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ON RECENT PROGRESS OF SINGLE-REALIZATION RECOVERIES OF RANDOM SCHRÖDINGER SYSTEMS

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Abstract. We consider the recovery of some statistical quantities by using the near-field or far-field data in quantum scattering generated under a single realization of the randomness. We survey the recent main progress in the literature and point out the similarity among the existing results. The methodologies in the reformulation of the forward problems are also investigated. We consider two separate cases of using the near-field and far-field data, and discuss the key ideas of obtaining some crucial asymptotic estimates. We pay special attention on the use of the theory of pseudodifferential operators and microlocal analysis needed in the proofs.

1. Introduction.

1.1. Mathematical formulations.

In this paper, we mainly focus on the random inverse problems associated with the following time-harmonic Schrödinger system

$$(-\Delta - E + \text{potential})\, u(x) = \text{source}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where E is the energy level, n is the dimension, and "source" and the "potential" in (1.1) shall be specified later. In some cases we may impose incident waves to the system in order to obtain more useful information, thus

$$u(x) = \alpha \cdot u^{in}(x) + u^{sc}(x) \tag{1.2}$$

where α takes the value of either 0 or 1 corresponding to impose or suppress the incident wave, respectively. The corresponding data are thus called passive or active measurements. Moreover, we shall impose the Sommerfeld radiation condition [10]

$$\lim_{r\to\infty} r \left(\frac{\partial u^{sc}}{\partial r} - i\sqrt{E} u^{sc} \right) = 0, \quad r := |x|, \tag{1.3}$$

that characterizes the outgoing nature of the scattered field u^{sc}. The system (1.1)-(1.3) describes the quantum scattering [13, 14] associated with a source and a potential at the energy level E. Later we follow the convention to use $k := \sqrt{E}$ to signify the frequency at which the system is acting on.

Under different assumptions of the potential and source, of the dimension, and of the incident wave, the regularity of the Schrödinger system (1.1)-(1.3) behaves differently and calls for different techniques for the recovery procedure. The randomness of the Schrödinger system (1.1)-(1.3) can present either in the potential,

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or in the source, or in both. In this paper we shall investigate all of these three cases, survey the results in the literature and give details of part of the proofs.

There are rich literature on the inverse scattering problem using either passive or active measurements as data. For a fixed potential, the recovery of the deterministic unknown source of the system is called the inverse source problem. For the theoretical analysis and computational methods of the inverse source problems, readers may refer to [3–5,9,31,34] and references therein. The simultaneous recovery of the deterministic unknown source and potential are also studied in the literature. In [17,26], the authors considered the simultaneous recovery of an unknown source and its surrounding medium parameter. This type of inverse problems also arises in the deterministic magnetic anomaly detections using geomagnetic monitoring [11, 12] with passive measurements. While [11, 12, 17, 26] focus on deterministic setting with passive measurements, the works [2,6,7,18–20,27,33] pay attention to random settings. We are particularly interested in the case with a *single* realization of the random sample. The single-realization recovery has been studied in the literature. In this paper we mainly focus on [8, 18–25].

In [18, 19], Lassas et. al. considered the inverse scattering problem for the two-dimensional random Schrödinger system $(-\Delta - k^2 - q(x,\omega))u(x,k,\omega) = \delta_y,\ x \in \mathbb{R}^2$ which is incited by point sources $u^{in}(x) = \frac{i}{4}H_0^{(1)}(k|x-y|)$; the $H_0^{(1)}$ is the Hankel function for the first kind, and the origin y of this source are located outside the support of the potential. The potential $q(x,\omega)$ is a micro-locally isotropic generalized Gaussian field (*migr* field) with compact support. The definition of the migr field can be found in Definition 1.1. They introduced the so-called rough strength $\mu(x)$, which is the informative part of the principal symbol $\mu(x)|\xi|^{-m}$ of the covariance operator. The $-m$ in $\mu(x)|\xi|^{-m}$ is the rough order of the random potential. The main result in their work states that under a single measurement of the random field inside a measurement domain, the rough strength can be recovered.

In 2019, Caro et. al. [8] considered an inverse scattering problem for an n-dimensional $(n \geq 2)$ random Schrödinger system $(-\Delta - k^2 - q(x,\omega))u(x,k,\omega) = 0,\ x \in \mathbb{R}^n$ with incident wave being the plane wave, i.e. u is incited by the point sources $u^{in}(x) = e^{ikd\cdot x}$; d is the incident direction. Again, the potential q is assumed to be a migr field with compact support. The main result is as follows: they used the backscattering far-field pattern and recovered the rough strength $\mu(x)$ almost surely, under a single realization of the randomness.

In [20], Li, et. al. studied the case where the potential is zero and the source is migr field. In [24] Li, et. al. studied the same setting but with the energy level E replaced by $(k^2 + i\sigma k)$ where the σ is the attenuation parameter. The random source term considered is constructed as a migr field. The system has been changed to Helmholtz system in [24] but the underlying equation is uniform with the Schrödinger's equation. The authors studied the regularity of the random source and gave the well-posedness of the direct problem. Then they represented the solution as the convolution between the fundamental solution and the random source. By truncating the fundamental solution, they indicated that the rough strength can be recovered by utilizing the correspondingly truncated solution. Further, the authors used calculus of symbols to recover the rough strength.

Then in [23], Li, et. al. further extended their study to Maxwell's equation. The recovery procedure in these three works share the same idea–the leading order term in the Bonn expansion gives the recovery of the desired statistics while the higher order terms converge to zero. The proof of these converges involve the utilization of Fourier integral operator. We shall give detailed explanations in Section 3.

In [21], the authors consider the direct and inverse scatterings for (1.1)–(1.3) with a deterministic potential and a random source. The random source is a generalized Gaussian random field with local mean value function and local variance function, which are assumed to be bounded and compactly supported. The well-posedness of the direct scattering has been formulated in some weighted L^2 space. Then the inverse scattering is studied and a recovery formula of the variance function is obtained, and the uniqueness recovery of the potential is given. The authors used both passive and active measurements to recover the unknowns. The passive measurements refer to the scattering data generated only by the unknown source (α is set to be 0 in (1.2)); active measurements refer to the scattering data generated by both the source and the incident wave (α is set to be 1 in (1.2)). To recover the variance function, only the passive measurements are needed, while the unique recovery of the potential needs active measurements.

In [25], the authors extended the work [21] to the case where the source is a migr field. The direct scattering problem is formulated in a similar manner as in [21], while the technique used in the inverse scattering problem differs from that of [21]. In order to analyze the asymptotics of higher order terms in the Bonn expansion corresponding to the migr fields, stationary phase lemma and pseudodifferential operator are utilized.

Then the authors extended the work [25] to the case where both the potential and the source are random (of migr type), and the extended result is presented in [22]. The results between [21] and [22, 25] have two major differences. First, in [21] the random part of the source is assumed to be a Gaussian white noise, while in [22] the potential and the source are assumed to be migr fields. The migr field can fit larger range of randomness by tuning its rough order and rough strength. Second, in [22] both the source and potential are random, while in [25] the potential is assumed to be deterministic. These two facts make [22] much more challenging than that in [25]. The techniques used in the estimates of higher order terms in [22] are pseudodifferential operators and microlocal analysis and we shall give a detailed treatment in Section 4.

Although the techniques used in [21,22,25] are different, the recovery formulae fall into the same pattern. The thesis [28] partially collected these three works [21,22,25] and readers may refer to the thesis for a more coherent discussion on this topic.

1.2. **Summarization of the main results.** In this paper we mainly pay attention to two types of random model, the Gaussian white noise and the migr field. The Gaussian white noise is well-known and readers may refer to [21, Section 2.1] for more details. Here we give a brief introduction to the migr field. We assume f to be a generalized Gaussian random distribution of the microlocally isotropic type (cf. Definition 1.1). It means that $f(\cdot,\omega)$ is a random distribution and the mapping

$$\omega \in \Omega \;\mapsto\; \langle f(\cdot,\omega), \varphi\rangle \in \mathbb{C}, \quad \varphi \in \mathscr{S}(\mathbb{R}^n),$$

is a Gaussian random variable whose probabilistic measure depends on the test function φ. Here and also in what follows, $\mathscr{S}(\mathbb{R}^n)$ stands for the Schwartz space. Since both $\langle f(\cdot,\omega), \varphi\rangle$ and $\langle f(\cdot,\omega), \psi\rangle$ are random variables for φ, $\psi \in \mathscr{S}(\mathbb{R}^n)$, from a statistical point of view, the covariance between these two random variables,

$$\mathbb{E}_\omega\big(\langle\overline{f(\cdot,\omega) - \mathbb{E}(f(\cdot,\omega))}, \varphi\rangle\langle f(\cdot,\omega) - \mathbb{E}(f(\cdot,\omega)), \psi\rangle\big), \tag{1.4}$$

can be understood as the covariance of f. Here $\mathbb{E}_\omega$ means to take expectation on the random variable ω. Hence, formula (1.4) defines an operator $\mathfrak{C}_f$,

$$\mathfrak{C}_f\colon \varphi \in \mathscr{S}(\mathbb{R}^n) \ \mapsto\ \mathfrak{C}_f(\varphi) \in \mathscr{S}'(\mathbb{R}^n),$$

in a way that $\mathfrak{C}_f(\varphi)\colon \psi \in \mathscr{S}(\mathbb{R}^n) \ \mapsto\ (\mathfrak{C}_f(\varphi))(\psi) \in \mathbb{C}$ where

$$(\mathfrak{C}_f(\varphi))(\psi) := \mathbb{E}_\omega\big(\langle \overline{f(\cdot,\omega) - \mathbb{E}(f(\cdot,\omega))}, \varphi\rangle \langle f(\cdot,\omega) - \mathbb{E}(f(\cdot,\omega)), \psi\rangle\big).$$

The operator $\mathfrak{C}_f$ is called the *covariance operator* of f.

Definition 1.1 (Migr field)**.** A generalized Gaussian random distribution f on $\mathbb{R}^n$ is called *microlocally isotropic* with *rough order* $-m$ and *rough strength* $\mu(x)$ in a bounded domain D, if the following conditions hold:

1. the expectation $\mathbb{E}(f)$ is in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\mathbb{E}(f) \subset D$;
2. f is supported in D a.s. (almost surely);
3. the covariance operator $\mathfrak{C}_f$ is a classical pseudodifferential operator of order $-m$;
4. $\mathfrak{C}_f$ has a principal symbol of the form $\mu(x)|\xi|^{-m}$ with $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n;\mathbb{R})$, $\operatorname{supp}\mu \subset D$ and $\mu(x) \geq 0$ for all $x \in \mathbb{R}^n$.

We call a microlocally isotropic Gaussian random distribution as an *migr* field.

For the case where both the source and the potential are deterministic and are L^∞ functions with compact supports, the well-posedness of the direct problem of system (1.1)–(1.3) is known; see, e.g., [10,13,29]. Moreover, there holds the following asymptotic expansion of the outgoing radiating field u^{sc} as $|x| \to +\infty$,

$$u^{sc}(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u^\infty(\hat{x}, k, d) + o(|x|^{-(n-1)/2}), \quad x \in \mathbb{R}^n.$$

$u^\infty(\hat{x}, k, d)$ is referred to as the far-field pattern, which encodes information of the potential and the source. $\hat{x} := x/|x|$ and d in $u^\infty(\hat{x}, k, d)$ are unit vectors and they respectively stand for the observation direction and the impinging direction of the incident wave. When $d = -\hat{x}$, $u^\infty(\hat{x}, k, -\hat{x})$ is called the backscattering far-field pattern. We shall see very soon that both the near-field u^{sc} and the far-field u^∞ can be used to achieve the recovery.

In (1.1), let us denote the source term as f and the potential term as q. In our study, both the source f and the potential q are assumed to be compactly supported. We shall treat [8,18–25] in more details. In [8,18,19], q is assumed to be a migr field while f is either zero or point a point source, i.e. $\delta_y(x)$. In [20,23,24], q is assumed to be zero and f is assumed to be a migr field. In [21], q is assumed to be unknown and deterministic and f is assumed to be a Gaussian white noise, while in [22,25], q is assumed to be deterministic or migr type and f is assumed to be a migr field.

In [18,19] the authors considered the inverse scattering problem for the two-dimensional random Schrödinger system $(-\Delta - k^2 - q(x,\omega))u(x,k,\omega) = \delta_y(x)$ ($x \in \mathbb{R}^2$) which is incited by point sources $u^{in}(x) = \frac{i}{4}H_0^{(1)}(k|x-y|)$; the $H_0^{(1)}$ is the Hankel function for the first kind, and the origin y of this source is located in U. The potential $q(x,\omega)$ is a migr field with compact support D and $\overline{U} \cap \overline{D} = \emptyset$. The main result is presented as follows (cf. [19, Theorem 7.1]).

Theorem 1.1. *In [18, 19], for $x, y \in U$ the limit*

$$R(x,y) = \lim_{K\to+\infty} \frac{1}{K-1}\int_1^K k^{2+m}|u^{sc}(x,y,k,\omega)|^2 \,\mathrm{d}k$$

holds almost surely where

$$R(x,x) := \frac{1}{2^{6+m}\pi^2}\int_{\mathbb{R}^2}\frac{\mu_q(z)}{|x-z|^2}\,\mathrm{d}z, \quad x\in U.$$

and the μ_q is the rough strength and $-m$ is the rough order of q.

In [8], the authors considered $(-\Delta - k^2 - q(x,\omega))u(x,k,\omega) = 0,\ x\in\mathbb{R}^n$ with incident plane wave $u^{in}(x) = e^{ikd\cdot s}$. The potential q is assumed to be a migr field with compact support. The main result (cf. [8, Corollary 4.4]) is as follows.

Theorem 1.2. *In [8], the limit*

$$\widehat{\mu}_q(2\tau\hat{x}) \simeq \lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^m u^\infty(\hat{x},-\hat{x},k)\overline{u^\infty(\hat{x},-\hat{x},k+\tau)}\,\mathrm{d}k, \quad \hat{x}\in\mathbb{S}^2,\ \tau>0.$$

holds almost surely.

We note that the near-field data are used in [18, 19], while in [8], the authors used the far-field data.

Part of the results in [20] and [23, 24] are similar to each other and we only survey the first result in [20]. In [20] the authors studied the Helmholtz equation $(-\Delta - k^2)u(x) = f$ where f is a source of migr type. Note that the potential equals zero. The main result (cf. [20, Theorem 3.9]) is similar to Theorem 1.1.

Theorem 1.3. *In [20], the limit*

$$\int\frac{\mu_f(z)}{|x-z|}\,\mathrm{d}z \simeq \lim_{K\to+\infty}\frac{1}{K-1}\int_1^K k^{1+m}|u^{sc}(x,k,\omega)|^2\,\mathrm{d}k, \quad x\in U,$$

holds almost surely.

In [21], the authors considered direct and inverse scattering for (1.1)–(1.3) with an unknown deterministic potential and a Gaussian noise source of the form $\sigma(x)\dot{B}_x(\omega)$, where $\sigma(x)$ is the variance and $\dot{B}_x(\omega)$ is the Gaussian white noise. The main result (cf. [21, Lemma 4.3]) is

Theorem 1.4. *In [21], the identity*

$$\widehat{\sigma^2}(x) = 4\sqrt{2\pi}\lim_{j\to+\infty}\frac{1}{K_j}\int_{K_j}^{2K_j}\overline{u^\infty(\hat{x},k,\omega)}\cdot u^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k.$$

holds almost surely.

The paper [25] extended the work [21] to the case where the source is a migr field f with μ_f as its rough strength and $-m$ as its rough order. For notational convenience, we shall use $\{K_j\}\in P(t)$ to signify a sequence $\{K_j\}_{j\in\mathbb{N}}$ satisfying $K_j \geq Cj^t$ $(j\in\mathbb{N})$ for some fixed constant $C>0$. Throughout the rest of the paper, γ stands for a fixed positive real number. The main result (cf. [25, Theorem 4.3]) is presented below.

Theorem 1.5. *In [25], assume $2<m<3$ and let $m^* = \max\{2/3,\,(3-m)^{-1}/2\}$. Assume that $\{K_j\}\in P(m^*+\gamma)$. Then $\exists\,\Omega_0\subset\Omega\colon \mathbb{P}(\Omega_0)=0$, Ω_0 depending only on $\{K_j\}_{j\in\mathbb{N}}$, such that for any $\omega\in\Omega\backslash\Omega_0$, there exists $S_\omega\subset\mathbb{R}^3\colon |S_\omega|=0$, it holds that for $\forall\tau\in\mathbb{R}_+$ and $\forall\hat{x}\in\mathbb{S}^2$ satisfying $\tau\hat{x}\in\mathbb{R}^3\backslash S_\omega$,*

$$\widehat{\mu}(\tau\hat{x}) = 4\sqrt{2\pi}\lim_{j\to+\infty}\frac{1}{K_j}\int_{K_j}^{2K_j} k^m\overline{u^\infty(\hat{x},k,\omega)}\cdot u^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k,$$

holds for $\forall\tau\in\mathbb{R}_+$ and $\forall\hat{x}\in\mathbb{S}^2$ satisfying $\tau\hat{x}\in\mathbb{R}^3\backslash S_\omega$.

Then in [22] the authors further extended the work [25] to the case where both the potential q and the source f are random of migr type. The f (resp. q) is assumed to be supported in the domain D_f (resp. D_q). In what follows, we assume that there is a positive distance between the convex hulls of the supports of f and q, i.e.,

$$\mathrm{dist}(\mathcal{CH}(D_f), \mathcal{CH}(D_q)) := \inf\{\, |x-y|\,;\, x \in \mathcal{CH}(D_f),\, y \in \mathcal{CH}(D_q)\,\} > 0, \tag{1.5}$$

where $\mathcal{CH}$ means taking the convex hull of a domain. Therefore, one can find a plane which separates D_f and D_q. In order to simplify the exposition, we assume that D_f and D_q are convex domains and hence $\mathcal{CH}(D_f) = D_f$ and $\mathcal{CH}(D_q) = D_q$. Moreover, we let $\boldsymbol{n}$ denote the unit normal vector of the aforementioned plane that separates D_f and D_q, pointing from the half-space containing D_f into the half-space containing D_q. Then the result of this work (cf. [22, Theorems 1.1 and 1.2]) is as follows.

Theorem 1.6. *In [22], suppose that f and q in system (1.1)-(1.3) are migr fields of order $-m_f$ and $-m_q$, respectively, satisfying*

$$2 < m_f < 4,\ \ m_f < 5m_q - 11.$$

Assume that (1.5) is satisfied and $\boldsymbol{n}$ is defined as above. Then, independent of μ_q, μ_f can be uniquely recovered almost surely and the recovering formula of μ_f is given by

$$\widehat{\mu}_f(\tau\hat{x}) = \begin{cases} \lim\limits_{K\to+\infty} \dfrac{4\sqrt{2\pi}}{K} \displaystyle\int_K^{2K} k^{m_f} \overline{u^\infty(\hat{x},k,\omega)} u^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k, & \hat{x}\cdot\boldsymbol{n} \geq 0, \\ \overline{\widehat{\mu}_f(-\tau\hat{x})}, \quad \hat{x}\cdot\boldsymbol{n} < 0, \end{cases} \tag{1.6}$$

where $\tau \geq 0$ and $u^\infty(\hat{x},k,\omega) \in \mathcal{M}_f(\omega) := \{\, u^\infty(\hat{x},k,\omega)\,;\, \forall\hat{x}\in\mathbb{S}^2,\, \forall k \in \mathbb{R}_+\,\}$.

When $m_q < m_f$, μ_q can be uniquely recovered almost surely by the data set $\mathcal{M}_q(\omega)$ for a fixed $\omega\in\Omega$. Moreover, the recovering formula is given by

$$\widehat{\mu}_q(\tau\hat{x}) = \begin{cases} \lim\limits_{K\to+\infty} \dfrac{4\sqrt{2\pi}}{K} \displaystyle\int_K^{2K} k^{m_q} \overline{u^\infty(\hat{x},k,-\hat{x},\omega)} u^\infty(\hat{x},k+\tfrac{\tau}{2},-\hat{x},\omega)\,\mathrm{d}k, & \hat{x}\cdot\boldsymbol{n} \geq 0, \\ \overline{\widehat{\mu}_f(-\tau\hat{x})}, \quad \hat{x}\cdot\boldsymbol{n} < 0, \end{cases} \tag{1.7}$$

where $\tau \geq 0$ and $u^\infty(\hat{x},k,-\hat{x},\omega) \in \mathcal{M}_q(\omega) := \{\, u^\infty(\hat{x},k,-\hat{x},\omega)\,;\, \forall\hat{x}\in\mathbb{S}^2,\, \forall k \in \mathbb{R}_+\,\}$.

Remark 1.1. In Theorem 1.6, the data sets $\mathcal{M}_f(\omega)$ and $\mathcal{M}_f(\omega)$ correspond to the case where the incident wave is passive and active, respectively. Readers may refer to [22, Section 1] for more details.

Readers should note that the recovery formulae in Theorems 1.1–1.6 only use a single realization of the randomness; the terms on the left-hand-sides are independent of the random sample ω, while these on the right-hand-sides are limits of terms depending on ω. This feature is also described as "statistically stable" in the literature. The key ingredient of making this single-realization recovery possible is ergodicity; on the right-hand-sides of these recoveries formulae in Theorems 1.1–1.6, the probabilistic expectation operation are replaced by the average in the frequency variable and then taking to the infinity of the frequency variable. Theorems 1.1 and 1.3 utilize the near-field data to achieve the recovery, while Theorem 1.2 and 1.4–1.6 use the far-field data. Due to this difference, the corresponding techniques required in the proofs are also different. We shall present these techniques separately in Sections 3 and 4.

The rest of this paper is organized as follows. In Section 2, we first give some preliminaries and present the well-posedness of the direct problems. In Section 3, we give the sketch of the proofs in [8, 18–20, 23, 24]. Section 4 is devoted to the details of the works [22,25]. We conclude the paper in Section 5 with some remarks and open problems.

2. Preliminaries and the direct problems.

Due to the presence of the randomness, the regularity of the potential and/or the source may be too bad to fall into the scenarios of standard PDEs techniques. In this section, we show some details used in reformulating the direct problems of (1.1)-(1.3) in a proper sense. Before that, we first present some preliminaries as well as some facts related to the migr field for the subsequent use.

2.1. Preliminary and auxiliary results.

For convenient reference and self-containedness, we first present some preliminary and auxiliary results. In this paper we mainly focus on the two- and three-dimensional cases. Nevertheless, some of the results derived also hold for higher dimensions and in those cases, we choose to present the results in the general dimension $n \geq 3$ since they might be useful in other studies. Here we follow closely [22].

Throughout the paper, we write $\mathcal{L}(\mathcal{A}, \mathcal{B})$ to denote the set of all the bounded linear mappings from a normed vector space $\mathcal{A}$ to a normed vector space $\mathcal{B}$. For any mapping $\mathcal{K} \in \mathcal{L}(\mathcal{A}, \mathcal{B})$, we denote its operator norm as $\|\mathcal{K}\|_{\mathcal{L}(\mathcal{A},\mathcal{B})}$. We also use C and its variants, such as C_D, $C_{D,f}$, to denote some generic constants whose particular values may change line by line. For two quantities, we write $\mathcal{P} \lesssim \mathcal{Q}$ to signify $\mathcal{P} \leq C\mathcal{Q}$ and $\mathcal{P} \simeq \mathcal{Q}$ to signify $\widetilde{C}\mathcal{Q} \leq \mathcal{P} \leq C\mathcal{Q}$, for some generic positive constants C and $\widetilde{C}$. We write "almost everywhere" as "a.e." and "almost surely" as "a.s." for short. We use $|\mathcal{S}|$ to denote the Lebesgue measure of any Lebesgue-measurable set $\mathcal{S}$.

The Fourier transform and inverse Fourier transform of a function φ are respectively defined as

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) := (2\pi)^{-n/2} \int e^{-ix\cdot\xi} \varphi(x)\,\mathrm{d}x,$$
$$\mathcal{F}^{-1}\varphi(\xi) := (2\pi)^{-n/2} \int e^{ix\cdot\xi} \varphi(x)\,\mathrm{d}x.$$

Set

$$\Phi(x, y) = \Phi_k(x, y) := \frac{e^{ik|x-y|}}{4\pi|x - y|}, \quad x \in \mathbb{R}^3 \backslash \{y\}.$$

Φ_k is the outgoing fundamental solution, centered at y, to the differential operator $-\Delta - k^2$. Define the resolvent operator $\mathcal{R}_k$,

$$(\mathcal{R}_k\varphi)(x) := \int_{\mathbb{R}^3} \Phi_k(x, y)\varphi(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{2.1}$$

where φ can be any measurable function on $\mathbb{R}^3$ as long as (2.1) is well-defined for almost all x in $\mathbb{R}^3$.

Write $\langle x\rangle := (1+|x|^2)^{1/2}$ for $x\in\mathbb{R}^n$, $n\geq 1$. We introduce the following weighted L^p-norm and the corresponding function space over $\mathbb{R}^n$ for any $\delta\in\mathbb{R}$,

$$\begin{aligned}\|\varphi\|_{L^p_\delta(\mathbb{R}^n)} &:= \|\langle\cdot\rangle^\delta\varphi(\cdot)\|_{L^p(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n}\langle x\rangle^{p\delta}|\varphi|^p\,\mathrm{d}x\Big)^{\frac{1}{p}},\\ L^p_\delta(\mathbb{R}^n) &:= \{\,\varphi\in L^1_{loc}(\mathbb{R}^n)\,;\,\|\varphi\|_{L^p_\delta(\mathbb{R}^n)}<+\infty\,\}.\end{aligned}\tag{2.2}$$

We also define $L^p_\delta(S)$ for any subset S in $\mathbb{R}^n$ by replacing $\mathbb{R}^n$ in (2.2) with S. In what follows, we may write $L^2_\delta(\mathbb{R}^n)$ as L^2_δ for short without ambiguities. Let I be the identity operator and define

$$\|f\|_{H^{s,p}_\delta(\mathbb{R}^n)} := \|(I-\Delta)^{s/2}f\|_{L^p_\delta(\mathbb{R}^n)},\ H^{s,p}_\delta(\mathbb{R}^n) = \{f\in\mathscr{S}';\|f\|_{H^{s,p}_\delta(\mathbb{R}^n)}<+\infty\},$$

where $\mathscr{S}'$ stands for the dual space of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. The space $H^{s,2}_\delta(\mathbb{R}^n)$ is abbreviated as $H^s_\delta(\mathbb{R}^n)$, and $H^{s,p}_0(\mathbb{R}^n)$ is abbreviated as $H^{s,p}(\mathbb{R}^n)$. It can be verified that

$$\|f\|_{H^s_\delta(\mathbb{R}^n)} = \|\langle\cdot\rangle^s\widehat{f}(\cdot)\|_{H^\delta(\mathbb{R}^n)}.\tag{2.3}$$

Let $m\in(-\infty,+\infty)$. We define S^m to be the set of all functions $\sigma(x,\xi)\in C^\infty(\mathbb{R}^n,\mathbb{R}^n;\mathbb{C})$ such that for any two multi-indices α and β, there is a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$\big|(D^\alpha_x D^\beta_\xi\sigma)(x,\xi)\big| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|},\quad \forall x,\xi\in\mathbb{R}^n.$$

We call any function σ in $\bigcup_{m\in\mathbb{R}}S^m$ a *symbol*. A *principal symbol* of σ is an equivalent class $[\sigma]=\{\tilde\sigma\in S^m\,;\,\sigma-\tilde\sigma\in S^{m-1}\}$. In what follows, we may use one representative $\tilde\sigma$ in $[\sigma]$ to represent the equivalent class $[\sigma]$. Let σ be a symbol. Then the *pseudo-differential operator* T, defined on $\mathscr{S}(\mathbb{R}^n)$ and associated with σ, is defined by

$$\begin{aligned}(T_\sigma\varphi)(x) &:= (2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\sigma(x,\xi)\hat\varphi(\xi)\,\mathrm{d}\xi\\ &= (2\pi)^{-n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}e^{i(x-y)\cdot\xi}\sigma(x,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi,\quad \forall\varphi\in\mathscr{S}(\mathbb{R}^n).\end{aligned}$$

Recall Definition 1.1. Lemma 2.1 below shows how the rough order of a migr field is related to its Sobolev regularity.

Lemma 2.1. *Let h be an migr distribution of rough order $-m$ in D_h. Then, $h\in H^{-s,p}(\mathbb{R}^n)$ almost surely for any $1<p<+\infty$ and $s>(n-m)/2$.*

Proof of Lemma 2.1. See [8, Proposition 2.4]. □

By the Schwartz kernel theorem (see [15, Theorem 5.2.1]), there exists a kernel $K_h(x,y)$ with $\operatorname{supp}K_h\subset D_h\times D_h$ such that

$$(\mathfrak{C}_h\varphi)(\psi) = \mathbb{E}_\omega(\langle\overline{h(\cdot,\omega)},\varphi\rangle\langle h(\cdot,\omega),\psi\rangle) = \iint K_h(x,y)\varphi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y,\tag{2.4}$$

for all φ, $\psi\in\mathscr{S}(\mathbb{R}^n)$. It is easy to verify that $K_h(x,y)=\overline{K_h(y,x)}$. Denote the symbol of $\mathfrak{C}_h$ as c_h, then it can be verified (see [8]) that the equalities

$$\begin{cases} K_h(x,y) = (2\pi)^{-n}\displaystyle\int e^{i(x-y)\cdot\xi}c_h(x,\xi)\,\mathrm{d}\xi, & \text{(2.5a)}\\ c_h(x,\xi) = \displaystyle\int e^{-i\xi\cdot(x-y)}K_h(x,y)\,\mathrm{d}x, & \text{(2.5b)}\end{cases}$$

hold in the distributional sense, and the integrals in (2.5) shall be understood as oscillatory integrals. Despite the fact that h usually is not a function, intuitively speaking, however, it is helpful to keep in mind the following correspondence,

$$K_h(x,y) \sim \mathbb{E}_\omega\big(\overline{h(x,\omega)}h(y,\omega)\big).$$

2.2. **Some techniques related to the direct problem.** One way to study the direct problem of (1.1)-(1.3) is to transform it into the Lippmann-Schwinger equation, and then use the Bonn expansion to define a solution. To that end, the estimate of the operator norm of the resolvent $\mathcal{R}_k$ is crucial. Among different types of the estimates in the literature, one of them is known as Agmon's estimate (cf. [13, §29]). Reformulating (1.1) into the Lippmann-Schwinger equation formally (cf. [10]), we obtain

$$(I - \mathcal{R}_k q)u^{sc} = \alpha\mathcal{R}_k q u^{in} - \mathcal{R}_k f.$$

We demonstrate two lemmas dealing with the lack of regularity when utilizing Agmon's estimates. Lemma 2.2 (cf. [25, Lemma 2.2]) shows the resolvent can take a migr field as an input without any trouble, while Lemma 2.3 (cf. [22, Theorem 2.1]) gives a variation of Agmon's estimate to fit our own problem settings.

Lemma 2.2. *Assume f is a migr field with rough order $-m$ and* $\operatorname{supp} f \subset D_f$ *almost surely, then we have $\mathcal{R}_k f \in L^2_{-1/2-\epsilon}$ for any $\epsilon > 0$ almost surely.*

Proof. We split $\mathcal{R}_k f$ into two parts, $\mathcal{R}_k(\mathbb{E}f)$ and $\mathcal{R}_k(f - \mathbb{E}f)$. [21, Lemma 2.1] gives $\mathcal{R}_k(\mathbb{E}f) \in L^2_{-1/2-\epsilon}$. For $\mathcal{R}_k(f - \mathbb{E}f)$, by using (2.4), (2.5) and (2.1), one can compute

$$\begin{aligned}
&\mathbb{E}(\|\mathcal{R}_k(f-\mathbb{E}f)(\cdot,\omega)\|^2_{L^2_{-1/2-\epsilon}}) \\
&= \int_{\mathbb{R}^3} \langle x\rangle^{-1-2\epsilon}\mathbb{E}(\langle\overline{f-\mathbb{E}f},\Phi_{-k,x}\rangle\langle f-\mathbb{E}f,\Phi_{k,x}\rangle)\,\mathrm{d}x = \int_{\mathbb{R}^3}\langle x\rangle^{-1-2\epsilon}\langle\mathfrak{C}_f\Phi_{-k,x},\Phi_{k,x}\rangle\,\mathrm{d}x \\
&\simeq \int \langle x\rangle^{-1-2\epsilon}\int_{D_f}\Big(\int_{D_f}\frac{\mathcal{I}(y,z)e^{-ik|x-z|}}{|x-z|\cdot|y-z|^2}\,\mathrm{d}z\Big)\cdot\frac{e^{ik|x-y|}}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x, \qquad (2.6)
\end{aligned}$$

where $c_f(y,\xi)$ is the symbol of the covariance operator $\mathfrak{C}_f$ and

$$\mathcal{I}(y,z) := \int_{\mathbb{R}^3} |y-z|^2 e^{i(y-z)\cdot\xi}c_f(y,\xi)\,\mathrm{d}\xi.$$

When $y = z$, we know $\mathcal{I}(y,z) = 0$ because the integrand is zero. Thanks to the condition $m > 2$, when $y \neq z$ we have

$$|\mathcal{I}(y,z)| = \Big|\sum_{j=1}^3\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi}(\partial^2_{\xi_j}c_f)(y,\xi)\,\mathrm{d}\xi\Big|\sum_{j=1}^3\int_{\mathbb{R}^3} C_j\langle\xi\rangle^{-m-2}\,\mathrm{d}\xi \leq C_0 < +\infty, \qquad (2.7)$$

for some constant C_0 independent of y and z. Note that if D_f is bounded, then for $j = 1, 2$ we have

$$\int_{D_f}|x-y|^{-j}\,\mathrm{d}y \leq C_{f,j}\langle x\rangle^{-j}, \quad \forall x \in \mathbb{R}^3, \qquad (2.8)$$

for some constant $C_{f,j}$ depending only on f, j and the dimension. The notation $\langle x\rangle$ in (2.8) stands for $(1+|x|^2)^{1/2}$ and readers may note the difference between the $\langle\cdot\rangle$ and the $\langle\cdot,\cdot\rangle$ appeared in (2.1). With the help of (2.7) and (2.8) and Hölder's inequality, we can continue (2.6) as

$$\mathbb{E}(\|\mathcal{R}_k(f-\mathbb{E}f)(\cdot,\omega)\|^2_{L^2_{-1/2-\epsilon}})$$

$$\lesssim \int \langle x\rangle^{-1-2\epsilon}\Big(\iint_{D_f\times D_f}(|x-z|\cdot|y-z|^2\cdot|x-y|)^{-1}\,\mathrm{d}z\,\mathrm{d}y\Big)\,\mathrm{d}x$$
$$\leq \int \langle x\rangle^{-1-2\epsilon}C_f\langle x\rangle^{-2}\,\mathrm{d}x \leq C_f < +\infty,$$

which gives

$$\mathbb{E}(\|\mathcal{R}_k(f-\mathbb{E}f)(\cdot,\omega)\|^2_{L^2_{-1/2-\epsilon}}) \leq C_f < +\infty. \tag{2.9}$$

By using the Hölder inequality applied to the probability measure, we obtain from (2.9) that

$$\mathbb{E}\|\mathcal{R}_k(f-\mathbb{E}f)\|_{L^2_{-1/2-\epsilon}} \leq [\mathbb{E}(\|\mathcal{R}_k(f-\mathbb{E}f)\|^2_{L^2_{-1/2-\epsilon}})]^{1/2} \leq C_f^{1/2} < +\infty, \tag{2.10}$$

for some constant C_f independent of k. The formula (2.10) gives that $\mathcal{R}_k(f-\mathbb{E}f) \in L^2_{-1/2-\epsilon}$ almost surely, and hence $\mathcal{R}_k f \in L^2_{-1/2-\epsilon}$ almost surely.

The proof is complete. □

Lemma 2.3. *For any* $0 < s < 1/2$ *and* $\epsilon > 0$, *when* $k > 2$,

$$\|\mathcal{R}_k\varphi\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^3)} \leq C_{\epsilon,s}k^{-(1-2s)}\|\varphi\|_{H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)}, \quad \varphi \in H^{-s}_{1/2+\epsilon}(\mathbb{R}^3).$$

Proof. We adopt the concept of *Limiting absorption principle* to first show desired results on a family of operator $\mathcal{R}_{k,\tau}$ controlled by a parameter τ, and then show that $\mathcal{R}_{k,\tau}$ converges in a proper sense as τ approaches zero. We sketch out the key steps in the proof and readers may refer to the proof of [22, Theorem 2.1] for complete details.

Define an operator

$$\mathcal{R}_{k,\tau}\varphi(x) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\frac{\hat{\varphi}(\xi)}{|\xi|^2-k^2-i\tau}\,\mathrm{d}\xi, \tag{2.11}$$

where $\tau \in \mathbb{R}_+$. Fix a function χ satisfying

$$\begin{cases}\chi \in C_c^\infty(\mathbb{R}^n),\ 0 \leq \chi \leq 1,\\ \chi(x) = 1 \text{ when } |x| \leq 1,\\ \chi(x) = 0 \text{ when } |x| \geq 2.\end{cases} \tag{2.12}$$

Write $\mathfrak{R}\psi(x) := \psi(-x)$. We have

$$\begin{aligned}
&(\mathcal{R}_{k,\tau}\varphi,\psi)_{L^2(\mathbb{R}^3)}\\
=\ &\int_{\mathbb{R}^3}\mathcal{R}_{k,\tau}\varphi(x)\overline{\psi(x)}\,\mathrm{d}x = \int_{\mathbb{R}^3}\mathcal{F}\{\mathcal{R}_{k,\tau}\varphi\}(\xi)\cdot\mathcal{F}\{\mathfrak{R}\overline{\psi}\}(\xi)\,\mathrm{d}\xi\\
=\ &\int_0^\infty \frac{(1-\chi^2(r-k))}{r^2-k^2-i\tau}\,\mathrm{d}r\cdot\int_{|\xi|=r}\hat{\varphi}(\xi)\cdot\widehat{\mathfrak{R}\overline{\psi}}(\xi)\,\mathrm{d}S(\xi)\\
&+\int_0^\infty \frac{\langle r\rangle^{1/p}r^2\chi^2(r-k)}{r^2-k^2-i\tau}\,\mathrm{d}r\times\int_{\mathbb{S}^2}[\langle k\rangle^{\frac{-1}{2p}}\hat{\varphi}(k\omega)][\langle k\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(k\omega)]\,\mathrm{d}S(\omega)\\
&+\int_0^\infty \frac{\langle r\rangle^{1/p}r^2\chi^2(r-k)}{r^2-k^2-i\tau}\,\mathrm{d}r\cdot\int_{\mathbb{S}^2}\{[\langle r\rangle^{\frac{-1}{2p}}\hat{\varphi}(r\omega)][\langle r\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(r\omega)]\\
&\qquad\qquad -[\langle k\rangle^{\frac{-1}{2p}}\hat{\varphi}(k\omega)][\langle k\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(k\omega)]\}\,\mathrm{d}S(\omega)\\
=:\ &I_1(\tau)+I_2(\tau)+I_3(\tau).
\end{aligned} \tag{2.13}$$

Here we divide $(\mathcal{R}_{k,\tau}\varphi,\psi)_{L^2(\mathbb{R}^3)}$ into three parts in order to deal with the singularity happened in the integral when $|\xi|$ is close to k. The integral in I_1 has avoided this singularity by the cutoff function χ. The singularity in I_2 is only contained in the integration w.r.t. r, and it can be shown that by using Cauchy's integral theorem and choosing a proper integral path w.r.t. r, the norm of the denominator $\tau^2-k^2-i\tau$ can always be bounded below by k, e.g. $|\tau^2-k^2-i\tau| \gtrsim k$. The singularity in I_3 is compensated by the difference $[\cdots]$ inside the integration $\int_{\mathbb{S}^2}[\cdots]\,\mathrm{d}S(\omega)$. In the following, we only show how to deal with I_2.

Now we estimate $I_1(\tau)$. By Young's inequality $ab \leq a^p/p + b^q/q$, for $a,b > 0$, $p,q > 1$, $1/p+1/q=1$ we have

$$(p^{1/p}q^{1/q})a^{1/p}b^{1/q} \leq a+b. \tag{2.14}$$

Note that $|r-k|>1$ in the support of the function $1-\chi^2(r-k)$ and $|\widehat{\Re\overline{\psi}}(\xi)| = |\hat{\psi}(\xi)|$, one can compute

$$\begin{aligned} |I_1(\tau)| &\leq \int_0^\infty \frac{1-\chi^2(r-k)}{1\cdot p^{1/p}q^{1/q}(r+1)^{1/p}(k-1)^{1/q}}\,\mathrm{d}r \cdot \int_{|\xi|=r} |\hat{\varphi}(\xi)|\cdot|\hat{\psi}(\xi)|\,\mathrm{d}S(\xi) \quad (\text{by } (2.14)) \\ &\leq C_p k^{1/p-1}\|\varphi\|_{H_\delta^{-1/(2p)}(\mathbb{R}^3)}\|\psi\|_{H_\delta^{-1/(2p)}(\mathbb{R}^3)}, \end{aligned} \tag{2.15}$$

where $1<p<+\infty$ and $\delta>0$ and the C_p is independent of τ.

We next estimate $I_2(\tau)$. One has

$$I_2(\tau) = \int_{\mathbb{S}^2}[\langle k\rangle^{\frac{-1}{2p}}\hat{\varphi}(k\omega)][\langle k\rangle^{\frac{-1}{2p}}\widehat{\Re\overline{\psi}}(k\omega)]\int_0^\infty \frac{\langle r\rangle^{\frac{1}{p}}r^2\chi^2(r-k)\,\mathrm{d}r}{r^2-k^2-i\tau}\,\mathrm{d}S(\omega). \tag{2.16}$$

It can be shown that, by choosing a fixed $\tau_0 \in (0,1)$ carefully, we can show that the denominator $p_\tau(r) := r^2-k^2-i\tau$ could satisfy

$$|p_\tau(r)| \geq \tau_0 k \text{ and } |r| \lesssim k,\ \forall r \in \{r; 2 \geq |r-k| \geq \tau_0\} \cup \Gamma_{k,\tau_0},\ \forall \tau \in (0,\tau_0), \tag{2.17}$$

where $\Gamma_{k,\tau_0} := \{r \in \mathbb{C}; |r-k| = \tau_0, \Im r \leq 0\}$. It is obvious that the purpose of (2.17) is to use Cauchy's integral theorem. By combining (2.17) with Cauchy's integral theorem, we can continue (2.16) as

$$\begin{aligned} |I_2(\tau)| \leq &\int_{|\xi|=k}\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\varphi}(\xi)|\cdot\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\psi}(\xi)|\Big(\int_{\{r\in\mathbb{R}_+\,;\,2\geq|r-k|\geq\tau_0\}}\frac{\langle r\rangle^{\frac{1}{p}}(r/k)^2}{\tau_0 k}\,\mathrm{d}r\Big)\,\mathrm{d}S(\xi) \\ &+ \int_{|\xi|=k}\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\varphi}(\xi)|\cdot\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\psi}(\xi)|\Big(\int_{\Gamma_{k,\tau_0}}\frac{(1+|r|^2)^{\frac{1}{2p}}(|r|/k)^2}{\tau_0 k}\,\mathrm{d}r\Big)\,\mathrm{d}S(\xi) \\ \leq &\, C_{\tau_0}\int_{|\xi|=k}\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\varphi}(\xi)|\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\psi}(\xi)|\Big(\int_{\Gamma_{k,\tau_0}\cup\{r\in\mathbb{R}_+;2\geq|r-k|\geq\tau_0\}}\frac{\langle k\rangle^{1/p}}{\tau_0 k}\,\mathrm{d}r\Big)\,\mathrm{d}S(\xi) \\ &+ C_{\tau_0}\int_{|\xi|=k}\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\varphi}(\xi)|\langle\xi\rangle^{\frac{-1}{2p}}|\hat{\psi}(\xi)|\Big(\int_{\Gamma_{k,\tau_0}}\frac{\langle k\rangle^{1/p}}{\tau_0 k}\,\mathrm{d}r\Big)\,\mathrm{d}S(\xi) \\ \leq &\, C_{\tau_0}k^{1/p-1}\Big(\int_{|\xi|=k}|\langle\xi\rangle^{\frac{-1}{2p}}\hat{h}(\xi)|^2\,\mathrm{d}S(\xi)\Big)^{\frac{1}{2}}\Big(\int_{|\xi|=k}|\langle\xi\rangle^{\frac{-1}{2p}}\hat{\psi}(\xi)|^2\,\mathrm{d}S(\xi)\Big)^{\frac{1}{2}} \\ \leq &\, C_{\tau_0,\epsilon}k^{1/p-1}\|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}\|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}, \end{aligned} \tag{2.18}$$

where the constant $C_{\tau_0,\epsilon}$ is independent of τ. Here, in deriving the last inequality in (2.18), we have made use of (2.3).

Finally, we estimate $I_3(\tau)$. Denote $\mathbb{F}(r\omega) = \mathbb{F}_r(\omega) := \langle r\rangle^{-1/(2p)}\hat{\varphi}(r\omega)$ and $\mathbb{G}(r\omega) = \mathbb{G}_r(\omega) := \langle r\rangle^{-1/(2p)}\widehat{\Re\overline{\psi}}(r\omega)$. One can compute

$$|I_3(\tau)| \leq \int_0^\infty \frac{\langle r\rangle^{1/p}\chi^2(r-k)}{|r^2-k^2|}\cdot\|\mathbb{F}_r\|_{L^2(\mathbb{S}_r^2)}\cdot\big(r^2\int_{\mathbb{S}^2}|\mathbb{G}_r-\mathbb{G}_k|^2\,\mathrm{d}S(\omega)\big)^{\frac{1}{2}}\,\mathrm{d}r$$

$$+ \int_0^\infty \frac{\langle r\rangle^{1/p}\chi^2(r-k)}{|r^2-k^2|} \cdot \big(r^2 \int_{\mathbb{S}^2} |\mathbb{F}_r - \mathbb{F}_k|^2 \,\mathrm{d}S(\omega)\big)^{\frac{1}{2}} \cdot \big(\frac{r}{k}\big)^2 \|\mathbb{G}_k\|_{L^2(\mathbb{S}_k^2)} \,\mathrm{d}r, \tag{2.19}$$

where $\mathbb{S}_r^2$ signifies the central sphere of radius r. Combining [13, Remark 13.1 and (13.28)] and (2.3) and (2.14), we can continue (2.19) as

$$\begin{aligned} |I_3(\tau)| &\leq C_{\alpha,\epsilon} \int_0^\infty \frac{\langle r\rangle^{1/p}\chi^2(r-k)}{|r-k|(r+k)} \cdot \|\mathbb{F}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \cdot |r-k|^\alpha \cdot \|\mathbb{G}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \,\mathrm{d}r \\ &\leq C_{\alpha,\epsilon,p} \int_0^\infty \frac{\langle r\rangle^{1/p}\chi^2(r-k)}{|r-k|^{1-\alpha}(r+1)^{1/p}(k-1)^{1-1/p}} \,\mathrm{d}r \cdot \|\mathbb{F}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \|\mathbb{G}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \\ &\leq C_{\alpha,\epsilon,p} k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \cdot \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}, \end{aligned} \tag{2.20}$$

where the ϵ can be any positive real number and the α satisfies $0 < \alpha < \epsilon$, and the constant $C_{\alpha,\epsilon,p}$ is independent of τ.

Combining (2.13), (2.15), (2.18) and (2.20), we arrive at

$$|(\mathcal{R}_{k,\tau}\varphi, \psi)_{L^2(\mathbb{R}^3)}| \leq |I_1(\tau)| + |I_2(\tau)| + |I_3(\tau)| \leq Ck^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)},$$

which implies that

$$\|\mathcal{R}_{k,\tau}\varphi\|_{H^{1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3)} \leq Ck^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \tag{2.21}$$

for some constant C independent of τ.

Next we investigate the limiting case $\lim\limits_{\tau\to 0^+} \mathcal{R}_{k,\tau}\varphi$. Following similar steps when dealing with I_1, I_2 and I_3, it can be shown that for any $\tilde{\tau} > 0$, we have

$$|I_j(\tau_1) - I_j(\tau_2)| \leq \tilde{\tau}^\beta k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}, \quad (j = 1, 2, 3)$$

holds for $\forall \tau_1, \tau_2 \in (0, \tilde{\tau})$. Therefore, we can conclude

$$\|\mathcal{R}_{k,\tau_1}\varphi - \mathcal{R}_{k,\tau_2}\varphi\|_{H^{-1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3)} \lesssim \tilde{\tau} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}, \quad \forall \tau_1, \tau_2 \in (0, \tilde{\tau}),$$

and thus $\mathcal{R}_{k,\tilde{\tau}}\varphi$ converges and

$$\lim_{\tilde{\tau}\to 0^+} \mathcal{R}_{k,\tilde{\tau}}\varphi = \mathcal{R}_k\varphi \quad \text{in} \quad H^{1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3). \tag{2.22}$$

Hence from (2.21) and (2.22) we conclude that

$$\|\mathcal{R}_k\varphi\|_{H^{1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3)} \leq C_{\epsilon,p} k^{-(1-1/p)} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}$$

holds for any $1 < p < +\infty$ and any $\epsilon > 0$.

The proof is complete. □

With the help of Lemmas 2.2 and 2.3, the direct problems can be reformulated. Readers may refer to [25, Theorem 2.1], [22, Theorem 2.3], [19, Theorem 4.3], [20, Theorem 3.3], and [24, Theorem 3.3] as examples of how to formulate the direct problems, and we omit the details here.

3. **Recovery by near-field data.**

In this section we consider key steps in the works [8, 18–20, 23, 24]. Lemma 3.3 is crucial in the key steps of the works, and its proof relies on Lemmas 3.1 and 3.2. We shall first investigate these useful lemmas.

3.1. **Useful lemmas.** Lemma 3.1 is a standard result in the field of oscillatory integral and microlocal analysis.

Lemma 3.1. *Assume α and β are multi-indexes, then the following identities hold in the oscillatory integral sense,*

$$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} e^{ix\cdot\xi}\,\mathrm{d}x\,\mathrm{d}\xi = (2\pi)^n, \tag{3.1}$$

$$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} e^{ix\cdot\xi} x^\alpha \xi^\beta\,\mathrm{d}x\,\mathrm{d}\xi = (2\pi)^n i^{|\alpha|} \alpha! \delta^{\alpha\beta}. \tag{3.2}$$

Here $\delta^{\alpha\beta}$ equals to 1 when $\alpha = \beta$ and equals to 0 otherwise.

Proof. The integral in (3.1) should be understood as oscillatory integral. Fix a cutoff function $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(0) = 1$, we can compute

$$\begin{aligned}\int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} e^{ix\cdot\xi}\,\mathrm{d}x\,\mathrm{d}\xi &= \lim_{\epsilon\to 0^+} \int e^{ix\cdot\xi} \chi(\epsilon x)\chi(\epsilon\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &= (2\pi)^{n/2} \lim_{\epsilon\to 0^+} \int \chi(\epsilon^2\xi)\widehat{\chi}(-\xi)\,\mathrm{d}\xi. \end{aligned}\tag{3.3}$$

Denote $M = \sup_{\mathbb{R}^n} \chi$. We have $|\chi(\epsilon^2\xi)| \leq M < \infty$. Note that $\chi \in C_c^\infty(\mathbb{R}^n)$, so $\widehat{\chi}$ is rapidly decaying, thus $\widehat{\chi}(-\xi)$ is Lebesgue integrable. Therefore, we can see that $\widehat{\chi}(-\xi)\chi(\epsilon^2\xi)$ is dominated by a Lebesgue integrable function. Thus by using Lebesgue Dominated Convergence Theorem, we can continue (3.3) as

$$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_\xi} e^{ix\cdot\xi}\,\mathrm{d}x\,\mathrm{d}\xi = (2\pi)^{n/2} \int \widehat{\chi}(-\xi)\,\mathrm{d}\xi = (2\pi)^n \chi(0) = (2\pi)^n.$$

We arrive at (3.1).

To show (3.2), we first show that

$$(2\pi)^{-n} \iint e^{-iy\cdot\eta} y^\alpha \eta^\beta\,\mathrm{d}y\,\mathrm{d}\eta = (2\pi)^{-n} \iint e^{-iy\cdot\eta} D_\eta^\alpha(\eta^\beta)\,\mathrm{d}y\,\mathrm{d}\eta, \tag{3.4}$$

where $D_{\eta_j} := \frac{1}{i}\partial_{\eta_j}$. Both the LHS and RHS in (3.4) should be understood as a oscillatory integral. Thus fix some $\chi \in \mathscr{D}(\mathbb{R}^n)$ such that $\chi(x) \equiv 1$ when $|x| \leq 1$, we have

$$\begin{aligned} \iint e^{-iy\cdot\eta} y^\alpha \eta^\beta\,\mathrm{d}y\,\mathrm{d}\eta &= \lim_{\epsilon\to 0^+} \iint e^{-iy\cdot\eta} y^\alpha \eta^\beta \chi(\epsilon y)\chi(\epsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\ &= \lim_{\epsilon\to 0^+} \iint (-D_\eta)^\alpha (e^{-iy\cdot\eta})\,\eta^\beta \chi(\epsilon y)\chi(\epsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\ &= \lim_{\epsilon\to 0^+} \iint e^{-iy\cdot\eta} \chi(\epsilon y) D_\eta^\alpha\big(\eta^\beta \chi(\epsilon\eta)\big)\,\mathrm{d}y\,\mathrm{d}\eta \\ &= \lim_{\epsilon\to 0^+} \sum_{0<\gamma\leq\alpha} \epsilon^{|\gamma|} \binom{\alpha}{\gamma} \iint e^{-iy\cdot\eta} \chi(\epsilon y) \cdot D_\eta^{\alpha-\gamma}(\eta^\beta) \cdot \big(\partial^\gamma \chi\big)(\epsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\ &\quad + \iint e^{-iy\cdot\eta} D_\eta^\alpha\big(\eta^\beta\big)\,\mathrm{d}y\,\mathrm{d}\eta. \end{aligned}\tag{3.5}$$

As ϵ goes to zero, we have

$$\iint e^{-iy\cdot\eta} \chi(\epsilon y) \cdot D_\eta^{\alpha-\gamma}(\eta^\beta) \cdot \big(\partial^\gamma \chi\big)(\epsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta \to D_\eta^{\alpha-\gamma}(\eta^\beta) \cdot \big(\partial^\gamma \chi\big)(\epsilon\eta)\big|_{\eta=0} \quad (\epsilon \to 0^+).$$

Because $\gamma > 0$, $\big(\partial^\gamma \chi\big)(\epsilon\eta)\big|_{\eta=0} = 0$. Therefore, we have

$$\lim_{\epsilon\to 0^+} \epsilon^{|\gamma|} \sum_{0<\gamma\leq\alpha} \binom{\alpha}{\gamma} \iint e^{-iy\cdot\eta}\chi(\epsilon y)\cdot D_\eta^{\alpha-\gamma}(\eta^\beta)\cdot\big(\partial^\gamma\chi\big)(\epsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta = 0. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at

$$\iint e^{-iy\cdot\eta}y^\alpha\eta^\beta\,\mathrm{d}y\,\mathrm{d}\eta = \lim_{\epsilon\to 0^+}\iint e^{-iy\cdot\eta}y^\alpha\eta^\beta\chi(\epsilon y)\chi(\epsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta = \iint e^{-iy\cdot\eta}D_\eta^\alpha\big(\eta^\beta\big)\,\mathrm{d}y\,\mathrm{d}\eta.$$

We proved (3.4).

Then, for multi-indexes α and β, if there exists i such that $\alpha_i \neq \beta_i$, say, $\alpha_i > \beta_i$, then $D_\xi^\alpha(\xi^\beta) = 0$ and so

$$\int e^{ix\cdot\xi}x^\alpha\xi^\beta\,\mathrm{d}x\,\mathrm{d}\xi = \int e^{ix\cdot\xi}(-D_\xi)^\alpha(\xi^\beta)\,\mathrm{d}x\,\mathrm{d}\xi = 0.$$

When $\alpha = \beta$, we have

$$\int e^{ix\cdot\xi}x^\alpha\xi^\beta\,\mathrm{d}x\,\mathrm{d}\xi = \int e^{ix\cdot\xi}(-D_\xi)^\alpha(\xi^\alpha)\,\mathrm{d}x\,\mathrm{d}\xi = \int e^{ix\cdot\xi}i^{|\alpha|}\alpha!\,\mathrm{d}x\,\mathrm{d}\xi = (2\pi)^n i^{|\alpha|}\alpha!.$$

We have arrived at (3.2). □

We also need [16, Lemma 18.2.1] and we present a proof below.

Lemma 3.2. *If $a \in S^m(\mathbb{R}^n\times\mathbb{R}^k)$ and u is defined by the oscillatory integral*

$$u(x) = \int e^{i\langle x',\xi'\rangle}a(x,\xi')\,\mathrm{d}\xi',$$

then there exists $\tilde{a} \in S^m(\mathbb{R}^{n-k}\times\mathbb{R}^k)$ such that

$$u(x) = \int e^{i\langle x',\xi'\rangle}\tilde{a}(x'',\xi')\,\mathrm{d}\xi',$$

and $\tilde{a}$ has the asymptotic expansion

$$\tilde{a}(x'',\xi') \sim \sum_\alpha i^{|\alpha|}\partial_{x'}^\alpha\partial_{\xi'}^\alpha a(0,x'',\xi')/\alpha!.$$

Remark 3.1. Note if $a(x,\xi') = 0$ near $\{x' = 0\}$, e.g. $a(x,\xi) = (1-\chi(x'))a'(x,\xi)$ for some a' and some cutoff function satisfying $\chi(y) \equiv 1$ near the origin, then Lemma 3.2 implies that $\tilde{a} \in S^{-\infty}$.

Proof. The $\tilde{a}(x'',\cdot)$ is the Fourier transform of $u(\cdot,x'')$ with some constants, i.e.

$$\tilde{a}(x'',\xi') = (2\pi)^{-k/2}\mathcal{F}_{x'}\{u(x',x'')\}(\xi') = (2\pi)^{-k}\int e^{-ix'\cdot\xi'}u(x',x'')\,\mathrm{d}x'.$$

Then we can have

$$\tilde{a}(x'',\xi') = (2\pi)^{-k}\int e^{-ix'\cdot\xi'}u(x)\,\mathrm{d}x' = (2\pi)^{-k}\int e^{ix'\cdot\theta}a(x,\xi'+\theta)\,\mathrm{d}\theta\,\mathrm{d}x'.$$

By adopting the way used in [1, §I.8.1] in computing the oscillatory integral, we can easily show that $|\partial_{x''}^\alpha\partial_{\xi'}^\beta\tilde{a}(x'',\xi)| \lesssim \langle\xi'\rangle^{m-|\beta|}$, and this can be seen by the fact that

$$|\partial_{\xi'}^\alpha[\chi(x,\theta)a(2^k x',x'',\xi'+2^k\theta)]| \lesssim 2^{mk}\langle\xi'\rangle^k,$$

so $\tilde{a} \in S^m(\mathbb{R}^{n-k}\times\mathbb{R}^k)$.

The idea of the proof is to expand $a(x', x'', \xi' + \theta)$ in terms of x' and θ by Taylor expansion

$$a(x', x'', \xi' + \theta) = \sum_{|\alpha|+|\beta|\leq 2N} \frac{x'^{\alpha}\theta^{\beta}}{\alpha!\beta!} \partial_{x'}^{\alpha} \partial_{\xi'}^{\beta} a(0, x'', \xi') + \sum_{|\alpha|+|\beta|=2N+1} \frac{x'^{\alpha}\theta^{\beta}}{\alpha!\beta!} \partial_{x'}^{\alpha} \partial_{\xi'}^{\beta} a(\eta x', x'', \xi' + \eta\theta), \quad 0 < \eta < 1,$$

and to use Lemma 3.1. We have

$$\begin{aligned}\tilde{a}(x'', \xi') &= (2\pi)^{-k} \int e^{ix'\cdot\theta} a(x', x'', \xi' + \theta) \,\mathrm{d}\theta \,\mathrm{d}x' \\ &= \sum_{|\alpha|\leq N} i^{|\alpha|} \partial_{x'}^{\alpha} \partial_{\xi'}^{\beta} a(0, x'', \xi')/\alpha! \\ &\quad + \sum_{\substack{|\alpha|+|\beta|=2N+1 \\ \gamma\leq\alpha\leq\beta+\gamma}} C_{\alpha,\beta,\gamma} \int e^{ix'\cdot\theta} (\partial_{x'}^{\alpha} \partial_{\xi'}^{\beta+\gamma} a)(\eta^{|\gamma|}) \partial_{\theta}^{\alpha-\gamma}(\theta^{\beta}) \,\mathrm{d}\theta \,\mathrm{d}x'. \end{aligned} \tag{3.7}$$

Note that the constraint $\alpha \leq \beta + \gamma$ in (3.7) comes from the fact that $\partial_{\theta}^{\alpha-\gamma}(\theta^{\beta}) = 0$ when $\alpha > \beta + \gamma$. Moreover, the constraint "$|\alpha| + |\beta| = 2N + 1$, $\gamma \leq \alpha \leq \beta + \gamma$" gives

$$2N + 1 = |\alpha| + |\beta| \leq 2|\beta| + |\gamma| \leq 2(|\beta| + |\gamma|) \quad \Rightarrow \quad |\beta + \gamma| \geq N + 1.$$

Now we show that each remainder term in (3.7) is controlled by $\langle \xi' \rangle^{m-N-1}$. Denote $b(x', x'', \theta; \xi', \eta) = \big(\partial_{x'}^{\alpha} \partial_{\xi'}^{\beta+\gamma} a(\eta x', x'', \xi' + \eta\theta)\big)(\eta^{|\gamma|}) \partial_{\theta}^{\alpha-\gamma}(\theta^{\beta})$ with underlining assumptions $\beta + \gamma \geq \alpha$ and $|\beta + \gamma| \geq N + 1$, and we have

$$\begin{aligned} &\tilde{a}(x'', \xi') - \sum_{|\alpha|\leq N} i^{|\alpha|} \partial_{x'}^{\alpha} \partial_{\xi'}^{\beta} a(0, x'', \xi')/\alpha! \\ &= \int e^{ix'\cdot\theta} \chi_0(x', \theta) b \,\mathrm{d}\theta \,\mathrm{d}x' \\ &\quad + \sum_{\ell\geq 1} \int e^{ix'\cdot\theta} \chi(x'/2^{\ell}, \theta/2^{\ell}) b(x', x'', \theta; \xi', \eta) \,\mathrm{d}\theta \,\mathrm{d}x', \end{aligned}$$

where χ_0 and χ is as in [1, §I.8.1]. Here we only show how the second term in the equation above is controlled by $\langle \xi' \rangle^{m-N-1}$. The computation is as follows,

$$\begin{aligned} &\int e^{ix'\cdot\theta} \chi(x'/2^{\ell}, \theta/2^{\ell}) b(x', x'', \theta; \xi', \eta) \,\mathrm{d}\theta \,\mathrm{d}x' \\ &\lesssim 2^{2\ell k} \int \big(\frac{(\theta, x') \cdot \nabla_{(x',\theta)}}{i2^{2\ell}(|x'|^2 + |\theta|^2)}\big)^L (e^{i2^{2\ell} x'\cdot\theta}) \cdot \chi(x', \theta) b(x', x'', \theta; \xi', 2^{\ell}\eta) \,\mathrm{d}\theta \,\mathrm{d}x' \\ &\lesssim \langle \xi' \rangle^{m-N-1} \cdot 2^{\ell(2k+1-2L)} \int_{\operatorname{supp}\chi} C_L 2^{\ell(|m-N-1|+L)} \,\mathrm{d}\theta \,\mathrm{d}x' \\ &\lesssim \langle \xi' \rangle^{m-N-1} \cdot 2^{\ell(2k+1+|m-N-1|-L)}, \end{aligned}$$

thus if we take L to be large enough such that $2k + 1 + |m - N - 1| - L < 0$, we can have

$$|\sum_{\ell\geq 1} \int e^{ix'\cdot\theta} \chi(x'/2^{\ell}, \theta/2^{\ell}) b \,\mathrm{d}\theta \,\mathrm{d}x'| \lesssim \sum_{\ell\geq 1} \langle \xi' \rangle^{m-N-1} 2^{\ell(2k+1+|m-N-1|-L)} \lesssim \langle \xi' \rangle^{m-N-1}.$$

This shows $|\tilde{a}(x'',\xi') - \sum_{|\alpha|\le N} i^{|\alpha|}\partial_{x'}^{\alpha}\partial_{\xi'}^{\alpha} a(0,x'',\xi')/\alpha!| \lesssim \langle\xi'\rangle^{m-N-1}$. Using the same procedure, we can show $\big|\partial_{x''}^{\kappa}\partial_{\xi'}^{\beta}[\tilde{a}(x'',\xi') - \sum_{|\alpha|\le N} i^{|\alpha|}\partial_{x'}^{\alpha}\partial_{\xi'}^{\alpha} a(0,x'',\xi')/\alpha!]\big| \lesssim \langle\xi'\rangle^{m-N-1-|\beta|}$, and hence

$$\tilde{a}(x'',\xi') - \sum_{|\alpha|\le N} i^{|\alpha|}\partial_{x'}^{\alpha}\partial_{\xi'}^{\alpha} a(0,x'',\xi')/\alpha! \in S^{m-N-1}(\mathbb{R}^{n-k}\times\mathbb{R}^k).$$

The proof is complete. □

We also need [16, Lemma 18.2.9] and we present a proof below.

Lemma 3.3. *Assume that $a \in S^m$ and*

$$u(x) = \int e^{i\langle x',\xi'\rangle} a(x,\xi')\,\mathrm{d}\xi', \quad \xi' \in \mathbb{R}^k,$$

and a C^∞ diffeomorphism $\rho\colon y \in \mathbb{R}^n \mapsto \rho(y) = (\rho_1(y),\rho_2(y)) \in \mathbb{R}^n$ preserving the hyperplane $S = \{x\,;\, x' = 0\}$. The ρ_1 is k-dimensional while ρ_2 is $(n-k)$-dimensional. Assume u and the pull-back $\rho^* u$ is C^∞-smooth in $\mathbb{R}^n\backslash S$, then there exists $\tilde{a} \in S^m(\mathbb{R}^{n-k}\times\mathbb{R}^k)$ such that $\rho^* u$ can be represented as

$$\rho^* u(y) = \int e^{i\langle y',\xi'\rangle}\tilde{a}(y'',\xi')\,\mathrm{d}\xi',$$

and

$$\tilde{a}(y'',\eta) - a(0,\rho_2(0,y''),(\psi(0,y''))^{T,-1}\eta)|\det\psi(0,y'')|^{-1} \in S^{m-1}(\mathbb{R}^{n-k}\times\mathbb{R}^k),$$

where $(*)^T$ and $(*)^{T,-1}$ signify the transpose and transpose with inverse of a matrix, respectively.

Remark 3.2. The condition "u and $\rho^* u$ is C^∞-smooth in $\mathbb{R}^n\backslash S$" is indispensable.

Proof. Because ρ preserves the hyperplane $\{x\,;\,x'=0\}$, there exists a C^∞ matrix-valued function ψ such that $\rho_1(y',y'') = \psi(y)\cdot y'$, where the dot operation "$\cdot$" here signifies the matrix multiplication. According to Lemma 3.2, there exist $\bar{a} \in S^m$ such that $u(x) = \int e^{i\langle x',\xi'\rangle}\bar{a}(x'',\xi')\,\mathrm{d}\xi'$. Hence we have

$$\begin{aligned}\tilde{u}(y) := \rho^* u(y) = u(\rho(y)) &= \int e^{i\langle \rho_1(y),\xi'\rangle}\bar{a}(\rho_2(y),\xi')\,\mathrm{d}\xi' \\ &= \int e^{i\langle \psi(y)\cdot y',\xi'\rangle}\bar{a}(\rho_2(y),\xi')\,\mathrm{d}\xi' = \int e^{i\langle y',(\psi(y))^T\xi'\rangle}\bar{a}(\rho_2(y),\xi')\,\mathrm{d}\xi',\end{aligned}$$

According to Remark 3.1, we could continue

$$\begin{aligned}\tilde{u}(y) &= \int e^{i\langle y',(\psi(y))^T\xi'\rangle}\chi(y')\bar{a}(\rho_2(y),(\psi(y))^{T,-1}(\psi(y))^T\xi')|\det\psi(y)|^{-1}\,\mathrm{d}((\psi(y))^T\xi') + v(y) \\ &= \int e^{i\langle y',\eta\rangle}\chi(y')\bar{a}(\rho_2(y),(\psi(y))^{T,-1}\eta)|\det\psi(y)|^{-1}\,\mathrm{d}\eta + v(y),\end{aligned}$$

where $\chi \in C_c^\infty(\mathbb{R}^k)$ with $\chi(y') \equiv 1$ in a neighborhood 0 such that the matrix $\psi(y)$ is invertible in $\operatorname{supp}\psi$, and $v(y) = \int e^{i\langle y',\eta\rangle} b(y'',\eta)\,\mathrm{d}\eta$ with $b \in S^{-\infty}$. Using Lemma 3.2, we obtain $\tilde{u}(y) = \int e^{i\langle y',\eta\rangle}\tilde{a}(y'',\eta)\,\mathrm{d}\eta$ where

$$\tilde{a}(y'',\eta) - \bar{a}(\rho_2(0,y''),(\psi(0,y''))^{T,-1}\eta)|\det\psi(0,y'')|^{-1} \in S^{m-1}(\mathbb{R}^{n-k}\times\mathbb{R}^k).$$

Note that $\bar{a}$ satisfies $\bar{a}(x'',\xi') - a(0,x'',\xi') \in S^{m-1}(\mathbb{R}^{n-k}\times\mathbb{R}^k)$, so

$$\tilde{a}(y'',\eta) - a(0,\rho_2(0,y''),(\psi(0,y''))^{T,-1}\eta)|\det\psi(0,y'')|^{-1} \in S^{m-1}(\mathbb{R}^{n-k}\times\mathbb{R}^k).$$

The proof is complete. □

Finally, we need Lemma 3.4.

Lemma 3.4. *For any stochastic process* $\{g(k,\omega)\}_{k\in\mathbb{R}_+}$ *satisfying*

$$\int_1^{+\infty} k^{m-1}\mathbb{E}(|g(k,\cdot)|)\,\mathrm{d}k < +\infty, \tag{3.8}$$

it holds that

$$\lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^m g(k,\omega)\,\mathrm{d}k = 0, \quad a.s.\ \omega\in\Omega. \tag{3.9}$$

Proof. Check [22, Lemma 4.1]. □

3.2. **Key steps in the proof.** Lemma 3.4 turns the justification of the ergodicity into the asymptotic analysis of the expectation of related terms.

With the help of Lemma 3.4, the most difficult part of the work [18–20, 23, 24] boils down to the estimate of the integral

$$\mathbb{I}(x,y,k_1,k_2) := \int e^{ik_1(|x-z_1|+|z_1-y|)-ik_2(|x-z_2|+|z_2-y|)}C(z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \tag{3.10}$$

where $C(z_1,z_2)=\int e^{i(z_1-z_2)\cdot\xi}c(z_1,\xi)\,\mathrm{d}\xi$ and $c\in S^{-m}$. Readers may refer to [19, (30)-(31)], [20, (3.21) and (3.24)], [24, (4.2) and (2.1)] as well as [23, Theorems 3.1 and 3.3] as examples.

One wonders the decaying rate of $\mathbb{I}$ in terms of k_1 and k_2, and after we got the decaying rate, we substitute this estimate into (3.8). If $\mathbb{I}$ decays fast enough in terms of k_1 and/or k_2, the corresponding integral in (3.8) will be finite and we can obtain some asymptotic ergodicity like (3.9). This is the principal idea in [18–20, 23, 24].

Proposition 3.1. *Assume* $\mathbb{I}$ *is defined as in* (3.10) *and* $C(z_1,z_2)=\int e^{i(z_1-z_2)\cdot\xi}$ $c(z_1,\xi)\,\mathrm{d}\xi$ *with* $c\in S^{-m}$ *is a symbol. Then for* $\forall N\in\mathbb{N}$ *there exists constants* $C_N>0$ *such that*

$$|\mathbb{I}(x,y,k_1,k_2)| \leq C_N\langle k_1-k_2\rangle^{-N}(k_1+k_2)^{-m},$$

holds uniformly for x, y.

Proof. Denote $\phi(z_1,z_2,x,y,k_1,k_2) := k_1(|x-z_1|+|z_1-y|)-k_2(|x-z_2|+|z_2-y|)$, then $\mathbb{I}=\int e^{i\phi}C\,\mathrm{d}z_1\,\mathrm{d}z_2$ and ϕ is the phase function. We have

$$\begin{aligned}\phi(z_1,z_2,x,y,k_1,k_2) = {} & \frac{k_1+k_2}{2}\big[(|x-z_1|+|z_1-y|)-(|x-z_2|+|z_2-y|)\big]\\ & + \frac{k_1-k_2}{2}\big[(|x-z_1|+|z_1-y|)+(|x-z_2|+|z_2-y|)\big].\end{aligned} \tag{3.11}$$

We note that the xyz part of the second term in (3.11) is always positive and the first term equals to zero when $z_1=z_2$. Also, the function C will be singular when $z_1=z_2$. Therefore, the situation near the hyperplane $S_0:=\{z_1=z_2\}$ is crucial for the behavior of $\mathbb{I}$ regarding the decaying rate in terms of k_1, k_2. Therefore, we are willing to do a change of variables inside the integral (3.10) such that the hyperplane S_0 can be featured by a single variable, i.e. $S_0=\{v=0\}$ for some variable v. To be specific, we choose the change of variables $\tau_1(z_1,z_2)=(v,w)$ where

$$\tau_1:\quad v=z_1-z_2,\quad w=z_1+z_2.$$

The pull-back of C under τ_1^{-1} is

$$C_1(v,w) := (\tau_1^{-1})^*C(v,w) = C(\tau_1^{-1}(v,w)) = \int e^{iv\cdot\xi}c((v+w)/2,\xi)\,\mathrm{d}\xi. \tag{3.12}$$

Second, in order to make the phase function ϕ more easy to handle, we are also willing to do another change of variables such that ϕ can be represented in the form of inner products, i.e. $\phi = s \cdot t$ for some s and t depending on x, y, z_1, z_2, k_1 and k_2. One of the choices is $\tau_2(z_1, z_2) = (s, t)$, $s = (s_1, \cdots, s_n) \in \mathbb{R}^n$ and $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ where

$$\tau_2\colon \quad \begin{cases} s_1 = (|x - z_1| + |z_1 - y|) - (|x - z_2| + |z_2 - y|), \\ t_1 = (|x - z_1| + |z_1 - y|) + (|x - z_2| + |z_2 - y|). \end{cases} \tag{3.13}$$

We comment that under (3.13), the phase function ϕ will only depend on s_1 and t_1, and the choice of s_j and t_j ($j = 2, \cdots, n$) is inessential as long as the change of variables τ_2 is a diffeomorphism. Hence we omit the precise definitions of s_j and t_j ($j > 1$) and readers may refer to [18–20, 23, 24] for more details. Another thing to note is the map $\tau_1 \circ \tau_2^{-1}$ preserves S_0, i.e. $\tau_1 \circ \tau_2^{-1}(0, t) = (0, w)$. By Lemma 3.3, there exists a symbol $c_2 \in S^{-m}$ such that the pull-back of C_1 under $\tau_1 \circ \tau_2^{-1}$ is

$$C_2(s, t) := (\tau_1 \circ \tau_2^{-1})^* C_1(s, t) = \int e^{is\cdot\xi} c_2(t, \xi) \,\mathrm{d}\xi, \tag{3.14}$$

By using Lemma 3.3, we can express c_2 by c, τ_1 and τ_2, which involves some detailed computations. Note that we only need the leading term of c_2 so the computations wouldn't be too complicated.

The relationship (3.14) also gives

$$C_2(s, t) = (\tau_1 \circ \tau_2^{-1})^* (\tau_1^{-1})^* C(s, t) = (\tau_2^{-1})^* C(s, t),$$

and hence we can do the change of variables τ_2 in (3.10) to obtain

$$\begin{aligned} &\mathbb{I}(x, y, k_1, k_2) \\ &= \int e^{ik_1(|x-z_1|+|z_1-y|) - ik_2(|x-z_2|+|z_2-y|)} C(\tau_2^{-1} \circ \tau_2(z_1, z_2)) \,\mathrm{d}(\tau_2^{-1} \circ \tau_2(z_1, z_2)) \\ &= \int e^{i(k_1+k_2)s_1/2 + i(k_1-k_2)t_1/2} C(\tau_2^{-1}(s, t)) |\det \tau_2^{-1}(s, t)| \,\mathrm{d}(s, t) \\ &= \int e^{i(k_1+k_2)s\cdot e_1/2 + i(k_1-k_2)t\cdot e_1/2} C_2(s, t) |\det \tau_2^{-1}(s, t)| \,\mathrm{d}s \,\mathrm{d}t. \end{aligned} \tag{3.15}$$

Here we need the help of Lemma 3.2 to deal with the $|\det \tau_2^{-1}(s, t)|$ term: there exists a symbol $\tilde{c}_2 \in S_{-m}$ such that

$$C_2(s, t) |\det \tau_2^{-1}(s, t)| = \int e^{is\cdot\xi} \tilde{c}_2(t, \xi) \,\mathrm{d}\xi. \tag{3.16}$$

The computation of the leading term of $\tilde{c}_2$ is straight forward,

$$\tilde{c}_2(t, \xi) - c_2(t, \xi) |\det \tau_2^{-1}(0, t)| \in S^{-m-1}.$$

Combining (3.15) and (3.16), we arrive at

$$\begin{aligned} \mathbb{I}(x, y, k_1, k_2) &= \int e^{i(k_1+k_2)s\cdot e_1/2 + i(k_1-k_2)t\cdot e_1/2} \int e^{is\cdot\xi} \tilde{c}_2(t, \xi) \,\mathrm{d}\xi \,\mathrm{d}s \,\mathrm{d}t \\ &\simeq \int e^{i(k_1-k_2)t\cdot e_1/2} \tilde{c}_2(t, -(k_1 + k_2)e_1/2) \,\mathrm{d}t. \end{aligned}$$

Now we can see $\mathbb{I}$ is decaying at the rate of $\langle k_1 - k_2 \rangle^{-N} (k_1 + k_2)^{-m}$ for arbitrary $N \in \mathbb{N}$. $\square$

We would like to comment that the estimation of $\mathbb{I}$ is difficult due to the presence of the norm inside the phase function ϕ. However, the designs of τ_1 and τ_2 in the arguments above are so peculiar that the estimate of $\mathbb{I}$ is possible.

4. Recovery by far-field data.

In this section we consider the key steps in the works [22,25]. In [22,25], the authors use far-field data to achieve the recovery, and this makes the derivations different from what has been discussed in Section 3. A different methodology is required to obtain accurate estimate of the decaying rate. Lemmas 4.1–4.3 plays key roles in the derivation. Before stepping into the key steps in the derivation, we shall first investigate some useful lemmas.

4.1. Useful lemmas.

First, let us recall the notion of the fractional Laplacian [30] of order $s \in (0,1)$ in $\mathbb{R}^n$ $(n \geq 3)$,

$$(-\Delta)^{s/2}\varphi(x) := (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi}|\xi|^s \varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{4.1}$$

where the integration is defined as an oscillatory integral. When $\varphi \in \mathscr{S}(\mathbb{R}^n)$, (4.1) can be understood as a usual Lebesgue integral if one integrates w.r.t. y first and then integrates w.r.t. ξ. By duality arguments, the fractional Laplacian can be generalized to act on wider range of functions and distributions (cf. [32]). It can be verified that the fractional Laplacian is self-adjoint.

In the following two lemmas, we present the results in a more general form where the space dimension n can be arbitrary but greater than 2, though only the case $n = 3$ shall be used subsequently.

Lemma 4.1. *For any $s \in (0,1)$, we have*

$$(-\Delta_\xi)^{s/2}(e^{ix\cdot\xi}) = |x|^s e^{ix\cdot\xi}$$

in the distributional sense.

Proof. Check [22, Lemma 3.1]. □

Lemma 4.2. *For any $m < 0$ and $s \in (0,1)$, we have*

$$\big((-\Delta_\xi)^{s/2}c\big)(x,\xi) \in S^{m-s} \quad \textit{for any} \quad c(x,\xi) \in S^m.$$

Proof. Check [22, Corollary 3.1]. □

In the sequel, we denote $\mathrm{diam}(\Omega) := \sup\limits_{x,x'\in\Omega} \{|x - x'|\}$.

Lemma 4.3. *Assume Ω is a bounded domain in $\mathbb{R}^n$. For $\forall\alpha,\beta \in \mathbb{R}$ such that $\alpha < n$ and $\beta < n$, and for $\forall p \in \mathbb{R}^n\backslash\{0\}$, there exists a constant $C_{\alpha,\beta}$ independent of p and Ω such that*

$$\int_\Omega |t|^{-\alpha}|t-p|^{-\beta}\,\mathrm{d}t \leq C_{\alpha,\beta} \times \begin{cases} |p|^{n-\alpha-\beta} + (\mathrm{diam}(\Omega))^{n-\alpha-\beta}, & \alpha+\beta \neq n, \\ \ln\frac{1}{|p|} + \ln(\mathrm{diam}(\Omega)) + C_{\alpha,\beta}, & \alpha+\beta = n. \end{cases}$$

Proof. Check [25, Lemma 3.5]. □

4.2. **Key steps in [22].** In this subsection we restrict ourselves to $\mathbb{R}^3$. One of the key difficulty in [22] is to obtain an asymptotics about a integral

$$\mathbb{J} := \int e^{ik\varphi(y,s,z,t)} \big(\int e^{i(z-y)\cdot\xi} c_q(z,\xi) \, \mathrm{d}\xi \big) \big(\int e^{i(t-s)\cdot\eta} c_f(t,\xi) \, \mathrm{d}\eta \big) \, \mathrm{d}(s,y,t,z), \quad (4.2)$$

in terms of k, where $\varphi(y,s,z,t) := -\hat{x}\cdot(y-z) - |y-s| + |z-t|$, $c_q \in S^{-m_q}$ and $c_f \in S^{-m_f}$ with m_q, m_f satisfying the requirement in Theorem 1.6, $\mathrm{d}(s,y,t,z)$ is a short notation for $\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}z$, and $y,z \in D_q$ and $s,t \in D_f$ two convex domains D_q and D_f satisfying (1.5). Recall the definition of the unit normal vector $\boldsymbol{n}$ after (1.5). We introduce two differential operators with C^∞-smooth coefficients as follows,

$$L_1 := \frac{(y-s)\cdot\nabla_s}{ik|y-s|}, \quad L_2 = L_{2,\hat{x}} := \frac{\nabla_y\varphi\cdot\nabla_y}{ik|\nabla_y\varphi|},$$

where $\nabla_y\varphi = \frac{s-y}{|s-y|} - \hat{x}$. The operator $L_{2,\hat{x}}$ depends on $\hat{x}$ because $\nabla_y\varphi$ does. Due to the fact that $y \in D_q$ while $s \in D_f$, the operator L_1 is well-defined. It can be verified there is a positive lower bound of $|\nabla_y\varphi|$ for all $\hat{x} \in \{\hat{x} \in \mathbb{S}^2 \colon \hat{x}\cdot\boldsymbol{n} \geq 0\}$. It can also be verified that

$$L_1(e^{ik\varphi(y,s,z,t)}) = L_2(e^{ik\varphi(y,s,z,t)}) = e^{ik\varphi(y,s,z,t)}.$$

In what follows, we shall use $\mathcal{C}(\cdot)$ and its variants, such as $\vec{\mathcal{C}}(\cdot)$, $\mathcal{C}_{a,b}(\cdot)$ etc., to represent some generic smooth scalar/vector functions, within $C_c^\infty(\mathbb{R}^3)$ or $C_c^\infty(\mathbb{R}^{3\times 4})$, whose particular definition may change line by line. By using integration by parts, one can compute

$$\begin{aligned}
\mathbb{J} = & \int (L_1^2 L_2^2)(e^{ik\varphi(y,s,z,t)}) \cdot \big(\int e^{i(z-y)\cdot\xi} c_q(z,\xi) \, \mathrm{d}\xi \big) \cdot \big(\int e^{i(t-s)\cdot\eta} c_f(t,\eta) \, \mathrm{d}\eta \big) \, \mathrm{d}(s,y,t,z) \\
\simeq & \; k^{-4} \int_{\mathcal{D}} e^{ik\varphi(y,s,z,t)} \big[\mathcal{J}_1 \, (\mathcal{K}_1 \, \mathcal{C} + \vec{\mathcal{K}}_2 \cdot \vec{\mathcal{C}} + \sum_{a,b=1,2,3} \mathcal{K}_{3;a,b} \, \mathcal{C}_{a,b}) \\
& + \sum_{c=1,2,3} \mathcal{J}_{2;c} \, (\mathcal{K}_1 \, \mathcal{C}_c + \vec{\mathcal{K}}_2 \cdot \vec{\mathcal{C}}_c + \sum_{a,b=1,2,3} \mathcal{K}_{3;a,b} \, \mathcal{C}_{a,b,c}) \\
& + \sum_{a',b'=1,2,3} \mathcal{J}_{3;a',b'} (\mathcal{K}_1 \, \mathcal{C}_{a',b'} + \vec{\mathcal{K}}_2 \cdot \vec{\mathcal{C}}_{a',b'} + \sum_{a,b=1,2,3} \mathcal{K}_{3;a,b} \, \mathcal{C}_{a,b,a',b'}) \big] \, \mathrm{d}(s,y,t,z),
\end{aligned} \quad (4.3)$$

where the integral domain $\mathcal{D} \subset \mathbb{R}^{3\times 4}$ is bounded and

$$\begin{aligned}
\mathcal{J}_1 &:= \int e^{i(t-s)\cdot\eta} \, c_f(t,\eta) \, \mathrm{d}\eta, & \mathcal{K}_1 &:= \int e^{i(z-y)\cdot\xi} \, c_q(z,\xi) \, \mathrm{d}\xi, \\
\vec{\mathcal{J}}_2 &:= \nabla_s \int e^{i(t-s)\cdot\eta} \, c_f(t,\eta) \, \mathrm{d}\eta, & \vec{\mathcal{K}}_2 &:= \nabla_y \int e^{i(z-y)\cdot\xi} \, c_q(z,\xi) \, \mathrm{d}\xi, \\
\mathcal{J}_{3;a,b} &:= \partial^2_{s_a,s_b} \int e^{i(t-s)\cdot\eta} \, c_f(t,\eta) \, \mathrm{d}\eta, & \mathcal{K}_{3;a,b} &:= \partial^2_{y_a,y_b} \int e^{i(z-y)\cdot\xi} \, c_q(z,\xi) \, \mathrm{d}\xi,
\end{aligned}$$

and $\mathcal{J}_{2;c}$ (resp. $\mathcal{K}_{2;c}$) is the c-th component of the vector $\vec{\mathcal{J}}_2$ (resp. $\vec{\mathcal{K}}_2$).

Here we only show how to estimate $\mathcal{J}_1$ and skip the details regarding $\vec{\mathcal{J}}_2$, $\mathcal{K}_1$, and $\vec{\mathcal{K}}_2$; readers may refer to the proof of [22, Lemma 3.3] for details. For the case where $s \neq t$, we have

$$\begin{aligned}
|\mathcal{J}_1| = & \, |\int e^{i(t-s)\cdot\eta} \, c_f(t,\eta) \, \mathrm{d}\eta| = |s-t|^{-2} \cdot |\int \Delta_\eta (e^{i(s-t)\cdot\eta}) \, c_f(t,\eta) \, \mathrm{d}\eta| \\
= & \, |s-t|^{-2} \cdot |\int e^{i(t-s)\cdot\eta} (\Delta_\eta c_f)(t,\eta) \, \mathrm{d}\eta| \leq |s-t|^{-2} \int |(\Delta_\eta c_f)(t,\eta)| \, \mathrm{d}\eta
\end{aligned}$$

$$\lesssim |s-t|^{-2} \int \langle \eta \rangle^{-m_f-2} \,\mathrm{d}\eta \lesssim |s-t|^{-2}. \tag{4.4}$$

Similarly, we can have

$$|\mathcal{J}_1|,\ |\vec{\mathcal{J}}_2|,\ |\mathcal{K}_1|,\ |\vec{\mathcal{K}}_2| \lesssim |y-z|^{-2}. \tag{4.5}$$

But for $J_{3;a,b}$, if we mimic the derivation (4.4), then

$$\begin{aligned}\mathcal{J}_{3;a,b} &\simeq \int e^{i(t-s)\cdot\eta} \cdot c_f(t,\eta)\eta_a\eta_b \,\mathrm{d}\eta \simeq |s-t|^{-2} \int \Delta_\eta(e^{i(t-s)\cdot\eta}) \cdot c_f(t,\eta)\eta_a\eta_b \,\mathrm{d}\eta \\ &= |s-t|^{-2} \int e^{i(t-s)\cdot\eta} \cdot \Delta_\eta(c_f(t,\eta)\eta_a\eta_b) \,\mathrm{d}\eta. \end{aligned} \tag{4.6}$$

Note that $\Delta_\eta(c_f(t,\eta)\eta_a\eta_b) \in S^{-m_f}$ and thus is not absolutely integrable in $\mathbb{R}^3$. If we further differentiate the term $e^{i(t-s)\cdot\eta}$ in (4.6) by $\frac{i(s-t)\cdot}{|s-t|^2}\nabla_\eta$ and then transfer the operator ∇_η onto $\Delta_\eta(c_f(t,\eta)\eta_a\eta_b)$ by using integration by parts, we would arrive at

$$|\mathcal{J}_{3;a,b}| \lesssim |s-t|^{-3} \int |\nabla_\eta \Delta_\eta(c_f(t,\eta)\eta_a\eta_b)| \,\mathrm{d}\eta \le |s-t|^{-3} \int \langle \eta \rangle^{-m_f-1} \,\mathrm{d}\eta.$$

The term $\int \langle \eta \rangle^{-m_f-1} \,\mathrm{d}\eta$ is absolutely integrable now, but the term $|s-t|^{-3}$ is not integrable at the hyperplane $s = t$ in $\mathbb{R}^3$. To circumvent this dilemma, the fractional Laplacian can be applied as follows. By using Lemma 4.1 and 4.2, we can continue (4.6) as

$$\begin{aligned}|\mathcal{J}_{3;a,b}| &\simeq |s-t|^{-2} \cdot \big||s-t|^{-s} \int (-\Delta_\eta)^{s/2}(e^{i(t-s)\cdot\eta}) \cdot \Delta_\eta(c_f(t,\eta)\eta_j\eta_\ell) \,\mathrm{d}\eta\big| \\ &= |s-t|^{-2-s} \cdot \big| \int e^{i(t-s)\cdot\eta} \cdot (-\Delta_\eta)^{s/2}\big(\Delta_\eta(c_f(t,\eta)\eta_j\eta_\ell)\big) \,\mathrm{d}\eta\big| \\ &\lesssim |s-t|^{-2-s} \int \langle \eta \rangle^{-m_f+2-2-s} \,\mathrm{d}\eta = |s-t|^{-2-s} \int \langle \eta \rangle^{-m_f-s} \,\mathrm{d}\eta, \end{aligned} \tag{4.7}$$

where the number s is chosen to satisfy $\max\{0, 3-m_f\} < s < 1$, and the existence of such a number s is guaranteed by noting that $m_f > 2$. Therefore, we have

$$\begin{cases} -m_f - s < -3, & (4.8a) \\ -2 - s > -3. & (4.8b) \end{cases}$$

Thanks to the condition (4.8a), we can continue (4.7) as

$$|\mathcal{J}_{3;a,b}| \lesssim |s-t|^{-2-s} \int \langle \eta \rangle^{-m_f-s} \,\mathrm{d}\eta \lesssim |s-t|^{-2-s}. \tag{4.9}$$

Using similar arguments, we can also conclude that $K_{3;a,b}| \lesssim |y-z|^{-2-s}$.

Combining (4.3), (4.5) and (4.9), we arrive at

$$\begin{aligned}|\mathbb{J}| &\lesssim k^{-4} \int_{\mathcal{D}} (|\mathcal{J}_1| + |\vec{\mathcal{J}}_2| + \sum_{a',b'=1,2,3} |\mathcal{J}_{3;a',b'}|) \cdot (|\mathcal{K}_1| + |\vec{\mathcal{K}}_2| + \sum_{a,b=1,2,3} |\mathcal{K}_{3;a,b}|) \,\mathrm{d}(s,y,t,z) \\ &\lesssim k^{-4} \int_{\widetilde{\mathcal{D}}} |s-t|^{-2-s} \,\mathrm{d}s \,\mathrm{d}t \cdot \int_{\widetilde{\mathcal{D}}} |y-z|^{-2-s} \,\mathrm{d}y \,\mathrm{d}z \end{aligned} \tag{4.10}$$

for some sufficiently large but bounded domain $\widetilde{\mathcal{D}} \subset \mathbb{R}^{3\times 2}$ satisfying $\mathcal{D} \subset \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$. Note that the integral (4.10) should be understood as a singular integral because of the presence of the singularities occurring when $s = t$ and $y = z$. By (4.10) and (4.8b), we can finally conclude $|\mathbb{J}| \lesssim k^{-4}$, as k be large enough.

4.3. **Key steps in [25].** In this subsection we restrict ourselves to $\mathbb{R}^3$. We note that in (4.2), the domains D_q and D_f are assumed to be separated by two convex hulls. This condition is relaxed in [25] and the corresponding details in the proof is also modified. One of the key difficulty in [25] is to obtain an asymptotics about a integral

$$\mathbb{K}(x,y) := \iint_{D_f \times D_f} K_f(s,t)\Phi(s-y;k_1)\overline{\Phi}(t-x;k_2)\,\mathrm{d}s\,\mathrm{d}t, \tag{4.11}$$

where K_f is the kernel of the covariance operator of the migr field f (cf. (2.4)), and Φ is defined in the beginning of Section 2.1. From (4.11) we have

$$\mathbb{K}(z,y) \simeq \iint_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} e^{ik_1|s-y|-ik_2|t-z|}\big(|s-y|^{-1}|t-z|^{-1}\int e^{i(s-t)\cdot\xi} c(s,\xi)\,\mathrm{d}\xi\big)\,\mathrm{d}s\,\mathrm{d}t. \tag{4.12}$$

Define two differential operators

$$L_1 := \frac{(s-y)\cdot\nabla_s}{ik_1|s-y|} \quad \text{and} \quad L_2 := \frac{(t-z)\cdot\nabla_t}{-ik_2|t-z|}.$$

It can be verified that

$$L_1 L_2(e^{ik_1|s-y|-ik_2|t-z|}) = e^{ik_1|s-y|-ik_2|t-z|}.$$

Hence, noting that the integrand is compactly supported in $\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$ and by using integration by part, we can continue (4.12) as

$$\begin{aligned}
&|\mathbb{K}(z,y)| \\
&\simeq |\iint_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} L_1L_2(e^{ik_1|s-y|-ik_2|t-z|})\big(|s-y|^{-1}|t-z|^{-1}\int e^{i(s-t)\cdot\xi} c_1(s,t,z,y,\xi)\,\mathrm{d}\xi\big)\,\mathrm{d}s\,\mathrm{d}t| \\
&\lesssim k_1^{-1}k_2^{-1}\iint_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} \big[|s-y|^{-2}|t-z|^{-2}\mathcal{J}_0 + |s-y|^{-2}|t-z|^{-1}(\max_a \mathcal{J}_{1;a}) \\
&\qquad + |s-y|^{-1}|t-z|^{-2}(\max_a \mathcal{J}_{1;a}) + |s-y|^{-1}|t-z|^{-1}(\max_{a,b} \mathcal{J}_{2;a,b})\big]\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned} \tag{4.13}$$

where a, b are indices running from 1 to 3, and

$$\begin{aligned}
\mathcal{J}_0 &:= |\int e^{i(s-t)\cdot\xi}\, c_1(s,t,z,y,\xi)\,\mathrm{d}\xi|, \\
\mathcal{J}_{1;a} &:= |\int e^{i(s-t)\cdot\xi}\, \xi_a c_1(s,t,z,y,\xi)\,\mathrm{d}\xi|, \\
\mathcal{J}_{2;a,b} &:= |\int e^{i(s-t)\cdot\xi}\, \xi_a\xi_b c_1(s,t,z,y,\xi)\,\mathrm{d}\xi|.
\end{aligned}$$

Because of the condition $m > 2$ (cf. Theorem 1.5), we can find a number $\tau \in (0,1)$ satisfying the inequalities $3 - m < \tau < 1$. Therefore, we have

$$\begin{cases} -m-\tau < -3, & (4.14\text{a}) \\ -2-\tau > -3. & (4.14\text{b}) \end{cases}$$

By using Lemmas 4.1 and 4.2, these quantities $\mathcal{J}_0$, $\mathcal{J}_{1;a}$ and $\mathcal{J}_{2;a,b}$ can be estimated as follows:

$$\begin{aligned}
\mathcal{J}_0 &= |s-t|^{-\tau}\cdot|\int(-\Delta_\xi)^{\tau/2}(e^{i(s-t)\cdot\xi})c_1(s,t,z,y,\xi)\,\mathrm{d}\xi| \\
&= |s-t|^{-\tau}\cdot|\int e^{i(s-t)\cdot\xi}\,(-\Delta_\xi)^{\tau/2}(c_1(s,t,z,y,\xi))\,\mathrm{d}\xi|
\end{aligned}$$

$$\lesssim |s-t|^{-\tau} \cdot \int \langle \xi \rangle^{-m-\tau} \, \mathrm{d}\xi \lesssim |s-t|^{-\tau}. \tag{4.15}$$

The last inequality in (4.15) makes use of the fact (4.14a). Similarly, by first using fractional Laplacian and then using first-order differential operator on $e^{i(s-t)\cdot\xi}$, we can have

$$\mathcal{J}_{1;a} \leq C|s-t|^{-1-\tau} \int \langle \xi \rangle^{-m+1-1-\tau} \, \mathrm{d}\xi \leq C|s-t|^{-1-\tau}, \tag{4.16}$$

$$\mathcal{J}_{2;a,b} \leq C|s-t|^{-2-\tau} \Big| \int \langle \xi \rangle^{-m+2-2-\tau} \, \mathrm{d}\xi \Big| \leq C|s-t|^{-2-\tau}, \tag{4.17}$$

where the constant C is independent of the indices a, b. Combining (4.13), (4.15), (4.16) and (4.17), we can rewrite (4.13) as

$$\begin{aligned} k_1 k_2 |\mathbb{K}(z,y)| \lesssim & \iint_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} \big[|s-y|^{-2}|t-z|^{-2}|s-t|^{-\tau} + |s-y|^{-2}|t-z|^{-1}|s-t|^{-1-\tau} \\ & + |s-y|^{-1}|t-z|^{-2}|s-t|^{-1-\tau} + |s-y|^{-1}|t-z|^{-1}|s-t|^{-2-\tau} \big] \, \mathrm{d}s \, \mathrm{d}t \\ & =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4. \end{aligned} \tag{4.18}$$

Denote $\mathbf{D} := \{x + x', x - x' \,;\, x, x' \in \widetilde{\mathcal{D}}\}$. Then we apply Lemma 4.3 to estimate $\mathbb{I}_1$ as follows,

$$\begin{aligned} \mathbb{I}_1 &= \iint_{\widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}} |s-y|^{-2}|t-z|^{-2}|s-t|^{-\tau} \, \mathrm{d}s \, \mathrm{d}t \\ &\leq \int_{\mathbf{D}} |s|^{-2} \Big(\int_{\mathbf{D}} |t|^{-2}|t-(s+y-z)|^{-\tau} \, \mathrm{d}t \Big) \, \mathrm{d}s \\ &\lesssim C_{\widetilde{\mathcal{D}}} + \int_{\mathbf{D}} |s|^{-2}|s-(z-y)|^{-(\tau-1)} \, \mathrm{d}s \\ &\simeq |z-y|^{2-\tau} + C_{\widetilde{\mathcal{D}}}. \end{aligned} \tag{4.19}$$

Note that in (4.19) we used Lemma 4.3 twice. Similarly,

$$\mathbb{I}_2,\, \mathbb{I}_3,\, \mathbb{I}_4 \lesssim |z-y|^{2-\tau} + C_{\widetilde{\mathcal{D}}}. \tag{4.20}$$

Recall that $\tau \in (0,1)$. By (4.18), (4.19) and (4.20) we arrive at

$$|\mathbb{K}(z,y)| \leq C k_1^{-1} k_2^{-1} (|z-y|^{2-\tau} + C_{\widetilde{\mathcal{D}}}) \leq C k^{-2} ((\operatorname{diam} D_V)^{2-\tau} + C_{\widetilde{\mathcal{D}}}) \lesssim k^{-2}.$$

5. Conclusions.

We have reviewed the recoveries of some statistics by using the near-field data as well as far-field data generated under a single realization of the randomness. In this paper we mainly focus on time-harmonic Schrödinger systems. One of the possible ways to extend the current works is to study the Helmholtz systems. It would be also interesting to conduct the work in the time domain. Moreover, the stability of the recovering procedure is also worth of investigation.

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