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REFINED ASYMPTOTIC BEHAVIOR AND UNIQUENESS OF LARGE SOLUTIONS TO A QUASILINEAR ELLIPTIC EQUATION IN A BORDERLINE CASE

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Abstract. This paper is considered with the quasilinear elliptic equation $\Delta_p u = b(x)f(u)$, $u(x) > 0$, $x \in \Omega$, where Ω is an exterior domain with compact smooth boundary, $b \in \mathrm{C}(\Omega)$ is non-negative in Ω and may be singular or vanish on $\partial\Omega$, $f \in C[0,\infty)$ is positive and increasing on $(0,\infty)$ which satisfies a generalized Keller-Osserman condition and is regularly varying at infinity with critical index $p-1$. By structuring a new comparison function, we establish the new asymptotic behavior of large solutions to the above equation in the exterior domain. We find that the lower term of f has an important influence on the asymptotic behavior of large solutions. And then we further establish the uniqueness of such solutions.

1. **Introduction and main results.** This presentation will investigate the influence of the lower term of nonlinearity f on the asymptotic behavior and uniqueness of large solutions $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ to the following quasilinear elliptic equation

$$\Delta_p u = b(x)f(u),\ u(x) > 0,\ x \in \Omega, \tag{1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for p-Laplacian operator with $1 < p < N$ and $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ is an exterior domain with compact smooth boundary. We say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a local weak solution to Eq. (1) in Ω means that for every sub-domain $D \Subset \Omega$, it holds

$$\int_D |\nabla u|^{p-2}\nabla u \nabla \varphi dx = -\int_D b(x)f(u(x))\varphi(x)dx,\ \forall\, \varphi \in W^{1,p}_{\text{loc}}(\Omega).$$

A local weak solution u is said to be a large solution if $u \in C(\Omega)$ and satisfies

$$u(x) \to \infty \text{ as } |x| \to \infty \text{ and } u(x) \to \infty \text{ as } d(x) := d(x, \partial\Omega) \to 0. \tag{2}$$

In this paper we assume that the nonlinearity f satisfies the following hypotheses:

($\mathbf{f_1}$): $f \in C[0,\infty)$, $f(0) = 0$, $f(t) > 0$, $t > 0$, and f is increasing on $[0,\infty)$;

($\mathbf{f_2}$): the following generalized Keller-Osserman condition holds,

$$\int_1^\infty (F(s))^{-1/p} ds < +\infty,\ \ F(t) = \int_0^t f(s)ds.$$

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The weight b satisfies the following hypotheses, not necessarily simultaneously:

($\mathbf{b_1}$): $b \in C(\Omega)$ is non-negative in Ω, where Ω is an exterior domain with compact smooth boundary;

($\mathbf{b_2}$): there exist $\theta \in \Lambda_1$ and positive constant b_0 such that

$$\lim_{|x|\to\infty} \frac{b(x)}{\theta^p(|x|)} = b_0,$$

where Λ_1 denotes the set of all positive non-increasing functions $\theta \in C^1[R_0,\infty) \cap L^1[R_0,\infty)$ $(R_0 > 0)$ which satisfy

$$\lim_{t\to\infty} \frac{d}{dt}\left(\frac{\Theta(t)}{\theta(t)}\right) := D_\theta \in (0,\infty),\ \Theta(t) := \int_t^\infty \theta(s)ds,\ t \geq R_0;$$

($\mathbf{b_3}$): there exist $k \in \Lambda_2$ and positive constant b_1 such that

$$\lim_{d(x)\to 0} \frac{b(x)}{k^p(d(x))} = b_1,$$

where Λ_2 denotes the set of all positive monotonic functions $k \in C^1(0,\delta_0) \cap L^1(0,\delta_0)$ $(\delta_0 > 0)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) := D_k \in [0,\infty),\quad K(t) := \int_0^t k(s)ds,$$

Λ_2 was first introduced by Cîrstea and Rădulescu [1] for non-decreasing functions and by Mohammed [11] for non-increasing functions.

When Ω is a bounded domain, $b \equiv 1$ in Ω and $f(u) = u^\gamma$ with $\gamma > p-1$, Diaz and Letelier [3] first studied the existence of large solutions to Eq. (1). Then, when f satisfies ($\mathbf{f_1}$) (or $f \in C(\mathbb{R})$ is positive and increasing on $\mathbb{R}$) and ($\mathbf{f_2}$), the existence and boundary behavior of large solutions to Eq. (1) are further investigated by Matero [9] in a bounded domain $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ with a C^2-boundary. In particular, the author obtained the following boundary behavior:

$$\lim_{d(x)\to 0} \frac{\Phi(u(x))}{d(x)} = 1,$$

where Φ is given by

$$\Phi(t) = \int_t^\infty ((p/(p-1))F(s))^{-1/p}ds.$$

If f further satisfies

$$\liminf_{t\to\infty} \frac{\Phi(\lambda t)}{\Phi(t)} > 1,\ \forall\, \lambda \in (0,1),$$

then u satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\Psi(d(x))} = 1,$$

where Ψ is the inverse of Φ, i.e., Ψ is given by

$$\int_{\Psi(t)}^\infty ((p/(p-1))F(s))^{-1/p}ds = t,\ t > 0. \tag{3}$$

Gladiali and Porru [6] showed that if $b \equiv 1$ in bounded domain $\Omega \subseteq \mathbb{R}^N$, f satisfies ($\mathbf{f_1}$) and $t \mapsto F(t)t^{-p}$ is increasing for large t, then any large solution u to Eq. (1) satisfies

$$|u(x) - \Psi(d(x))| < cd(x)\Psi(d(x)) \text{ near } \partial\Omega.$$

Furthermore, under the additional assumption $F(t)t^{-2p} \to \infty$ as $t \to \infty$, they obtained

$$u(x) - \Psi(d(x)) \to 0 \text{ as } d(x) \to 0.$$

Based on a comparison principle, Du and Guo [4] discussed the existence, uniqueness and asymptotic behavior of various boundary blow-up solutions for a class of quasilinear elliptic equations. Let Ω be a bounded domain, b satisfy $(\mathbf{b_1})$ and

(C): If there exists $x_0 \in \Omega$ such that $b(x_0) = 0$, then there exists a bounded domain Ω_0 ($\Omega_0 \Subset \Omega$) containing x_0 such that $b(x) > 0$ for all $x \in \partial\Omega_0$,

f satisfy $(\mathbf{f_1})$ and $(\mathbf{f_2})$. Mohammed [10] showed that if the Poisson problem

$$-\Delta_p v(x) = b(x),\ x \in \Omega,\ v|_{\partial\Omega} = 0$$

has a weak solution, then Eq. (1) admits a non-negative boundary blow-up solution. Moreover, the author further established the asymptotic boundary estimates of such blow-up solutions. Later, Mohammed [11] showed that if $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with C^2-boundary, b satisfies $(\mathbf{b_1})$ and $(\mathbf{b_3})$ (k is non-increasing on $(0, \delta_0)$), f satisfies $(\mathbf{f_1})$ and $f \in RV_{\rho+1}$ (please refer to Definition 2.1) with $\rho > p-2$, then any large solution $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ to Eq. (1) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\Psi(K(d(x)))} = \left(\frac{p + D_k(2+\rho-p)}{b_1(2+\rho)}\right)^{1/(2+\rho-p)},$$

where Ψ is given by (3). By introducing some structure condition, the result of Mohammed in [11] was extended by Zhang [18] from $f \in RV_{\rho+1}$ with $\rho > p-2$ to the case that $f \in RV_{\rho+1}$ with $\rho \geq p-2$ or f is rapidly varying at infinity. Moreover, the author also studied the boundary behavior of large solutions to Eq. (1) when b is critical singular on the boundary. Inspired by the above results, by using Karamata regular varying theory, Wan [14] investigated the asymptotic behavior and uniqueness of entire large solutions to Eq. (1) in $\mathbb{R}^N$. For other related insight on Eq. (1.1), please refer to [2], [8]-[7], [15]-[16].

In this paper, by structuring a comparison function, we establish the new asymptotic behavior of large solutions to problem (1)-(2) (including the case of $p = 2$) when $f \in NRV_{p-1}$. Our results imply that the lower term of f has an important influence on the asymptotic behavior of large solutions to the above exterior domain problem. Then, we further establish the uniqueness of the solutions to problem (1)-(2).

To obtain our results, we further assume that f satisfies

$(\mathbf{f_3})$**:** there exist some constant $t_0 > 0$ and two functions f_1 and f_2 such that

$$f(t) := f_1(t) + f_2(t),\ t \geq t_0,$$

where $f_1 \in C^2[t_0, \infty)$. If we denote $g(t) := \frac{tf_1'(t)}{f_1(t)} - (p-1)$, $t \geq t_0$ and g and f_2 satisfy the following conditions:

$(\mathbf{f_4})$**:**

$$g(t) > 0,\ t \geq t_0,\ \lim_{t\to\infty} g(t) = 0,\ \lim_{t\to\infty} \frac{tg'(t)}{g(t)} = 0,$$

$$\lim_{t\to\infty} \frac{tg'(t)}{g^2(t)} = \kappa_g \in \mathbb{R},\ \lim_{t\to\infty} \frac{t^{p-1}}{f_1(t)g^p(t)} = 0;$$

$(\mathbf{f_5})$**:** for any $\xi > 0$

$$\lim_{t\to\infty} \frac{f_2(\xi t)}{f_2(t)} = \xi^{p-1},$$

and there exists $E_1 \neq 0$ such that

$$\lim_{t\to\infty} \frac{f_2(t)}{g(t)f_1(t)} = E_1,$$

or

$(\mathbf{f_6})$:

$$\lim_{t\to\infty} \frac{f_2(t)}{g(t)f_1(t)} = 0$$

and there exists $\mu \leq p-1$ such that for any $\xi > 0$

$$\lim_{t\to\infty} \frac{f_2(\xi t)}{f_2(t)} = \xi^{\mu}.$$

Our results are summarized as follows.

Theorem 1.1. *Let f satisfy $(\mathbf{f_1})$-$(\mathbf{f_6})$, b satisfy $(\mathbf{b_1})$-$(\mathbf{b_2})$. If*

$$(N-p)D_\theta - p + 1 > 0 \text{ and } \frac{1}{p} + \kappa_g > 0,$$

then any solution u to problem (1)-(2) satisfies

$$\lim_{|x|\to\infty} \frac{u(x)}{\psi(b_0^{1/p}\Theta(|x|))} = \exp(\xi_0), \tag{4}$$

where ψ is uniquely determined by

$$\int_{\psi(t)}^{\infty} (sf_1(s))^{-1/p} ds = \frac{t}{(p-1)^{1/p}} \tag{5}$$

and

$$\xi_0 = \frac{1}{p} - E_2 + \frac{1}{p-1}\left(\frac{1}{p} + \kappa_g\right)((N-p)D_\theta - p + 1)$$

with

$$E_2 := \begin{cases} E_1, & \text{if } (\mathbf{f_5}) \text{ holds;} \\ 0, & \text{if } (\mathbf{f_6}) \text{ holds.} \end{cases} \tag{6}$$

Theorem 1.2. *Let f satisfy $(\mathbf{f_1})$-$(\mathbf{f_6})$, b satisfy $(\mathbf{b_1})$ and $(\mathbf{b_3})$ with one of the following conditions:*

(I): *k is non-decreasing on $(0, \delta_0)$;*

(II): *k is non-increasing on $(0, \delta_0)$ with $D_k > 1$.*

When (**II**) *holds, we further assume $\frac{1}{p} + \kappa_g > 0$. Then any solution u to problem (1)-(2) satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(b_1^{1/p}K(d(x)))} = \exp(\xi_1), \tag{7}$$

where ψ is uniquely determined by (5) and

$$\xi_1 = \frac{1}{p} - E_2 - \left(\frac{1}{p} + \kappa_g\right)(1 - D_k),$$

where E_2 is defined by (6).

Remark 1.3. If we replace $D_k > 1$ by $D_k = 1$ in (**II**) of Theorem 1.2 and further assume that

$$\limsup_{t\to 0^+} k'(t)/k(t) \le -\frac{(N-1)\tilde{C}}{(p-1)^{1/p}} \text{ and } \inf_{\bar{x}\in\partial\Omega} H(\bar{x}) > -\tilde{C},$$

where $\tilde{C}$ is a positive constant and H is the mean curvature of $\partial\Omega$, then Theorem 1.2 still holds.

Remark 1.4. In Theorems 1.1 and 1.2, the comparison function ψ given by (5) can not be replaced by Ψ given by the following integral equation

$$\int_{\Psi(t)}^{\infty} (sf_1(s))^{-1/p} ds = t,\ t > 0.$$

Remark 1.5. Some basic examples which satisfy all of our requirements in Theorems 1.1-1.2 are the following:

(**i**): $f(t) = c_1 t^{p-1}(\ln t)^{p\alpha} + c_2 t^{\alpha_1}(\ln t)^{\alpha_2}$, $t \ge t_0$, where $\alpha > 1$, $c_1 > 0$, $\alpha_1 \le p-1$ and c_2, $\alpha_2 \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$g(t) = p\alpha(\ln t)^{-1},\ t \ge t_0,\ \kappa_g = -\frac{1}{p\alpha}$$

and

$$E_2 = \begin{cases} \frac{c_2}{c_1 p\alpha}, & \alpha_1 = p-1,\ \alpha_2 = p\alpha - 1; \\ 0, & \alpha_1 < p-1 \text{ or } \alpha_1 = p-1,\ \alpha_2 < p\alpha - 1; \end{cases}$$

$$\psi(t) = \exp\left(\frac{(\alpha-1)c_1^{1/p}}{(p-1)^{1/p}}\right)^{1/(1-\alpha)} t^{1/(1-\alpha)},\ t > 0.$$

(**ii**): $f(t) = c_1 t^{p-1}\exp((\ln t)^q) + c_2 t^{\alpha_1}(\ln t)^{\alpha_2}\exp((\ln t)^{\alpha_3})$, $t \ge t_0$, where $c_1 > 0$, $\alpha_1 \le p-1$, q, $\alpha_3 \in (0,1)$, c_2, $\alpha_2 \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$g(t) = q(\ln t)^{q-1},\ t \ge t_0,\ \kappa_g = 0$$

and

$$E_2 = \begin{cases} \frac{c_2}{c_1 q}, & \alpha_1 = p-1,\ \alpha_2 = q-1 \text{ and } \alpha_3 = q; \\ 0, & \alpha_1 < p-1 \text{ or } \alpha_1 = p-1,\ \alpha_2 < q-1 \text{ and } \alpha_3 = q; \\ 0, & \alpha_1 = p-1 \text{ and } \alpha_3 < q. \end{cases}$$

(**iii**): $f(t) = c_1 t^{p-1}(\ln t)^p(\ln(\ln t))^{p\alpha} + c_2 t^{\alpha_1}(\ln t)^{\alpha_2}(\ln(\ln t))^{\alpha_3}$, $t \ge t_0$, where $c_1 > 0$, $\alpha > 1$ and $\alpha_1 \le p-1$, c_2, α_2, $\alpha_3 \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$g(t) = p(\ln t)^{-1}(1 + \alpha(\ln(\ln t))^{-1}),\ t \ge t_0 \text{ and } \kappa_g = -\frac{1}{p}$$

and

$$E_2 = \begin{cases} \frac{c_2}{c_1 p}, & \alpha_1 = \alpha_2 = p-1 \text{ and } \alpha_3 = p\alpha; \\ 0, & \alpha_1 < p-1 \text{ or } \alpha_1 = p-1,\ \alpha_2 < p-1 \text{ and } \alpha_3 = q; \\ 0, & \alpha_1 = \alpha_2 = p-1 \text{ and } \alpha_3 < p\alpha; \end{cases}$$

$$\psi(t) = \exp\left(\exp\left(\left(\frac{(\alpha-1)c_1^{1/p}}{(p-1)^{1/p}}\right)^{1/(1-\alpha)} t^{1/(1-\alpha)}\right)\right),\ t > 0.$$

Theorem 1.6. *Let f satisfy* $(\mathbf{f_1})$ *and*

$(\mathbf{f_8})$**:** *$t \mapsto f(t)t^{1-p}$ is non-decreasing on $(0, \infty)$,*

b satisfy $(\mathbf{b_1})$, and u_1, u_2 be arbitrary positive solutions to problem (1)-(2) and satisfy

$$\lim_{|x|\to\infty} \frac{u_1(x)}{u_2(x)} = 1, \quad \lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1,$$

then $u_1 = u_2$ in Ω.

Corollary 1.7. *If b, f satisfy the hypotheses in Theorems 1.1-1.2 and $(\mathbf{f_8})$ holds, then the solution to problem (1)-(2) is unique.*

The paper is organized as follows. In Section 2, we give some bases of Karamata regular variation theory. In Section 3, we collect some preliminary considerations. The proofs of our Theorems are given in Sections 4-6, respectively.

2. **Some basic facts from Karamata regular variation theory.** In this section, we introduce some preliminaries of Karamata regular variation theory which come from [12]-[13].

Definition 2.1. A positive continuous function f defined on $[a, \infty)$, for some $a > 0$, is called **regularly varying at infinity** with index μ, denoted by $f \in RV_\mu$, if for each $\xi > 0$ and some $\mu \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^\mu. \tag{8}$$

In particular, when $\mu = 0$, f is called **slowly varying at infinity**.

Clearly, if $f \in RV_\mu$, then $L(t) := f(t)/t^\mu$ is slowly varying at infinity.
We also see that a positive continuous function h defined on $(0, a)$ for some $a > 0$, is **regularly varying at zero** with index μ (written as $h \in RVZ_\mu$) if $t \to h(1/t) \in RV_{-\mu}$.

Proposition 2.1. (*Uniform Convergence Theorem*) *If $f \in RV_\mu$, then (8) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2.2. (*Representation Theorem*) *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(t) = \varphi(t)\exp\left(\int_{a_1}^t \frac{y(s)}{s} ds\right), \; t \geq a_1,$$

for some $a_1 \geq a$, where the functions φ and y are continuous and for $t \to \infty$, $y(t) \to 0$ and $\varphi(t) \to c_0$, with $c_0 > 0$. If $\varphi \equiv c_0$, then L is called **normalized** *slowly varying at infinity and*

$$f(t) = t^\mu \hat{L}(t), \; t \geq a_1,$$

is called **normalized** *regularly varying at infinity with index μ (written as $f \in NRV_\mu$).*

A function $f \in C^1[a_1, \infty)$ for some $a_1 > 0$ belongs to NRV_μ if and only if

$$\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = \mu.$$

Similarly, $h \in C^1(0, a_1]$ for some $a_1 > 0$ belongs to $NRVZ_\mu$ if and only if

$$\lim_{t\to 0^+} \frac{th'(t)}{h(t)} = \mu.$$

3. **Auxiliary results.** In this section, we collect some useful results.

Lemma 3.1. *Let $\theta \in \Lambda_1$, then*

(i): $\Theta'(t) = -\theta(t)$, $t \geq R_0$, $\lim_{t\to\infty} \frac{\Theta(t)}{t\theta(t)} = D_\theta$, *i.e.*, $\Theta \in NRV_{-1/D_\theta}$ *and* $\lim_{t\to\infty} \frac{\Theta(t)}{\theta(t)} = \infty$;

(ii): $\lim_{t\to\infty} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = -1 - D_\theta$ *and* $\theta \in NRV_{-(1+D_\theta)/D_\theta}$.

Proof. (**i**) By the definition of Λ_1 and the l'Hospital's rule, we obtain (**i**) holds.
(**ii**) A straightforward calculation shows that $\lim_{t\to\infty} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = -1 - D_\theta$. This combined with (**i**) implies that $\theta \in NRV_{-(1+D_\theta)/D_\theta}$. □

Lemma 3.2. ([17] , *Lemma 2.1*) *Let $k \in \Lambda_2$, then*

(i): $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$ *and* $\lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - D_k$;

(ii): *when k is non-decreasing, $D_k \in [0,1]$; when k is non-increasing, $D_k \geq 1$;*

(iii): *when $D_k > 0$, $k \in NRVZ_{(1-D_k)/D_k}$;*

(iv): *when $D_k = 0$, then $\lim_{t\to 0^+} t^{-m}K(t) = 0$ for any $m > 0$.*

Lemma 3.3. *Let f satisfy* ($\mathbf{f_1}$)-($\mathbf{f_6}$), *ψ be the unique solution of* (5), *then*

(i): $\psi'(t) = -\frac{1}{(p-1)^{1/p}}(\psi(t)f_1(\psi(t)))^{1/p}$, $t > 0$ *and* $\lim_{t\to 0^+} \psi(t) = \infty$;

(ii): $(-\psi'(t))^{p-2}\psi''(t) = \frac{1}{p(p-1)}\big(f_1(\psi(t)) + \psi(t)f_1'(\psi(t))\big)$, $t > 0$;

(iii): $\lim_{t\to\infty}(g(t))^{-1}\big(\frac{f_1(\xi t)}{\xi^{p-1}f_1(t)} - 1\big) = \ln\xi$, $\xi > 0$;

(iv): $\lim_{t\to\infty} \frac{f_2(\xi t)}{\xi^{p-1}g(t)f_1(t)} = E_2$, $\xi > 0$;

(v): $\lim_{t\to\infty} \frac{1}{p-1}\frac{(tf_1(t))^{(p-1)/p}}{g(t)f_1(t)\int_t^\infty (sf_1(s))^{-1/p}ds} = \frac{1}{p-1}\big(\frac{1}{p} + \kappa_g\big)$;

(vi): $\lim_{t\to 0^+}(g(\psi(t)))^{-1}\left[\frac{1}{p}\left(1 + \frac{\psi(t)f_1'(\psi(t))}{f_1(\psi(t))}\right) - \frac{f_1(\xi\psi(t))}{\xi^{p-1}f_1(\psi(t))}\right] = \frac{1}{p} - \ln\xi$, $\xi > 0$;

(vii): $\lim_{t\to 0^+} \frac{f_2(\xi\psi(t))}{\xi^{p-1}g(\psi(t))f_1(\psi(t))} = E_2$, $\xi > 0$;

(viii): $\lim_{t\to 0^+} \frac{1}{(p-1)^{(p-1)/p}}\frac{(\psi(t)f_1(\psi(t)))^{(p-1)/p}}{tg(\psi(t))f_1(\psi(t))} = \frac{1}{p-1}\big(\frac{1}{p} + \kappa_g\big)$.

Proof. (**i**) By the definition of ψ and a direct calculation, we see that (**i**)-(**ii**) hold.
(**iii**) If $\xi = 1$, the result is obvious. Otherwise, by $f \in NRV_{p-1}$, we have

$$\frac{f_1(\xi t)}{\xi^{p-1}f_1(t)} - 1 = \exp\left(\int_t^{\xi t} \frac{g(\tau)}{\tau}d\tau\right) - 1,\ t \geq t_0. \tag{9}$$

It follows by ($\mathbf{f_4}$) and Proposition 2.1 that $\lim_{t\to\infty} \frac{g(ts)}{s} = 0$ and $\lim_{t\to\infty} \frac{g(ts)}{g(t)} = 1$ uniformly with respect to $s \in [c_1, c_2]$. Hence, we have

$$\lim_{t\to\infty} \int_t^{t\xi} \frac{g(\tau)}{\tau}d\tau = \lim_{t\to\infty} \int_1^{\xi} \frac{g(st)}{s}ds = 0.$$

By the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{t\to\infty} \int_1^{\xi} \frac{g(st)}{sg(t)}ds = \int_1^{\xi} s^{-1}ds = \ln\xi. \tag{10}$$

On the other hand, we see that

$$\exp(t) - 1 \simeq t \text{ as } t \to 0 \tag{11}$$

and

$$\lim_{t\to\infty} (g(t))^{-1}\left(\exp\left(\int_t^{\xi t} \frac{g(\tau)}{\tau}d\tau\right) - 1 - \int_t^{\xi t} \frac{g(\tau)}{\tau}d\tau\right) = 0. \tag{12}$$

It follows by (9)-(12) that (**iii**) holds.
(**iv**) Since

$$\lim_{t\to\infty}\frac{f_2(\xi t)}{\xi^{p-1}g(t)f_1(t)}=\lim_{t\to\infty}\frac{f_2(\xi t)}{\xi^{p-1}f_2(t)}\lim_{t\to\infty}\frac{f_2(t)}{g(t)f_1(t)},$$

we see that if ($\mathbf{f_5}$) holds, then

$$\lim_{t\to\infty}\frac{f_2(\xi t)}{\xi^{p-1}f_2(t)}=1 \text{ and } \lim_{t\to\infty}\frac{f_2(t)}{g(t)f_1(t)}=E_1;$$

if ($\mathbf{f_6}$) holds, then

$$\lim_{t\to\infty}\frac{f_2(\xi t)}{\xi^{p-1}f_2(t)}=\xi^{\mu-p+1} \text{ and } \lim_{t\to\infty}\frac{f_2(t)}{g(t)f_1(t)}=0.$$

(**v**) By ($\mathbf{f_3}$)-($\mathbf{f_4}$) and the l'Hospital's rule, we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{p-1}\frac{(tf_1(t))^{(p-1)/p}}{g(t)f_1(t)\int_t^\infty(sf_1(s))^{-1/p}ds}\\
&=\lim_{t\to\infty}-\frac{1}{p-1}\frac{\frac{p-1}{p}g(t)-g'(t)t-\frac{1}{p}g(t)\frac{f_1'(t)t}{f_1(t)}}{(g(t))^2}\\
&=\lim_{t\to\infty}\frac{1}{p-1}\left(\frac{g'(t)t}{(g(t))^2}+\frac{1}{p}\frac{\frac{f_1'(t)t}{f_1(t)}-(p-1)}{g(t)}\right)=\frac{1}{p-1}\left(\frac{1}{p}+\kappa_g\right).
\end{aligned}$$

(**vi**)-(**viii**) We conclude by ($\mathbf{f_3}$) and (**iii**)-(**v**) that (**vi**)-(**viii**) hold. □

Lemma 3.4. ([11], *Lemma 2.2*) *Let Ω be a bounded domain and $G:\Omega\times\mathbb{R}\to\mathbb{R}$ be non-increasing in the second variable and continuous. Let u, $w\in W^{1,p}(\Omega)$ satisfy the respective inequalities*

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx\le\int_\Omega G(x,u)\varphi dx;$$
$$\int_\Omega|\nabla w|^{p-2}\nabla w\cdot\nabla\varphi dx\ge\int_\Omega G(x,w)\varphi dx,$$

for all non-negative $\varphi\in W_0^{1,p}(\Omega)$. Then the inequality $u\le w$ on $\partial\Omega$ implies $u\le w$ in Ω.

4. Proof of Theorem 1.1.

Proof. Take $\varepsilon\in(0,\min\{\xi_0,b_0\}/2)$ and

$$b_{+\varepsilon}=(b_0+\varepsilon)^{1/p},\ b_{-\varepsilon}=(b_0-\varepsilon)^{1/p},$$
$$\tau_{+\varepsilon}=\exp(\xi_0+\varepsilon),\ \tau_{-\varepsilon}=\exp(\xi_0-\varepsilon).$$

A simple calculation shows that

$$\exp(\xi_0/2)<\tau_{-\varepsilon}<\tau_{+\varepsilon}<\exp(3\xi_0/2),\ (b_0/2)^{1/p}<b_{-\varepsilon}<b_{+\varepsilon}<(3b_0/2)^{1/p}.$$

For any constant $R>R_0$, we define

$$\Omega_R:=\{x\in\mathbb{R}^N:|x|>R\},\tag{13}$$

where R_0 is given by the definition Λ_1 in ($\mathbf{b_2}$).

By Lemma 3.1 and Lemma 3.3 (**vi**)-(**viii**), we see that

$$\begin{aligned}
\lim_{(t,r)\to(0,\infty)}I_\pm(t,r)&=\frac{1}{p}-\ln\tau_{\pm\varepsilon}-E_2+\frac{1}{p-1}\left(\frac{1}{p}+\kappa_g\right)((N-p)D_\theta-p+1)\\
&=\xi_0-\ln\tau_{\pm\varepsilon}=\mp\varepsilon,
\end{aligned}$$

where

$$I_\pm(t,r) := (g(\psi(t)))^{-1}\left(\frac{1}{p}\left(1+\frac{\psi(t)f_1'(\psi(t))}{f_1(\psi(t))}\right) - \frac{f_1(\tau_{\pm\varepsilon}\psi(t))}{\tau_{\pm\varepsilon}^{p-1}f_1(\psi(t))}\right)$$
$$- \frac{f_2(\tau_{\pm\varepsilon}\psi(t))}{\tau_{\pm\varepsilon}^{p-1}g(\psi(t))f_1(\psi(t))} + \frac{1}{(p-1)^{(p-1)/p}}\frac{(\psi(t)f_1(\psi(t)))^{(p-1)/p}}{tg(\psi(t))f_1(\psi(t))}$$
$$\times\left((p-1)\frac{\Theta(r)\theta'(r)}{\theta^2(r)} + \frac{(N-1)\Theta(r)}{r\theta(r)}\right).$$

This implies that there exist a large constant $R_\varepsilon > R_0$ and a small constant $\delta_\varepsilon > 0$ corresponding to ε such that

$$(3b_0/2)^{1/p}\Theta(|x|) < \delta_\varepsilon,\ x \in \Omega_{R_\varepsilon} \subseteq \Omega$$

and for any $(t,x) \in (0, 2\delta_\varepsilon) \times \Omega_{R_\varepsilon}$, the following hold

$$I_+(t,|x|) \le 0 \text{ and } I_-(t,|x|) \ge 0.$$

In fact, we can always adjust R_ε such that for any $x \in \Omega_{R_\varepsilon}$, it holds

$$b_0 - \varepsilon \le \frac{b(x)}{\theta^p(|x|)} \le b_0 + \varepsilon.$$

Let u be the solution of problem (1)-(2) and take

$$\sigma < \min\{\delta_\varepsilon, (b_0/2)^{1/p}\Theta(R_\varepsilon)\}.$$

Set

$$D^\sigma_{R_\varepsilon -} := \Omega_{R_\varepsilon} \setminus \Omega^\sigma_{R_\varepsilon -},\ D^\sigma_{R_\varepsilon +} := \Omega_{R_\varepsilon} \setminus \Omega^\sigma_{R_\varepsilon +},$$

where

$$\Omega^\sigma_{R_\varepsilon -} := \{x \in \Omega_{R_\varepsilon} : b_{-\varepsilon}\Theta(|x|) \le \sigma\}$$

and

$$\Omega^\sigma_{R_\varepsilon +} := \{x \in \Omega_{R_\varepsilon + r_0} : \tau_{-\varepsilon}\psi(b_{+\varepsilon}\Theta(|x|) + \sigma) \le u(x)\}, \tag{14}$$

where r_0 is a large enough constant such that $D^\sigma_{R_\varepsilon +}$ is an annular domain. Moreover, by the definition of $\Omega^\sigma_{R_\varepsilon -}$, we see that $D^\sigma_{R_\varepsilon -}$ is also an annular domain.

Define

$$\overline{u}_\varepsilon(x) := \tau_{+\varepsilon}\psi(b_{-\varepsilon}\Theta(|x|) - \sigma),\ x \in D^\sigma_{R_\varepsilon -},\ \underline{u}_\varepsilon(x) := \tau_{-\varepsilon}\psi(b_{+\varepsilon}\Theta(|x|) + \sigma),\ x \in D^\sigma_{R_\varepsilon +}.$$

By a straightforward calculation, we have for any $x \in D^\sigma_{R_\varepsilon -}$,

$$\begin{aligned}
&\Delta_p\overline{u}_\varepsilon(x) - b(x)f(\overline{u}_\varepsilon(x))\\
&= (p-1)\tau_{+\varepsilon}^{p-1}b_{-\varepsilon}^p(-\psi'(b_{-\varepsilon}\Theta(|x|) - \sigma))^{p-2}\psi''(b_{-\varepsilon}\Theta(|x|) - \sigma)\theta^p(|x|)\\
&+ \tau_{+\varepsilon}^{p-1}b_{-\varepsilon}^{p-1}(-\psi'(b_{-\varepsilon}\Theta(|x|)))^{p-1}\theta^p(|x|)\left(\frac{(p-1)\theta'(|x|)}{\theta^2(|x|)} + \frac{N-1}{\theta(|x|)|x|}\right)\\
&- b(x)\big(f_1(\tau_{+\varepsilon}\psi(b_{-\varepsilon}\Theta(|x|) - \sigma)) + f_2(\tau_{+\varepsilon}\psi(b_{-\varepsilon}\Theta(|x|) - \sigma))\big)\\
&\le \tau_{+\varepsilon}^{p-1}b_{-\varepsilon}^p f_1(\psi(b_{-\varepsilon}\Theta(|x|) - \sigma))g(\psi(b_{-\varepsilon}\Theta(|x|) - \sigma))\theta^p(|x|)
\end{aligned}$$

$$\times \Bigg[\big(g(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))\big)^{-1}\Bigg(\frac{1}{p}\Bigg(1+\frac{\psi(b_{-\varepsilon}\Theta(|x|)-\sigma)f_1'(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}{f_1(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}\Bigg)$$
$$-\frac{f_1(\tau_{+\varepsilon}\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}{\tau_{+\varepsilon}^{p-1}f_1(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}\Bigg)-\frac{f_2(\tau_{+\varepsilon}\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}{\tau_{+\varepsilon}^{p-1}g(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))f_1(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}$$
$$+\frac{1}{(p-1)^{(p-1)/p}}\frac{\big(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma)f_1(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))\big)^{(p-1)/p}}{(b_{-\varepsilon}\Theta(|x|)-\sigma)g(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))f_1(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))}$$
$$\times\frac{b_{-\varepsilon}\Theta(|x|)-\sigma}{b_{-\varepsilon}\Theta(|x|)}\Bigg((p-1)\frac{\Theta(|x|)\theta'(|x|)}{\theta^2(|x|)}+\frac{(N-1)\Theta(|x|)}{|x|\theta(|x|)}\Bigg)\Bigg]\le\tau_{+\varepsilon}^{p-1}b_{-\varepsilon}^{p}$$
$$\times f_1(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))g(\psi(b_{-\varepsilon}\Theta(|x|)-\sigma))\theta^p(|x|)I_+\big(b_{-\varepsilon}\Theta(|x|)-\sigma,|x|\big)\le 0,$$

i.e., $\overline{u}_\varepsilon$ is an upper solution to Eq. (1) in $D^\sigma_{R_\varepsilon-}$. In a similar way, we can show that $\underline{u}_\varepsilon$ is a lower solution to Eq. (1) in $D^\sigma_{R_\varepsilon+}$.

We can choose a positive constant M independent of σ such that

$$u(x)\le\overline{u}_\varepsilon(x)+M \text{ and } \underline{u}_\varepsilon(x)\le u(x)+M \text{ on } \partial\Omega_{R_\varepsilon}. \tag{15}$$

Next, we prove

$$u(x)\le\overline{u}_\varepsilon(x)+M,\ x\in D^\sigma_{R_\varepsilon-} \tag{16}$$

and

$$\underline{u}_\varepsilon(x)\le u(x)+M,\ x\in\Omega_{R_\varepsilon}. \tag{17}$$

Since

$$u(x)<\overline{u}_\varepsilon(x)=\infty \text{ on } \big\{x\in\mathbb{R}^N: b_{-\varepsilon}\Theta(|x|)=\sigma\big\},$$

we take a small enough positive constant ρ such that

$$\sup_{x\in D^\sigma_{R_\varepsilon-}} u(x)\le\overline{u}_\varepsilon(x),\ x\in D^\sigma_{R_\varepsilon-}\setminus\tilde{D}^\sigma_{R_\varepsilon-}, \tag{18}$$

where

$$\tilde{D}^\sigma_{R_\varepsilon-}:=\Omega_{R_\varepsilon}\setminus\tilde{\Omega}^\sigma_{R_\varepsilon-}$$

and

$$\tilde{\Omega}^\sigma_{R_\varepsilon-}:=\big\{x\in\Omega_{R_\varepsilon}: b_{-\varepsilon}\Theta(|x|)\le\sigma(1+\rho)\big\}.$$

By (15) and (18), we have

$$u(x)\le\overline{u}_\varepsilon(x)+M,\ x\in\partial(\tilde{D}^\sigma_{R_\varepsilon-}).$$

On the other hand, combining (14) and (15), we obtain

$$\underline{u}_\varepsilon(x)\le u(x)+M,\ x\in\partial(D^\sigma_{R_\varepsilon+}).$$

Since f is increasing on $[0,\infty)$, we see that $\overline{u}_\varepsilon+M$ and $u+M$ are both upper solutions in $\tilde{D}^\sigma_{R_\varepsilon-}$ and $D^\sigma_{R_\varepsilon+}$, respectively. By Lemma 3.4, we have

$$u(x)\le\overline{u}_\varepsilon(x)+M,\ x\in\tilde{D}^\sigma_{R_\varepsilon-} \tag{19}$$

and

$$\underline{u}_\varepsilon(x)\ge u(x)+M,\ x\in D^\sigma_{R_\varepsilon+}. \tag{20}$$

By (18)-(19), we obtain (16) holds. By (14) and (20), we obtain (17) holds. So, passing to $\sigma\to 0$, we have for $x\in\Omega_{R_\varepsilon}$,

$$\frac{u(x)}{\psi(b_{-\varepsilon}\Theta(|x|))}\le\tau_{+\varepsilon}+\frac{M}{\psi(b_{-\varepsilon}\Theta(|x|))} \text{ and } \frac{u(x)}{\psi(b_{+\varepsilon}\Theta(|x|))}\ge\tau_{-\varepsilon}-\frac{M}{\psi(b_{+\varepsilon}\Theta(|x|))}.$$

We obtain by Lemma 3.3 (**i**) that

$$\limsup_{|x|\to\infty}\frac{u(x)}{\psi(b_{-\varepsilon}\Theta(|x|))}\le\tau_{+\varepsilon} \text{ and } \liminf_{|x|\to\infty}\frac{u(x)}{\psi(b_{+\varepsilon}\Theta(|x|))}\ge\tau_{-\varepsilon}.$$

Passing to $\varepsilon \to 0$, we obtain (4). □

5. **Proof of Theorem 1.2.**

Proof. Take $\varepsilon \in (0, \min\{\xi_1, b_1\}/2)$ and

$$\tilde{b}_{+\varepsilon} = (b_1 + \varepsilon)^{1/p},\ \tilde{b}_{-\varepsilon} = (b_1 - \varepsilon)^{1/p},$$

$$\tilde{\tau}_{+\varepsilon} = \exp(\xi_1 + \varepsilon),\ \tilde{\tau}_{-\varepsilon} = \exp(\xi_1 - \varepsilon).$$

A simple calculation shows that

$$\exp(\xi_1/2) < \tilde{\tau}_{-\varepsilon} < \tilde{\tau}_{+\varepsilon} < \exp(3\xi_1/2),\ (b_1/2)^{1/p} < \tilde{b}_{-\varepsilon} < \tilde{b}_{+\varepsilon} < (3b_1/2)^{1/p}.$$

For any $\delta > 0$, we define

$$\mathfrak{D}_\delta := \{x \in \Omega : 0 < d(x) < \delta\}. \tag{21}$$

Since Ω is a smooth exterior domain in $\mathbb{R}^N$, there exists $\delta_1 > 0$ such that (please refer to Lemmas 14.16 and 14.17 in [5])

$$d \in C^2(\mathfrak{D}_{\delta_1}),\ |\nabla d(x)| = 1,\ \Delta d(x) = -(N-1)H(\bar{x}) + o(1),\ x \in \mathfrak{D}_{\delta_1},$$

where for all $x \in \mathfrak{D}_{\delta_1}$ near the boundary of Ω, $\bar{x} \in \partial\Omega$ is the nearest point to x, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at $\bar{x}$.

Case 1. k is non-decreasing on $(0, \delta_0)$. By Lemma 3.2 (**i**) and Lemma 3.3 (**vi**)-(**viii**), we see that

$$\lim_{d(x)\to 0} J_\pm(d(x)) = \frac{1}{p} - \ln \tau_{\pm\varepsilon} - E_2 - \left(\frac{1}{p} + \kappa_g\right)(1 - D_k) = \mp\varepsilon,$$

where

$$\begin{aligned} J_\pm(d(x)) :=\ & (g(\psi(\tilde{b}_{\mp\varepsilon}K(d(x)))))^{-1}\left[\frac{1}{p}\left(1 + \frac{\psi(\tilde{b}_{\mp\varepsilon}K(d(x)))f_1'(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}{f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}\right)\right. \\ & \left. - \frac{f_1(\tilde{\tau}_{\pm\varepsilon}\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}{\tilde{\tau}_{\pm\varepsilon}^{p-1}f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}\right] - \frac{f_2(\tilde{\tau}_{\pm\varepsilon}\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}{\tilde{\tau}_{\pm\varepsilon}^{p-1}g(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))} \\ & - \frac{p-1}{(p-1)^{(p-1)/p}}\frac{\left(\psi(\tilde{b}_{\mp\varepsilon}K(d(x)))f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))\right)^{(p-1)/p}}{\tilde{b}_{\mp\varepsilon}K(d(x))g(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}\frac{K(d(x))k'(d(x))}{k^2(d(x))} \\ & - \frac{\left(\psi(\tilde{b}_{\mp\varepsilon}K(d(x)))f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))\right)^{(p-1)/p}}{\tilde{b}_{\mp\varepsilon}K(d(x))g(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))f_1(\psi(\tilde{b}_{\mp\varepsilon}K(d(x))))}\frac{K(d(x))}{k(d(x))}\Delta d(x). \end{aligned}$$

This implies that there exists a sufficiently small constant $0 < \delta_\varepsilon < \frac{\delta_1}{2}$ corresponding to ε such that for any $x \in \mathfrak{D}_{2\delta_\varepsilon}$, the following hold

$$J_+(d(x)) \le 0 \text{ and } J_-(d(x)) \ge 0.$$

As before, we can always adjust δ_ε such that for any $x \in \mathfrak{D}_{2\delta_\varepsilon}$, it holds

$$b_1 - \varepsilon \le \frac{b(x)}{k^p(d(x))} \le b_1 + \varepsilon. \tag{22}$$

Set $\sigma \in (0, \delta_\varepsilon)$ and define

$$D_-^\sigma := \mathfrak{D}_{2\delta_\varepsilon} \setminus \bar{\mathfrak{D}}_\sigma,\ D_+^\sigma := \mathfrak{D}_{2\delta_\varepsilon - \sigma} \tag{23}$$

and

$$d_1(x) := d(x) - \sigma,\ x \in D_-^\sigma,\ \ d_2(x) := d(x) + \sigma,\ x \in D_+^\sigma.$$

Let

$$\overline{u}_\varepsilon(x) := \tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K(d_1(x))),\ x \in D_-^\sigma,\ \underline{u}_\varepsilon(x) := \tilde{\tau}_{-\varepsilon}\psi(\tilde{b}_{+\varepsilon}K(d_2(x))),\ x \in D_+^\sigma.$$

By a straightforward calculation, we obtain that for any $x \in D_-^\sigma$,

$$\begin{aligned}
&\Delta\overline{u}_\varepsilon(x) - b(x)f(\overline{u}_\varepsilon(x))\\
&= (p-1)\tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p}(-\psi'(\tilde{b}_{-\varepsilon}K(d_1(x))))^{p-2}\psi''(\tilde{b}_{-\varepsilon}K(d_1(x)))k^p(d_1(x))\\
&- (p-1)\tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p-1}(-\psi'(\tilde{b}_{-\varepsilon}K(d_1(x))))^{p-1}k^{p-2}(d_1(x))k'(d_1(x))\\
&- \tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p-1}(-\psi'(\tilde{b}_{-\varepsilon}K(d_1(x))))^{p-1}k^{p-1}(d_1(x))\Delta d(x)\\
&- b(x)\big(f_1(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K(d_1(x)))) + f_2(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K(d_1(x))))\big)\\
&\le \tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p}f_1(\psi(\tilde{b}_{-\varepsilon}K(d_1(x))))g(\psi(\tilde{b}_{-\varepsilon}K(d_1(x))))k^p(d_1(x))J_+(d_1(x)) \le 0,
\end{aligned}$$

i.e., $\overline{u}_\varepsilon$ is an upper solution to Eq. (1) in D_-^σ. In a similar way, we can show that $\underline{u}_\varepsilon$ is a lower solution to Eq. (1) in D_+^σ.

Case 2. k is non-increasing on $(0, \delta_0)$. As before, by Lemma 3.2 (**i**) and Lemma 3.3 (**vi**)-(**viii**), we obtain

$$\lim_{(r,d(x))\to(0,0)} \tilde{J}_\pm(r, d(x)) = \frac{1}{p} - \ln\tilde{\tau}_{\pm\varepsilon} - E_2 - \left(\frac{1}{p} + \kappa_g\right)(1 - D_k) = \mp\varepsilon, \tag{24}$$

where

$$\begin{aligned}
&\tilde{J}_\pm(r, d(x)) := (g(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x)))))^{-1}\\
&\times\left(\frac{1}{p}\left(1 + \frac{\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x)))f_1'(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}{f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}\right)\right.\\
&\left.- \frac{f_1(\tilde{\tau}_{\pm\varepsilon}\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}{\tilde{\tau}_{\pm\varepsilon}^{p-1}f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}\right) - \frac{f_2(\tilde{\tau}_{\pm\varepsilon}\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}{\tilde{\tau}_{\pm\varepsilon}^{p-1}g(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}\\
&- \frac{p-1}{(p-1)^{(p-1)/p}}\frac{\big(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x)))f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))\big)^{(p-1)/p}}{\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))g(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}\\
&\times\frac{K(d(x))k'(d(x))}{k^2(d(x))} - \frac{K(d(x))}{k(d(x))}\Delta d(x)\\
&\times\frac{\big(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x)))f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))\big)^{(p-1)/p}}{\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))g(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))f_1(\psi(\tilde{b}_{\mp\varepsilon}K_r^{\mp}(d(x))))}
\end{aligned}$$

with

$$K_r^{\mp}(d(x)) := K(d(x)) \mp K(r) > 0.$$

By (24), we see that there exists a small enough constant $\delta_\varepsilon \in (0, \delta_1/2)$ corresponding to ε such that for any $(r, x) \in (0, \delta_\varepsilon) \times \mathfrak{D}_{2\delta_\varepsilon}$, the following hold

$$\tilde{J}_+(r, d(x)) \le 0 \text{ and } \tilde{J}_-(r, d(x)) \ge 0$$

and (22) holds here for any $x \in \mathfrak{D}_{2\delta_\varepsilon}$.

Take $\sigma \in (0, \delta_\varepsilon)$ and let

$$\overline{u}_\varepsilon(x) := \tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))),\ x \in D_-^\sigma,\ \underline{u}_\varepsilon(x) := \tilde{\tau}_{-\varepsilon}\psi(\tilde{b}_{+\varepsilon}K_\sigma^+(d(x))),\ x \in D_+^\sigma,$$

where $D^\sigma_\mp$ are defined as (23). A straightforward calculation shows that for any $x \in D^\sigma_-$,

$$
\begin{aligned}
&\Delta \overline{u}_\varepsilon(x) - b(x) f(\overline{u}_\varepsilon(x)) \\
&= (p-1)\tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p}(-\psi'(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))^{p-2}\psi''(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))k^p(d(x)) \\
&- (p-1)\tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p-1}(-\psi'(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))^{p-1}k^{p-2}(d(x))k'(d(x)) \\
&- \tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p-1}(-\psi'(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))^{p-1}k^{p-1}(d(x))\Delta d(x) \\
&- b(x)\big(f_1(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))) + f_2(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))\big) \\
&\le \tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p} f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))k^p(d(x)) \\
&\times \Bigg[(g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))))^{-1}\Bigg(\frac{1}{p}\Bigg(1+\frac{\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))f_1'(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}{f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}\Bigg) \\
&- \frac{f_1(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}{\tilde{\tau}_{+\varepsilon}^{p-1}f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}\Bigg) - \frac{f_2(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}{\tilde{\tau}_{+\varepsilon}^{p-1}g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))} \\
&- \frac{p-1}{(p-1)^{(p-1)/p}}\frac{\big(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))\big)^{(p-1)/p}}{\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))} \\
&\times \frac{K_\sigma^-(d(x))k'(d(x))}{k^2(d(x))} - \frac{K_\sigma^-(d(x))}{k(d(x))}\Delta d(x) \\
&\times \frac{\big(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))\big)^{(p-1)/p}}{\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}\Bigg] \\
&\le \tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p} f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))k^p(d(x)) \\
&\times \Bigg[(g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))))^{-1}\Bigg(\frac{1}{p}\Bigg(1+\frac{\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))f_1'(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}{f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}\Bigg) \\
&- \frac{f_1(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}{\tilde{\tau}_{+\varepsilon}^{p-1}f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}\Bigg) - \frac{f_2(\tilde{\tau}_{+\varepsilon}\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))}{\tilde{\tau}_{+\varepsilon}^{p-1}g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))} \\
&+ \frac{\big(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x)))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))\big)^{(p-1)/p}}{\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))} \\
&\times \Bigg(-\frac{p-1}{(p-1)^{(p-1)/p}}\frac{d(x)k'(d(x))}{k(d(x))} - d(x)\Delta d(x)\Bigg)\frac{K_\sigma^-(d(x))}{d(x)k(d(x))}\Bigg] \\
&\le \tilde{\tau}_{+\varepsilon}^{p-1}\tilde{b}_{-\varepsilon}^{p} f_1(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))g(\psi(\tilde{b}_{-\varepsilon}K_\sigma^-(d(x))))k^p(d(x))\tilde{J}_+(\sigma, d(x)) \le 0,
\end{aligned}
$$

i.e., $\overline{u}_\varepsilon$ is an upper solution to Eq. (1) in D^σ_-. In a similar way, we can show that $\underline{u}_\varepsilon$ is a lower solution to Eq. (1) in D^σ_+.

For case 1 and case 2, let u be an arbitrary solution of problem (1)-(2). Next, we prove that there exists a large constant $M > 0$ such that

$$u(x) \le \overline{u}_\varepsilon(x) + M,\ x \in D^\sigma_- \text{ and } \underline{u}_\varepsilon(x) \le u(x) + M,\ x \in D^\sigma_+. \tag{25}$$

Obviously, we can always take a constant $M > 0$ independent of σ such that

$$
\begin{aligned}
&u(x) \le \overline{u}_\varepsilon(x) + M,\ x \in \{x \in \Omega : d(x) = 2\delta_\varepsilon\}, \\
&\underline{u}_\varepsilon(x) \le u(x) + M,\ x \in \{x \in \Omega : d(x) = 2\delta_\varepsilon - \sigma\}.
\end{aligned} \tag{26}
$$

On the other hand, we have

$$u(x) < \overline{u}_\varepsilon(x) = \infty,\ x \in \{x \in \Omega : d(x) = \sigma\} \text{ and } \underline{u}_\varepsilon(x) < u(x) = \infty,\ x \in \partial\Omega.$$

This implies that we can take a small enough positive constant ρ with $0 < \rho < \delta_\varepsilon$ such that

$$\sup_{x\in D_-^\sigma} u(x) \le \overline{u}_\varepsilon(x),\ x \in D_-^\sigma \setminus \tilde{D}_-^\sigma,\quad \sup_{x\in D_+^\sigma} \underline{u}_\varepsilon(x) \le u(x),\ x \in D_+^\sigma \setminus \tilde{D}_+^\sigma, \tag{27}$$

where

$$\tilde{D}_-^\sigma := \mathfrak{D}_{2\delta_\varepsilon} \setminus \bar{\mathfrak{D}}_{(1+\rho)\sigma},\ \tilde{D}_-^\sigma := \mathfrak{D}_{2\delta_\varepsilon-\sigma} \setminus \bar{\mathfrak{D}}_\rho.$$

Since f is increasing on $[0,\infty)$, we see that $\overline{u}_\varepsilon + M$ and $u + M$ are both upper solutions in $\tilde{D}_-^\sigma$ and $\tilde{D}_+^\sigma$, respectively. We conclude by (26)-(27) and Lemma 3.4 that

$$u(x) \le \overline{u}_\varepsilon(x) + M,\ x \in \tilde{D}_-^\sigma,\ \ \underline{u}_\varepsilon(x) \le u(x) + M,\ x \in \tilde{D}_+^\sigma.$$

This fact, combined with (27), shows that (25) holds. So, passing to $\sigma \to 0$, we have for $x \in \mathfrak{D}_{2\delta_\varepsilon}$,

$$\frac{u(x)}{\psi(\tilde{b}_{-\varepsilon}K(d(x)))} \le \tilde{\tau}_{+\varepsilon}+\frac{M}{\psi(\tilde{b}_{-\varepsilon}K(d(x)))} \text{ and } \frac{u(x)}{\psi(\tilde{b}_{+\varepsilon}K(d(x)))} \ge \tilde{\tau}_{-\varepsilon}-\frac{M}{\psi(\tilde{b}_{+\varepsilon}K(d(x)))}.$$

We obtain by Lemma 3.3 (**i**) that

$$\limsup_{d(x)\to 0} \frac{u(x)}{\psi(\tilde{b}_{-\varepsilon}K(d(x)))} \le \tilde{\tau}_{+\varepsilon} \text{ and } \liminf_{d(x)\to 0} \frac{u(x)}{\psi(\tilde{b}_{+\varepsilon}K(d(x)))} \ge \tilde{\tau}_{-\varepsilon}.$$

Passing to $\varepsilon \to 0$, we obtain (7). □

6. **Proof of Theorem 1.6.**

Proof. Let u_1 and u_2 be two positive solutions of problem (1)-(2). By

$$\lim_{|x|\to\infty} \frac{u_1(x)}{u_2(x)} = 1 \text{ and } \lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1,$$

we see that for fixed $\varepsilon > 0$, there exist a sufficiently large constant R_ε and a sufficiently small constant δ_ε such that

$$\overline{(\mathbb{R}^N \setminus \Omega) \cup \mathfrak{D}_{\delta_\varepsilon}} \subseteq \mathbb{R}^N \setminus \bar{\Omega}_{R_\varepsilon}$$

and

$$(1-\varepsilon)u_2(x) \le u_1(x) \le (1+\varepsilon)u_2(x),\ x \in \Omega_{R_\varepsilon} \cup \mathfrak{D}_{\delta_\varepsilon}, \tag{28}$$

where Ω_{R_ε} and $\mathfrak{D}_{\delta_\varepsilon}$ are defined as (13) and (21), respectively.

Let

$$u^\pm(x) := (1 \pm \varepsilon)u_2(x),\ x \in \Omega.$$

The condition ($\mathbf{f_8}$) implies that

$$\Delta_p u^+ \le b(x)f(u^+) \text{ and } \Delta_p u^- \le b(x)f(u^-) \text{ in } \Omega.$$

Assume that u_0 is the unique solution for

$$\Delta_p u_0 = b(x)f(u_0),\ x \in \Omega_0,\ u|_{\partial\Omega_0} = u_1,$$

where

$$\Omega_0 = \Omega \setminus (\Omega_{R_\varepsilon} \cup \mathfrak{D}_{\delta_\varepsilon}).$$

It follows by Lemma 3.4 that

$$u^-(x) \le u_0(x) \le u^+(x),\ x \in \Omega_0. \tag{29}$$

Since $u_0 \equiv u_1$ in Ω_0, by (28)-(29) we have

$$(1-\varepsilon)u_2(x) \le u_1(x) \le (1+\varepsilon)u_2(x),\ x \in \Omega = \Omega_0 \cup \Omega_{R_\varepsilon} \cup \mathfrak{D}_{\delta_\varepsilon}.$$

It follows by passing to $\varepsilon \to 0$ that $u_1 = u_2$ in Ω. □

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