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THE BREZIS-NIRENBERG TYPE DOUBLE CRITICAL PROBLEM FOR A CLASS OF SCHRÖDINGER-POISSON EQUATIONS

Li Cai and Fubao Zhang*

School of Mathematics, Southeast University
Nanjing 210096, China

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Abstract. In this paper, we study the following Schrödinger-Poisson equations with double critical exponents:

$$\begin{cases} -\Delta u = |u|^4u + \phi|u|^3u + \lambda u, & in\ \Omega, \\ -\Delta\phi = |u|^5, & in\ \Omega, \\ u = \phi = 0, & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary, λ is a real parameter satisfying suitable conditions. Using variational methods, we show the existence and nonexistence of nontrivial solutions for the Schrödinger-Poisson equations.

1. **Introduction.** In this paper, we are concerned with the existence of solutions for the following Schrödinger-Poisson equations

$$\begin{cases} -\Delta u = |u|^4u + \phi|u|^3u + \lambda u, & in\ \Omega, \\ -\Delta\phi = |u|^5, & in\ \Omega, \\ u = \phi = 0, & on\ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary, λ is real parameter. In recent years, many people have studied the Brezis-Nirenberg type problem. For example, the following Brezis-Nirenberg type equation

$$-\Delta u = |u|^{2^*-2}u + \lambda u, \quad in\ \Omega, \tag{2}$$

where $2^* = \frac{2N}{N-2}$ is the critical exponent for the embedding of $H_0^1(\Omega)$ to $L^p(\Omega)$. In [6], H. Brezis and L. Nirenberg showed that equation (2) has a nontrivial solution if $N \geq 4, \lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta$ in the bounded domain Ω. While, if $N = 3$, then there exists a constant $\lambda_* \in (0, \lambda_1)$ such that for any $\lambda \in (\lambda_*, \lambda_1)$, equation (2) with Dirichlet boundary data has a positive solution. Moreover, if Ω is a ball, equation (2) has a positive solution if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$. Other conclusions about equation (2) can be founded in [8, 9]. In [5],

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the following form of Schrödinger-Poisson equations with a critical exponent on a bounded domain were studied :

$$\begin{cases} -\Delta u = q\phi|u|^3 u + \lambda u, & in\ B_R, \\ -\Delta \phi = q|u|^5, & in\ B_R, \\ u = \phi = 0, & on\ \partial B_R, \end{cases} \tag{3}$$

where $B_R(0)$ is a ball centered in 0 in $\mathbb{R}^3$ with radius R. They obtained a positive ground state solution for $\lambda \in (\frac{3}{10}\lambda_1, \lambda_1)$ and $q > 0$. In addition, Qi Zhang [29] studied the following equations

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-\gamma}, & in\ \Omega, \\ -\Delta \phi = u^2, & in\ \Omega, \\ u > 0, & in\ \Omega, \\ u = \phi = 0 & on\ \partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^3$. By using the variational method and the Nehari manifold, the author proved the existence and uniqueness of positive solution for $\mu > 0$, $\eta = 1$ and the existence of two positive solutions for $\mu > 0$, $\eta = -1$ respectively.

Further, Fashun Gao and Minbo Yang [12] considered the following general Choquard equation in a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary,

$$-\Delta u = (I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu - 2}u + \lambda u,\ in\,\Omega. \tag{5}$$

They proved that if $N \geq 4$, then for each $\lambda > 0$ equation (5) has a nontrivial solution in $H_0^1(\Omega)$, and if $N = 3$, then there exists λ_* such that equation (5) has a nontrivial solution for $\lambda > \lambda_*$, where λ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data. In general, equation (5) is related to the following nonlinear Choquard equation

$$-\Delta u + V(x)u = (\frac{1}{|x|^\mu} * |u|^p)|u|^{p-2}u, \quad in\,\mathbb{R}^3. \tag{6}$$

A large number of results have been obtained about the existence and qualitative properties of solutions to equation (6) in recent decades. For example, the existence and uniqueness of the ground states about equation (6) have been obtained in [18]. Later, in [19], Lions proved the existence of a sequence of radially symmetric solutions. The regularity, positivity and radial symmetry of the ground states and decay property at infinity about equation (6) have been studied in [10, 20, 21]. Other results about equation (6) are available in [1, 2, 3].

Also, a great deal of work on elliptic equations can been founded in [4, 11, 13, 14, 15, 16, 17, 23, 25, 28, 30].

Recently, in [7], we studied the Brezis-Nirenberg type double critical problem for the Choquard equation

$$-\Delta u = \alpha|u|^{2^*-2}u + \beta(I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu-2}u + \lambda u, \quad in\ \Omega, \tag{7}$$

where α and β satisfy suitable conditions, we obtained the existence of nontrivial solutions and nonexistence. As far as we know, there are only a few results on the Brezis-Nirenberg type for the Schrödinger-Poisson equations with double critical exponents, the main difficulty is the lack of compactness of (PS) sequence.

In the following, we will give some notations which will be used throughout this paper. Let $L^p(\Omega)(1 \leq p < +\infty)$ be the usual Lebesgue space with the norm $|u|_p = (\int_\Omega |u|^p\,dx)^{\frac{1}{p}}$, $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^1(\mathbb{R}^3)$, where

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}.$$

$\|u\| := (\int_\Omega |\nabla u|^2 \, dx)^{\frac{1}{2}}$ is the norm of $H_0^1(\Omega)$. $D_0^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{1,2}(\mathbb{R}^3)$, where

$$D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with the corresponding norm $(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx)^{\frac{1}{2}}$.

Define

$$S := \inf_{D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^3} |u|^6 \, dx)^{\frac{1}{3}}}.$$

By [26], we have

$$S(\Omega) := \inf_{D_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |u|^6 \, dx)^{\frac{1}{3}}} = S.$$

Firstly, we observe that by the Lax-Milgram theorem , for given $u \in H_0^1(\Omega)$, there exists a unique solution $\phi = \phi_u \in D_0^{1,2}(\Omega)$ satisfying $-\Delta\phi_u = |u|^5$ in a weak sense. By [5, 14], it is easy to obtain the following properties.

Lemma 1.1. *The following properties hold:*

(i) $\phi_u \geq 0$ *for all* $u \in H_0^1(\Omega)$;

(ii) for each $u \in H_0^1(\Omega)$,

$$|\phi_u|_{D^{1,2}} \leq S^{-\frac{1}{2}} |u|_6^5$$

and

$$\int_\Omega \phi_u |u|^5 \leq S^{-1} |u|_6^{10};$$

(iii) if $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $D_0^{1,2}(\Omega)$.

Next we will denote the sequence of eigenvalues of the operator $-\Delta$ on Ω with homogeneous Dirichlet boundary data by

$$0 < \lambda_1 \leq \cdots \leq \lambda_i \leq \lambda_{i+1} \leq \cdots$$

and $\lambda_i \to +\infty$ as $i \to +\infty$. $\{e_i\}_{i\in\mathbb{N}^+} \subset L^\infty(\Omega)$ will be the sequence of eigenfunctions corresponding to $\{\lambda_i\}$. We recall that this sequence is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$. Denote

$$E_{i+1} := \{u \in H_0^1(\Omega) : (u, e_j)_{H_0^1} = 0, \forall j = 1, 2, \cdots, i\}, \tag{8}$$

while $Y_i := span\{e_1, e_2, \cdots, e_i\}$ denote the linear subspace generated by the first i eigenfunctions of $-\Delta$ for any $i \in \mathbb{N}^+$. It is easy to see that Y_i is finite dimensional and $Y_i \oplus E_{i+1} = H_0^1(\Omega)$.

Now we introduce the energy functional associated to (1) by

$$\begin{aligned} I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{6}\int_\Omega |u|^6 \, dx - \frac{1}{10}\int_\Omega \phi_u |u|^5 \, dx \\ - \frac{\lambda}{2}\int_\Omega |u|^2 \, dx. \end{aligned}$$

It is easy to know that I belongs to $C^1(H_0^1(\Omega), \mathbb{R})$. Moreover, u is a weak solution of (1) if and only if u is a critical point of functional I.

Definition 1.2. Let I be a C^1 functional defined on Banach space X, if $I(u_n) \to c$, and $I'(u_n) \to 0$, as $n \to +\infty$, then $\{u_n\}$ is called a Palais-Smale sequence of I at c. If every Palais-Smale sequence at c has a convergent subsequence, then we call that I satisfies the Palais-Smale condition at the level c.

Now we state the main results of this paper.

Theorem 1.3. (i) *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded domain with Lipschitz boundary, if* Ω *contains a ball with radius* R *such that* $\frac{13-\sqrt{5}}{40}\frac{\pi^2}{R^2} < \lambda_1$, *then for* $\lambda \in (\frac{13-\sqrt{5}}{40}\frac{\pi^2}{R^2}, \lambda_1)$, (1) *has a nontrivial solution.*

(ii) *There exists a constant* λ_* *such that* (1) *has a nontrivial solution for* $\lambda > \lambda_* \geq \lambda_1$, *provided* λ *is not an eigenvalue of* $-\Delta$ *with homogeneous Dirichlet boundary data.*

Remark 1. If Ω is the ball in $\mathbb{R}^3$ centered in 0 with radius R, then $\frac{\pi^2}{R^2} = \lambda_1$, the first eigenvalue of $-\Delta$ in $B_R(0)$, thus we obtain that if $\lambda \in (\frac{13-\sqrt{5}}{40}\lambda_1, \lambda_1)$, (1) has a nontrivial solution. Furthermore, if Ω has s a good approximation of a ball, then the condition of (ii) can be satisfied, due to the continuous dependence of λ_1 on Ω.

Theorem 1.4. *If* $\lambda < 0$, Ω *is a smooth and strictly star-shaped domain with respect to the origin in* $\mathbb{R}^3$, *then* (1) *admits no nontrivial solution.*

This paper is organized as follows. In Section 2, we give some preliminaries which can be applied to Theorem 1.3. In Sections 3, 4, we will prove Theorem 1.3. Section 5 is devoted to the proof of Theorem 1.4.

2. **Preliminaries.**

Lemma 2.1. [27] *Let* $N \geq 3$, $q \in (1, +\infty)$ *and* $\{u_n\}$ *be a bounded sequence in* $L^q(\mathbb{R}^N)$. *If* $u_n \to u$ *almost everywhere in* $\mathbb{R}^N$ *as* $n \to \infty$, *then* $u_n \rightharpoonup u$ *weakly in* $L^q(\mathbb{R}^N)$.

Lemma 2.2. *Let* $\lambda > 0$. *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* I, *then* $\{u_n\}$ *is bounded in* $H_0^1(\Omega)$. *Let* $u_0 \in H_0^1(\Omega)$ *be the weak limit of* $\{u_n\}$, *then* u_0 *is a weak solution of* (1).

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence of I, we know that there exists $C > 0$ such that

$$|I(u_n)| \leq C, \ |(I'(u_n), \frac{u_n}{\|u_n\|})| \leq C.$$

In order to prove the boundedness of $\{u_n\}$, we will consider the two cases: $0 < \lambda < \lambda_1$ and $\lambda \in [\lambda_i, \lambda_{i+1})$ for some $i \in \mathbb{N}^+$.

Case 1. $0 < \lambda < \lambda_1$.

For n large enough, we get

$$\begin{aligned} C(1 + \|u_n\|) &\geq I(u_n) - \frac{1}{6}(I'(u_n), u_n) \\ &\geq \frac{1}{3}(\|u_n\|^2 - \lambda |u_n|_2^2) \\ &\geq \frac{\delta_1}{3}\|u_n\|^2 \end{aligned}$$

for $\delta_1 = \frac{\lambda_1 - \lambda}{\lambda_1} > 0$. Thus $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Case 2. $\lambda \in [\lambda_i, \lambda_{i+1})$ for some $i \in \mathbb{N}^+$.

Let $\theta \in (\frac{1}{6}, \frac{1}{2})$. For n large enough, we have

$$\begin{aligned} C(1 + \|u_n\|) &\geq I(u_n) - \theta(I'(u_n), u_n) \\ &\geq (\frac{1}{2} - \theta)(\|u_n\|^2 - \lambda |u_n|_2^2) + (\theta - \frac{1}{6})|u_n|_6^6 \end{aligned}$$

$$
\begin{aligned}
&+(\theta-\frac{1}{10})\int_\Omega \phi_u|u|^5\,dx\\
\geq\;& (\frac{1}{2}-\theta)(\|x_n\|^2+\|y_n\|^2-\lambda|x_n|_2^2-\lambda|y_n|_2^2)\\
&+(\theta-\frac{1}{6})|u_n|_6^6\\
\geq\;& (\frac{1}{2}-\theta)(\delta_2\|x_n\|^2+(\lambda_1-\lambda)|y_n|_2^2)\\
&+(\theta-\frac{1}{6})|u_n|_6^6\\
\geq\;& (\frac{1}{2}-\theta)(\delta_2\|x_n\|^2+(\lambda_1-\lambda)|y_n|_2^2)+(\theta-\frac{1}{6})|u_n|_6^6
\end{aligned}
$$

for $\delta_2=\frac{\lambda_{i+1}-\lambda}{\lambda_{i+1}}>0$, $u_n=x_n+y_n$, $x_n\in E_{i+1}$, $y_n\in Y_i$. Then we have

$$
\begin{aligned}
&(\frac{1}{2}-\theta)(\lambda-\lambda_1)|y_n|_2^2+C(1+\|u_n\|)\\
&\geq (\frac{1}{2}-\theta)\delta_2\|x_n\|^2+(\theta-\frac{1}{6})|u_n|_6^6\\
&\geq (\frac{1}{2}-\theta)\delta_2\|x_n\|^2+C(\theta-\frac{1}{6})|u_n|_2^6\\
&\geq (\frac{1}{2}-\theta)\delta_2\|x_n\|^2+C(\theta-\frac{1}{6})|y_n|_2^6.
\end{aligned}
$$

So, from the fact that Y_i is finite dimensional, we can deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive, up to a sequence, still denoted by $\{u_n\}$, there exists $u_0\in H_0^1(\Omega)$ such that $u_n\rightharpoonup u_0$ in $H_0^1(\Omega)$. Since the proof that u_0 is a weak solution is almost the same as that of Lemma 2.4 of [12], so we omit it here. Especially, we take $h=u_0\in H_0^1(\Omega)$ as a test function in (1), then

$$
\int_\Omega|\nabla u_0|^2\,dx=\lambda\int_\Omega u_0^2\,dx+\int_\Omega\phi_u|u|^5\,dx+\int_\Omega u_0^6\,dx.
$$

And we can deduce that

$$
I(u_0)\geq\frac{2}{5}\int_\Omega\phi_u|u|^5\,dx+\frac{1}{3}\int_\Omega u_0^6\,dx\geq 0.
$$

This completes the proof. □

Lemma 2.3. *Let $\lambda>0$. If $\{u_n\}$ is a (PS) sequence at c with*

$$
0<c<\frac{13-\sqrt{5}}{30}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}},
$$

then there exists $u_0\in H_0^1(\Omega)$ such that $u_0\not\equiv 0$ is a weak limit of $\{u_n\}$.

Proof. From Lemma 2.2, we know that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, we can assume that it is weakly convergent. Suppose by contradiction that $u_n\rightharpoonup 0$ in $H_0^1(\Omega)$. Then we have

$$
c\leftarrow I(u_n)=\frac{1}{2}\int_\Omega|\nabla u_n|^2\,dx-\frac{1}{6}\int_\Omega u_n^6\,dx-\frac{1}{10}\int_\Omega\phi_{u_n}|u_n|^5\,dx+o_n(1) \tag{9}
$$

and

$$
o_n(1)=(I'(u_n),u_n)=\int_\Omega|\nabla u_n|^2\,dx-\int_\Omega u_n^6\,dx-\int_\Omega\phi_{u_n}|u_n|^5\,dx+o_n(1). \tag{10}
$$

From (9), up to subsequences, we know that there exists a non-negative constant a such that

$$\int_\Omega |\nabla u_n|^2\,dx \to a$$

and

$$\int_\Omega u_n^6\,dx + \int_\Omega \phi_{u_n}|u_n|^5\,dx \to a,$$

as $n\to+\infty$. If $a=0$, then by $c>0$, we know that this is a contradiction. If $a\neq 0$, from (9) and (10), for convenience, let $b_n=\int_\Omega u_n^6\,dx$, $l_n=\int_\Omega \phi_{u_n}|u_n|^5\,dx$. Without loss of generality, we may assume $b_n\to b$ and $l_n\to l$, as $n\to\infty$. Recall that, $\forall t>0$,

$$\begin{aligned}\int_\Omega |u_n|^6\,dx &= \int_\Omega \nabla\phi_{u_n}\nabla u_n\,dx\\ &\le \frac{t^2}{2}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{2t^2}\int_\Omega |\nabla\phi_{u_n}|^2\,dx\\ &= \frac{1}{2t^2}\int_\Omega \phi_{u_n}|u_n|^5\,dx + \frac{t^2}{2}\int_\Omega |\nabla u_n|^2\,dx,\end{aligned}$$

thus, as $n\to\infty$, we have

$$b\le \frac{1}{2t^2}l+\frac{t^2}{2}a.$$

Taking $t^2=\frac{\sqrt5-1}{2}$, since $b+l=a$, we get

$$l\ge \frac{3-\sqrt5}{2}a.$$

Then from (9), we can deduce

$$c=\frac25 l+\frac13 b=\frac13 a+\frac1{15}l\ge \frac{13-\sqrt5}{30}a.$$

On the other hand, it follows from Lemma 1.1 that

$$a\le S^{-6}a^5+S^{-3}a^3.$$

Hence, we have either $a=0$ or $a^2\ge \frac{-1+\sqrt5}{2}S^3$. In each case it will come to a contradiction. So the proof is completed. □

3. **The case** $0<\lambda<\lambda_1$ **in** (1).

Lemma 3.1. *If* $\lambda\in(0,\lambda_1)$, *then the functional* I *satisfies the following properties:*
(*i*) *There exist* $\gamma_1,\rho_1>0$ *such that* $I(u)\ge\gamma_1$ *for* $\|u\|=\rho_1$.
(*ii*) *There exists* $u_0\in H_0^1(\Omega)$ *with* $\|u_0\|\ge\rho_1$ *such that* $I(u_0)<0$.

Proof. (*i*) Since $\lambda\in(0,\lambda_1)$, by the Sobolev embedding, for all $u\in H_0^1(\Omega)\backslash\{0\}$ we get

$$\begin{aligned}I(u)&\ge \frac12\int_\Omega|\nabla u|^2\,dx-\frac{\lambda}{2\lambda_1}\int_\Omega|\nabla u|^2\,dx-\frac16|u|_6^6-\frac1{10}S^{-1}|u|_6^{10}\\ &\ge \frac12(1-\frac{\lambda}{\lambda_1})\|u\|^2-\frac16 S^{-3}\|u\|^6-\frac1{10}S^{-6}\|u\|^{10}.\end{aligned}$$

So we can choose some $\gamma_1,\rho_1>0$ such that $I(u)\ge\gamma_1$ for $\|u\|=\rho_1$.

(ii) For each $u_1 \in H_0^1(\Omega)\backslash\{0\}$, we have

$$
\begin{aligned}
I(tu_1) \leq \frac{t^2}{2}\int_\Omega |\nabla u_1|^2\,dx - \frac{\lambda t^2}{2}\int_\Omega u_1^2\,dx - \frac{t^6}{6}\int_\Omega u_1^6\,dx \\
- \frac{t^{10}}{10}\int_\Omega \phi_{u_1}|u_1|^5\,dx < 0
\end{aligned}
$$

for $t > 0$ large enough. Hence, we can take $u_0 = t_* u_1$ for $t_* > 0$ large to end the proof. □

By Lemma 3.1 and the mountain pass theorem without (PS) condition, there exists a (PS) sequence $\{u_n\}$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ at the minimax level

$$c^* := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Without loss of generality, we may assume that $0 \in \Omega$ and $B_R \subset \Omega \subset B_{kR}$ for some positive R, k. By [6] and [5], we consider a fixed function ψ such that $\psi(r) = cos(\frac{\pi r}{2R})$, $\forall r \in [0, R)$, and $\psi(r) = 0$, $\forall r \in [R, kR]$. Set $r = |x|$ and

$$u_\epsilon(r) = \frac{\psi(r)}{(\epsilon + r^2)^{\frac{1}{2}}}.$$

Then we have

$$
\begin{aligned}
\int_\Omega |\nabla u_\epsilon|^2\,dx = \int_{B_{kR}} |\nabla u_\epsilon|^2\,dx = \int_{B_R} |\nabla u_\epsilon|^2\,dx \\
= S\frac{K}{\epsilon^{\frac{1}{2}}} + w\int_0^R |\psi'(r)|^2\,dr + O(\epsilon^{\frac{1}{2}}),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
(\int_\Omega u_\epsilon^6\,dx)^{\frac{1}{3}} = (\int_{B_{kR}} u_\epsilon^6\,dx)^{\frac{1}{3}} = (\int_R u_\epsilon^6\,dx)^{\frac{1}{3}} \\
= \frac{K}{\epsilon^{\frac{1}{2}}} + O(\epsilon^{\frac{1}{2}}),
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
\int_\Omega u_\epsilon^2\,dx = \int_{B_{kR}} u_\epsilon^2\,dx = \int_{B_{kR}} u_\epsilon^2\,dx \\
= w\int_0^R \psi^2(r)\,dr + O(\epsilon^{\frac{1}{2}}),
\end{aligned}
\tag{13}
$$

where K is a positive constant and w is the area of the unitary sphere in $\mathbb{R}^3$. Note that

$$\int_\Omega u^6\,dx \leq \frac{1}{\sqrt{5}-1}\int_\Omega \phi|u|^5\,dx + \frac{\sqrt{5}-1}{4}\int_\Omega |\nabla u|^2\,dx,$$

hence we can introduce the new functional $J : H^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \frac{13-\sqrt{5}}{20}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{2+3\sqrt{5}}{30}\int_\Omega u^6\,dx.$$

Obviously we have $I(u) \leq J(u)$ for any $u \in H^1(\Omega)$.

Proof of Theorem 1.3(i). Firstly, for $v \in H_0^1(\Omega)\backslash\{0\}$, we have

$$0 < \max_{t\geq 0} I(tv) \leq \max_{t\geq 0} J(tv)$$
$$= \max_{t\geq 0}\{\frac{13-\sqrt{5}}{20}t^2\int_\Omega |\nabla v|^2\,dx - \frac{\lambda t^2}{2}\int_\Omega v^2\,dx - \frac{2+3\sqrt{5}}{30}t^6\int_\Omega v^6\,dx\},$$

Set

$$y(t) := \frac{13-\sqrt{5}}{20}t^2 b_1 - \frac{t^2}{2}c_1 - \frac{2+3\sqrt{5}}{30}t^6 d_1,$$

where

$$b_1 := \int_\Omega |\nabla v|^2\,dx,$$

$$c_1 = \lambda\int_\Omega v^2\,dx$$

$$d_1 := \int_\Omega v^6\,dx.$$

Let

$$y' = \frac{13-\sqrt{5}}{10}b_1 t - c_1 t - \frac{2+3\sqrt{5}}{5}d_1 t^5 = 0,$$

which yields that there exists unique $t_* > 0$ such that $t_*^4 = \frac{\frac{13-\sqrt{5}}{10}b_1 - c_1}{\frac{2+3\sqrt{5}}{5}d_1}$ and $y(t_*)$ is the maximum of y. If we can verify that $y_{\max} < \frac{13-\sqrt{5}}{30}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}}$, by the definition of c^*, we know that $c^* < \frac{13-\sqrt{5}}{30}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}}$. Then let $\{u_n\}$ be the (PS) sequence, which follows from Lemma 2.3 that $u_n \rightharpoonup u_0 \not\equiv 0$. Thus (1) has a nontrivial solution. So it is enough to verify that

$$y_{\max} < \frac{13-\sqrt{5}}{30}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}}.$$

Set $v = u_\epsilon$, compute $\max_{t\geq 0} J(tu_\epsilon) = J(t_\epsilon u_\epsilon)$, where t_ϵ is the unique positive solution of the equation $\frac{d}{dt}J(tu_\epsilon) = 0$, we obtain that

$$t_\epsilon^4 = \frac{\frac{13-\sqrt{5}}{10}\int_{B_R}|\nabla u_\epsilon|^2\,dx - \lambda\int_{B_R}u_\epsilon^2\,dx}{\frac{2+3\sqrt{5}}{5}\int_{B_R}u_\epsilon^6\,dx},$$

which yields that

$$J(t_\epsilon u_\epsilon) \leq \frac{1}{3}\frac{(\frac{13-\sqrt{5}}{10}\int_{B_R}|\nabla u_\epsilon|^2\,dx - \lambda\int_{B_R}u_\epsilon^2\,dx)^{\frac{3}{2}}}{(\frac{2+3\sqrt{5}}{5}\int_{B_R}u_\epsilon^6\,dx)^{\frac{1}{2}}}. \tag{14}$$

In addition, when $r \in [0, R]$, we get

$$\int_0^R |\psi'(r)|^2\,dr = \frac{\pi^2}{4R^2}\int_0^R \psi^2(r)\,dr. \tag{15}$$

Then we take (11), (12), (13) and (15) into (14), by the simple computation, we have if $\lambda \in (\frac{13-\sqrt{5}}{40}\frac{\pi^2}{R^2}, \lambda_1)$, we can deduce that $c < \frac{13-\sqrt{5}}{30}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}}$ for ϵ small enough. Then by using Lemma 2.3, the conclusion is true in case $\lambda \in (\frac{13-\sqrt{5}}{40}\frac{\pi^2}{R^2}, \lambda_1)$. $\square$

4. **The case $\lambda \geq \lambda_1$ in (1).** In this section, we will consider the cross eigenvalue case, i.e., suppose $\lambda \in [\lambda_i, \lambda_{i+1})$ for some $i \in \mathbb{N}^+$, where λ_i is the i-th eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$, and e_i is the i-th eigenfunctions corresponding to the eigenvalue λ_i. For the cross eigenvalue case, we need to use the following linking theorem [22].

Theorem 4.1. *Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$, satisfies (PS), and*

(i) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$;

(ii) there is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\bar{B}_R \cap V) \oplus \{re | 0 < r < R\}$, then $I|_{\partial Q} \leq 0$.

Then I possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{Q}, E) | h = id \, on \, \partial Q\}.$$

Lemma 4.2. *If $\lambda \in [\lambda_i, \lambda_{i+1})$ for some $i \in \mathbb{N}$, then the functional I satisfies the following properties:*

(i) there exist $\gamma_2, \rho_2 > 0$ such that $I \geq \gamma_2$, for any $u \in E_{i+1}$ with $\|u\| = \rho_2$;

(ii) $I(u) < 0$ for any $u \in Y_i \backslash \{0\}$;

(iii) let F be a finite dimensional subspace of $H_0^1(\Omega)$, then there exists $R > \rho_2$ such that $I(u) \leq 0$, for any $u \in F$ with $\|u\| \geq R$.

Proof. (i) Since $\lambda \in [\lambda_i, \lambda_{i+1})$, by Lemma 1.1 and the Sobolev embedding, for all $u \in E_{i+1} \backslash \{0\}$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2\lambda_{i+1}} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{6} S^{-3} \|u\|^6 - \frac{1}{10} S^{-6} \|u\|^{10} \\ &\geq \frac{1}{2}(1 - \frac{\lambda}{\lambda_{i+1}}) \|u\|^2 - \frac{1}{6} S^{-3} \|u\|^6 - \frac{1}{10} S^{-6} \|u\|^{10}. \end{aligned}$$

We can choose some $\gamma_2, \rho_2 > 0$ such that $I \geq \gamma_2$ for $u \in E_{i+1}$ with $\|u\| = \rho_2$.

(ii) Let $u \in Y_i$, i.e., $u = \sum_{j=1}^i l_j e_j$, where $l_j \in \mathbb{R}$, $j = 1, \cdots, i$. Since $\{e_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$, we get

$$\int_\Omega |\nabla u|^2 \, dx = \sum_{j=1}^i l_j^2 |\nabla e_j|_2^2, \int_\Omega u^2 \, dx = \sum_{j=1}^i l_j^2.$$

Then we can get

$$\begin{aligned} I(u) &= \frac{1}{2} \sum_{j=1}^i l_j^2 (|\nabla e_j|_2^2 - \lambda) - \frac{1}{6} \int_\Omega u^6 \, dx - \frac{1}{10} \int_\Omega \phi_u |u|^5 \, dx \\ &< \frac{1}{2} \sum_{j=1}^i l_j^2 (\lambda_j - \lambda) \\ &\leq 0. \end{aligned}$$

(iii) Choose $u \in F\backslash\{0\}$. Since all norms on finite dimensional space are equivalent, then

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{6}|u|_6^6 - \frac{1}{10}\int_\Omega \phi_u |u|^5\,dx \\ \leq \frac{1}{2}\|u\|^2 - C^*\|u\|^6,$$

where C^* is a positive constant. So, $I \to -\infty$ as $\|u\| \to +\infty$. Hence, there exists $R > \rho_2$ such that $I(u) \leq 0$, for any $u \in F$ with $\|u\| \geq R$ and the proof of (iii) ends. □

It is well known that $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ is a minimizer for S.

Without loss of generality, we may assume that $0 \in \Omega$ and $B_\delta \subset \Omega \subset B_{k\delta}$ for some positive k. Let $\varphi \in C_0^\infty$ such that if $x \in B_\delta$, $\varphi = 1$; and if $x \in \mathbb{R}^3\backslash B_{k\delta}$, $\varphi = 0$. $\forall x \in \mathbb{R}^3$, $0 \leq \varphi \leq 1$ and $|D\varphi(x)| \leq C = const.$

For $\epsilon > 0$, define

$$U_\epsilon(x) := \epsilon^{-\frac{1}{2}}U(\frac{x}{\epsilon}),\ u_\epsilon(x) := \varphi(x)U_\epsilon(x),$$

then in the view of [12], we have

$$|\nabla U_\epsilon|_2^2 = |U_\epsilon|_6^6 = S^{\frac{3}{2}},$$

and as $\epsilon \to 0^+$, we can have

$$\int_\Omega |\nabla u_\epsilon|^2\,dx = S^{\frac{3}{2}} + O(\epsilon), \tag{16}$$

$$\int_\Omega |u_\epsilon|^6\,dx = S^{\frac{3}{2}} + O(\epsilon^3), \tag{17}$$

and

$$\int_\Omega |u_\epsilon|^2\,dx \geq C_0\epsilon, \tag{18}$$

where C_0 is a positive constant.

Define the linear space for any $i \in \mathbb{N}^+$,

$$G_{i,\epsilon} := span\{e_1, \cdots, e_i, u_\epsilon\}. \tag{19}$$

Now we give the proof of Theorem 1.3 for the general case.

Proof of Theorem 1.3(ii). Suppose that $\lambda \in [\lambda_i, \lambda_{i+1})$ for $i \in N^+$, then from the definition of $G_{i,\epsilon}$, we know

$$u = v + tu_\epsilon, \forall\, u \in G_{i,\epsilon},$$

where $v \in Y_i$. Then

$$G_{i,\epsilon} = Y_i \oplus \mathbb{R}u_\epsilon.$$

From Lemma 4.2, we know that I satisfies the geometric structure of the linking theorem, i.e.,

$$\inf_{u\in E_{i+1}, \|u\|=\rho_2} I(u) \geq \gamma_2 > 0,\ \sup_{u\in Y_i\backslash\{0\}} I(u) < 0,\ \sup_{u\in G_{i,\epsilon}, \|u\|\geq R} I(u) \leq 0,$$

where γ_2 and R are defined in Lemma 4.2. Define the linking critical level of I, i.e.,

$$\bar{c} = \inf_{\gamma\in\Gamma}\max_{u\in V} I(\gamma(u)) > 0,$$

where

$$\Gamma := \{\gamma \in C(\bar{V}, H_0^1(\Omega)) : \gamma = id \, on \, \partial V\}$$

and

$$V := (\bar{B}_R \cap Y_i) \oplus \{ru_\epsilon : r \in (0, R)\}.$$

For any $\gamma \in \Gamma$, we get

$$\bar{c} \leq \max_{u \in V} I(\gamma(u))$$

and in particular, if we take $\gamma = id$ on $\bar{V}$, then

$$\bar{c} \leq \max_{u \in V} I(u).$$

Then we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{6}\int_\Omega u^6\,dx - \frac{1}{10}\int_\Omega \phi_u |u|^5\,dx \\
&= \frac{1}{2}\int_\Omega |\nabla(v + tu_\epsilon)|^2\,dx - \frac{\lambda}{2}\int_\Omega |v + tu_\epsilon|^2\,dx - \frac{1}{6}\int_\Omega |v + tu_\epsilon|^6\,dx \\
&\quad - \frac{1}{10}\int_\Omega \phi_{v+tu_\epsilon} |v + tu_\epsilon|^5\,dx \\
&\leq \frac{(\lambda_i - \lambda)}{2}|v|_2^2 + \frac{t^2}{2}(\int_\Omega |\nabla u_\epsilon|^2\,dx - \lambda\int_\Omega |u_\epsilon|^2\,dx) + C_1 t |u_\epsilon|_1 |v|_2 \\
&\quad - \frac{t^6}{6}\int_\Omega |u_\epsilon|^6\,dx - \int_\Omega |tu_\epsilon|^5 v(x)\,dx - \frac{t^{10}}{10}\int_\Omega \phi_{u_\epsilon}|u_\epsilon|^5\,dx - \int_\Omega \phi_{tu_\epsilon}|tu_\epsilon|^4 v\,dx \\
&\leq \frac{(\lambda_i - \lambda)}{2}|v|_2^2 + C_1 t |v|_2 O(\epsilon^{\frac{1}{2}}) + \frac{t^2}{2}(\int_\Omega |\nabla u_\epsilon|^2\,dx - \lambda\int_\Omega u_\epsilon^2\,dx) \\
&\quad - \frac{t^6}{6}\int_\Omega u_\epsilon^6\,dx - \frac{t^{10}}{10}\int_\Omega \phi_{u_\epsilon}|u_\epsilon|^5\,dx + |v|_2 t^5 O(\epsilon^{\frac{1}{2}}) + |v|_2 t^9 O(\epsilon^{\frac{1}{2}}) \\
&\leq \frac{C_2}{2(\lambda - \lambda_i)}O(\epsilon) + \frac{t^2}{2}(\int_\Omega |\nabla u_\epsilon|^2\,dx - \lambda\int_\Omega u_\epsilon^2\,dx) \\
&\quad - \frac{t^6}{6}\int_\Omega u_\epsilon^6\,dx - \frac{t^{10}}{10}\int_\Omega \phi_{u_\epsilon}|u_\epsilon|^5\,dx,
\end{aligned} \tag{20}$$

where in the view of boundedness of t, by calculating the maximum value of the quadratic function, we get

$$\begin{aligned}
&\frac{(\lambda_i - \lambda)}{2}|v|_2^2 + C_1 t |v|_2 O(\epsilon^{\frac{1}{2}}) + |v|_2 t^5 O(\epsilon^{\frac{1}{2}}) + |v|_2 t^9 O(\epsilon^{\frac{1}{2}}) \\
&\leq \frac{(\lambda_i - \lambda)}{2}|v|_2^2 + C|v|_2 O(\epsilon^{\frac{1}{2}}) \\
&\leq \frac{C_2}{2(\lambda - \lambda_i)}O(\epsilon).
\end{aligned}$$

Then take (16), (17), and (18) into (20), we have

$$\int_\Omega |\nabla u_\epsilon|^2\,dx - \lambda\int_\Omega u_\epsilon^2\,dx \leq S^{\frac{3}{2}} + O(\epsilon) - \lambda C_0 \epsilon.$$

Since $-\Delta\phi_{u_\epsilon} = |u_\epsilon|^5$, we deduce that

$$\begin{aligned}\int_\Omega u_\epsilon^6\,dx &= \int_\Omega \nabla\phi_{u_\epsilon}\nabla u_\epsilon\,dx\\ &\le \frac{1}{2}\int_\Omega |\nabla\phi_{u_\epsilon}|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2\,dx\\ &= \frac{1}{2}\int_\Omega \phi_{u_\epsilon}|u_\epsilon|^5\,dx + \frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2\,dx.\end{aligned}$$

For $\epsilon > 0$ sufficiently small, we deduce

$$\begin{aligned}\int_\Omega \phi_{u_\epsilon}|u_\epsilon|^5\,dx &\ge 2\int_\Omega u_\epsilon^6\,dx - \int_\Omega |\nabla u_\epsilon|^2\,dx\\ &= S^{\frac{3}{2}} + O(\epsilon).\end{aligned}$$

We have

$$\begin{aligned}&\frac{C_2}{2(\lambda-\lambda_i)}O(\epsilon) + \frac{t^2}{2}(\int_\Omega |\nabla u_\epsilon|^2\,dx - \lambda\int_\Omega u_\epsilon^2\,dx)\\ &\quad - \frac{t^6}{6}\int_\Omega u_\epsilon^6\,dx - \frac{t^{10}}{10}\int_\Omega \phi_{u_\epsilon}|u_\epsilon|^5\,dx\\ &\le C_3\epsilon + \frac{t^2}{2}S^{\frac{3}{2}} - \frac{t^6}{6}S^{\frac{3}{2}} - \frac{t^{10}}{10}S^{\frac{3}{2}}\\ &\quad + \frac{t^2}{2}(O(\epsilon) - \lambda C_0\epsilon) - \frac{t^6}{6}O(\epsilon^3) + \frac{t^{10}}{10}O(\epsilon)\\ &< \frac{t^2}{2}S^{\frac{3}{2}} - \frac{t^6}{6}S^{\frac{3}{2}} - \frac{t^{10}}{10}S^{\frac{3}{2}}\\ &\le \frac{13-\sqrt{5}}{10}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}}\end{aligned}$$

for ϵ small enough. Hence there exists $\lambda_* > 0$ such that $\lambda > \lambda_*$, we can get

$$\max_{u\in G_{i,\epsilon}} I(u) < \frac{13-\sqrt{5}}{30}(\frac{\sqrt{5}-1}{2})^{\frac{1}{2}}S^{\frac{3}{2}}.$$

Taking $\lambda_* \ge \lambda_1$, it follows from the linking theorem and Lemma 2.3 that (1) has a nontrivial solution $u \in H_0^1(\Omega)$ in case $\lambda > \lambda_*$ and $\lambda \ne \lambda_i$, where λ_i is an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data. □

5. **Nonexistence of nontrivial solutions.** In this section, we will prove the nonexistence of nontrivial solutions for (1) in case $\lambda < 0$.

Proof of Theorem 1.4. By [26, 24], similar to the proof in [5], let $\Omega \subset \mathbb{R}^3$ be a star shaped domain and $u, \phi \in C^2(\Omega) \cap C^1(\Omega)$ be a nontrivial solution of (1). We multiply the first equation of (1) by $x\cdot\nabla u$ and the second one by $x\cdot\nabla\phi$, we obtain that

$$\begin{aligned}0 &= (\Delta u + \lambda u + \phi|u|^3u + |u|^4u)(x\cdot\nabla u)\\ &= div[(\nabla u)(x\cdot\nabla u) - x\frac{|\nabla u|^2}{2} + \frac{\lambda}{2}xu^2 + \frac{1}{5}x\phi|u|^5 + x|u|^6] + \frac{1}{2}|\nabla u|^2 - \frac{3}{2}\lambda u^2\\ &\quad - \frac{3}{5}\phi|u|^5 - \frac{1}{5}(x\cdot\nabla\phi)|u|^5 - \frac{1}{2}|u|^6\end{aligned}$$

and

$$0 = (\Delta\phi + |u|^5)(x \cdot \nabla\phi)$$
$$= div[(\nabla\phi)(x \cdot \nabla\phi) - x\frac{|\nabla\phi|^2}{2}] + \frac{1}{2}|\nabla\phi|^2 + (x \cdot \nabla\phi)|u|^5.$$

Assume that ν is the unit exterior normal to $\partial\Omega$, integrating on Ω, by boundary conditions $\nabla u = \frac{\partial u}{\partial \nu}\nu$ and $\nabla\phi = \frac{\partial \phi}{\partial \nu}\nu$ on $\partial\Omega$, we have

$$-\frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\partial\Omega}|\frac{\partial u}{\partial \nu}|^2 x\cdot\nu = -\frac{3\lambda}{2}|u|_2^2 - \frac{3}{5}\int_\Omega \phi|u|^5\,dx - \frac{1}{5}\int_\Omega (x\cdot\nabla\phi)|u|^5 - \frac{1}{2}|u|^6 \tag{21}$$

and

$$-\frac{1}{2}\|\nabla\phi\|_2^2 - \frac{1}{2}\int_{\partial\Omega}|\frac{\partial \phi}{\partial \nu}|^2 x\cdot\nu = \int_\Omega (x\cdot\nabla\phi)|u|^5. \tag{22}$$

Taking (22) into (21), we get

$$-\frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\partial\Omega}|\frac{\partial u}{\partial \nu}|^2 x\cdot\nu$$
$$= -\frac{3\lambda}{2}|u|_2^2 - \frac{3}{5}\int_\Omega \phi|u|^5\,dx + \frac{1}{10}\|\nabla\phi\|_2^2 + \frac{1}{10}\int_{\partial\Omega}|\frac{\partial \phi}{\partial \nu}|^2 x\cdot\nu - \frac{1}{2}\int_\Omega |u|^6.$$

In addition, since (u,ϕ) is a nontrivial solution for (1), we see that

$$\|\nabla u\|_2^2 = \lambda|u|_2^2 + \int_\Omega \phi|u|^5 + |u|_6^6$$

and $\|\nabla\phi\|_2^2 = \int_\Omega \phi|u|^5$. Combing the above argument, we deduce that

$$-\lambda|u|_2^2 + \frac{1}{2}\int_{\partial\Omega}|\frac{\partial u}{\partial \nu}|^2 x\cdot\nu + \frac{1}{10}\int_{\partial\Omega}|\frac{\partial \phi}{\partial \nu}|^2 x\cdot\nu = 0.$$

In the view of $\lambda < 0$, we know that this is a contradiction. So we have $u = 0$, that is, (1) admits no nontrivial solution. □

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E-mail address: 1967724371@qq.com
E-mail address: zhangfubao@seu.edu.cn