

CONTINUOUS SUBSONIC-SONIC FLOWS IN A TWO-DIMENSIONAL SEMI-INFINITELY LONG NOZZLE

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ABSTRACT. This paper focuses on two-dimensional continuous subsonic-sonic potential flows in a semi-infinitely long nozzle with a straight lower wall and an upper wall which is convergent at the outlet while straight at the far fields. It is proved that if the variation rate of the cross section of the nozzle is suitably small, there exists a unique continuous subsonic-sonic flows in the nozzle such that the sonic curve intersects the upper wall at a fixed point and the velocity of the flow is along the normal direction at the sonic curve. Furthermore, the sonic curve is free, where the flow is singular in the sense that the flow speed is only Hölder continuous and the flow acceleration blows up. Additionally, the asymptotic behaviors of the flow speed at the far fields is shown.

1. Introduction. The Euler system

$$\begin{cases} \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \\ \frac{\partial}{\partial x}(p + \rho u^2) + \frac{\partial}{\partial y}(\rho u v) = 0, \\ \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(p + \rho v^2) = 0 \end{cases} \quad (1)$$

is usually used to describe the two-dimensional steady isentropic inviscid compressible flow, where (u, v) , p and ρ represent the velocity, pressure and density of the flow, respectively, and $p(\rho) = \rho^\gamma/\gamma$ for a polytropic gas with the adiabatic exponent $\gamma > 1$ after the nondimensionalization. Suppose that the flow is irrotational, i.e.,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \quad (2)$$

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Then the density ρ can be formulated as a function of the flow speed $q = \sqrt{u^2 + v^2}$ according to the Bernoulli law ([2]):

$$\rho(q^2) = \left(1 - \frac{\gamma - 1}{2}q^2\right)^{1/(\gamma-1)}, \quad 0 < q < \sqrt{2/(\gamma - 1)}. \quad (3)$$

The sound speed c is defined as $c^2 = p'(\rho)$. At the sonic state, the flow speed is $c_* = \sqrt{2/(\gamma + 1)}$, which is critical in the sense that the flow is subsonic when $q < c_*$, sonic when $q = c_*$, and supersonic when $q > c_*$. The system (1), (2) can be transformed into the full potential equation

$$\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, \quad (4)$$

where φ is the velocity potential with $\nabla\varphi = (u, v)$, ρ is the function given by (3). It is noted that (4) is elliptic in the subsonic region, degenerate at the sonic state, while hyperbolic in the supersonic region.

Subsonic-sonic flow is one of the most interesting aspects in the mathematical theory of compressible flows. The related problems are usually raised in physical experiments and engineering designs, and there are a lot of numerical simulations and rigorous theory involved in this field (see, e.g., [2, 8, 15]). Two kinds of subsonic-sonic flows have been intensively studied for decades: the flow past a profile and the flow in a nozzle. The outstanding work [1] by L. Bers proved that there exists a unique two-dimensional subsonic potential flow past a profile provided that the freestream Mach number is less than a critical value and the maximum flow speed tends to the sound speed as the freestream Mach number tends to the critical value. Later, the similar results for multi-dimensional cases were established in [13, 9] by G. Dong, R. Finn and D. Gilbarg. These three works did not cover the flow with the critical freestream Mach number. It was shown in [3] based on a compensated compactness framework that the two-dimensional flow with sonic points past a profile may be realized as the weak limit of a sequence of strictly subsonic flows. However, all the subsonic-sonic flows above are obtained in the weak sense and their smoothness and uniqueness are unknown yet, so are the subsonic-sonic flows in an infinitely long nozzle. For a two-dimensional infinitely long nozzle, C. Xie et al. ([22]) proved that there exists a critical value such that a strictly subsonic flow exists uniquely as long as the incoming mass flux is less than the critical value, and a subsonic-sonic flow exists as the weak limit of a sequence of strictly subsonic flows. The multi-dimensional cases were investigated in [24, 12, 14]. A typical subsonic-sonic flow with precise regularity is a radially symmetric subsonic-sonic flow in a convergent straight nozzle. The structural stability was initially proved in [20] for the case of two-dimensional finitely long nozzle, and some new results can be found in [16, 17, 18, 21, 19]. In the recent decade, there are also some studies on rotational subsonic and subsonic-sonic flows, see [4, 6, 11, 7, 5, 23] and the references therein.

In the present paper, we would like to investigate the subsonic-sonic flow in a class of semi-infinitely long nozzles. Assume precisely that $l_0, l_1 > 0$ and $\alpha \in (0, 1)$ are constants, and $f \in C^{2,\alpha}((-\infty, 0])$ satisfies

$$f''(0) < f(0) = 0, \quad (-x)^{-1/2}f'' \in L^\infty((-l_0, 0]), \quad (5)$$

$$f(x) > 0 \text{ for } x \in (-\infty, 0), \quad f'(x) = 0 \text{ for } x \in (-\infty, -l_0]. \quad (6)$$

The upper and lower wall of the nozzle are described as

$$\Gamma_{\text{up}} : y = f_k(x) \quad (x \in (-\infty, 0]), \quad \text{and} \quad \Gamma_{\text{low}} : y = -l_1 \quad (x \in \mathbb{R}),$$

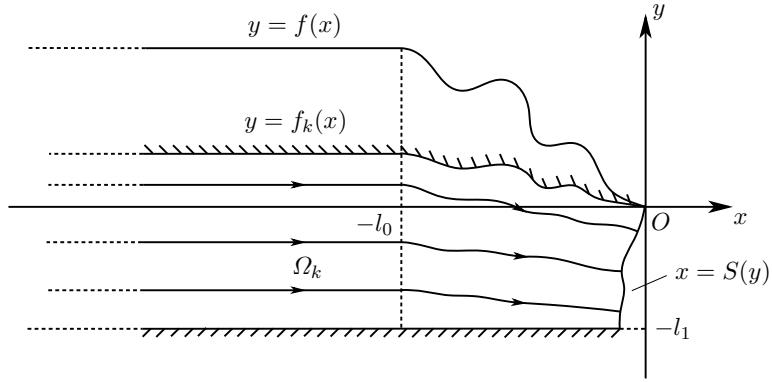
respectively, where $k \in (0, 1]$ and

$$f_k(x) = kf(x), \quad x \in (-\infty, 0].$$

The sonic curve of the flow is a free boundary intersecting the upper wall at the origin, which is chosen as the outlet of the nozzle and is denoted by

$$\Gamma_{\text{out}} : x = S(y), \quad y \in [-l_1, 0], \quad S(0) = 0.$$

It is assumed further that the subsonic-sonic flow satisfies the slip conditions at Γ_{up} and Γ_{low} , and its velocity is along the normal direction at Γ_{out} . See the following figure for an intuition.



As in [18, 21], the subsonic-sonic flow problem can be formulated in the physical plane as

$$\operatorname{div}(\rho(|\nabla \varphi|^2) \nabla \varphi) = 0, \quad (x, y) \in \Omega_k, \quad (7)$$

$$\frac{\partial \varphi}{\partial y}(x, -l_1) = 0, \quad x \in (-\infty, S(-l_1)), \quad (8)$$

$$\frac{\partial \varphi}{\partial y}(x, f_k(x)) - f'_k(x) \frac{\partial \varphi}{\partial x}(x, f_k(x)) = 0, \quad x \in (-\infty, 0), \quad (9)$$

$$|\nabla \varphi(S(y), y)| = c_*, \quad \varphi(S(y), y) = 0, \quad y \in (-l_1, 0), \quad (10)$$

where (φ, S) is a solution and Ω_k is the semi-infinitely long nozzle bounded by Γ_{up} , Γ_{low} and Γ_{out} . The problem (7)–(10) is a free boundary problem of a quasilinear degenerate elliptic equation in an unbounded domain, whose degeneracy occurs at the free boundary and is characteristic. As mentioned in Remark 1.6 of [22], one can not require in advance that the flow tends to be uniformly subsonic at the far fields, otherwise, the elliptic problem may be overdetermined. In the paper, we prove that the subsonic-sonic flow in the nozzle is uniformly subsonic at the far fields, and the uniqueness of the flow results from this property. Similar to [20, 18, 16, 17, 21], we still solve the problem in the potential plane for the reason that the shape of the sonic curve is unknown in the physical plane while known in the potential plane, and the estimates of the flow speed can be made conveniently. In the potential plane, the subsonic-sonic flow problem (7)–(10) can be transformed into a quasilinear degenerate elliptic problem with free parameters and nonlocal boundary conditions in unbounded domain. The unboundedness of the domain makes the problem more difficult than the ones in [20, 18, 16, 17, 21]. The Schauder

fixed point theorem is employed to prove the existence of subsonic-sonic flows. For a given incoming mass flux and flow speed at the upper wall, we solve a fixed boundary problem of a quasilinear degenerate elliptic equation. If the solved incoming mass flux and flow speed at the upper wall are just the given ones, we get the solution. Note that the problem we concerns is in unbounded domain, we get the solution to the fixed boundary problem by taking limits of the sequences of the solutions to the truncated problems. Like that in [21], it seems very hard to construct appropriate super and sub solutions to prove the existence of solutions to truncated problems without sufficiently small $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$. The method in [21] is used here: we first solve every regularized truncated problem when the flow speed at the outlet is suitable small and get priori estimates for the average and the derivatives of the solution, then we show the existence of the solution to the regularized truncated problem by use of the preliminaries obtained above, and finally we prove that their limit as the flow speed tends to be sonic at the outlet is a desired solution to the truncated problem. The difficulty here is that in order to get the solution to the fixed boundary problem by taking the limit of the solutions to the truncated problems, we must seek a suitable variation rate k_0 such that the solutions to all the truncated problems exit provided that $k \in (0, k_0]$. We overcome this difficulty by constructing complicated super and sub solutions to all the truncated problems. The Harnack's inequality is used to achieve the regularities and the asymptotic behaviors of the solution to the fixed boundary problem. As to the uniqueness of the subsonic-sonic flow, we first fix the free boundaries into fixed ones and transform the nonlocal boundary conditions into common ones by a proper coordinates transformation, and then we estiblish the uniqueness theorem by the energy estimates. Summing up, it is proved in this paper that if f satisfies (5) and (6), then there exists a unique subsonic-sonic flow to the problem (7)–(10) for suitably small k , and the flow speed is only $C^{1/2}$ Hölder continuous and the flow acceleration blows up at the sonic curve. Furthermore, the flow is uniformly subsonic at the far fields.

The paper is arranged as follows. In Section 2, we formulate the subsonic-sonic flow problem (7)–(10) in the potential plane. Then in Section 3, we solve the fixed boundary problem of a quasilinear degenerate elliptic equation in an unbounded domain. Finally in Section 4, we establish the well-posedness of the subsonic-sonic flow, and prove that the flow is uniformly subsonic at the far fields.

2. Formulation of the subsonic-sonic flow problem in the potential plane.

Define a velocity potential φ and a stream function ψ , respectively, by

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= u = q \cos \theta, & \frac{\partial \varphi}{\partial y} &= v = q \sin \theta, \\ \frac{\partial \psi}{\partial x} &= -\rho v = -\rho q \sin \theta, & \frac{\partial \psi}{\partial y} &= \rho u = \rho q \cos \theta, \end{aligned} \tag{11}$$

where θ is the flow angle. The system (1), (2) can be reduced to the Chaplygin equations ([2]):

$$\frac{\partial \theta}{\partial \psi} + \frac{\rho(q^2) + 2q^2 \rho'(q^2)}{q \rho^2(q^2)} \frac{\partial q}{\partial \varphi} = 0, \quad \frac{1}{q} \frac{\partial q}{\partial \psi} - \frac{1}{\rho(q^2)} \frac{\partial \theta}{\partial \varphi} = 0 \tag{12}$$

in the potential-stream coordinates (φ, ψ) . The coordinates transformation (11) between the two coordinate systems are valid at least in the absence of stagnation points. Eliminating θ from (12) yields the following quasilinear equation of second

order

$$\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0,$$

where

$$A(q) = \int_{c_*}^q \frac{\rho(s^2) + 2s^2 \rho'(s^2)}{s \rho^2(s^2)} ds, \quad B(q) = \int_{c_*}^q \frac{\rho(s^2)}{s} ds, \quad 0 < q < \sqrt{2/(\gamma - 1)}.$$

It is obvious that $B(\cdot)$ is strictly increasing in $(0, \sqrt{2/(\gamma - 1)})$, while $A(\cdot)$ is strictly increasing in $(0, c_*]$ and strictly decreasing in $[c_*, \sqrt{2/(\gamma - 1)})$. It follows from [21] that there exist two constants $0 < N_1 < N_2$ depending only on γ such that for $c_*/6 \leq q \leq c_*$,

$$N_1(c_* - q) \leq A'(q) \leq N_2(c_* - q), \quad N_1 \leq B'(q), -A''(q), -B''(q) \leq N_2, \quad (13)$$

$$N_1(c_* - q) \leq E'(B(q)) \leq N_2(c_* - q), \quad -N_2 \leq E''(B(q)), E'''(B(q)) \leq -N_1, \quad (14)$$

where $E = A \circ B^{-1}$ and B^{-1} is the inverse function of B . We use $A^{-1}(\cdot)$ to denote the inverse function of $A(\cdot)|_{(0, c_*]}$ in this paper. Additionally, the flow angle at the upper and the lower wall are

$$\Theta_{\text{up}}(x) = \arctan f'_k(x), \quad x \in [-l_0, 0] \quad \text{and} \quad \Theta_{\text{low}}(x) \equiv 0, \quad x \in (-\infty, 0),$$

respectively.

As in [18, 21], in order to describe the problem in the potential plane, we denote the flow speed at the upper wall by

$$Q_{\text{up}}(x) = q(x, f_k(x)), \quad x \in (-\infty, 0],$$

then the potential function at the upper wall is expressed by

$$\begin{aligned} \Phi_{\text{up}}(x) &= \int_0^x Q_{\text{up}}(s)(1 + (f'_k(s))^2)^{1/2} ds \\ &= \begin{cases} \int_0^x Q_{\text{up}}(s)(1 + (f'_k(s))^2)^{1/2} ds, & \text{if } x \in [-l_0, 0], \\ \zeta_0 + \int_{-l_0}^x Q_{\text{up}}(s) ds, & \text{if } x \in (-\infty, -l_0) \end{cases} \end{aligned} \quad (15)$$

with

$$\zeta_0 = \int_0^{-l_0} Q_{\text{up}}(s)(1 + (f'_k(s))^2)^{1/2} ds.$$

The inverse function of Φ_{up} is denoted by X_{up} . The subsonic-sonic flow problem (7)–(10) can be formulated in the potential plane as follows:

$$\frac{\partial^2 A(q)}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(q)}{\partial \psi^2}(\varphi, \psi) = 0, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m), \quad (16)$$

$$\frac{\partial q}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (-\infty, 0), \quad (17)$$

$$\frac{\partial B(q)}{\partial \psi}(\varphi, m) = \frac{f''_k(x)}{(1 + (f'_k(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)}, \quad \varphi \in (-\infty, 0), \quad (18)$$

$$q(0, \psi) = c_*, \quad \psi \in (0, m), \quad (19)$$

$$Q_{\text{up}}(x) = q(\varphi, m) \Big|_{\varphi=\Phi_{\text{up}}(x)}, \quad x \in (-\infty, 0], \quad (20)$$

where (q, m) is a solution with $m > 0$ being the incoming mass flux. Solutions to the problem (16)–(19) are defined as follows.

Definition 2.1. For $m > 0$, a function $q \in L^\infty((-\infty, 0) \times (0, m))$ is called a solution to the fixed boundary problem (16)–(19), if

$$0 < \inf_{(-\infty, 0) \times (0, m)} q \leq \sup_{(-\infty, 0) \times (0, m)} q \leq c_*$$

such that the integral equation

$$\begin{aligned} & \int_{-\infty}^0 \int_0^m \left(A(q(\varphi, \psi)) \frac{\partial^2 \xi}{\partial \varphi^2}(\varphi, \psi) + B(q(\varphi, \psi)) \frac{\partial^2 \xi}{\partial \psi^2}(\varphi, \psi) \right) d\psi d\varphi \\ & + \int_{-\infty}^0 \frac{f_k''(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)} \xi(\varphi, m) d\varphi = 0 \end{aligned}$$

holds for any $\xi \in C^2((-\infty, 0) \times [0, m])$ which vanishes for large $|\varphi|$ with

$$\frac{\partial \xi}{\partial \psi}(\cdot, 0) \Big|_{(-\infty, 0)} = \frac{\partial \xi}{\partial \psi}(\cdot, m) \Big|_{(-\infty, 0)} = \xi(0, \cdot) \Big|_{(0, m)} = 0.$$

The existence of solutions to the problem (16)–(20) will be proved by a fixed point argument. Give m and Q_{up} in advance as follows:

$$\delta_1 \leq m \leq \delta_2 \quad (21)$$

with

$$\delta_1 = \frac{c_* \rho (c_*^2/4) l_1}{2}, \quad \delta_2 = c_* \rho (c_*^2) (l_1 + f(-l_0)),$$

while $Q_{\text{up}} \in C^{1/4}((-\infty, 0])$ satisfies

$$\max \left\{ \frac{c_*}{2}, c_* - k^{1/4} \right\} \leq Q_{\text{up}}(x) \leq c_* \text{ for } x \in (-\infty, 0], \quad [Q_{\text{up}}]_{C^{1/4}((-\infty, 0])} \leq 1. \quad (22)$$

For such Q_{up} , it is clear that Φ_{up} and X_{up} are well determined. Direct calculations yield that

$$-\delta_4 \leq \zeta_0 \leq -\delta_3, \quad \frac{c_*}{2} \leq \Phi'_{\text{up}}(x) \leq \delta_5, \quad x \in (-\infty, 0], \quad (23)$$

$$\left| \frac{f_k''(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)} \right| \leq k \delta_6 (-\varphi)^{1/2} \chi_{[\zeta_0, 0]}(\varphi), \quad \varphi \in (-\infty, 0], \quad (24)$$

where $\chi_{[\zeta_0, 0]}(\varphi)$ is the characteristic function of the interval $[\zeta_0, 0]$, and

$$\begin{aligned} \delta_3 &= \frac{c_* l_0}{2}, \quad \delta_4 = c_* l_0 \left(1 + \|f'\|_{L^\infty((-l_0, 0))}^2 \right)^{1/2}, \\ \delta_5 &= c_* \left(1 + \|f'\|_{L^\infty((-l_0, 0))}^2 \right)^{1/2}, \quad \delta_6 = \|(-x)^{1/2} f''\|_{L^\infty((-l_0, 0))} \left(\frac{2}{c_*} \right)^{3/2}. \end{aligned}$$

For $f \in C^{2,\alpha}((-\infty, 0])$ satisfying (5) and (6), it follows from [21] that there exists a constant $\tilde{l}_0 \in (0, l_0)$ depending only on $f'(0)$ and $\|(-x)^{-1/2} f''\|_{L^\infty(-l_0, 0)}$ such that $2f'(0) \leq f'(x) \leq f'(0)/2$ for $x \in [-\tilde{l}_0, 0]$, and hence there exist two constants $0 < \tau_1 \leq \tau_2$ depending only on \tilde{l}_0 , l_0 , $f'(0)$, $\inf_{(-l_0, -\tilde{l}_0)} f$ and $\sup_{(-l_0, -\tilde{l}_0)} f$ such that

$$-\tau_1 x \leq f(x) \leq -\tau_2 x, \quad x \in [-l_0, 0]. \quad (25)$$

3. Fixed boundary problem of a quasilinear degenerate elliptic equation in an unbounded domain. In this section, we deal with the well-posedness of the fixed boundary problem. For the given m and $Q_{\text{up}} \in C^{1/4}((-\infty, 0])$ satisfy (21) and (22), respectively, we solve the degenerate elliptic problem (16)–(19). Since the problem is in an unbounded domain, we first deal with the truncated problem in $[\zeta_0 - n, 0] \times [0, m]$ with any sufficient large positive integer n , and make some useful compact estimates. Then we solve the problem (16)–(19) by a limit process. The key of the proof is seeking the variation rate k , which ensures the solutions to the truncated problems exist, is independent of n .

3.1. Well-posedness of the truncated problem. The truncated problem is written as

$$\frac{\partial^2 A(q_n)}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(q_n)}{\partial \psi^2}(\varphi, \psi) = 0, \quad (\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m), \quad (26)$$

$$\frac{\partial A(q_n)}{\partial \varphi}(\zeta_0 - n, \psi) = 0, \quad \psi \in (0, m), \quad (27)$$

$$\frac{\partial q_n}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (\zeta_0 - n, 0), \quad (28)$$

$$\frac{\partial B(q_n)}{\partial \psi}(\varphi, m) = \frac{f_k''(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)}, \quad \varphi \in (\zeta_0 - n, 0), \quad (29)$$

$$q_n(0, \psi) = c_*, \quad \psi \in (0, m). \quad (30)$$

Note that (26) is degenerate at $q_n = c_*$, we replace (30) with the following boundary condition

$$q_n(0, \psi) = c, \quad \psi \in (0, m), \quad (31)$$

where $c \in [c_*/3, c_*]$ is a constant, and consider the regularized truncated problem (26)–(29), (31). Then we solve the problem (26)–(30) by a limit process.

The proof can be divided into four steps.

Step 1. Well-posedness of the problem (26)–(29), (31) for $c \in [c_*/3, c_*/2]$.

Lemma 3.1. *Assume that $n \geq 2\delta_4 + 1$ and $c \in [c_*/3, c_*/2]$. There exists a constant $k_1 \in (0, 1]$ depending only on γ , l_0 , l_1 , $f(-l_0)$, $\|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$, such that if $k \in (0, k_1]$, then the problem (26)–(29), (31) admits a unique solution $q_{n,c} \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \cap C([\zeta_0 - n, 0] \times [0, m])$. Furthermore, $q_{n,c}$ satisfies*

$$c_*/6 \leq q_{n,c}(\varphi, \psi) < c_*, \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m], \quad (32)$$

$$q_{n,c}(\zeta_0 - n, \psi) \leq c_* - k^{3/4}, \quad \psi \in [0, m]. \quad (33)$$

Proof. The uniqueness result follows from Proposition 3.2 in [20]. Set

$$\begin{aligned} k_1 = \min \Big\{ & \left(\frac{c_*}{6} \right)^{4/3}, \left(\frac{c_*}{48\delta_2^2\delta_4^3} \right)^2, \left(\frac{c_*}{96\delta_4^3} \right)^4, \left(\frac{A(c_*/4) - A(c_*/6)}{8\delta_2^2\delta_4^3} \right)^2, \\ & \left(\frac{A(c_*/3) - A(c_*/4)}{16\delta_4^3} \right)^4, \left(\frac{2\delta_1\delta_4^{5/2} B'(5c_*/6)}{\delta_6} \right)^2, \left(\frac{2\delta_2\delta_4^{5/2} B'(c_*/6)}{\delta_6 A'(c_*/6)} \right)^2, \\ & \left(\frac{1}{\delta_2^2 e^{2\delta_4}} \right)^4, \left(\frac{3A'(5c_*/6)}{4\delta_4^2 e^{2\delta_4} B'(5c_*/6)} \right)^2 \Big\}. \end{aligned}$$

For $k \in (0, k_1]$, define

$$\begin{aligned}\bar{q}_{n,c}(\varphi, \psi) &= \frac{2}{3}c_* + \left(k^{1/2}\psi^2 + k^{1/4}(\varphi - 2)e^\varphi\right)\Lambda(\varphi), \\ &\quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m], \\ \underline{q}_{n,c}(\varphi, \psi) &= A^{-1}\left(A(c_*/4) - \left(k^{1/2}\psi^2 + k^{1/4}(\varphi - 2)e^\varphi\right)\Lambda(\varphi)\right), \\ &\quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m],\end{aligned}$$

where

$$\Lambda(\varphi) = \max\{0, (\varphi + 2\delta_4)^3\}, \quad \varphi \in (-\infty, 0].$$

Thanks to (13), (14), (23) and (24), direct calculations show that

$$\begin{aligned}\frac{c_*}{2} \leq \bar{q}_{n,c}(\varphi, \psi) &\leq \frac{5c_*}{6}, \quad \frac{c_*}{6} \leq \underline{q}_{n,c}(\varphi, \psi) \leq \frac{c_*}{3}, \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m], \\ \frac{\partial A(\bar{q}_{n,c})}{\partial \varphi}(\zeta_0 - n, \psi) &= \frac{\partial A(\underline{q}_{n,c})}{\partial \varphi}(\zeta_0 - n, \psi) = 0, \quad \psi \in (0, m), \\ \frac{\partial \bar{q}_{n,c}}{\partial \psi}(\varphi, 0) &= \frac{\partial \underline{q}_{n,c}}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (\zeta_0 - n, 0), \\ \frac{\partial B(\bar{q}_{n,c})}{\partial \psi}(\varphi, m) &= 2k^{1/2}mB'(\bar{q}_{n,c}(\varphi, m))\Lambda(\varphi) \\ &\geq 2k^{1/2}\delta_2\delta_4^3B'(5c_*/6)\chi_{[\zeta_0, 0]}(\varphi) \\ &\geq k\delta_6(-\varphi)^{1/2}\chi_{[\zeta_0, 0]}(\varphi), \quad \varphi \in (\zeta_0 - n, 0), \\ \frac{\partial B(\underline{q}_{n,c})}{\partial \psi}(\varphi, m) &= -2k^{1/2}m\frac{B'(\underline{q}_{n,c}(\varphi, m))}{A'(\underline{q}_{n,c}(\varphi, m))}\Lambda(\varphi) \\ &\leq -2k^{1/2}\delta_2\delta_4^3\frac{B'(c_*/6)}{A'(c_*/6)}\chi_{[\zeta_0, 0]}(\varphi) \\ &\leq -k\delta_6(-\varphi)^{1/2}\chi_{[\zeta_0, 0]}(\varphi), \quad \varphi \in (\zeta_0 - n, 0), \\ \frac{\partial^2 A(\bar{q}_{n,c})}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(\bar{q}_{n,c})}{\partial \psi^2}(\varphi, \psi) &\leq B'(\bar{q}_{n,c}(\varphi, \psi))\left(\frac{A'(\bar{q}_{n,c}(\varphi, \psi))}{B'(\bar{q}_{n,c}(\varphi, \psi))}\frac{\partial^2 \bar{q}_{n,c}}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 \bar{q}_{n,c}}{\partial \psi^2}(\varphi, \psi)\right) \\ &\leq 2k^{1/4}B'(\bar{q}_{n,c}(\varphi, \psi))(\varphi + 2\delta_4) \\ &\quad \times \left(\frac{A'(5c_*/6)}{B'(5c_*/6)}(3k^{1/4}\delta_2^2 - 6e^{-2\delta_4}) + 4k^{1/4}\delta_4^2\right)\chi_{[-2\delta_4, 0]}(\varphi) \\ &\leq 2k^{1/4}B'(\bar{q}_{n,c}(\varphi, \psi))(\varphi + 2\delta_4) \\ &\quad \times \left(-3e^{-2\delta_4}\frac{A'(5c_*/6)}{B'(5c_*/6)} + 4k^{1/4}\delta_4^2\right)\chi_{[-2\delta_4, 0]}(\varphi) \\ &\leq 0, \quad (\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m),\end{aligned}$$

and

$$\frac{\partial^2 A(\underline{q}_{n,c})}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(\underline{q}_{n,c})}{\partial \psi^2}(\varphi, \psi)$$

$$\begin{aligned}
&\geq \frac{\partial^2 A(\underline{q}_{n,c})}{\partial \varphi^2}(\varphi, \psi) + \frac{B'(\underline{q}_{n,c}(\varphi, \psi))}{A'(\underline{q}_{n,c}(\varphi, \psi))} \frac{\partial^2 A(\underline{q}_{n,c})}{\partial \psi^2}(\varphi, \psi) \\
&\geq 2k^{1/4}(\varphi + 2\delta_4) \left(6e^{-2\delta_4} - 3k^{1/4}\delta_2^2 - 4k^{1/4}\delta_4^2 \frac{B'(c_*/3)}{A'(c_*/3)} \right) \chi_{[-2\delta_4, 0]}(\varphi) \\
&\geq 2k^{1/4}(\varphi + 2\delta_4) \left(3e^{-2\delta_4} - 4k^{1/4}\delta_4^2 \frac{B'(c_*/3)}{A'(c_*/3)} \right) \chi_{[-2\delta_4, 0]}(\varphi) \\
&\geq 0, \quad (\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m),
\end{aligned}$$

where $\chi_{[-2\delta_4, 0]}(\varphi)$ is the characteristic function of the interval $[-2\delta_4, 0]$. Therefore, $\bar{q}_{n,c}$ and $\underline{q}_{n,c}$ are a supersolution and a subsolution to the problem (26)–(29), (31), respectively. Thanks to the comparison principle (Proposition 3.2 in [20]) and a standard argument in the classical theory for elliptic equations, one can complete the lemma. \square

Step 2. A priori estimates of the average of solutions to the problem (26)–(29), (31).

Lemma 3.2. *Assume that $n \geq 2\delta_4 + 1$, $c \in [c_*/3, c_*]$ and $q_{n,c} \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \cap C([\zeta_0 - n, 0] \times [0, m])$ is a solution to the problem (26)–(29), (31). Then*

$$\frac{1}{m} \int_0^m A(q_{n,c}(\varphi, \psi)) d\psi = \frac{1}{m} \int_0^m A(q_{n,c}(\zeta_0, \psi)) d\psi, \quad \varphi \in [\zeta_0 - n, \zeta_0]. \quad (34)$$

Furthermore, there exist three constants $k_2 \in (0, 1]$ and $0 < \sigma_1 \leq \sigma_2$ depending only on γ , τ_1 , τ_2 and $\|f'\|_{L^\infty((-l_0, 0))}$ such that if $k \in (0, k_2]$, then

$$\begin{aligned}
A(c) - k\sigma_2 \min\{-\varphi, -\zeta_0\} \leq \frac{1}{m} \int_0^m A(q_{n,c}(\varphi, \psi)) d\psi \leq A(c) - k\sigma_1 \min\{-\varphi, -\zeta_0\}, \\
\varphi \in [\zeta_0 - n, 0].
\end{aligned} \quad (35)$$

Proof. The proof is similar to the proof of Lemma 3.2 in [21]. Integrating (26) over $(0, m)$ with respect to ψ and using (28) and (29) show that

$$\frac{d^2}{d\varphi^2} \int_0^m A(q_{n,c}(\varphi, \psi)) d\psi = - \frac{f_k''(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)}, \quad \varphi \in (\zeta_0 - n, 0). \quad (36)$$

And (27) yields that

$$\frac{d}{d\varphi} \int_0^m A(q_{n,c}(\zeta_0 - n, \psi)) d\psi = 0. \quad (37)$$

One gets from (6), (36) and (37) that

$$\frac{d}{d\varphi} \int_0^m A(q_{n,c}(\varphi, \psi)) d\psi = 0, \quad \varphi \in [\zeta_0 - n, \zeta_0], \quad (38)$$

and

$$\begin{aligned}
\frac{d}{d\varphi} \int_0^m A(q_{n,c}(\varphi, \psi)) d\psi &= - \int_{\zeta_0}^{\varphi} \frac{f_k''(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(s)} ds \\
&= - \int_{-l_0}^{X_{\text{up}}(\varphi)} \frac{f_k''(x) \Phi'_{\text{up}}(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} dx
\end{aligned}$$

$$\begin{aligned}
&= - \int_{-l_0}^{X_{\text{up}}(\varphi)} (\arctan f'_k(x))' dx \\
&= - \arctan f'_k(X_{\text{up}}(\varphi)), \quad \varphi \in [\zeta_0, 0].
\end{aligned} \tag{39}$$

Thus (34) follows from (38). As in the proof of Lemma 3.2 in [21], it follows from (15) and (39) that

$$\begin{aligned}
&\frac{1}{m} \int_0^m A(q_{n,c}(\varphi, \psi)) d\psi \\
&= \frac{1}{m} \int_0^m A(q_{n,c}(0, \psi)) d\psi + \frac{1}{m} \int_{\varphi}^0 \arctan f'_k(X_{\text{up}}(\tilde{\varphi})) d\tilde{\varphi} \\
&= A(c) + \frac{1}{m} \int_{\varphi}^0 \arctan f'_k(X_{\text{up}}(\tilde{\varphi})) d\tilde{\varphi} \\
&= A(c) - kc_* f(X_{\text{up}}(\varphi)) + O(k^{5/4}), \quad \varphi \in [\zeta_0, 0],
\end{aligned} \tag{40}$$

where $O(\cdot)$ depend only on $\|f'\|_{L^\infty((-l_0, 0))}$. Using (25), (34) and (40), we can obtain (35). \square

Step 3. A priori derivative estimates of solutions to the problem (26)–(29), (31).

Lemma 3.3. *Assume that $n \geq 2\delta_4 + 1$, $c \in [c_*/3, c_*]$, and $q_{n,c} \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \cap C([\zeta_0 - n, 0] \times [0, m])$ is a solution to the problem (26)–(29), (31) satisfying (32) and (33). Then for $k \in (0, 1]$,*

$$\left| \frac{\partial q_{n,c}}{\partial \psi}(\varphi, \psi) \right| \leq k\sigma_3(\min\{-\varphi, -\zeta_0\})^{1/2}, \quad (\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m), \tag{41}$$

$$|A(q_{n,c}(\varphi_1, \psi_1)) - A(q_{n,c}(\varphi_2, \psi_2))| \leq k\sigma_4(|\varphi_1 - \varphi_2|^{1/2} + |\psi_1 - \psi_2|), \quad (\varphi_1, \psi_1), (\varphi_2, \psi_2) \in [\zeta_0 - n, 0] \times [0, m], \tag{42}$$

where σ_3 and σ_4 are positive constants depending only on γ , l_0 , l_1 , $f(-l_0)$, $\|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$.

Proof. The proof is similar to Proposition 3.2 in [20]. Set

$$z(\varphi, \psi) = \frac{\partial B(q_{n,c})}{\partial \psi}(\varphi, \psi), \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m].$$

Then $z \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C([\zeta_0 - n, 0] \times [0, m])$ solves the problem

$$\begin{aligned}
j_1(\varphi, \psi) \frac{\partial^2 z}{\partial \varphi^2} + \frac{\partial^2 z}{\partial \psi^2} + j_2(\varphi, \psi) \frac{\partial z}{\partial \varphi} + j_3(\varphi, \psi) \frac{\partial z}{\partial \psi} + j_4(\varphi, \psi) z = 0, \\
(\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m),
\end{aligned} \tag{43}$$

$$\frac{\partial z}{\partial \varphi}(\zeta_0 - n, \psi) = 0, \quad \psi \in (0, m), \tag{44}$$

$$z(\varphi, 0) = 0, \quad \varphi \in (\zeta_0 - n, 0), \tag{45}$$

$$z(\varphi, m) = \frac{f''_k(x)}{(1 + (f'_k(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)}, \quad \varphi \in (\zeta_0 - n, 0), \tag{46}$$

$$z(0, \psi) = 0, \quad \psi \in (0, m), \tag{47}$$

where $j_i \in C^\infty((\zeta_0, 0) \times (0, m))$ ($1 \leq i \leq 4$) are defined by

$$j_1 = E'(B(q_{n,c})) > 0,$$

$$\begin{aligned}
j_2 &= \frac{E''(B(q_{n,c}))}{E'(B(q_{n,c}))} \frac{\partial A(q_{n,c})}{\partial \varphi}, \\
j_3 &= -\frac{E''(B(q_{n,c}))}{E'(B(q_{n,c}))} \frac{\partial B(q_{n,c})}{\partial \psi}, \\
j_4 &= \left(\frac{E'''(B(q_{n,c}))}{(E'(B(q_{n,c})))^2} - \frac{(E''(B(q_{n,c})))^2}{(E'(B(q_{n,c})))^3} \right) \left(\frac{\partial A(q_{n,c})}{\partial \varphi} \right)^2 \\
&\leq -\frac{(E''(B(q_{n,c})))^2}{(E'(B(q_{n,c})))^3} \left(\frac{\partial A(q_{n,c})}{\partial \varphi} \right)^2 \leq 0
\end{aligned}$$

and E satisfies (14). It is clear that

$$\begin{aligned}
&\frac{1}{4}j_1(\varphi, \psi)(-\varphi)^{-3/2} - j_4(\varphi, \psi)(-\varphi)^{1/2} \\
&\geq \sqrt{-j_1(\varphi, \psi)j_4(\varphi, \psi)}(-\varphi)^{-1/2} \\
&\geq -\frac{1}{2}j_2(\varphi, \psi)(-\varphi)^{-1/2}, \quad (\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m).
\end{aligned}$$

Due to (24), one can show that

$$z_{\pm}(\varphi, \psi) = \pm k\delta_6(-\varphi)^{1/2}, \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m]$$

are a supersolution and a subsolution to the problem (43)–(47), respectively. The comparison principle (Proposition 3.2 in [20]) implies that

$$|z(\varphi, \psi)| \leq k\delta_6(-\varphi)^{1/2}, \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m]. \quad (48)$$

Define

$$\tilde{z}_{\pm}(\varphi, \psi) = \pm k\delta_6(-\zeta_0)^{1/2}, \quad (\varphi, \psi) \in [\zeta_0 - n, \zeta_0] \times [0, m].$$

It is easy to verify that \tilde{z}_{\pm} are a supersolution and subsolution to the following problem

$$\begin{aligned}
&j_1(\varphi, \psi) \frac{\partial^2 z}{\partial \varphi^2} + \frac{\partial^2 z}{\partial \psi^2} + j_2(\varphi, \psi) \frac{\partial z}{\partial \varphi} + j_3(\varphi, \psi) \frac{\partial z}{\partial \psi} + j_4(\varphi, \psi)z = 0, \\
&(\varphi, \psi) \in (\zeta_0 - n, \zeta_0) \times (0, m), \\
&\frac{\partial z}{\partial \varphi}(\zeta_0 - n, \psi) = 0, \quad \psi \in (0, m), \\
&z(\varphi, 0) = 0, \quad \varphi \in (\zeta_0 - n, \zeta_0), \\
&z(\varphi, m) = 0, \quad \varphi \in (\zeta_0 - n, \zeta_0), \\
&z(\zeta_0, \psi) = z(\zeta_0, \psi), \quad \psi \in (0, m),
\end{aligned}$$

respectively. The comparison principle shows that

$$|z(\varphi, \psi)| \leq k\delta_6(-\zeta_0)^{1/2}, \quad (\varphi, \psi) \in [\zeta_0 - n, \zeta_0] \times [0, m],$$

which, together with (48), leads to (41). Finally, (42) can be proved in the same way as the proof of Proposition 3.2 in [20]. \square

Step 4. Well-posedness of the truncated problem (26)–(30).

Lemma 3.4. *Assume that $n \geq 2\delta_4 + 1$. There exists a constant $0 < k_3 \leq \min\{k_1, k_2\}$ depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and*

$\|(-x)^{-1/2}f''\|_{L^\infty((-l_0,0))}$, such that if $k \in (0, k_3]$, then the problem (26)–(30) admits a unique solution $q_n \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \cap C([\zeta_0 - n, 0] \times [0, m])$ satisfies

$$\left| \frac{\partial q_n}{\partial \psi}(\varphi, \psi) \right| \leq k\sigma_3(\min\{-\varphi, -\zeta_0\})^{1/2}, \quad (\varphi, \psi) \in (\zeta_0 - n, 0) \times (0, m), \quad (49)$$

$$|A(q_n(\varphi_1, \psi_1)) - A(q_n(\varphi_2, \psi_2))| \leq k\sigma_4(|\varphi_1 - \varphi_2|^{1/2} + |\psi_1 - \psi_2|), \\ (\varphi_1, \psi_1), (\varphi_2, \psi_2) \in [\zeta_0 - n, 0] \times [0, m], \quad (50)$$

$$c_* - \sigma_6 k^{1/2}(\min\{-\varphi, -\zeta_0\})^{1/2} \leq q_n(\varphi, \psi) \leq c_* - \sigma_5 k^{1/2}(\min\{-\varphi, -\zeta_0\})^{1/2}, \\ (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m], \quad (51)$$

where $0 < \sigma_5 \leq \sigma_6$ are constants depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0,0))}$ and $\|(-x)^{-1/2}f''\|_{L^\infty((-l_0,0))}$.

Proof. The uniqueness result follows from Proposition 3.2 in [20]. For $0 < k \leq \min\{k_1, k_2\}$, set

$$\mathcal{C}_k = \left\{ c \in [c_*/3, c_*] : \text{the problem (26)–(29), (31) admits a solution} \right. \\ q_{n,c} \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \\ \left. \text{with (32) and (33)} \right\}.$$

It follows from Lemma 3.1 and the comparison principle (Proposition 3.2 in [20]) that \mathcal{C}_k is a nonempty interval. Assume that $c \in \mathcal{C}_k$. For $\varphi \in [\zeta_0 - n, 0]$, thanks to $c \in \mathcal{C}_k$, (13) and (35), there exists a number $\psi_\varphi \in (0, m)$ such that

$$q_{n,c}(\varphi, \psi_\varphi) \leq c_* - \left(\frac{k\sigma_1}{N_2} \right)^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2},$$

which, together with (41), yields

$$q_{n,c}(\varphi, \psi) = q_{n,c}(\varphi, \psi_\varphi) + \int_{\psi_\varphi}^{\psi} \frac{\partial q_{n,c}}{\partial \psi}(\varphi, \tilde{\psi}) d\tilde{\psi} \\ \leq c_* - \left(\left(\frac{\sigma_1}{N_2} \right)^{1/2} - k^{1/2} \sigma_3 \delta_2 \right) k^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2}, \\ (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m]. \quad (52)$$

Choose

$$\sigma_5 = \left(\frac{\sigma_1}{4N_2} \right)^{1/2}, \quad k_3 = \min \left\{ k_1, k_2, \frac{\sigma_1}{4\sigma_3^2 \delta_2^2 N_2}, \frac{\sigma_5^4 \delta_4^2}{16} \right\}.$$

For $0 < k \leq k_3$, one gets from $c \in \mathcal{C}_k$, (23) and (52) that

$$c_*/4 \leq q_{n,c}(\varphi, \psi) \leq c_* - \sigma_5 k^{1/2}(\min\{-\varphi, -\zeta_0\})^{1/2}, \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m], \quad (53)$$

$$q_{n,c}(\zeta_0 - n, \psi) \leq c_* - 2k^{3/4}, \quad \psi \in [0, m]. \quad (54)$$

It follows from $c \in \mathcal{C}_k$, (31) and (42) that

$$|A(q_{n,c}(\varphi, \psi)) - A(c)| \leq k\sigma_4(-\varphi)^{1/2}, \quad \psi \in [0, m]. \quad (55)$$

Thanks to (53)–(55), one can prove from the comparison principle (Proposition 3.2 in [20]) and the continuous dependence of solutions to the problem (26)–(29), (31) that $\mathcal{C}_k = [c_*/3, c_*]$ for $0 < k \leq k_3$.

Let $0 < k \leq k_3$. For $c_*/3 \leq c_1 < c_2 < c_*$, the comparison principle (Proposition 3.2 in [20]) gives

$$q_{n,c_1}(\varphi, \psi) \leq q_{n,c_2}(\varphi, \psi), \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m].$$

Set

$$q_n(\varphi, \psi) = \lim_{c \rightarrow c_*^-} q_{n,c}(\varphi, \psi), \quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m].$$

Due to (41), (42) and (53), it is clear that q_n is a solution to the problem (26)–(30), and q_n satisfies (49), (50) and the second inequality in (51). For $\varphi \in [\zeta_0 - n, 0]$, it follows from (35) and (13) that there exists a number $\tilde{\psi}_\varphi \in (0, m)$ such that

$$q_n(\varphi, \tilde{\psi}_\varphi) \geq c_* - \left(\frac{k\sigma_2}{N_1} \right)^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2}.$$

This estimate above and (49) yield

$$\begin{aligned} q_n(\varphi, \psi) &= q_n(\varphi, \tilde{\psi}_\varphi) + \int_{\tilde{\psi}_\varphi}^{\psi} \frac{\partial q_n}{\partial \psi}(\varphi, \tilde{\psi}) d\tilde{\psi} \\ &\geq c_* - \left(\left(\frac{\sigma_2}{N_1} \right)^{1/2} + k^{1/2} \sigma_3 \delta_2 \right) k^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2}, \\ &\quad (\varphi, \psi) \in [\zeta_0 - n, 0] \times [0, m]. \end{aligned}$$

Hence the first inequality in (51) holds for $\sigma_6 = (\sigma_2/N_1)^{1/2} + \sigma_3 \delta_2$. Finally, the Schauder theory for elliptic equations shows that $q_n \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \cap C([\zeta_0 - n, 0] \times [0, m])$. \square

3.2. Well-posedness of the fixed boundary problem. Let us establish the existence of the solution to the problem (16)–(19).

Proposition 1. *Assume that $k \in (0, k_3]$, then the problem (16)–(19) admits a solution $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m])$ satisfies*

$$\left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq k\sigma_3 (\min\{-\varphi, -\zeta_0\})^{1/2}, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m), \quad (56)$$

$$\begin{aligned} |A(q(\varphi_1, \psi_1)) - A(q(\varphi_2, \psi_2))| &\leq k\sigma_4 (|\varphi_1 - \varphi_2|^{1/2} + |\psi_1 - \psi_2|), \\ &\quad (\varphi_1, \psi_1), (\varphi_2, \psi_2) \in (-\infty, 0] \times [0, m], \end{aligned} \quad (57)$$

$$\begin{aligned} c_* - \sigma_6 k^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2} &\leq q(\varphi, \psi) \leq c_* - \sigma_5 k^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2}, \\ &\quad (\varphi, \psi) \in (-\infty, 0] \times [0, m], \end{aligned} \quad (58)$$

where $\sigma_3, \sigma_4, \sigma_5$ and σ_6 are given in Lemmas 3.3 and 3.4. Furthermore,

$$\frac{1}{m} \int_0^m A(q(\varphi, \psi)) d\psi = A(q_\infty), \quad \varphi \in (-\infty, \zeta_0], \quad (59)$$

where

$$q_\infty = A^{-1} \left(\frac{1}{m} \int_0^m A(q(\zeta_0, \psi)) d\psi \right) \in [c_* - \sigma_6 k^{1/2} (-\zeta_0)^{1/2}, c_* - \sigma_5 k^{1/2} (-\zeta_0)^{1/2}]. \quad (60)$$

Proof. For any $n > 2\delta_4 + 1$, the truncated problem (26)–(30) admits a unique solution

$$q_n \in C^\infty((\zeta_0 - n, 0) \times (0, m)) \cap C^1([\zeta_0 - n, 0] \times [0, m]) \cap C^{1/2}([\zeta_0 - n, 0] \times [0, m])$$

satisfying (49)–(51). Therefore, there exists a subsequence of $\{q_n\}$ weakly star converging to a function q in $L^\infty((-\infty, 0) \times (0, m))$, and q satisfies (58). It is not hard to check that q is a solution to the problem (16)–(19), and q satisfies (56)–(58). Finally, the Schauder theory for elliptic equations yields that

$$q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m]).$$

Integrating (16) over $(0, m)$ with respect to ψ and using (6), (17) and (18) lead to that

$$\frac{d^2}{d\varphi^2} \int_0^m A(q(\varphi, \psi)) d\psi = 0, \quad \varphi \in (-\infty, \zeta_0),$$

and then there exists some constant C such that

$$\frac{d}{d\varphi} \int_0^m A(q(\varphi, \psi)) d\psi = C, \quad \varphi \in (-\infty, \zeta_0), \quad (61)$$

which implies that

$$\int_0^m A(q(\varphi, \psi)) d\psi = \int_0^m A(q(\zeta_0, \psi)) d\psi + C(\varphi - \zeta_0), \quad \varphi \in (-\infty, \zeta_0). \quad (62)$$

It follows from (57) and (62) that

$$\begin{aligned} |C||\varphi - \zeta_0| &\leq \int_0^m |A(q(\varphi, \psi)) - A(q(\zeta_0, \psi))| d\psi \\ &\leq k\sigma_4\delta_2|\varphi - \zeta_0|^{1/2}, \quad \varphi \in (-\infty, \zeta_0), \end{aligned}$$

that is,

$$|C| \leq k\sigma_4\delta_2|\varphi - \zeta_0|^{-1/2}, \quad \varphi \in (-\infty, \zeta_0). \quad (63)$$

One can get $C = 0$ by taking $\varphi \rightarrow -\infty$ in (63), and then (61) implies that

$$\frac{1}{m} \int_0^m A(q(\varphi, \psi)) d\psi = \frac{1}{m} \int_0^m A(q(\zeta_0, \psi)) d\psi, \quad \varphi \in (-\infty, \zeta_0).$$

Therefore, (59) holds. \square

The solution to the problem (16)–(19) has the following regularity and asymptotic behavior.

Proposition 2. *Assume that q is a solution to the problem (16)–(19) satisfying Proposition 1. Then $q \in C^{1/2}([2\zeta_0, 0] \times [0, m])$ and*

$$\left| \frac{\partial q}{\partial \varphi}(\varphi, \psi) \right| \leq \sigma_7 k^{1/4}(-\varphi)^{-1/2}, \quad (\varphi, \psi) \in [2\zeta_0, 0) \times (0, m), \quad (64)$$

where σ_7 is a positive constants depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2}f''\|_{L^\infty((-l_0, 0))}$. Moreover, it holds that

$$\begin{aligned} \left| \frac{\partial q}{\partial \varphi}(\varphi, \psi) \right| &\leq \sigma_8 k^{1/2}(-\varphi)^{-2}, \quad \left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq \sigma_8 k(-\varphi)^{-2}, \\ &(\varphi, \psi) \in (-\infty, 2\zeta_0) \times (0, m), \end{aligned} \quad (65)$$

and hence

$$\|q(\varphi, \psi) - q_\infty\|_{L^\infty((-\infty, \zeta) \times (0, m))} \leq \sigma_9 k(-\zeta)^{-2}, \quad \zeta \in (-\infty, 2\zeta_0), \quad (66)$$

where q_∞ is given in (60), and $\sigma_8, \sigma_9 > 0$ depend only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2}f''\|_{L^\infty((-l_0, 0))}$.

Proof. Similarly to the proof of Proposition 4.1 in [18], one can prove that $q \in C^{1/2}([2\zeta_0, 0] \times [0, m])$ and satisfies (64).

In the remaining of the proof, we use μ_i ($1 \leq i \leq 11$) to denote a generic positive constant depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$. It follows from (59) that for any $\varphi \in (-\infty, \zeta_0]$, there exists a number $\psi_\varphi \in (0, m)$ such that

$$q(\varphi, \psi_\varphi) = q_\infty,$$

which, together with (56), yields

$$\|q(\varphi, \psi) - q_\infty\|_{L^\infty((-\infty, \zeta_0) \times (0, m))} \leq \int_0^m \left\| \frac{\partial q}{\partial \psi} \right\|_{L^\infty((-\infty, \zeta_0) \times (0, m))} d\psi \leq \mu_1 k. \quad (67)$$

Note that $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m])$ solves

$$\begin{aligned} \frac{\partial}{\partial \varphi} \left(a(\varphi, \psi) \frac{\partial q}{\partial \varphi} \right) + \frac{\partial}{\partial \psi} \left(b(\varphi, \psi) \frac{\partial q}{\partial \psi} \right) &= 0, & (\varphi, \psi) \in (-\infty, \zeta_0) \times (0, m), \\ \frac{\partial q}{\partial \psi}(\varphi, 0) &= 0, & \varphi \in (-\infty, \zeta_0), \\ \frac{\partial q}{\partial \psi}(\varphi, m) &= 0, & \varphi \in (-\infty, \zeta_0), \end{aligned}$$

where

$$a(\varphi, \psi) = A'(q(\varphi, \psi)), \quad b(\varphi, \psi) = B'(q(\varphi, \psi)), \quad (\varphi, \psi) \in (-\infty, \zeta_0) \times (0, m).$$

Fix integer $n \geq 2$. Introducing

$$\begin{cases} \hat{\varphi} = k^{-1/4}(\varphi - n\zeta_0)/n, & \varphi \in [4n\zeta_0, n\zeta_0/2], \\ \hat{\psi} = \psi/n, & \psi \in [0, m], \end{cases}$$

and setting

$$\hat{q}(\hat{\varphi}, \hat{\psi}) = q(n\zeta_0 + k^{1/4}n\hat{\varphi}, n\hat{\psi}) - q_\infty, \quad (\hat{\varphi}, \hat{\psi}) \in [3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2] \times [0, m/n].$$

One can verify that

$$\hat{q} \in C^\infty((3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2) \times (0, m/n)) \cap C^1([3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2] \times [0, m/n])$$

solves

$$\begin{aligned} \frac{\partial}{\partial \hat{\varphi}} \left(k^{-1/2} \hat{a}(\hat{\varphi}, \hat{\psi}) \frac{\partial \hat{q}}{\partial \hat{\varphi}} \right) + \frac{\partial}{\partial \hat{\psi}} \left(\hat{b}(\hat{\varphi}, \hat{\psi}) \frac{\partial \hat{q}}{\partial \hat{\psi}} \right) &= 0, \\ (\hat{\varphi}, \hat{\psi}) \in (3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2) \times (0, m/n), \end{aligned} \quad (68)$$

$$\frac{\partial \hat{q}}{\partial \hat{\psi}}(\hat{\varphi}, 0) = 0, \quad \hat{\varphi} \in (3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2), \quad (69)$$

$$\frac{\partial \hat{q}}{\partial \hat{\psi}}(\hat{\varphi}, m/n) = 0, \quad \hat{\varphi} \in (3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2), \quad (70)$$

where

$$\begin{aligned} \hat{a}(\hat{\varphi}, \hat{\psi}) &= a(n\zeta_0 + k^{1/4}n\hat{\varphi}, n\hat{\psi}), & \hat{b}(\hat{\varphi}, \hat{\psi}) &= b(n\zeta_0 + k^{1/4}n\hat{\varphi}, n\hat{\psi}), \\ (\hat{\varphi}, \hat{\psi}) &\in [3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2] \times [0, m/n]. \end{aligned}$$

Extending the problem (68)–(70) into the domain $[3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2] \times [0, 2m]$ yields

$$\begin{aligned} \frac{\partial}{\partial \check{\varphi}} \left(k^{-1/2} \check{a}(\check{\varphi}, \check{\psi}) \frac{\partial \check{q}}{\partial \check{\varphi}} \right) + \frac{\partial}{\partial \check{\psi}} \left(\check{b}(\check{\varphi}, \check{\psi}) \frac{\partial \check{q}}{\partial \check{\psi}} \right) &= 0, \\ (\check{\varphi}, \check{\psi}) &\in (3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2) \times (0, 2m), \\ \frac{\partial \check{q}}{\partial \check{\psi}}(\check{\varphi}, 0) &= 0, \quad \check{\varphi} \in (3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2), \\ \frac{\partial \check{q}}{\partial \check{\psi}}(\check{\varphi}, 2m) &= 0, \quad \check{\varphi} \in (3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2), \end{aligned}$$

where for $(\check{\varphi}, \check{\psi}) \in [3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2] \times [(i-1)m/n, im/n]$ ($1 \leq i \leq 2n$),

$$\begin{aligned} \check{a}(\check{\varphi}, \check{\psi}) &= \begin{cases} \hat{a}(\check{\varphi}, \check{\psi} - (i-1)m/n), & \text{if } i \text{ is odd,} \\ \hat{a}(\check{\varphi}, im/n - \check{\psi}), & \text{if } i \text{ is even,} \end{cases} \\ \check{b}(\check{\varphi}, \check{\psi}) &= \begin{cases} \hat{b}(\check{\varphi}, \check{\psi} - (i-1)m/n), & \text{if } i \text{ is odd,} \\ \hat{b}(\check{\varphi}, im/n - \check{\psi}), & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Duo to (13), (51) and (67), one gets that

$$\begin{aligned} \mu_2 k^{1/2} \leq \check{a}(\check{\varphi}, \check{\psi}) \leq \mu_3 k^{1/2}, \quad \mu_2 \leq \check{b}(\check{\varphi}, \check{\psi}) \leq \mu_3, \\ (\check{\varphi}, \check{\psi}) \in [-4k^{-1/4}, 3k^{-1/4}\varepsilon/(4n)] \times [0, 2m], \end{aligned}$$

and

$$\|\check{q}\|_{L^\infty((3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2) \times (0, 2m))} \leq \mu_1 k.$$

It follows from the Hölder continuity estimates for uniformly elliptic equations that there exists a number $\beta \in (0, 1)$ such that

$$[\check{q}]_{\beta; (5k^{-1/4}\zeta_0/2, -k^{-1/4}\zeta_0/4) \times (0, 2m)} \leq \mu_4 \|\check{q}\|_{L^\infty((3k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/2) \times (0, 2m))} \leq \mu_5 k,$$

which implies

$$\begin{aligned} [\check{a}]_{\beta; (5k^{-1/4}\zeta_0/2, -k^{-1/4}\zeta_0/4) \times (0, 2m)} &\leq \mu_6 k, \\ [\check{b}]_{\beta; (5k^{-1/4}\zeta_0/2, -k^{-1/4}\zeta_0/4) \times (0, 2m)} &\leq \mu_6 k. \end{aligned}$$

The Schauder estimates on uniformly elliptic equations imply that

$$\begin{aligned} \|\check{q}\|_{C^{1,\beta}((2k^{-1/4}\zeta_0, -k^{-1/4}\zeta_0/8) \times (0, 2m))} &\leq \mu_7 \|\check{q}\|_{L^\infty((5k^{-1/4}\zeta_0/2, -k^{-1/4}\zeta_0/4) \times (0, 2m))} \\ &\leq \mu_8 k. \end{aligned} \tag{71}$$

Transforming (71) into the (φ, ψ) plane, one can get that

$$\begin{aligned} \left\| \frac{\partial q}{\partial \varphi} \right\|_{L^\infty((3n\zeta_0, 3n\zeta_0/4) \times (0, m))} &\leq \mu_9 k^{3/4} n^{-1}, \\ \left\| \frac{\partial q}{\partial \psi} \right\|_{L^\infty((3n\zeta_0, 3n\zeta_0/4) \times (0, m))} &\leq \mu_9 k n^{-1}. \end{aligned} \tag{72}$$

Similar to (67), we have from (72) that

$$\begin{aligned} \|q(\varphi, \psi) - q_\infty\|_{L^\infty((3n\zeta_0, 3n\zeta_0/4) \times (0, m))} &\leq \int_0^m \left\| \frac{\partial q_n}{\partial \psi} \right\|_{L^\infty((3n\zeta_0, 3n\zeta_0/4) \times (0, m))} d\psi \\ &\leq \mu_{10} k n^{-1}. \end{aligned} \tag{73}$$

Using (73) and the same operation on q leads to that

$$\begin{aligned}\left\|\frac{\partial q}{\partial \varphi}\right\|_{L^\infty((2n\zeta_0, n\zeta_0) \times (0, m))} &\leq \mu_{11} k^{1/2} n^{-2}, \\ \left\|\frac{\partial q}{\partial \psi}\right\|_{L^\infty((2n\zeta_0, n\zeta_0) \times (0, m))} &\leq \mu_{11} k n^{-2},\end{aligned}$$

Then the arbitrariness of $n \geq 2$ leads to (65), and hence (66) holds. \square

Remark 1. Through the similar process of the proof of Proposition 2, one can show that for any positive integer λ , it holds that

$$\begin{aligned}\left|\frac{\partial q}{\partial \varphi}(\varphi, \psi)\right| &\leq \sigma'_8 k^{1-\lambda/4} (-\varphi)^{-\lambda}, \quad \left|\frac{\partial q}{\partial \psi}(\varphi, \psi)\right| \leq \sigma'_8 k (-\varphi)^{-\lambda}, \\ (\varphi, \psi) &\in (-\infty, 2\zeta_0) \times (0, m)\end{aligned}$$

and

$$\|q(\varphi, \psi) - q_\infty\|_{L^\infty((-\infty, \zeta) \times (0, m))} \leq \sigma'_9 k (-\zeta)^{-\lambda}, \quad \zeta \in (-\infty, 2\zeta_0),$$

where $\sigma'_8, \sigma'_9 > 0$ depend only on $\lambda, \gamma, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$.

The solution to the problem (16)–(19) is also unique for small k .

Proposition 3. *There exists a constant $k_4 \in (0, 1]$ depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$, such that if $k \in (0, k_4]$, then the problem (16)–(19) admits at most one solution $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m])$ satisfying (58).*

Proof. In the proof, we use ν_i ($1 \leq i \leq 5$) to denote a generic positive constant depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$. Let $q^{(1)}, q^{(2)} \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m])$ be two solution to the problem (16)–(19) satisfying (58). Define

$$w_i(\varphi, \psi) = A(q^{(i)})(\varphi, \psi), \quad (\varphi, \psi) \in (-\infty, 0] \times [0, m], \quad i = 1, 2.$$

Then w_i ($i = 1, 2$) solves

$$\frac{\partial^2 w_i}{\partial \varphi^2} + \frac{\partial^2 B(A^{-1}(w_i))}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m),$$

$$\frac{\partial w_i}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (-\infty, 0),$$

$$\frac{\partial B(A^{-1}(w_i))}{\partial \psi}(\varphi, m) = \frac{f_k''(x)}{(1 + (f_k'(x))^2)^{3/2} Q_{\text{up}}(x)} \Big|_{x=X_{\text{up}}(\varphi)}, \quad \varphi \in (-\infty, 0),$$

$$w_i(0, \psi) = 0, \quad \psi \in (0, m).$$

Set

$$w(\varphi, \psi) = w_1(\varphi, \psi) - w_2(\varphi, \psi), \quad (\varphi, \psi) \in (-\infty, 0] \times [0, m].$$

It is easy to show that w solves

$$\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2}(h(\varphi, \psi)w) = 0, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m), \quad (74)$$

$$\frac{\partial w}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (-\infty, 0), \quad (75)$$

$$\frac{\partial(hw)}{\partial\psi}(\varphi, m) = 0, \quad \varphi \in (-\infty, 0), \quad (76)$$

$$w(0, \psi) = 0, \quad \psi \in (0, m), \quad (77)$$

where

$$h(\varphi, \psi) = \int_0^1 \frac{B'(A^{-1}(\eta w_1(\varphi, \psi) + (1 - \eta)w_2((\varphi, \psi))))}{A'(A^{-1}(\eta w_1(\varphi, \psi) + (1 - \eta)w_2((\varphi, \psi))))} d\eta, \\ (\varphi, \psi) \in (-\infty, 0) \times (0, m).$$

Thanks to (56), (58), (64) and (65), direct calculations yield

$$\nu_1 k^{1/2} \langle -\varphi \rangle^{1/2} \leq h(\varphi, \psi) \leq \nu_1 k^{1/2} \langle -\varphi \rangle^{1/2}, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m), \quad (78)$$

$$\left| \frac{\partial h}{\partial \psi}(\varphi, \psi) \right| \leq \begin{cases} \nu_2 \langle -\varphi \rangle^{-1/2}, & (\varphi, \psi) \in [2\zeta_0, 0) \times (0, m), \\ \nu_2 \langle -\varphi \rangle^{-2}, & (\varphi, \psi) \in (-\infty, 2\zeta_0) \times (0, m), \end{cases} \quad (79)$$

$$\left| \frac{\partial w}{\partial \varphi}(\varphi, \psi) \right| \leq \nu_2 k \langle -\varphi \rangle^{-2}, \quad (\varphi, \psi) \in (-\infty, 2\zeta_0) \times (0, m), \quad (80)$$

where $\langle -\varphi \rangle = \min\{-\varphi, -2\zeta_0\}$. Fix $\zeta < 2\zeta_0 - 1$. Multiplying (74) by $-w$, then integrating over $(\zeta, 0) \times (0, m)$ by parts and using (75)–(77), we have

$$\int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + \int_{\zeta}^0 \int_0^m h(\varphi, \psi) \left(\frac{\partial w}{\partial \psi} \right)^2 d\psi d\varphi \\ = - \int_{\zeta}^0 \int_0^m \frac{\partial h}{\partial \psi}(\varphi, \psi) w \frac{\partial w}{\partial \psi} d\psi d\varphi - \int_0^m w(\zeta, \psi) \frac{\partial w}{\partial \varphi}(\zeta, \psi) d\psi,$$

which, together with (78)–(80), yields

$$\int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + k^{-1/2} \int_{\zeta}^0 \int_0^m \langle -\varphi \rangle^{1/2} \left(\frac{\partial w}{\partial \psi} \right)^2 d\psi d\varphi \\ \leq \nu_3 \int_{2\zeta_0}^0 \int_0^m \langle -\varphi \rangle^{-1/2} \left| w \frac{\partial w}{\partial \psi} \right| d\psi d\varphi + \nu_3 \int_{\zeta}^{2\zeta_0} \int_0^m \langle -\varphi \rangle^{-2} \left| w \frac{\partial w}{\partial \psi} \right| d\psi d\varphi \\ + \nu_3 k \langle -\zeta \rangle^{-2} \int_0^m |w(\zeta, \psi)| d\psi.$$

Then the Hölder's inequality gives

$$\int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + k^{-1/2} \int_{\zeta}^0 \int_0^m \langle -\varphi \rangle^{1/2} \left(\frac{\partial w}{\partial \psi} \right)^2 d\psi d\varphi \\ \leq \nu_4 k^{1/2} \int_{2\zeta_0}^0 \int_0^m \langle -\varphi \rangle^{-1/2} w^2 d\psi d\varphi + \nu_4 k^{1/2} \int_{\zeta}^{2\zeta_0} \int_0^m \langle -\varphi \rangle^{-4} w^2 d\psi d\varphi \\ + \nu_4 k \langle -\zeta \rangle^{-2} \int_0^m |w(\zeta, \psi)| d\psi. \quad (81)$$

It follows from the Hölder's inequality and Cauchy inequality that

$$\int_{2\zeta_0}^0 \int_0^m \langle -\varphi \rangle^{-1/2} w^2 d\psi d\varphi \leq \int_{2\zeta_0}^0 \int_0^m \langle -\varphi \rangle^{-1/2} \left(\int_{\varphi}^0 \frac{\partial w}{\partial \varphi}(s, \psi) ds \right)^2 d\psi d\varphi \\ \leq \int_{2\zeta_0}^0 \langle -\varphi \rangle^{1/2} d\varphi \int_{\zeta_0}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\varphi d\psi \\ \leq (-2\zeta_0)^{3/2} \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\varphi d\psi, \quad (82)$$

$$\begin{aligned}
\int_{\zeta}^{2\zeta_0} \int_0^m (-\varphi)^{-4} w^2 d\psi d\varphi &\leq \int_{\zeta}^{2\zeta_0} \int_0^m (-\varphi)^{-4} \left(\int_{\varphi}^0 \frac{\partial w}{\partial \varphi}(s, \psi) ds \right)^2 d\psi d\varphi \\
&\leq \int_{\zeta}^{2\zeta_0} (-\varphi)^{-3} d\varphi \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi \\
&\leq (-2\zeta_0)^{-2} \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi,
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
\int_0^m |w(\zeta, \psi)| d\psi &\leq \frac{m}{2} + \frac{1}{2} \int_0^m w^2(\zeta, \psi) d\psi \\
&\leq \frac{\delta_2}{2} + \int_0^m \left(\int_{\zeta}^0 \left| \frac{\partial w}{\partial \varphi} \right| d\varphi \right)^2 d\psi \\
&\leq \frac{\delta_2}{2} + (-\zeta) \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\varphi d\psi.
\end{aligned} \tag{84}$$

Substituting (82)–(84) into (81) to get

$$\begin{aligned}
&\int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + k^{-1/2} \int_{\zeta}^0 \int_0^m \langle -\varphi \rangle^{1/2} \left(\frac{\partial w}{\partial \psi} \right)^2 d\psi d\varphi \\
&\leq \nu_5 k^{1/2} \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + \nu_5 k (-\zeta)^{-2} + \nu_5 k (-\zeta)^{-1} \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\varphi d\psi \\
&\leq 2\nu_5 k^{1/2} \int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + \nu_5 k (-\zeta)^{-2}.
\end{aligned} \tag{85}$$

Choose $k_4 = 1/(16\nu_5^2 + 1)$. For any $k \in (0, k_4]$, (85) implies

$$\int_{\zeta}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + k^{-1/2} \int_{\zeta}^0 \int_0^m \langle -\varphi \rangle^{1/2} \left(\frac{\partial w}{\partial \psi} \right)^2 d\psi d\varphi \leq 2\nu_5 k^{1/2} (-\zeta)^{-2}. \tag{86}$$

Taking $\zeta \rightarrow -\infty$ in (86) to get

$$\int_{-\infty}^0 \int_0^m \left(\frac{\partial w}{\partial \varphi} \right)^2 d\psi d\varphi + k^{-1/2} \int_{-\infty}^0 \int_0^m \langle -\varphi \rangle^{1/2} \left(\frac{\partial w}{\partial \psi} \right)^2 d\psi d\varphi \leq 0,$$

which implies

$$\frac{\partial w}{\partial \varphi}(\varphi, \psi) = \frac{\partial w}{\partial \psi}(\varphi, \psi) = 0, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m). \tag{87}$$

It follows (77) and (87) that

$$w(\varphi, \psi) = 0, \quad (\varphi, \psi) \in (-\infty, 0] \times [0, m].$$

Therefore, $q^{(1)} = q^{(2)}$. \square

4. Well-posedness of the subsonic-sonic flow problem. First we prove the existence of the solution to the problem (16)–(20) by a fixed point argument.

Theorem 4.1. *Assume that $f \in C^{2,\alpha}([-l_0, 0])$ satisfies (5) and (6). There exists a constant $k_0 \in (0, 1]$ depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$*

and $\|(-x)^{-1/2}f''\|_{L^\infty((-l_0,0))}$, such that if $k \in (0, k_0]$, then the problem (16)–(20) admits a solution (q, m) satisfying

$$q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C^{1/2}((-\infty, 0] \times [0, m])$$

$$\left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq k\sigma_3(\min\{-\varphi, -\zeta_0\})^{1/2}, \quad (\varphi, \psi) \in (-\infty, 0) \times (0, m), \quad (88)$$

$$|A(q(\varphi_1, \psi_1)) - A(q(\varphi_2, \psi_2))| \leq k\sigma_4(|\varphi_1 - \varphi_2|^{1/2} + |\psi_1 - \psi_2|), \\ (\varphi_1, \psi_1), (\varphi_2, \psi_2) \in (-\infty, 0] \times [0, m], \quad (89)$$

$$c_* - \sigma_6 k^{1/2}(\min\{-\varphi, -\zeta_0\})^{1/2} \leq q(\varphi, \psi) \leq c_* - \sigma_5 k^{1/2}(\min\{-\varphi, -\zeta_0\})^{1/2}, \\ (\varphi, \psi) \in (-\infty, 0] \times [0, m], \quad (90)$$

where

$$m = q_\infty \rho(q_\infty^2)(f_k(-l_0) + l_1), \quad c_* - \sigma_6 k^{1/2}(-\zeta_0)^{1/2} \leq q_\infty \leq c_* - \sigma_5 k^{1/2}(-\zeta_0)^{1/2}, \quad (91)$$

and $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ are given in Lemmas 3.3 and 3.4. Furthermore,

$$\left| \frac{\partial q}{\partial \varphi}(\varphi, \psi) \right| \leq \sigma_7 k^{1/4}(-\varphi)^{-1/2}, \quad (\varphi, \psi) \in [2\zeta_0, 0) \times (0, m), \quad (92)$$

and for any positive integer λ , it holds that

$$\left| \frac{\partial q}{\partial \varphi}(\varphi, \psi) \right| \leq \sigma'_8 k^{1-\lambda/4}(-\varphi)^{-\lambda}, \quad \left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq \sigma'_8 k(-\varphi)^{-\lambda}, \\ (\varphi, \psi) \in (-\infty, 2\zeta_0) \times (0, m), \quad (93)$$

and

$$\|q(\varphi, \psi) - q_\infty\|_{L^\infty((-\infty, \zeta) \times (0, m))} \leq \sigma'_9 k(-\zeta)^{-\lambda}, \quad \zeta \in (-\infty, 2\zeta_0), \quad (94)$$

where σ_7, σ'_8 and σ'_9 are given in Proposition 2 and Remark 1. Therefore, the flow is uniformly subsonic at the far fields.

Proof. Choose

$$k_0 = \min \left\{ k_3, k_4, \frac{c_*^2}{4\sigma_6^2\delta_4}, \frac{1}{\sigma_6^4\delta_4^2}, \frac{N_1}{2\sigma_4\delta_5^{1/2}} \right\}.$$

For $k \in (0, k_0]$, set

$$\mathcal{Q} = \left\{ (m, Q_{\text{up}}) \in [\delta_1, \delta_2] \times C^{1/4}((-\infty, 0]) : Q_{\text{up}} \text{ satisfies (22)} \right\}$$

with the norm

$$\|(m, Q_{\text{up}})\|_{\mathcal{Q}} = \max \left\{ m, \|Q_{\text{up}}\|_{L^\infty(-\infty, 0)} \right\}.$$

For a given $(m, Q_{\text{up}}) \in \mathcal{Q}$, it is clear that Φ_{up} , X_{up} and q_∞ are well determined, and it follows from Propositions 1–3 that the problem (16)–(19) admits a unique solution $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m])$ satisfying (56)–(58) and (66). Set

$$\hat{m} = q_\infty \rho(q_\infty^2)(f_k(-l_0) + l_1), \quad \hat{Q}_{\text{up}}(x) = q(\Phi_{\text{up}}(x), m), \quad x \in (-\infty, 0].$$

From (56)–(58), (66) and the choice of k_0 , it is easy to verify that $(\hat{m}, \hat{Q}_{\text{up}}) \in \mathcal{Q}$ and

$$\mathcal{K} : \mathcal{Q} \rightarrow \mathcal{Q}, \quad (m, Q_{\text{up}}) \mapsto (\hat{m}, \hat{Q}_{\text{up}}).$$

is a self-mapping. Furthermore, one can prove the compactness of \mathcal{K} by using (56)–(58), and the continuity of \mathcal{K} by using its compactness and the uniqueness result for

the problem (16)–(19). Therefore, the Schauder fixed point theorem shows that the problem (16)–(20) admits a solution (q, m) such that $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C^{1/2}((-\infty, 0] \times [0, m])$ satisfies (88)–(94). \square

From Theorem 4.1, for $k \in (0, k_0]$, the problem (16)–(20) admits a solution (q, m) satisfying $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C^{1/2}((-\infty, 0] \times [0, m])$,

$$\begin{aligned} & \max \left\{ \frac{c_*}{2}, c_* - M_1 k^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2} \right\} \\ & \leq q(\varphi, \psi) \leq c_* - M_2 k^{1/2} (\min\{-\varphi, -\zeta_0\})^{1/2}, \end{aligned} \quad (95)$$

$(\varphi, \psi) \in (-\infty, 0) \times (0, m)$

and

$$\|q(\varphi, \psi) - q_\infty\|_{L^\infty((-\infty, \zeta) \times (0, m))} \leq M_3 k (-\zeta)^{-2}, \quad \zeta < 2\zeta_0,$$

where

$$\begin{aligned} m &= q_\infty \rho(q_\infty^2) (f_k(-l_0) + l_1), \\ \max \left\{ \frac{c_*}{2}, c_* - M_1 k^{1/2} (-\zeta_0)^{1/2} \right\} &\leq q_\infty \leq c_* - M_2 k^{1/2} (-\zeta_0)^{1/2}, \end{aligned}$$

and M_1, M_2, M_3 are positive constants. Indeed, this solution is also unique if k is suitably small.

Theorem 4.2. *Assume that $f \in C^{2,\alpha}([-l_0, 0])$ satisfies (5) and (6). There exists a constant $k'_0 \in (0, 1]$ depending only on $\gamma, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}, \|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}, M_1$ and M_2 , such that if $k \in (0, k'_0]$, then there is at most one solution (q, m) to the problem (16)–(20) such that $q \in C^\infty((-\infty, 0) \times (0, m)) \cap C^1((-\infty, 0) \times [0, m]) \cap C((-\infty, 0] \times [0, m])$ and q satisfies (95).*

Proof. In the proof, we use C_i ($1 \leq i \leq 5$) to denote a generic positive constant depending only on $\gamma, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}, \|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}, M_1$ and M_2 . Let $(q^{(1)}, m^{(1)})$ and $(q^{(2)}, m^{(2)})$ be two solutions to the problem (16)–(20) such that $q^{(i)} \in C^\infty((-\infty, 0) \times (0, m^{(i)})) \cap C^1((-\infty, 0) \times [0, m^{(i)}]) \cap C((-\infty, 0] \times [0, m^{(i)}])$ and satisfies (95) for $i = 1, 2$. Denote $\Phi_{\text{up},i}$ and $X_{\text{up},i}$ to be the associated functions defined in Section 2 corresponding to $q^{(i)}$ for $i = 1, 2$. For $i = 1, 2$, introduce the new coordinates transformations

$$\begin{cases} x = X_{\text{up},i}(\varphi), & \varphi \in (-\infty, 0], \\ y = \frac{\psi}{m^{(i)}}, & \psi \in [0, m^{(i)}], \end{cases} \quad \begin{cases} \varphi = \Phi_{\text{up},i}(x), & x \in (-\infty, 0], \\ \psi = m^{(i)}y, & y \in [0, 1]. \end{cases}$$

Define

$$W_i(x, y) = A(q^{(i)}(\Phi_{\text{up},i}(x), m^{(i)}y)), \quad (x, y) \in (-\infty, 0] \times [0, 1], \quad i = 1, 2.$$

Then W_i satisfies

$$\begin{aligned} \frac{\partial}{\partial x} \left(m^{(i)} X_i(x) \frac{\partial W_i}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{m^{(i)} X_i(x)} \frac{\partial B(A^{-1}(W_i))}{\partial y} \right) &= 0, \\ (x, y) \in (-\infty, 0) \times (0, 1), \end{aligned} \quad (96)$$

$$\frac{\partial W_i}{\partial y}(x, 0) = 0, \quad x \in (-\infty, 0), \quad (97)$$

$$\frac{1}{m^{(i)} X_i(x)} \frac{\partial B(A^{-1}(W_i))}{\partial y}(x, 1) = \frac{f_k''(x)}{1 + (f_k'(x))^2}, \quad x \in (-\infty, 0), \quad (98)$$

$$W_i(0, y) = 0, \quad y \in (0, 1), \quad (99)$$

where

$$X_i(x) = \frac{1}{(1 + (f'_k(x))^2)^{1/2} A^{-1}(W_i(x, f_k(-L_0)))}, \quad x \in (-\infty, 0].$$

Set

$$W(x, y) = W_1(x, y) - W_2(x, y), \quad (x, y) \in (-\infty, 0] \times [0, 1].$$

One can verify from that W satisfies

$$\begin{aligned} & \frac{\partial}{\partial x} \left(m^{(1)} X_1(x) \frac{\partial W}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{m^{(1)} X_1(x)} H(x, y) \frac{\partial W}{\partial y} \right) \\ & + \frac{\partial}{\partial x} \left(m^{(1)} X(x) \frac{\partial W_2}{\partial x} \right) + \frac{\partial}{\partial x} \left(m X_2(x) \frac{\partial W_2}{\partial x} \right) \\ & + \frac{\partial}{\partial y} \left(\frac{1}{m^{(1)} X_1(x)} \frac{\partial Z}{\partial y}(x, y) W \right) - \frac{\partial}{\partial y} \left(\frac{m}{m^{(1)} m^{(2)} X_1(x)} \frac{\partial B(A^{-1}(W_2))}{\partial y} \right) \\ & - \frac{\partial}{\partial y} \left(\frac{X(x)}{m^{(2)} X_1(x) X_2(x)} \frac{\partial B(A^{-1}(W_2))}{\partial y} \right) = 0, \quad (x, y) \in (-\infty, 0) \times (0, 1), \end{aligned} \quad (100)$$

where

$$\begin{aligned} m &= m^{(1)} - m^{(2)}, \\ X(x) &= X_1(x) - X_2(x), \quad x \in (-\infty, 0], \\ H(x, y) &= \int_0^1 \frac{B'(A^{-1}(\eta W_1(x, y) + (1 - \eta) W_2(x, y)))}{A'(A^{-1}(\eta W_1(x, y) + (1 - \eta) W_2(x, y)))} d\eta, \quad (x, y) \in (-\infty, 0) \times (0, 1). \end{aligned}$$

It follows from (13), (59), (88) and (90)–(93) that

$$C_1 k^{-1/2} \langle -x \rangle^{-1/2} \leq H(x, y) \leq C_2 k^{-1/2} \langle -x \rangle^{-1/2}, \quad (x, y) \in (-\infty, 0) \times (0, 1), \quad (101)$$

$$\left| \frac{\partial H}{\partial y}(x, y) \right| \leq \begin{cases} C_2 (-x)^{-1/2}, & (x, y) \in [-L_0, 0) \times (0, 1), \\ C_2 (-x)^{-2}, & (x, y) \in (-\infty, -L_0) \times (0, 1), \end{cases} \quad (102)$$

$$\left| \frac{\partial W_i}{\partial x}(x, y) \right| \leq \begin{cases} C_2 k^{3/4}, & (x, y) \in [-L_0, 0) \times (0, 1), \\ C_2 k (-x)^{-2}, & (x, y) \in (-\infty, -L_0) \times (0, 1), \end{cases} \quad i = 1, 2, \quad (103)$$

$$\left| \frac{\partial B(A^{-1}(W_2))}{\partial y}(x, y) \right| \leq \begin{cases} C_2 k (-x)^{1/2}, & (x, y) \in [-L_0, 0) \times (0, 1), \\ C_2 k (-x)^{-2}, & (x, y) \in (-\infty, -L_0) \times (0, 1), \end{cases} \quad (104)$$

$$|X(x)| \leq \begin{cases} C_2 k^{-1/2} \langle -x \rangle^{-1/2} |W(x, 1)|, & (x, y) \in [-L_0, 0) \times (0, 1), \\ C_2 k^{-1/2} |W(x, 1)|, & (x, y) \in (-\infty, -L_0) \times (0, 1), \end{cases} \quad (105)$$

$$|m| \leq C_2 \left(\int_{-L_0}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \right)^{1/2}, \quad (106)$$

where

$$\langle -x \rangle = \min\{-x, L_0\}, \quad L_0 = 3l_0 \left(1 + \|f'\|_{L^\infty((-l_0, 0))}^2 \right)^{1/2}.$$

Fix $L > L_0$. Multiplying (100) by $-W$ and then integrating by parts over $(-L, 0) \times (0, 1)$, one gets from (96)–(99) that

$$\int_{-L}^0 \int_0^1 m^{(1)} X_1(x) \left(\frac{\partial W}{\partial x} \right)^2 dy dx + \int_{-L}^0 \int_0^1 \frac{1}{m^{(1)} X_1(x)} H(x, y) \left(\frac{\partial W}{\partial y} \right)^2 dy dx$$

$$\begin{aligned}
&= - \int_{-L}^0 \int_0^1 m^{(1)} X(x) \frac{\partial W}{\partial x} \frac{\partial W_2}{\partial x} dy dx - \int_{-L}^0 \int_0^1 m X_2(x) \frac{\partial W}{\partial x} \frac{\partial W_2}{\partial x} dy dx \\
&\quad - \int_{-L}^0 \int_0^1 \frac{1}{m^{(1)} X_1(x)} \frac{\partial H}{\partial y}(x, y) W \frac{\partial W}{\partial y} dy dx \\
&\quad + \int_{-L}^0 \int_0^1 \frac{m}{m^{(1)} m^{(2)} X_1(x)} \frac{\partial B(A^{-1}(W_2))}{\partial y} \frac{\partial W}{\partial y} dy dx \\
&\quad + \int_{-L}^0 \int_0^1 \frac{X(x)}{m^{(2)} X_1(x) X_2(x)} \frac{\partial B(A^{-1}(W_2))}{\partial y} \frac{\partial W}{\partial y} dy dx \\
&\quad + \int_0^1 W(-L, y) \left(m^{(1)} X_1(-L) \frac{\partial W_1}{\partial x}(-L, y) - m^{(2)} X_2(-L) \frac{\partial W_2}{\partial x}(-L, y) \right) dy,
\end{aligned}$$

which, together with (23), (90), (101) and (103), yields

$$\begin{aligned}
&\int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{-1/2} \int_{-L}^0 \int_0^1 (-x)^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\
&\leq C_3 \underbrace{\int_{-L}^0 \int_0^1 \left| X(x) \frac{\partial W}{\partial x} \frac{\partial W_2}{\partial x} \right| dy dx}_{J_1} + C_3 \underbrace{\int_{-L}^0 \int_0^1 \left| m \frac{\partial W}{\partial x} \frac{\partial W_2}{\partial x} \right| dy dx}_{J_2} \\
&\quad + C_3 \underbrace{\int_{-L}^0 \int_0^1 \left| \frac{\partial H}{\partial y}(x, y) W \frac{\partial W}{\partial y} \right| dy dx}_{J_3} \\
&\quad + C_3 \underbrace{\int_{-L}^0 \int_0^1 \left| m \frac{\partial B(A^{-1}(W_2))}{\partial y} \frac{\partial W}{\partial y} \right| dy dx}_{J_4} \\
&\quad + C_3 \underbrace{\int_{-L}^0 \int_0^1 \left| X(x) \frac{\partial B(A^{-1}(W_2))}{\partial y} \frac{\partial W}{\partial y} \right| dy dx}_{J_5} \\
&\quad + C_3 k(-L)^{-2} \underbrace{\int_0^1 |W(-L, y)| dy}_{I_L}.
\end{aligned} \tag{107}$$

Below, let us make estimates on J_i ($1 \leq i \leq 5$) and I_L in (107). The following five inequalities are necessary. From the Hölder's inequality and (99), it follows

$$\begin{aligned}
&\int_{-L_0}^0 \int_0^1 (-x)^{-\vartheta_1} W^2 dy dx \\
&\leq \int_{-L_0}^0 \int_0^1 (-x)^{-\vartheta_1} \left(\int_x^0 \left| \frac{\partial W}{\partial x}(s, y) \right| ds \right)^2 dy dx \\
&\leq \int_{-L_0}^0 (-x)^{1-\vartheta_1} dx \int_{-L_0}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\
&\leq \frac{L_0^{2-\vartheta_1}}{2-\vartheta_1} \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx, \quad \vartheta_1 \in [0, 2),
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
& \int_{-L}^{-L_0} \int_0^1 (-x)^{-\vartheta_2} W^2 dy dx \\
& \leq \int_{-L}^{-L_0} \int_0^1 (-x)^{-\vartheta_2} \left(\int_x^0 \left| \frac{\partial W}{\partial x}(s, y) \right| ds \right)^2 dy dx \\
& \leq \int_{-L}^{-L_0} (-x)^{1-\vartheta_2} dx \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\
& \leq \frac{L_0^{2-\vartheta_2}}{\vartheta_2 - 2} \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx, \quad \vartheta_2 \in (2, +\infty).
\end{aligned} \tag{109}$$

Then from the Cauchy's inequality, (108) and (109), we have

$$\begin{aligned}
& \int_{-L_0}^0 W^2(x, 1) dx \\
& \leq \int_{-L_0}^0 \int_0^1 W^2 dy dx + 2 \int_{-L_0}^0 \int_0^1 \left| W \frac{\partial W}{\partial y} \right| dy dx \\
& \leq L_0^2 \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{1/2} L_0^{1/2} \int_{-L_0}^0 \int_0^1 W^2 dy dx \\
& \quad + k^{-1/2} \int_{-L_0}^0 \int_0^1 (-x)^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\
& \leq (L_0^2 + L_0^{5/2}) \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\
& \quad + k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx,
\end{aligned} \tag{110}$$

$$\begin{aligned}
& \int_{-L_0}^0 (-x)^{-1} W^2(x, 1) dx \\
& \leq \int_{-L_0}^0 \int_0^1 (-x)^{-1} W^2 dy dx + 2 \int_{-L_0}^0 \int_0^1 (-x)^{-1} \left| W \frac{\partial W}{\partial y} \right| dy dx \\
& \leq L_0 \int_{-L_0}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{1/2} \int_{-L_0}^0 \int_0^1 (-x)^{-3/2} W^2 dy dx \\
& \quad + k^{-1/2} \int_{-L_0}^0 \int_0^1 (-x)^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\
& \leq (L_0 + 2L_0^{1/2}) \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\
& \quad + k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx,
\end{aligned} \tag{111}$$

and

$$\begin{aligned}
& \int_{-L}^{-L_0} (-x)^{-4} W^2(x, 1) dx \\
& \leq \int_{-L}^{-L_0} \int_0^1 (-x)^{-4} W^2 dy dx + 2 \int_{-L}^{-L_0} \int_0^1 (-x)^{-4} \left| W \frac{\partial W}{\partial y} \right| dy dx
\end{aligned}$$

$$\begin{aligned}
&\leq L_0^{-2} \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{1/2} L_0^{1/2} \int_{-L}^{-L_0} \int_0^1 (-x)^{-8} W^2 dy dx \\
&\quad + k^{-1/2} L_0^{-1/2} \int_{-L}^{-L_0} \int_0^1 \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\
&\leq (L_0^{-2} + L_0^{-5/2}) \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\
&\quad + k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx. \tag{112}
\end{aligned}$$

It follows from Cauchy's inequality with ε , (102)–(106) and (108)–(112) that

$$\begin{aligned}
J_1 &\leq \varepsilon \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + \frac{1}{\varepsilon} \int_{-L}^0 \int_0^1 |X(x)|^2 \left| \frac{\partial W_2}{\partial x} \right|^2 dy dx \\
&\leq \varepsilon \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + \frac{C_2^2 k^{1/2}}{\varepsilon} \int_{-L_0}^0 (-x)^{-1} W^2(x, 1) dy dx \\
&\quad + \frac{C_2^2 k}{\varepsilon} \int_{-L}^{-L_0} (-x)^{-4} W^2(x, 1) dy dx \\
&\leq C_4 \left(\varepsilon + \frac{k^{1/2}}{\varepsilon} \right) \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\
&\quad + \frac{C_4 k^{1/2}}{\varepsilon} \cdot k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx, \tag{113}
\end{aligned}$$

$$\begin{aligned}
J_2 &\leq \varepsilon \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + \frac{1}{\varepsilon} \int_{-L}^0 \int_0^1 m^2 \left(\frac{\partial W_2}{\partial x} \right)^2 dy dx \\
&\leq \varepsilon \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + \frac{C_2^2}{\varepsilon} m^2 \left(k^{3/2} + L_0 k^2 \int_{-L}^{-L_0} (-x)^{-4} dx \right) \\
&\leq C_4 \left(\varepsilon + \frac{k^{1/2}}{\varepsilon} \right) \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx, \tag{114}
\end{aligned}$$

$$\begin{aligned}
J_3 &\leq C_2 \int_{-L_0}^0 \int_0^1 (-x)^{-1/2} \left| W \frac{\partial W}{\partial y} \right| dy dx + C_2 \int_{-L}^{-L_0} \int_0^1 (-x)^{-2} \left| W \frac{\partial W}{\partial y} \right| dy dx \\
&\leq \frac{C_2 k^{1/2}}{\varepsilon} \int_{-L_0}^0 \int_0^1 (-x)^{-1/2} W^2 dy dx + \frac{C_2 L_0^{1/2} k^{1/2}}{\varepsilon} \int_{-L}^{-L_0} \int_0^1 (-x)^{-4} W^2 dy dx \\
&\quad + C_2 \varepsilon k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\
&\leq \frac{C_4 k^{1/2}}{\varepsilon} \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + C_4 \varepsilon k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx, \tag{115}
\end{aligned}$$

$$\begin{aligned}
J_4 &\leq C_2 k \int_{-L_0}^0 \int_0^1 (-x)^{1/2} \left| m \frac{\partial W}{\partial y} \right| dy dx + C_2 k \int_{-L}^{-L_0} \int_0^1 (-x)^{-2} \left| m \frac{\partial W}{\partial y} \right| dy dx \\
&\leq \frac{C_2 k^{5/2}}{\varepsilon} m^2 \left(\int_{-L_0}^0 (-x)^{3/2} dx + \int_{-L}^{-L_0} (-x)^{-4} dx \right) \\
&\quad + C_2 \varepsilon k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx
\end{aligned}$$

$$\leq \frac{C_4 k^{1/2}}{\varepsilon} \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + C_4 \varepsilon k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx, \quad (116)$$

and

$$\begin{aligned} J_5 &\leq C_2^2 k^{1/2} \int_{-L_0}^0 \int_0^1 \left| W(x, 1) \frac{\partial W}{\partial y} \right| dy dx \\ &\quad + C_2^2 k^{1/2} \int_{-L}^{-L_0} \int_0^1 \langle -x \rangle^{-2} \left| W(x, 1) \frac{\partial W}{\partial y} \right| dy dx \\ &\leq \frac{C_2^2 L_0^{1/2} k^{3/2}}{\varepsilon} \int_{-L_0}^0 W^2(x, 1) dx + \frac{C_2^2 L_0^{1/2} k^{3/2}}{\varepsilon} \int_{-L}^{-L_0} \langle -x \rangle^{-4} W^2(x, 1) dx \\ &\quad + C_2^2 \varepsilon k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\ &\leq \frac{C_4 k^{1/2}}{\varepsilon} \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx \\ &\quad + C_4 \left(\varepsilon + \frac{k^{1/2}}{\varepsilon} \right) k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx, \end{aligned} \quad (117)$$

where $\varepsilon > 0$ is to be determined. Additionally,

$$\begin{aligned} I_L &\leq 1 + \int_0^1 W^2(-L, y) dy \\ &\leq 1 + \int_0^1 \left(\int_{-L}^0 \left| \frac{\partial W}{\partial x} \right| dx \right)^2 dy \\ &\leq 1 + (-L) \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx. \end{aligned} \quad (118)$$

Substituting (113)–(118) into (107) to get

$$\begin{aligned} &\int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \\ &\leq C_5 \left(\varepsilon + \frac{k^{1/2}}{\varepsilon} \right) \left(\int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \right) \\ &\quad + C_5 (-L)^{-1} + C_5 k \int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx. \end{aligned} \quad (119)$$

Choose $\varepsilon = (4C_5)^{-1}$ and $k'_0 = \min\{(16C_5^2 + 1)^{-1}, (4C_5 + 1)^{-1}\}$. For any $k \in (0, k'_0]$, (119) implies

$$\int_{-L}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{-1/2} \int_{-L}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \leq 2C_5 (-L)^{-1}. \quad (120)$$

Taking $L \rightarrow +\infty$ in (120), we obtain that

$$\int_{-\infty}^0 \int_0^1 \left(\frac{\partial W}{\partial x} \right)^2 dy dx + k^{-1/2} \int_{-\infty}^0 \int_0^1 \langle -x \rangle^{-1/2} \left(\frac{\partial W}{\partial y} \right)^2 dy dx \leq 0,$$

which shows that

$$\frac{\partial W}{\partial x}(x, y) = \frac{\partial W}{\partial y}(x, y) = 0, \quad (x, y) \in (-\infty, 0) \times (0, 1).$$

Then $W(x, y) = 0$ follows from (99), and hence $(q^{(1)}, m^{(1)}) = (q^{(2)}, m^{(2)})$. \square

In terms of the physical variables, Theorems 4.1 and 4.2 can be transformed as

Theorem 4.3. *Assume that $f \in C^{2,\alpha}([-l_0, 0])$ satisfies (5) and (6). There exist four constants $\tilde{k}_0 \in (0, 1]$ and $\tilde{M}_1, \tilde{M}_2 > 0$ depending only on $\gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$, such that if $k \in (0, \tilde{k}_0]$ then the problem (7)–(10) admits a unique solution (φ, S, m) satisfying $\varphi \in C^3(\Omega_k) \cap C^2(\overline{\Omega}_k \setminus S) \cap C^1(\overline{\Omega}_k)$, $S \in C^1([-l_1, 0])$,*

$$\max \left\{ \frac{c_*}{2}, c_* - \tilde{M}_2(k \operatorname{dist}_S(\langle x \rangle, y))^{1/2} \right\} \leq |\nabla \varphi(x, y)| \leq c_* - \tilde{M}_1(k \operatorname{dist}_S(\langle x \rangle, y))^{1/2},$$

$$(x, y) \in \Omega_k,$$

where $\operatorname{dist}_S(x, y)$ is the distance from (x, y) to S and $\langle x \rangle = \max\{x, -l_0\}$. Moreover, for any positive integer λ , there exists a constant $\tilde{M}_3 > 0$ depending only on $\lambda, \gamma, \tau_1, \tau_2, l_0, l_1, f(-l_0), \|f'\|_{L^\infty((-l_0, 0))}$ and $\|(-x)^{-1/2} f''\|_{L^\infty((-l_0, 0))}$, such that

$$\|\varphi(x, y) - q_\infty x\|_{C^1(\Omega_k \cap \{x < -R\})} \leq \tilde{M}_3 k R^{-\lambda}, \quad R > l_0,$$

where

$$\max \left\{ \frac{c_*}{2}, c_* - \tilde{M}_2(k l_0)^{1/2} \right\} \leq q_\infty \leq c_* - \tilde{M}_1(k l_0)^{1/2}.$$

Therefore, the flow is uniformly subsonic at the far fields.

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