)+g(v_1(x,t-\tau_2)). \end{cases} \tag{4}$$

The existence and stability of traveling waves of (4) have been extensively studied, see [7, 19, 21, 24] and references therein. Note that system (1) is also a delay version of the following system

$$\begin{cases} \partial_t v_1(x,t)=d_1\mathcal{D}[v_1](x,t)-a_1v_1(x,t)+h(v_2(x,t)),\\ \partial_t v_2(x,t)=d_2\mathcal{D}[v_2](x,t)-a_2v_2(x,t)+g(v_1(x,t)). \end{cases} \tag{5}$$

When system (5) is a quasi-monotone system, Yu, Wan and Hsu [27] established the existence and stability of traveling waves of (5). To the best of our knowledge, when systems (1) and (5) are non-quasi-monotone systems, no result on the existence and stability of traveling waves has been reported. We should point out that the existence of traveling waves of (1) can be easily obtained. Hence, the main purpose of the current paper is to establish the stability of traveling waves of (1).

The stability of traveling waves for the classical reaction-diffusion equations with and without time delay has been extensively investigated, see e.g., [4, 9, 10, 12, 13, 14, 16, 22, 24]. Compared to the rich results for the classical reaction-diffusion equations, limited results exist for the spatial discrete diffusion equations. Chen and Guo [1] took the squeezing technique to prove the asymptotic stability of traveling waves for discrete quasilinear monostable equations without time delay. Guo and Zimmer [5] proved the global stability of traveling wavefronts for spatially discrete equations with nonlocal delay effects by using a combination of the weighted energy method and the Green function technique. Tian and Zhang [19] investigated the global stability of traveling wavefronts for a discrete diffusive Lotka-Volterra competition system with two species by the weighted energy method together with the comparison principle. Later on, Chen, Wu and Hsu [2] employed the similar method to show the global stability of traveling wavefronts for a discrete diffusive Lotka-Volterra competition system with three species. We should point out that the methods for the above stability results heavily depend on the monotonicity of equations and the comparison principle. However, the most interesting cases are the equations without monotonicity. It is known that when the evolution equations are non-monotone, the comparison principle is not applicable. Thus, the methods, such as the squeezing technique, the weighted energy method combining with the comparison principle are not valid for the stability of traveling waves of the spatial discrete diffusion equations without monotonicity.

Recently, the technical weighted energy method without the comparison principle has been used to prove the stability of traveling waves of nonmonotone equations, see Chern et al. [3], Lin et al. [10], Wu et al. [22], Yang et al. [24]. In particular, Tian et al. [20] and Yang et al. [26], respectively, applied this method to prove the local stability of traveling waves for nonmonotone traveling waves for spatially discrete reaction-diffusion equations with time delay. Later, Yang and Zhang [25] established the stability of non-monotone traveling waves for a discrete diffusion equation with monostable convolution type nonlinearity. Unfortunately, the local stability (the initial perturbation around the traveling wave is properly small in a weighted norm) of traveling waves can only be obtained. Very recently, Mei et al. [15] developed a new method to prove the global stability of the oscillatory traveling waves of local Nicholson's blowflies equations. This method is based on some key observations for the structure of the govern equations and the anti-weighted energy method together with the Fourier transform. Later on, Zhang [28] and Xu et al. [23], respectively, applied this method successfully to a nonlocal dispersal equation with time delay and obtained the global stability of traveling waves. More recently, Su and Zhang [17] further studied a discrete diffusion equation with a monostable convolution type nonlinearity and established the global stability of traveling waves with large speed. Motivated by the works [15, 28, 23, 17, 18], in this paper, we shall extend this method to study the global stability of traveling waves of spatial discrete diffusion system (1) without quasi-monotonicity.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and summarize our main results. Section 3 is dedicated to the global stability of traveling waves of (1) by the Fourier transform and the weighted energy method, when $h(u)$ and $g(u)$ are not monotone.

2. Preliminaries and main results.

In this section, we first give the equivalent integral form of the initial value problem of (1) with (2), then recall the existence of traveling waves of (1), and finally state the main result on the global stability of traveling waves of (1). Throughout this paper, we assume $\tau_1 = \tau_2 = \tau$.

First of all, we consider the initial value problem (1) with (2), i.e.,

$$\begin{cases} \partial_t v_1(x,t) = d_1\mathcal{D}[v_1](x,t) - \alpha v_1(x,t) + h(v_2(x,t-\tau)), \\ \partial_t v_2(x,t) = d_2\mathcal{D}[v_2](x,t) - \beta v_2(x,t) + g(v_1(x,t-\tau)), \\ v_i(x,s) = v_{i0}(x,s), \ x\in\mathbb{R}, \ s\in[-\tau,0], \ i=1,2. \end{cases} \tag{6}$$

According to [8], with aid of modified Bessel functions, the solution to the initial value problem

$$\begin{cases} \partial_t u(x,t) = d[u(x+1,t) - 2u(x,t) + u(x-1,t)], \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}, \end{cases}$$

can be expressed by

$$u(x,t) = (S(t)u_0)(x) = e^{-2dt} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2dt) u_0(x-m),$$

where $u_0(\cdot) \in L^\infty(\mathbb{R})$, $\mathbf{I}_m(\cdot)$, $m \geq 0$ are defined as

$$\mathbf{I}_m(t) = \sum_{k=0}^{\infty} \frac{(t/2)^{m+2k}}{k!(m+k)!},$$

and $\mathbf{I}_m(t) = \mathbf{I}_{-m}(t)$ for $m < 0$. Moreover,

$$\mathbf{I}'_m(t) = \frac{1}{2}[\mathbf{I}_{m+1}(t) + \mathbf{I}_{m-1}(t)], \ \forall t > 0, m \in \mathbb{Z}, \tag{7}$$

and $\mathbf{I}_m(0) = 0$ for $m \neq 0$ while $\mathbf{I}_0 = 1$, and $\mathbf{I}_m(t) \geq 0$ for any $t \geq 0$. In addition, one has

$$e^{-t} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(t) = e^{-t}[\mathbf{I}_0(t) + 2\mathbf{I}_1(t) + 2\mathbf{I}_2(t) + \mathbf{I}_3(t) + \cdots] = 1. \tag{8}$$

Thus, the solution $(v_1(x,t), v_2(x,t))$ of (6) can be expressed as

$$\begin{cases} v_1(x,t) = & e^{-(2d_1+\alpha)t} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2d_1 t) v_{10}(x-m,0) \\ & + \sum_{m=-\infty}^{\infty} \int_0^t e^{-(2d_1+\alpha)(t-s)} \mathbf{I}_m(2d_1(t-s))(h(v_2(x-m,s-\tau)))ds, \\ v_2(x,t) = & e^{-(2d_2+\beta)t} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2d_2 t) v_{20}(x-m,0) \\ & + \sum_{m=-\infty}^{\infty} \int_0^t e^{-(2d_2+\beta)(t-s)} \mathbf{I}_m(2d_2(t-s))(g(v_1(x-m,s-\tau)))ds. \end{cases} \tag{9}$$

In fact, by [8, Lemma 2.1], we can differentiate the series on t variable in (9). Using the recurrence relation (7), we obtain

$$\begin{aligned} &\partial_t v_1(x,t) \\ &= -(2d_1+\alpha)e^{-(2d_1+\alpha)t} \sum_{m=-\infty}^{\infty} \mathbf{I}_m(2d_1 t) v_{10}(x-m,0) \\ &\quad + e^{-(2d_1+\alpha)t} \sum_{m=-\infty}^{\infty} 2d_1 \mathbf{I}'_m(2d_1 t) v_{10}(x-m,0) \\ &\quad + \sum_{m=-\infty}^{\infty} \mathbf{I}_m(0)(h(v_2(x-m,t-\tau))) \\ &\quad - (2d_1+\alpha) \sum_{m=-\infty}^{\infty} \int_0^t e^{-(2d_1+\alpha)(t-s)} \mathbf{I}_m(2d_1(t-s))(h(v_2(x-m,s-\tau)))ds \\ &\quad + \sum_{m=-\infty}^{\infty} \int_0^t e^{-(2d_1+\alpha)(t-s)} 2d_1 \mathbf{I}'_m(2d_1(t-s))(h(v_2(x-m,s-\tau)))ds \\ &= d_1[v_1(x+1,t) - 2v_1(x,t) + v_1(x-1,t)] - \alpha v_1(x,t) + h(v_2(x,t-\tau)) \end{aligned}$$

and

$$
\begin{aligned}
&\partial_t v_2(x,t)\\
&= -(2d_2+\beta)e^{-(2d_2+\beta)t}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_2t)v_{20}(x-m,0)\\
&\quad + e^{-(2d_2+\beta)t}\sum_{m=-\infty}^{\infty}2d_2\mathbf{I}'_m(2d_2t)v_{20}(x-m,0)\\
&\quad + \sum_{m=-\infty}^{\infty}\mathbf{I}_m(0)(g(v_1(x-m,t-\tau)))\\
&\quad - (2d_2+\beta)\sum_{m=-\infty}^{\infty}\int_0^t e^{-(2d_2+\beta)(t-s)}\mathbf{I}_m(2d_2(t-s))(g(v_1(x-m,s-\tau)))ds\\
&\quad + \sum_{m=-\infty}^{\infty}\int_0^t e^{-(2d_2+\beta)(t-s)}2d_2\mathbf{I}'_m(2d_2(t-s))(g(v_1(x-m,s-\tau)))ds\\
&= d_2[v_2(x+1,t)-2v_2(x,t)+v_2(x-1,t)]-\beta v_2(x,t)+g(v_1(x,t-\tau)).
\end{aligned}
$$

Next we investigate the characteristic roots of the linearized system for the wave profile system (3) at the trivial equilibrium $\mathbf{0}$. Clearly, the characteristic function of (3) at $\mathbf{0}$ is

$$\mathcal{P}_1(c,\lambda) := f_1(c,\lambda) - f_2(c,\lambda)$$

for $c \geq 0$ and $\lambda \in \mathbb{C}$, where

$$f_1(c,\lambda) := \Delta_1(c,\lambda)\Delta_2(c,\lambda), \quad f_2(c,\lambda) := h'(0)g'(0)e^{-2c\lambda\tau},$$

with

$$\Delta_1(c,\lambda) = d_1(e^{\lambda}+e^{-\lambda}-2) - c\lambda - \alpha, \quad \Delta_2(c,\lambda) = d_2(e^{\lambda}+e^{-\lambda}-2) - c\lambda - \beta.$$

It is easy to see that $\Delta_1(c,\lambda)=0$ admits two roots $\lambda_1^- < 0 < \lambda_1^+$, and $\Delta_2(c,\lambda)=0$ has two roots $\lambda_2^- < 0 < \lambda_2^+$. We denote $\lambda_m^+ = \min\{\lambda_1^+, \lambda_2^+\}$.

Similar to [27, Lemma 3.1], we can obtain the following result.

Lemma 2.1. *There exists a positive constant c_* such that if $c > c_*$, then $\mathcal{P}_1(c,\lambda) = 0$ has two distinct positive real roots $\lambda_1 := \lambda_1(c)$ and $\lambda_2 := \lambda_2(c)$ with $\lambda_1(c) < \lambda_2(c) < \lambda_m^+$, i.e. $\mathcal{P}_1(c,\lambda_1) = \mathcal{P}_1(c,\lambda_2) = 0$, and $\mathcal{P}(c,\lambda) > 0$ for $\lambda \in (\lambda_1(c), \lambda_2(c))$. In addition, $\lim_{c\to c_*}\lambda_1(c) = \lim_{c\to c_*}\lambda_2(c) = \lambda_* > 0$, i.e., $\mathcal{P}_1(c_*,\lambda_*) = 0$.*

Furthermore, we show the existence of traveling wave of (1). When system (1) is a quasi-monotone system, the existence of traveling wavefronts follows from [6, Theorem 1.1]. When system (1) is a non-quasi-monotone system, the existence of traveling waves can also be obtained by using auxiliary equations and Schauder's fixed point theorem [21, 24], if we assume the following assumptions:

(H2): There exist $\mathbf{K}^{\pm} = (K_1^{\pm}, K_2^{\pm}) \gg 0$ with $\mathbf{K}^- < \mathbf{K} < \mathbf{K}^+$ and four continuous and twice piecewise continuous differentiable functions $g^{\pm} : [0, K_1^+] \to \mathbb{R}$ and $h^{\pm} : [0, K_2^+] \to \mathbb{R}$ such that
(i) $K_2^{\pm} = g^{\pm}(K_1^{\pm})/\beta$, $h^{\pm}(\frac{1}{\beta}g^{\pm}(K_1^{\pm})) = \alpha K_1^{\pm}$, and $h^{\pm}(\frac{1}{\beta}g^{\pm}(v)) > \alpha v$ for $v \in (0, K_1^{\pm})$;
(ii) $g^{\pm}(u)$ and $h^{\pm}(v)$ are non-decreasing on $[0, K_1^+]$ and $[0, K_2^+]$, respectively;

(iii) $(g^{\pm})'(0) = g'(0)$, $(h^{\pm})'(0) = h'(0)$ and

$$0 < g^{-}(u) \leq g(u) \leq g^{+}(u) \leq g'(0)u \text{ for } u \in [0, K_1^{+}],$$
$$0 < h^{-}(v) \leq h(v) \leq h^{+}(v) \leq h'(0)v \text{ for } v \in [0, K_2^{+}].$$

Proposition 1. *Assume that* (H1) *and* (H2) *hold,* $\tau \geq 0$*, and let* c_* *be defined as in Lemma 2.1. Then for every* $c > c_*$*, system* (1) *has a traveling wave* $(\phi_1(\xi), \phi_2(\xi))$ *satisfying* $(\phi_1(-\infty), \phi_2(-\infty)) = (0, 0)$ *and*

$$K_1^{-} \leq \liminf_{\xi \to +\infty} \phi_1(\xi) \leq \limsup_{\xi \to +\infty} \phi_1(\xi) \leq K_1^{+},$$
$$0 \leq \liminf_{\xi \to +\infty} \phi_2(\xi) \leq \limsup_{\xi \to +\infty} \phi_2(\xi) \leq K_2^{+}.$$

Finally, we shall state the stability result of traveling waves derived in Proposition 1. Before that, let us introduce the following notations.

Notations. $C > 0$ denotes a generic constant, while $C_i (i = 1, 2, \dots)$ represents a specific constant. Let $\|\cdot\|$ and $\|\cdot\|_\infty$ denote 1-norm and ∞-norm of the matrix (or vector), respectively. Let I be an interval, typically $I = \mathbb{R}$. Denote by $L^1(I)$ the space of integrable functions defined on I, and $W^{k,1}(I)(k \geq 0)$ the Sobolev space of the L^1-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^n}{dx^n} f (n = 1, \dots, k)$ also belong to $L^1(I)$. Let $L^1_w(I)$ be the weighted L^1-space with a weight function $w(x) > 0$ and its norm is defined by

$$||f||_{L^1_w(I)} = \int_I w(x)|f(x)|dx,$$

$W^{k,1}_w(I)$ be the weighted Sobolev space with the norm given by

$$||f||_{W^{k,1}_w(I)} = \sum_{i=0}^{k} \int_I w(x) \left| \frac{d^i f(x)}{dx^i} \right| dx.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0, T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0, T]$, and by $L^1([0, T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued L^1-functions on $[0, T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0, \infty)$ are defined similarly. For any function $f(x)$, its Fourier transform is defined by

$$\mathcal{F}[f](\eta) = \widehat{f}(\eta) = \int_{\mathbb{R}} e^{-\boldsymbol{i}x\eta} f(x)dx$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\widehat{f}](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\boldsymbol{i}x\eta} \widehat{f}(\eta)d\eta,$$

where $\boldsymbol{i}$ is the imaginary unit, $\boldsymbol{i}^2 = -1$.

To guarantee the global stability of traveling waves of (1), we need the following additional assumptions.

(H3): $|g'(u)| \leq g'(0)$ and $|h'(v)| \leq h'(0)$ for $u, v \in [0, +\infty)$.
(H4): $d_2 > d_1$, $\alpha > \beta$, $d_2 - d_1 < \frac{\alpha - \beta}{2}$ and $\max\{h'(0), g'(0)\} > \beta$.
(H5): The initial data $(v_{10}(x, s), v_{20}(x, s)) \geq (0, 0)$ satisfies

$$\lim_{x \to \pm\infty} (v_{10}(x, s), v_{20}(x, s)) = (v_{1\pm}, v_{2\pm}) \text{ uniformly in } \ s \in [-\tau, 0].$$

Consider the following function

$$\mathcal{P}_2(\lambda, c) = d_2(e^{\lambda} + e^{-\lambda} - 2) - c\lambda - \beta + \max\{h'(0), g'(0)\}e^{-\lambda c\tau}.$$

Since $\max\{h'(0), g'(0)\} > \beta$, it then follows from [20, Lemma 2.1] that there exists $\lambda^* > 0$ and $c^* > 0$, such that $\mathcal{P}_2(\lambda^*, c^*) = 0$ and $\frac{\partial \mathcal{P}_2(\lambda,c)}{\partial \lambda}|_{(\lambda^*,c^*)} = 0$. When $c > c^*$, the equation $\mathcal{P}_2(\lambda, c) = 0$ has two positive real roots $\lambda_1^{\natural}(c)$ and $\lambda_2^{\natural}(c)$ with $0 < \lambda_1^{\natural}(c) < \lambda^* < \lambda_2^{\natural}(c)$. When $\lambda \in (\lambda_1^{\natural}(c), \lambda_2^{\natural}(c))$, $\mathcal{P}_2(\lambda, c) < 0$. Moreover, $(\lambda_1^{\natural})'(c) < 0$ and $(\lambda_2^{\natural})'(c) > 0$.

We select the weight function $w(\xi) > 0$ as the form

$$w(\xi) = e^{-2\lambda\xi},$$

where $\lambda > 0$ satisfies $\lambda_1^{\natural}(c) < \lambda < \lambda_2^{\natural}(c)$. Now we are ready to present the main result of this paper.

Theorem 2.2 (Global stability of traveling waves)**.** *Assume that* (H1)*,* (H3)-(H5) *hold. For any given traveling wave* $(\phi_1(x+ct), \phi_2(x+ct))$ *of* (1) *with speed* $c > \max\{c_*, c^*\}$ *connecting* $(0,0)$ *and* (K_1, K_2)*, whether it is monotone or non-monotone, if the initial data satisfy*

$$v_{i0}(x, s) - \phi_i(x + cs) \in C_{unif}[-\tau, 0] \cap C([-\tau, 0]; W_w^{1,1}(\mathbb{R})),\ \ i = 1, 2,$$
$$\partial_s(v_{i0} - \phi_i) \in L^1([-\tau, 0]; L_w^1(\mathbb{R})),\ \ i = 1, 2,$$

then there exists $\tau_0 > 0$ *such that for any* $\tau \le \tau_0$*, the solution* $(v_1(x,t), v_2(x,t))$ *of* (1)-(2) *converges to the traveling wave* $(\phi_1(x+ct), \phi_2(x+ct))$ *as follows:*

$$\sup_{x\in\mathbb{R}} |v_i(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}, \quad t > 0,$$

where C *and* μ *are two positive constants, and* $C_{unif}[r, T]$ *is the uniformly continuous space, for* $0 < T \le \infty$*, defined by*

$$\begin{aligned} &C_{unif}[r, T] \\ =&\big\{u \in C([r, T] \times \mathbb{R}) \text{ such that } \lim_{x\to+\infty} v(x,t) \text{ exists uniformly in } t \in [r, T]\big\}. \end{aligned}$$

3. **Global stability of traveling waves.** This section is devoted to proving the stability theorem, i.e., Theorem 2.2. Let $(\phi_1(x+ct), \phi_2(x+ct)) = (\phi_1(\xi), \phi_2(\xi))$ be a given traveling wave solution with speed $c \ge c_*$ and define

$$\begin{cases} V_i(\xi, t) := v_i(x,t) - \phi_i(x+ct) = v_i(\xi - ct, t) - \phi_i(\xi),\ \ i = 1, 2, \\ V_{i0}(\xi, s) := v_{i0}(x, s) - \phi_i(x+cs) = v_{i0}(\xi - cs, s) - \phi(\xi),\ \ i = 1, 2. \end{cases}$$

Then it follows from (1) and (3) that $V_i(\xi, t)$ satisfies

$$\begin{cases} V_{1t} + cV_{1\xi} - d_1\mathcal{D}[V_1] + \alpha V_1 = Q_1(V_2(\xi - c\tau, t - \tau)), \\ V_{2t} + cV_{2\xi} - d_2\mathcal{D}[V_2] + \beta V_2 = Q_2(V_1(\xi - c\tau, t - \tau)), \\ V_i(\xi, s) = V_{i0}(\xi, s),\ \ (\xi, s) \in \mathbb{R} \times [-\tau, 0],\ \ i = 1, 2. \end{cases} \tag{10}$$

The nonlinear terms Q_1 and Q_2 are given by

$$\begin{cases} Q_1(V_2) := h(\phi_2 + V_2) - h(\phi_2) = h'(\tilde{\phi}_2)V_2, \\ Q_2(V_1) := g(\phi_1 + V_1) - g(\phi_1) = g'(\tilde{\phi}_1)V_1, \end{cases} \tag{11}$$

for some $\tilde{\phi}_i$ between ϕ_i and $\phi_i + V_i$, with $\phi_i = \phi_i(\xi - c\tau_i)$ and $V_i = V_i(\xi - c\tau_i, t - \tau_i)$.

We first prove the existence and uniqueness of solution $(V_1(\xi,t),V_2(\xi,t))$ to the initial value problem (10) in the uniformly continuous space $C_{unif}[-\tau,+\infty)\times C_{unif}[-\tau,+\infty)$.

Lemma 3.1. *Assume that* (H1) and (H3) *hold. If the initial perturbation* $(V_{10},V_{20}) \in C_{unif}[-\tau,0]\times C_{unif}[-\tau,0]$ *for* $c\ge c_*$*, then the solution* (V_1,V_2) *of the perturbed equation* (10) *is unique and time-globally exists in* $C_{unif}[-\tau,+\infty)\times C_{unif}[-\tau,+\infty)$.

Proof. Let $U_i(x,t)=v_i(x,t)-\phi_i(x+ct)$, $i=1,2$. It is clear that $U_i(x,t)=V_i(\xi,t)$, $i=1,2$, and satisfies

$$\begin{cases} U_{1t}-d_1\mathcal{D}[U_1]+\alpha U_1=Q_1(U_2(x,t-\tau)),\\ U_{2t}-d_2\mathcal{D}[U_2]+\beta U_2=Q_2(U_1(x,t-\tau)),\\ U_i(x,s)=v_{i0}(x,s)-\phi_i(x+cs):=U_{i0}(x,s),\ (x,s)\in\mathbb{R}\times[-\tau,0],\ i=1,2. \end{cases} \tag{12}$$

Thus, the global existence and uniqueness of solution of (10) are transformed into that of (12).

When $t\in[0,\tau]$, we have $t-\tau\in[-\tau,0]$ and $U_i(x,t-\tau)=U_{i0}(x,t-\tau)$, $i=1,2$, which imply that (12) is linear. Thus, the solution of (12) can be explicitly and uniquely solved by

$$\begin{cases} U_1(x,t)=e^{-(2d_1+\alpha)t}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_1t)U_{10}(x-m,0)\\ \qquad\qquad +\sum_{m=-\infty}^{\infty}\int_0^t e^{-(2d_1+\alpha)(t-s)}\mathbf{I}_m(2d_1(t-s))Q_1(U_{20}(x-m,s-\tau))ds,\\ U_2(x,t)=e^{-(2d_2+\beta)t}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_2t)U_{20}(x-m,0)\\ \qquad\qquad +\sum_{m=-\infty}^{\infty}\int_0^t e^{-(2d_2+\beta)(t-s)}\mathbf{I}_m(2d_2(t-s))Q_2(U_{10}(x-m,s-\tau))ds \end{cases} \tag{13}$$

for $t\in[0,\tau]$.

Since $V_{i0}(\xi,t)\in C_{unif}[-\tau,0]$, $i=1,2$, namely, $\lim_{\xi\to+\infty}V_{i0}(\xi,t)$ exist uniformly in $t\in[-\tau,0]$, which implies $\lim_{x\to+\infty}U_{i0}(x,t)$ exist uniformly in $t\in[-\tau,0]$. Denote $U_{i0}(\infty,t)=\lim_{x\to+\infty}U_{i0}(x,t)$, $i=1,2$. Taking the limit $x\to+\infty$ to (13) yields

$$\begin{aligned} &\lim_{x\to+\infty}U_1(x,t)\\ =&e^{-(2d_1+\alpha)t}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_1t)\lim_{x\to+\infty}U_{10}(x-m,0)\\ &+\sum_{m=-\infty}^{\infty}\int_0^t e^{-(2d_1+\alpha)(t-s)}\mathbf{I}_m(2d_1(t-s))\lim_{x\to+\infty}Q_1(U_{20}(x-m,s-\tau))ds\\ =&e^{-\alpha t}U_{10}(\infty,0)+\int_0^t e^{-\alpha(t-s)}Q_1(U_{20}(\infty,s-\tau))\sum_{m=-\infty}^{\infty}e^{-2d_1(t-s)}\mathbf{I}_m(2d_1(t-s))ds\\ =:&\,\mathcal{U}_1(t)\ \text{ uniformly in } t\in[0,\tau] \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\lim_{x\to+\infty}U_2(x,t)\\ =&e^{-(2d_2+\beta)t}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_2t)\lim_{x\to+\infty}U_{20}(x-m,0) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=-\infty}^{\infty}\int_0^t e^{-(2d_2+\beta)(t-s)}\mathbf{I}_m(2d_2(t-s))\lim_{x\to+\infty}Q_2(U_{10}(x-m,s-\tau))ds\\
=&e^{-\beta t}U_{20}(\infty,0)+\int_0^t e^{-\beta(t-s)}Q_2(U_{10}(\infty,s-\tau))\sum_{m=-\infty}^{\infty}e^{-2d_2(t-s)}\mathbf{I}_m(2d_2(t-s))ds\\
=:&\,\mathcal{U}_2(t)\quad\text{uniformly in } t\in[0,\tau],
\end{aligned}
\tag{15}
$$

where we have used (8). Thus, we obtain that $(U_1,U_2)\in C_{unif}[-\tau,\tau)\times C_{unif}[-\tau,\tau)$.

When $t\in[\tau,2\tau]$, system (12) with the initial data $U_i(x,s)$ for $s\in[0,\tau]$ is still linear, because the source term $Q_1(U_2(x,t-\tau))$ and $Q_2(U_1(x,t-\tau))$ is known due to $t-\tau\in[0,\tau]$ and $U_i(s,t-\tau)$ is solved in (13). Hence, the solution $U_i(x,t)$ for $t\in[\tau,2\tau]$ is uniquely and explicitly given by

$$
\begin{aligned}
U_1(x,t)=&e^{-(2d_1+\alpha)(t-\tau)}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_1(t-\tau))U_1(x-m,\tau)\\
&+\sum_{m=-\infty}^{\infty}\int_\tau^t e^{-(2d_1+\alpha)(t-s)}\mathbf{I}_m(2d_1(t-s))Q_1(U_2(x-m,s-\tau))ds,\\
U_2(x,t)=&e^{-(2d_2+\beta)(t-\tau)}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_2(t-\tau))U_2(x-m,\tau)\\
&+\sum_{m=-\infty}^{\infty}\int_\tau^t e^{-(2d_2+\beta)(t-s)}\mathbf{I}_m(2d_2(t-s))Q_2(U_1(x-m,s-\tau))ds.
\end{aligned}
$$

Similarly, by (14) and (15), we have

$$
\begin{aligned}
&\lim_{x\to+\infty}U_1(x,t)\\
=&e^{-(2d_1+\alpha)(t-\tau)}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_1(t-\tau))\lim_{x\to+\infty}U_1(x-m,\tau)\\
&+\sum_{m=-\infty}^{\infty}\int_\tau^t e^{-(2d_1+\alpha)(t-s)}\mathbf{I}_m(2d_1(t-s))\lim_{x\to+\infty}Q_1(U_2(x-m,s-\tau))ds\\
=&e^{-\alpha(t-\tau)}\mathcal{U}_1(\tau)+\int_\tau^t e^{-\alpha(t-s)}Q_1(\mathcal{U}_1(s-\tau))\sum_{m=-\infty}^{\infty}e^{-2d_1(t-s)}\mathbf{I}_m(2d_1(t-s))ds\\
=:&\,\bar{\mathcal{U}}_1(t)\quad\text{uniformly in } t\in[\tau,2\tau],
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{x\to+\infty}U_2(x,t)\\
=&e^{-(2d_2+\beta)(t-\tau)}\sum_{m=-\infty}^{\infty}\mathbf{I}_m(2d_2(t-\tau))\lim_{x\to+\infty}U_2(x-m,\tau)\\
&+\sum_{m=-\infty}^{\infty}\int_\tau^t e^{-(2d_2+\beta)(t-s)}\mathbf{I}_m(2d_2(t-s))\lim_{x\to+\infty}Q_2(U_1(x-m,s-\tau))ds\\
=&e^{-\beta(t-\tau)}\mathcal{U}_2(\tau)+\int_\tau^t e^{-\beta(t-s)}Q_2(\mathcal{U}_2(s-\tau))\sum_{m=-\infty}^{\infty}e^{-2d_2(t-s)}\mathbf{I}_m(2d_2(t-s))ds
\end{aligned}
$$

$$=: \bar{\mathcal{U}}_2(t) \;\text{ uniformly in } t \in [\tau, 2\tau].$$

By repeating this procedure for $t \in [n\tau, (n+1)\tau]$ with $n \in \mathbb{Z}_+$, we prove that there exists a unique solution $(V_1, V_2) \in C_{unif}[-\tau, (n+1)\tau] \times C_{unif}[-\tau, (n+1)\tau]$ for (10), and step by step, we finally prove the uniqueness and time-global existence of the solution $(V_1, V_2) \in C_{unif}[-\tau, \infty) \times C_{unif}[-\tau, \infty)$ for (10). The proof is complete. □

Now we state the stability result for the perturbed system (10), which automatically implies Theorem 2.2.

Proposition 2. *Assume that* (H1), (H3)-(H5) *hold. If*

$$V_{i0} \in C_{unif}[-\tau, 0] \cap C([-\tau, 0]; W_w^{1,1}(\mathbb{R})),\ i = 1, 2,$$

and

$$\partial_s V_{i0} \in L^1([-\tau, 0]; L_w^1(\mathbb{R})),\ i = 1, 2,$$

then there exists $\tau_0 > 0$ such that for any $\tau \le \tau_0$, when $c > \max\{c_*, c^*\}$, it holds

$$\sup_{\xi \in \mathbb{R}} |V_i(\xi, t)| \le Ce^{-\mu t}, \quad t > 0,\ i = 1, 2, \tag{16}$$

for some $\mu > 0$ and $C > 0$.

In order to prove Proposition 2, we first investigate the decay estimate of $V_i(\xi, t)$ at $\xi = +\infty$, $i = 1, 2$.

Lemma 3.2. *Assume that $V_{i0} \in C_{unif}[-\tau, 0]$, $i = 1, 2$. Then, there exist $\tau_0 > 0$ and a large number $x_0 \gg 1$ such that when $\tau \le \tau_0$, the solution $V_i(\xi, t)$ of* (10) *satisfies*

$$\sup_{\xi \in [x_0, +\infty)} |V_i(\xi, t)| \le Ce^{-\mu_1 t},\ t > 0,\ i = 1, 2,$$

for some $\mu_1 > 0$ and $C > 0$.

Proof. Denote

$$z_i^+(t) := V_i(\infty, t),\ z_{i0}^+(s) := V_{i0}(\infty, s),\ s \in [-\tau, 0],\ i = 1, 2.$$

Since $V_{i0} \in C_{unif}[-\tau, 0]$, $i = 1, 2$, by Lemma 3.1, we have $V_i \in C_{unif}[-\tau, +\infty)$, which implies

$$\lim_{\xi \to +\infty} V_i(\xi, t) = z_i^+(t)$$

exists uniformly for $t \in [-\tau, +\infty)$. Taking the limit $\xi \to +\infty$ to (10), we obtain

$$\begin{cases} \frac{dz_1^+}{dt} + \alpha z_1^+ - h'(v_{2+}) z_2^+(t-\tau) = P_1(z_2^+(t-\tau)), \\ \frac{dz_2^+}{dt} + \beta z_2^+ - g'(v_{1+}) z_1^+(t-\tau) = P_2(z_1^+(t-\tau)), \\ z_i^+(s) = z_{i0}^+(s),\ s \in [-\tau, 0],\ i = 1, 2, \end{cases}$$

where

$$\begin{cases} P_1(z_2^+) = h(v_{2+} + z_2^+) - h(v_{2+}) - h'(v_{2+}) z_2^+, \\ P_2(z_1^+) = g(v_{1+} + z_1^+) - g(v_{1+}) - g'(v_{1+}) z_1^+. \end{cases}$$

Then by [9, Lemma 3.8], there exist positive constants τ_0, μ_1 and C such that when $\tau \le \tau_0$,

$$|V_i(\infty, t)| = |z_i^+(t)| \le Ce^{-\mu_1 t},\ t > 0,\ i = 1, 2, \tag{17}$$

provided that $|z_{i0}^{+}| \ll 1$, $i = 1, 2$.

By the continuity and the uniform convergence of $V_i(\xi, t)$ as $\xi \to +\infty$, there exists a large $x_0 \gg 1$ such that (17) implies

$$\sup_{\xi \in [x_0, +\infty)} |V_i(\xi, t)| \le Ce^{-\mu_1 t}, \ t > 0, \ i = 1, 2,$$

provided that $\sup_{\xi \in [x_0, +\infty)} |V_{i0}(\xi, s)| \ll 1$ for $s \in [-\tau, 0]$. Such a smallness for the initial perturbation (V_{10}, V_{20}) near $\xi \to +\infty$ can be easily verified, since

$$\lim_{x \to +\infty} (v_{10}(x, s), v_{20}(x, s)) = (K_1, K_2) \text{ uniformly in } s \in [-\tau, 0],$$

which implies

$$\lim_{\xi \to +\infty} V_{i0}(\xi, s) = \lim_{\xi \to +\infty} [v_{i0}(\xi, s) - \phi_i(\xi)] = K_i - K_i = 0$$

uniformly for $s \in [-\tau, 0]$, $i = 1, 2$. The proof is complete. □

Next we are going to establish the a priori decay estimate of $\sup_{\xi \in (-\infty, x_0]} |V_i(\xi, t)|$ by using the anti-weighted technique [3] together with the Fourier transform. First of all, we shift $V_i(\xi, t)$ to $V_i(\xi + x_0, t)$ by the constant x_0 given in Lemma 3.2, and then introduce the following transformation

$$\widetilde{V}_i(\xi, t) = \sqrt{w(\xi)} V_i(\xi + x_0, t) = e^{-\lambda \xi} V_i(\xi + x_0, t), \ i = 1, 2.$$

Substituting $V_i = w^{-1/2} \widetilde{V}_i$ to (10) yields

$$\begin{cases} \widetilde{V}_{1t} + c\widetilde{V}_{1\xi} + c_1 \widetilde{V}_1(\xi, t) - d_1 e^{\lambda} \widetilde{V}_1(\xi + 1, t) - d_1 e^{-\lambda} \widetilde{V}_1(\xi - 1, t) \\ = \widetilde{Q}_1(\widetilde{V}_2(\xi - c\tau, t - \tau)), \\ \widetilde{V}_{2t} + c\widetilde{V}_{2\xi} + c_2 \widetilde{V}_2(\xi, t) - d_2 e^{\lambda} \widetilde{V}_2(\xi + 1, t) - d_2 e^{-\lambda} \widetilde{V}_2(\xi - 1, t) \\ = \widetilde{Q}_2(\widetilde{V}_1(\xi - c\tau, t - \tau)), \\ \widetilde{V}_i(\xi, s) = \sqrt{w(\xi)} V_{i0}(\xi + x_0, s) =: \widetilde{V}_{i0}(\xi, s), \ \xi \in \mathbb{R}, s \in [-\tau, 0], \ i = 1, 2, \end{cases} \tag{18}$$

where

$$c_1 = c\lambda + 2d_1 + \alpha, \quad c_2 = c\lambda + 2d_2 + \beta$$

and

$$\widetilde{Q}_1(\widetilde{V}_2) = e^{-\lambda \xi} Q_1(V_2), \quad \widetilde{Q}_2(\widetilde{V}_1) = e^{-\lambda \xi} Q_2(V_1).$$

By (11), $\tilde{Q}_1(\tilde{V}_2)$ satisfies

$$\begin{aligned} \widetilde{Q}_1(\widetilde{V}_2(\xi - c\tau, t - \tau)) =& e^{-\lambda \xi} Q_1(V_2(\xi - c\tau + x_0, t - \tau)) \\ =& e^{-\lambda \xi} h'(\tilde{\phi}_2) V_2(\xi - c\tau + x_0, t - \tau) \\ =& e^{-\lambda c\tau} h'(\tilde{\phi}_2) \widetilde{V}_2(\xi - c\tau, t - \tau) \end{aligned} \tag{19}$$

and $\widetilde{Q}_2(\widetilde{V}_1)$ satisfies

$$\widetilde{Q}_2(\widetilde{V}_1(\xi - c\tau, t - \tau)) = e^{-\lambda c\tau} g'(\tilde{\phi}_1) \widetilde{V}_1(\xi - c\tau, t - \tau). \tag{20}$$

By (H3), we further obtain

$$\begin{aligned} |\widetilde{Q}_1(\widetilde{V}_2(\xi - c\tau, t - \tau))| &\le h'(0) e^{-\lambda c\tau} |\widetilde{V}_2(\xi - c\tau, t - \tau)|, \\ |\widetilde{Q}_2(\widetilde{V}_1(\xi - c\tau, t - \tau))| &\le g'(0) e^{-\lambda c\tau} |\widetilde{V}_1(\xi - c\tau, t - \tau)|. \end{aligned}$$

Taking (19) and (20) into (18), one can see that the coefficients $h'(\tilde{\phi}_2)$ and $g'(\tilde{\phi}_1)$ on the right side of (18) are variable and can be negative. Thus, the classical methods, such as the monotone technique and the Fourier transform cannot be applied directly to establish the decay estimate for $(\widetilde{V}_1, \widetilde{V}_2)$. Motivated by [15, 28, 17, 23], we introduce a new method which can be described as follows.

∘ By replacing $h'(\tilde{\phi}_2)$ in the first equation of (18) with a constant $h'(0)$, and $g'(\tilde{\phi}_1)$ in the second equation of (18) with a constant $g'(0)$, we can obtain a linear delayed reaction-diffusion system

$$\begin{cases} V_{1t}^+ + cV_{1\xi}^+ + c_1V_1^+(\xi,t) - d_1e^{\lambda}V_1^+(\xi+1,t) - d_1e^{-\lambda}V_1^+(\xi-1,t) \\ = h'(0)e^{-\lambda c\tau}V_2^+(\xi - c\tau, t-\tau), \\ V_{2t}^+ + cV_{2\xi}^+ + c_2V_2^+(\xi,t) - d_2e^{\lambda}V_2^+(\xi+1,t) - d_2e^{-\lambda}V_2^+(\xi-1,t) \\ = g'(0)e^{-\lambda c\tau}V_1^+(\xi - c\tau, t-\tau), \end{cases} \tag{21}$$

with

$$V_i^+(\xi, s) = \sqrt{w(\xi)}V_{i0}(\xi + x_0, s) =: V_{i0}^+(\xi, s), \ i = 1, 2,$$

where $\xi \in \mathbb{R}$, $t \in (0, +\infty]$ and $s \in [-\tau, 0]$. Then we investigate the decay estimate of (V_1^+, V_2^+) by applying the Fourier transform to (21);

∘ We prove that the solution $(\widetilde{V}_1, \widetilde{V}_2)$ of (18) can be bounded by the solution (V_1^+, V_2^+) of (21).

Now we are in a position to derive the decay estimate of (V_1^+, V_2^+) for the linear system (21). We first recall some properties of the solutions to the delayed ODE system.

Lemma 3.3. ([11, Lemma 3.1]) *Let $z(t)$ be the solution to the following scalar differential equation with delay*

$$\begin{cases} \frac{d}{dt}z(t) = Az(t) + Bz(t-\tau), \quad t \geq 0, \tau > 0, \\ z(s) = z_0(s), \quad s \in [-\tau, 0]. \end{cases} \tag{22}$$

where $A, B \in \mathbb{C}^{N\times N}$, $N \geq 2$, and $z_0(s) \in C^1([-\tau, 0], \mathbb{C}^N)$. Then

$$z(t) = e^{A(t+\tau)}e_\tau^{B_1t}z_0(-\tau) + \int_{-\tau}^0 e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}[z_0'(s) - Az_0(s)]ds,$$

where $B_1 = Be^{-A\tau}$ and $e_\tau^{B_1t}$ is the so-called delayed exponential function in the form

$$e_\tau^{B_1t} = \begin{cases} 0, & -\infty < t < -\tau, \\ I, & -\tau \leq t < 0, \\ I + B_1\frac{t}{1!}, & 0 \leq t < \tau, \\ I + B_1\frac{t}{1!} + B_1^2\frac{(t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ I + B_1\frac{t}{1!} + B_1^2\frac{(t-\tau)^2}{2!} + \cdots + B_1^m\frac{[t-(m-1)\tau]^m}{m!}, & (m-1)\tau \leq t < m\tau, \\ \vdots & \vdots \end{cases}$$

where $0, I \in \mathbb{C}^{N\times N}$, and 0 is zero matrix and I is unit matrix.

Lemma 3.4. ([11, Theorem 3.1]) *Suppose $\mu(A) := \frac{\mu_1(A)+\mu_\infty(A)}{2} < 0$, where $\mu_1(A)$ and $\mu_\infty(A)$ denote the matrix measure of A induced by the matrix 1-norm $\|\cdot\|_1$ and ∞-norm $\|\cdot\|_\infty$, respectively. If $\nu(B) := \frac{\|B\|+\|B\|_\infty}{2} \leq -\mu(A)$, then there exists a* *decreasing function $\varepsilon_\tau = \varepsilon(\tau) \in (0,1)$ for $\tau > 0$ such that any solution of system (22) satisfies*

$$\|z(t)\| \le C_0 e^{-\varepsilon_\tau \sigma t}, \quad t > 0,$$

where C_0 is a positive constant depending on initial data $z_0(s), s \in [-\tau, 0]$ and $\sigma = |\mu(A)| - \nu(B)$. In particular,

$$\|e^{At} e_\tau^{B_1 t}\| \le C_0 e^{-\varepsilon_\tau \sigma t}, \quad t > 0,$$

where $e_\tau^{B_1 t}$ is defined in Lemma 3.3.

From the proof of [11, Theome 3.1], one can see that

$$\mu_1(A) = \lim_{\theta \to 0^+} \frac{\|I + \theta A\| - 1}{\theta} = \max_{1 \le j \le N} \left[Re(a_{jj}) + \sum_{j \ne i}^{N} |a_{ij}| \right]$$

and

$$\mu_\infty(A) = \lim_{\theta \to 0^+} \frac{\|I + \theta A\|_\infty - 1}{\theta} = \max_{1 \le i \le N} \left[Re(a_{ii}) + \sum_{i \ne j}^{N} |a_{ij}| \right].$$

Taking the Fourier transform to (21) and denoting the Fourier transform of $V^+(\xi,t) := (V_1^+(\xi,t), V_2^+(\xi,t))^T$ by $\hat{V}^+(\eta,t) := (\hat{V}_1^+(\eta,t), \hat{V}_2^+(\eta,t))^T$, we obtain

$$\begin{cases} \frac{\partial}{\partial t}\hat{V}_1^+(\eta,t) = \left(-c_1 + d_1(e^{\lambda + \boldsymbol{i}\eta} + e^{-(\lambda+\boldsymbol{i}\eta)}) - \boldsymbol{i}c\eta\right)\hat{V}_1^+(\eta,t) \\ \qquad\qquad + h'(0)e^{-c\tau(\lambda+\boldsymbol{i}\eta)}\hat{V}_2^+(\eta, t-\tau), \\ \frac{\partial}{\partial t}\hat{V}_2^+(\eta,t) = \left(-c_2 + d_2(e^{\lambda + \boldsymbol{i}\eta} + e^{-(\lambda+\boldsymbol{i}\eta)}) - \boldsymbol{i}c\eta\right)\hat{V}_2^+(\eta,t) \\ \qquad\qquad + g'(0)e^{-c\tau(\lambda+\boldsymbol{i}\eta)}\hat{V}_1^+(\eta, t-\tau), \\ \hat{V}_i^+(\eta,s) = \hat{V}_{i0}^+(\eta,s), \ \eta \in \mathbb{R}, \ s \in [-\tau, 0], \ i = 1,2. \end{cases} \tag{23}$$

Let

$$A(\eta) = \begin{pmatrix} -c_1 + d_1(e^{\lambda+\boldsymbol{i}\eta} + e^{-(\lambda+\boldsymbol{i}\eta)}) - \boldsymbol{i}c\eta & 0 \\ 0 & -c_2 + d_2(e^{\lambda+\boldsymbol{i}\eta} + e^{-(\lambda+\boldsymbol{i}\eta)}) - \boldsymbol{i}c\eta \end{pmatrix}$$

and

$$B(\eta) = \begin{pmatrix} 0 & h'(0)e^{-c\tau(\lambda+\boldsymbol{i}\eta)} \\ g'(0)e^{-c\tau(\lambda+\boldsymbol{i}\eta)} & 0 \end{pmatrix}.$$

Then system (23) can be rewritten as

$$\hat{V}_t^+(\eta,t) = A(\eta)\hat{V}^+(\eta,t) + B(\eta)\hat{V}^+(\eta, t-\tau). \tag{24}$$

By Lemma 3.3, the linear delayed system (24) can be solved by

$$\begin{aligned} \hat{V}^+(\eta,t) =& e^{A(\eta)(t+\tau)} e_\tau^{B_1(\eta)t} \hat{V}_0^+(\eta, -\tau) \\ &+ \int_{-\tau}^0 e^{A(\eta)(t-s)} e_\tau^{B_1(\eta)(t-s-\tau)} \left[\partial_s \hat{V}_0^+(\eta,s) - A(\eta)\hat{V}_0^+(\eta,s)\right] ds \\ :=& I_1(\eta,t) + \int_{-\tau}^0 I_2(\eta, t-s) ds, \end{aligned} \tag{25}$$

where $B_1(\eta) = B(\eta)e^{A(\eta)\tau}$. Then by taking the inverse Fourier transform to (25), one has

$$\begin{aligned} &V^+(\xi,t) \quad (26)\\ &= \mathcal{F}^{-1}[I_1](\xi,t) + \int_{-\tau}^{0} \mathcal{F}^{-1}[I_2](\xi,t-s)ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\boldsymbol{i}\xi\eta} e^{A(\eta)(t+\tau)} e_{\tau}^{B_1(\eta)t}\hat{V}_0^+(\eta,-\tau)d\eta \\ &\quad + \frac{1}{2\pi}\int_{-\tau}^{0}\int_{-\infty}^{\infty} e^{\boldsymbol{i}\xi\eta} e^{A(\eta)(t-s)} e_{\tau}^{B_1(\eta)(t-s-\tau)} \left[\partial_s \hat{V}_0^+(\eta,s) - A(\eta)\hat{V}_0^+(\eta,s)\right] d\eta ds. \end{aligned} \quad (27)$$

Lemma 3.5. *Let the initial data $V_{i0}^+(\xi,s)$, $i=1,2$, be such that*

$$V_{i0}^+ \in C([-\tau,0];W^{1,1}(\mathbb{R})),\ \partial_s V_{i0}^+ \in L^1([-\tau,0];L^1(\mathbb{R})),\ i=1,2.$$

Then

$$\|V_i^+(t)\|_{L^\infty(\mathbb{R})} \le Ce^{-\mu_2 t} \ \textit{for}\ c \ge \max\{c_*,c^*\},\ i=1,2,$$

where $\mu_2 > 0$ and $C > 0$.

Proof. According to (26), we shall estimate $\mathcal{F}^{-1}[I_1](\xi,t)$ and $\int_{-\tau}^0 \mathcal{F}^{-1}[I_2](\xi,t-s)ds$, respectively. By the definition of $\mu(\cdot)$ and $\nu(\cdot)$, we have

$$\begin{aligned} \mu(A(\eta)) =& \frac{\mu_1(A(\eta)) + \mu_\infty(A(\eta))}{2} \\ =& \max\left\{-c_1 + d_1(e^\lambda \cos\eta + e^{-\lambda}\cos\eta), -c_2 + d_2(e^\lambda\cos\eta + e^{-\lambda}\cos\eta)\right\} \\ =& -c_2 + d_2(e^\lambda\cos\eta + e^{-\lambda}\cos\eta) \\ =& -c_2 + d_2(e^\lambda + e^{-\lambda})\cos\eta \\ =& -c\lambda + d_2(e^\lambda + e^{-\lambda} - 2) - \beta - m(\eta), \end{aligned}$$

where $c_2 = c\lambda + 2d_2 + \beta$ and

$$m(\eta) = d_2(1-\cos\eta)(e^\lambda + e^{-\lambda}) \ge 0,$$

since $d_2 > d_1$, $\alpha > \beta$ and $d_2 - d_1 < \frac{\alpha-\beta}{2}$, and

$$\nu(B(\eta)) = \max\{h'(0), g'(0)\}e^{-\lambda c\tau}.$$

By considering $\lambda \in (\lambda_1^\natural(c), \lambda_2^\natural(c))$, we get $\mu(A(\eta)) < 0$ and

$$\mu(A(\eta)) + \nu(B(\eta)) = -c\lambda + d_2(e^\lambda + e^{-\lambda} - 2) - \beta - m(\eta) + \max\{h'(0), g'(0)\}e^{-\lambda c\tau} < 0.$$

Furthermore, we obtain

$$\begin{aligned} |\mu(A(\eta))| - \nu(B(\eta)) =& c\lambda - d_2(e^\lambda + e^{-\lambda} - 2) + \beta + m(\eta) - \max\{h'(0), g'(0)\}e^{-\lambda c\tau} \\ =& -\mathcal{P}_2(\lambda,c) + m(\eta), \end{aligned}$$

where $\mathcal{P}_2(\lambda,c) = d_2(e^\lambda + e^{-\lambda} - 2) - c\lambda - \beta + \max\{h'(0), g'(0)\}e^{-\lambda c\tau} < 0$ for $c > \max\{c_*, c^*\}$. It then follows from Lemma 3.4 that there exists a decreasing function $\varepsilon_\tau = \varepsilon(\tau) \in (0,1)$ such that

$$\|e^{A(\eta)(t+\tau)}e^{B_1(\eta)t}\| \le C_1 e^{-\varepsilon_\tau(|\mu(A(\eta))| - \nu(B(\eta)))t} \le C_1 e^{-\varepsilon_\tau \mu_0 t} e^{-\varepsilon_\tau m(\eta)t}, \quad (28)$$

where C_1 is a positive constant and $\mu_0 := -\mathcal{P}_2(\lambda, c) > 0$ with $c > c^*$. By the definition of Fourier's transform, we have

$$\sup_{\eta\in\mathbb{R}} \|\hat{V}_0^+(\eta, -\tau)\| \le \int_{\mathbb{R}} \|V_0^+(\xi, -\tau)\| d\xi = \sum_{i=1}^{2} \|V_{i0}^+(\cdot, -\tau)\|_{L^1(\mathbb{R})}.$$

Applying (28), we derive

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[I_1](\xi, t)\| =& \sup_{\xi\in\mathbb{R}} \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\boldsymbol{i}\xi\eta} e^{A(\eta)(t+\tau)} e^{B_1(\eta)t} \hat{V}_0^+(\eta, -\tau) d\eta \right\| \\
\le& C \int_{-\infty}^{\infty} e^{-\varepsilon_\tau m(\eta) t} e^{-\varepsilon_\tau \mu_0 t} \|\hat{V}_0^+(\eta, -\tau)\| d\eta \\
\le& C e^{-\varepsilon_\tau \mu_0 t} \sup_{\eta\in\mathbb{R}} \|\hat{V}_0^+(\eta, -\tau)\| \int_{-\infty}^{\infty} e^{-\varepsilon_\tau m(\eta) t} d\eta \\
\le& C e^{-\mu_2 t} \sum_{i=1}^{2} \|V_{i0}^+(\cdot, -\tau)\|_{L^1(\mathbb{R})}, \qquad (29)
\end{aligned}$$

with $\mu_2 := \varepsilon_\tau \mu_0$.

Note that

$$\sup_{\eta\in\mathbb{R}} \|A(\eta)\hat{V}_0^+(\eta, s)\| \le C \sum_{i=1}^{2} \|V_{i0}^+(\cdot, s)\|_{W^{1,1}(\mathbb{R})}.$$

Similarly, we can obtain

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[I_2](\xi, t-s)\| \\
&= \sup_{\xi\in\mathbb{R}} \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\boldsymbol{i}\xi\eta} e^{A(\eta)(t-s)} e^{B_1(\eta)(t-s-\tau)} \left[\partial_s \hat{V}_0^+(\eta, s) - A(\eta)\hat{V}_0^+(\eta, s)\right] d\eta \right\| \\
&\le C \int_{-\infty}^{\infty} e^{-\varepsilon_\tau m(\eta)(t-s)} e^{-\varepsilon_\tau \mu_0 (t-s)} \left\| \partial_s \hat{V}_0^+(\eta, s) - A(\eta)\hat{V}_0^+(\eta, s) \right\| d\eta \\
&\le C e^{-\varepsilon_\tau \mu_0 t} e^{\varepsilon_\tau \mu_0 s} \sup_{\eta\in\mathbb{R}} \left\| \partial_s \hat{V}_0^+(\eta, s) - A(\eta)\hat{V}_0^+(\eta, s) \right\| \int_{-\infty}^{\infty} e^{-\varepsilon_\tau m(\eta)(t-s)} d\eta.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\int_{-\tau}^{0} \sup_{\xi\in\mathbb{R}} \|\mathcal{F}^{-1}[I_2](\xi, t-s)\| ds \\
&\le C e^{-\varepsilon_\tau \mu_0 t} \int_{-\tau}^{0} e^{\varepsilon_\tau \mu_0 s} \sup_{\eta\in\mathbb{R}} \left\| \partial_s \hat{V}_0^+(\eta, s) - A(\eta)\hat{V}_0^+(\eta, s) \right\| \int_{-\infty}^{\infty} e^{-\varepsilon_\tau m(\eta)(t-s)} d\eta ds \\
&\le C e^{-\varepsilon_\tau \mu_0 t} \int_{-\tau}^{0} \|\partial_s V_0^+(\cdot, s)\|_{L^1(\mathbb{R})} + \|V_0^+(\cdot, s)\|_{W^{1,1}(\mathbb{R})} ds \\
&\le C e^{-\varepsilon_\tau \mu_0 t} \left(\|\partial_s V_0^+(s)\|_{L^1([-\tau,0];L^1(\mathbb{R}))} + \|V_0^+(s)\|_{L^1([-\tau,0];W^{1,1}(\mathbb{R}))} \right). \qquad (30)
\end{aligned}$$

Substituting (29) and (30) to (26), we obtain the following the decay rate

$$\sum_{i=1}^{2} \|V_i^+(t)\|_{L^\infty(\mathbb{R})} \le C e^{-\mu_2 t}.$$

This proof is complete. □

The following maximum principle is needed to obtain the crucial boundedness estimate of $(\widetilde{V}_1, \widetilde{V}_2)$, which has been proved in [17, Lemma 3.4].

Lemma 3.6. *Let $T > 0$. For any $a_1, a_2 \in \mathbb{R}$ and $\nu > 0$, if the bounded function v satisfies*

$$\begin{cases} \frac{\partial v}{\partial t} + a_1 \frac{\partial v}{\partial \xi} + a_2 v - de^{\nu} v(t, \xi + 1) - de^{-\nu} v(t, \xi - 1) \geq 0, \ (t, \xi) \in (0, T] \times \mathbb{R}, \\ v(0, \xi) \geq 0, \quad \xi \in \mathbb{R}, \end{cases} \tag{31}$$

then $v(t, \xi) \geq 0$ for all $(t, \xi) \in (0, T] \times \mathbb{R}$.

Lemma 3.7. *When $(V_{10}^+(\xi, s), V_{20}^+(\xi, s)) \geq (0, 0)$ for $(\xi, s) \in \mathbb{R} \times [-\tau, 0]$, then $(V_1^+(\xi, t), V_2^+(\xi, t)) \geq (0, 0)$ for $(\xi, t) \in \mathbb{R} \times [0, +\infty)$.*

Proof. When $t \in [0, \tau]$, we have $t - \tau \in [-\tau, 0]$ and

$$h'(0) e^{-\lambda c\tau} V_2^+(\xi - c\tau, t - \tau) = h'(0) e^{-\lambda c\tau} V_{20}^+(\xi - c\tau, t - \tau) \geq 0. \tag{32}$$

Applying (32) to the first equation of (21), we get

$$\begin{cases} V_{1t}^+ + cV_{1\xi}^+ + c_1 V_1^+(\xi, t) - d_1 e^{\lambda} V_1^+(\xi + 1, t) - d_1 e^{-\lambda} V_1^+(\xi - 1, t) \\ \geq 0, \ (\xi, t) \in \mathbb{R} \times [0, \tau], \\ V_{10}^+(\xi, s) \geq 0, \ \xi \in \mathbb{R}, \ s \in [-\tau, 0]. \end{cases}$$

By Lemma 3.6, we derive

$$V_1^+(\xi, t) \geq 0, \quad (\xi, t) \in \mathbb{R} \times [0, \tau]. \tag{33}$$

Similarly, we obtain

$$\begin{cases} V_{2t}^+ + cV_{2\xi}^+ + c_2 V_2^+(\xi, t) - d_2 e^{\lambda} V_2^+(\xi + 1, t) - d_2 e^{-\lambda} V_2^+(\xi - 1, t) \\ \geq 0, \ (\xi, t) \in \mathbb{R} \times [0, \tau], \\ V_{20}^+(\xi, s) \geq 0, \ \xi \in \mathbb{R} \ s \in [-\tau, 0]. \end{cases}$$

Using Lemma 3.6 again, we obtain

$$V_2^+(\xi, t) \geq 0, \quad (\xi, t) \in \mathbb{R} \times [0, \tau]. \tag{34}$$

When $t \in [n\tau, (n+1)\tau]$, $n = 1, 2, \cdots$, repeating the above procedure step by step, we can similarly prove

$$(V_1^+(\xi, t), V_2^+(\xi, t)) \geq (0, 0), \quad (\xi, t) \in \mathbb{R} \times [n\tau, (n+1)\tau]. \tag{35}$$

Combining (33), (34) and (35), we obtain $(V_1^+(\xi, t), V_2^+(\xi, t)) \geq (0, 0)$ for $(\xi, t) \in \mathbb{R} \times [0, +\infty)$. The proof is complete. □

Now we establish the following crucial boundedness estimate for $(\widetilde{V}_1, \widetilde{V}_2)$.

Lemma 3.8. *Let $(\widetilde{V}_1(\xi, t), \widetilde{V}_2(\xi, t))$ and $(V_1^+(\xi, t), V_2^+(\xi, t))$ be the solutions of (18) and (21), respectively. When*

$$|\widetilde{V}_{i0}(\xi, s)| \leq V_{i0}^+(\xi, s) \quad \text{for} \quad (\xi, s) \in \mathbb{R} \times [-\tau, 0], \ i = 1, 2, \tag{36}$$

then

$$|\widetilde{V}_i(\xi, t)| \leq V_i^+(\xi, t) \quad \text{for} \quad (\xi, t) \in \mathbb{R} \times [0, +\infty), \ i = 1, 2.$$

Proof. First of all, we prove $|\widetilde{V}_i(\xi,t)| \le V_i^+(\xi,t)$ for $t\in[0,\tau], i=1,2$. In fact, when $t\in[0,\tau]$, namely, $t-\tau\in[-\tau,0]$, it follows from (36) that

$$\begin{aligned}|\widetilde{V}_i(\xi-c\tau,t-\tau)| &= |\widetilde{V}_{i0}(\xi-c\tau,t-\tau)|\\ &\le V_{i0}^+(\xi-c\tau,t-\tau)\\ &= V_i^+(\xi-c\tau,t-\tau) \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau]. \end{aligned} \tag{37}$$

Then by $|h'(\tilde\phi_2)|<h'(0)$ and $|g'(\tilde\phi_1)|<g'(0)$ and (37), we get

$$\begin{aligned}& h'(0)e^{-\lambda c\tau}V_2^+(\xi-c\tau,t-\tau) \pm h'(\tilde\phi_2)e^{-\lambda c\tau}\widetilde{V}_2(\xi-c\tau,t-\tau)\\ &\ge h'(0)e^{-\lambda c\tau}V_2^+(\xi-c\tau,t-\tau) - |h'(\tilde\phi_2)|e^{-\lambda c\tau}|\widetilde{V}_2(\xi-c\tau,t-\tau)|\\ &\ge 0 \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau]\end{aligned} \tag{38}$$

and

$$\begin{aligned}& g'(0)e^{-\lambda c\tau}V_1^+(\xi-c\tau,t-\tau) \pm g'(\tilde\phi_1)e^{-\lambda c\tau}\widetilde{V}_1(\xi-c\tau,t-\tau)\\ &\ge 0 \quad \text{for } (\xi,t)\in\mathbb{R}\times[0,\tau].\end{aligned} \tag{39}$$

Let

$$U_i^-(\xi,t) := V_i^+(\xi,t) - \widetilde{V}_i(\xi,t) \quad \textit{and} \quad U_i^+(\xi,t) := V_i^+(\xi,t) + \widetilde{V}_i(\xi,t), \quad i=1,2.$$

We are going to estimate $U_i^{\pm}(\xi,t)$ respectively.

From (18), (19), (21) and (38), we see that $U_1^-(\xi,t)$ satisfies

$$\begin{cases} U_{1t}^- + cU_{1\xi}^- + c_1U_1^-(\xi,t) - d_1e^{\lambda}U_1^-(\xi+1,t) - d_1e^{-\lambda}U_1^-(\xi-1,t)\\ \ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],\\ U_{10}^-(\xi,s) = V_{10}^+(\xi,s) - \widetilde{V}_{10}(\xi,s) \ge 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0].\end{cases}$$

By Lemma 3.6, we obtain

$$U_1^-(\xi,t)\ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

namely,

$$\widetilde{V}_1(\xi,t) \le V_1^+(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{40}$$

Similarly, one has

$$\begin{cases} U_{2t}^- + cU_{2\xi}^- + c_2U_2^-(\xi,t) - d_2e^{\lambda}U_2^-(\xi+1,t) - d_2e^{-\lambda}U_2^-(\xi-1,t)\\ \ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],\\ U_{20}^-(\xi,s) = V_{20}^+(\xi,s) - \widetilde{V}_{20}(\xi,s) \ge 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0].\end{cases}$$

Applying Lemma 3.6 again, we have

$$U_2^-(\xi,t)\ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

i.e.,

$$\widetilde{V}_2(\xi,t) \le V_2^+(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{41}$$

On the other hand, $U_1^+(\xi,t)$ satisfies

$$\begin{cases} U_{1t}^+ + cU_{1\xi}^+ + c_1U_1^+(\xi,t) - d_1e^{\lambda}U_1^+(\xi+1,t) - d_1e^{-\lambda}U_1^+(\xi-1,t)\\ \ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],\\ U_{10}^-(\xi,s) = V_{10}^+(\xi,s) - \widetilde{V}_{10}(\xi,s) \ge 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0].\end{cases}$$

Then Lemma 3.6 implies that

$$U_1^+(\xi,t) = V_1^+(\xi,t) + \widetilde{V}_1(\xi,t) \ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

that is,

$$-V_1^+(\xi,t) \le \widetilde{V}_1(\xi,t), \quad (\xi,t) \in \mathbb{R}\times[0,\tau]. \tag{42}$$

Similarly, $U_2^+(\xi,t)$ satisfies

$$\begin{cases} U_{2t}^+ + cU_{2\xi}^+ + c_2U_2^+(\xi,t) - d_2e^{\lambda}U_2^+(\xi+1,t) - d_2e^{-\lambda}U_2^+(\xi-1,t) \\ \ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau], \\ U_{20}^-(\xi,s) = V_{20}^+(\xi,s) - \widetilde{V}_{10}(\xi,s) \ge 0, \quad \xi\in\mathbb{R},\ s\in[-\tau,0]. \end{cases}$$

Therefore, we can prove that

$$U_2^+(\xi,t) = V_2^+(\xi,t) + \widetilde{V}_2(\xi,t) \ge 0, \quad (\xi,t)\in\mathbb{R}\times[0,\tau],$$

namely

$$-V_2^+(\xi,t) \le \widetilde{V}_2(\xi,t), \quad (\xi,t) \in \mathbb{R}\times[0,\tau]. \tag{43}$$

Combining (40) and (42), we obtain

$$|\widetilde{V}_1(\xi,t)| \le V_1^+(\xi,t) \quad \text{for} \quad (\xi,t)\in\mathbb{R}\times[0,\tau], \tag{44}$$

and combining (41) and (43), we prove

$$|\widetilde{V}_2(\xi,t)| \le V_2^+(\xi,t) \quad \text{for} \quad (\xi,t)\in\mathbb{R}\times[0,\tau]. \tag{45}$$

Next, when $t\in[\tau,2\tau]$, namely, $t-\tau\in[0,\tau]$, based on (44) and (45), we can similarly prove

$$|\widetilde{V}_i(\xi,t)| \le V_i^+(\xi,t) \quad \text{for} \quad (\xi,t)\in\mathbb{R}\times[\tau,2\tau], \quad i=1,2.$$

Repeating this procedure, we then further prove

$$|\widetilde{V}_i(\xi,t)| \le V_i^+(\xi,t),\ (\xi,t)\in\mathbb{R}\times[n\tau,(n+1)\tau], \quad n=1,2,\cdots,$$

which implies

$$|\widetilde{V}_i(\xi,t)| \le V_i^+(\xi,t) \quad \text{for} \quad (\xi,t)\in\mathbb{R}\times[0,\infty), \quad i=1,2.$$

The proof is complete. □

Let us choose $V_{i0}^+(\xi,s)$ such that

$$V_{i0}^+ \in C([-\tau,0];W^{1,1}(\mathbb{R})), \quad \partial_s V_{i0}^+ \in L^1([-\tau,0];L^1(\mathbb{R})),$$

and

$$V_{i0}^+(\xi,s) \ge |V_{i0}(\xi,s)|, \quad (\xi,s)\in\mathbb{R}\times[-\tau,0],\ i=1,2.$$

Combining Lemmas 3.5 and 3.8, we can get the convergence rates for $\widetilde{V}(\xi,t)$.

Lemma 3.9. *When $\widetilde{V}_{i0}\in C([-\tau,0];W^{1,1}(\mathbb{R}))$ and $\partial_s\widetilde{V}_{i0}\in L^1([-\tau,0];L^1(\mathbb{R}))$, then*

$$\|\widetilde{V}_i(t)\|_{L^\infty(\mathbb{R})} \le Ce^{-\mu_2 t},$$

for some $\mu_2>0$, $i=1,2$.

Lemma 3.10. *It holds that*

$$\sup_{\xi\in(-\infty,x_0]} |V_i(\xi,t)| \le Ce^{-\mu_2 t},\ i=1,2,$$

for some $\mu_2>0$.

Proof. Since $\widetilde{V}_i(\xi,t) = \sqrt{w(\xi)}V_i(\xi+x_0,t) = e^{-\lambda\xi}V_i(\xi+x_0,t)$ and $\sqrt{w(\xi)} = e^{-\lambda\xi} \geq 1$ for $\xi \in (-\infty,0]$, then we obtain

$$\sup_{\xi\in(-\infty,0]} |V_i(\xi+x_0,t)| \leq \|\widetilde{V}_i(t)\|_{L^\infty(\mathbb{R})} \leq Ce^{-\mu_2 t},$$

which implies

$$\sup_{\xi\in(-\infty,x_0]} |V_i(\xi,t)| \leq Ce^{-\mu_2 t}.$$

Thus, the estimate for the unshifted $V(\xi,t)$ is obtained. The proof is complete. □

Proof of Proposition 3.2. By Lemmas 3.2 and 3.10, we immediately obtain (16) for $0 < \mu < \min\{\mu_1,\mu_2\}$. □

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