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TRAVELING WAVE SOLUTION FOR A DIFFUSION SEIR EPIDEMIC MODEL WITH SELF-PROTECTION AND TREATMENT

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Abstract. A reaction-diffusion SEIR model, including the self-protection for susceptible individuals, treatments for infectious individuals and constant recruitment, is introduced. The existence of traveling wave solution, which is determined by the basic reproduction number R_0 and wave speed c, is firstly proved as $R_0 > 1$ and $c \geq c^*$ via the Schauder fixed point theorem, where c^* is minimal wave speed. Asymptotic behavior of traveling wave solution at infinity is also proved by applying the Lyapunov functional. Furthermore, when $R_0 \leq 1$ or $R_0 > 1$ with $c \in (0,\ c^*)$, we derive the non-existence of traveling wave solution with utilizing two-sides Laplace transform. We take advantage of numerical simulations to indicate the existence of traveling wave, and show that self-protection and treatment can reduce the spread speed at last.

1. **Introduction.** Epidemics are widespread around the world, and always jeopardize public lives and health, such as COVID-19, malaria, severe acute respiratory syndrome (SARS), influenza and so on. The public has taken all kinds of measures to struggle against the different infectious epidemics. Various of mathematical models are established to research dynamics and influences of prevention and control for different epidemics [12, 23, 22, 10, 9, 15]. Recently, both the popularization of knowledge about infectious diseases and the improvement of treatment have built a solid foundation for epidemics' prevention. This popularization can also lead to the strengthening in public self-protection. For instance, since Chinese had owned strongly self-protection and governments of China had taken efficient treatments, COVID-19 was first under effective control in China [11, 17, 13]. As a result, it is necessary to integrate self-protection and treatment into epidemic models. Because epidemics are always discovered at one location and then spread to other areas [21], reaction-diffusion models become essential to describe this spatial spread. Simultaneously, epidemic models with spatial spread usually can result in a development from a diseases-free state to an infective state, which can be predicted a wave for

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this evolution of epidemics. Consequently, traveling wave solution becomes critical to study the spatial spread of epidemics [7, 8, 18, 20].

Many people have analyzed traveling wave solution and the asymptotic speed of propagation of classic compartmental epidemic models [16, 26, 1, 4, 30, 28, 32, 25]. Ducrot and Magal [3] studied the existence and the non-existence of traveling wave solution satisfying a diffusive epidemic model with age-structure and constant recruitment, and constructed a suitable Lyapunov functional to discuss their asymptotic behavior at infinity. Li et al. [14] found that the existence and the non-existence of traveling wave solution for the system which is a nonlocal dispersal delayed SIR model with constant recruitment and Holling-slowromancapii@ incidence rate are determined via both the minimal wave speed and the basic reproduction number R_0 defined by the corresponding reaction system. Zhang et al.[29] considered the existence of traveling wave solution satisfying a influenza model with treatment for infectious individuals. Zhao et al. [31] proved the existence and non-existence of traveling wave solution for a SIR model with multiple parallel infectious stages which own same diffusion coefficients. Then Zhao et al. [33] established a two-groups reaction-diffusion epidemic model to research the influence of host heterogeneity in spread of disease by applying the traveling wave solution. However, human self-protection for susceptible individuals and treatment for infectious individuals were not considered together in a SEIR diffusion epidemic model.

In this paper, we firstly integrate self-protection into the classical reaction-diffusion SEIR model with constant recruitment, so susceptible individuals are divided into two groups: susceptible individuals S_1 without self-protection and susceptible people S_2 with self-protection. Next, the treatment for infectious individuals is considered such that I_1 represents the infectious with treatment and I_2 means the infectious without it. Therefore, a reaction-diffusion SEIR model is constructed as system (1), where $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. In this model, $S_i(x,\ t)$ and $I_i(x,\ t)$ $(i = 1,\ 2)$ represent the density of two groups of susceptible and infectious individuals above determined, respectively. $E(x,\ t)$ represents the density of individuals who are infected without infectivity. $R(x,\ t)$ represents the density of individuals who recover and own permanent immunity. $d_1,\ d_2,\ D_0,\ D_1,\ D_2$, and D represent diffusion coefficients of $S_1,\ S_2,\ E,\ I_1,\ I_2$ and R, respectively. Λ is constant recruitment. β is bilinear incidence. μ is the sum of the mortality rate. α_i is the recovery rate of I_i $(i = 1,\ 2)$, respectively. γ is the reciprocal of latency. q_i is the proportion of individuals entering I_i $(i = 1,\ 2)$, respectively. ρ is the migration rate from S_1 to S_2. σ is the rate which susceptible individuals with self-protection reduce contacts with infectious. θ represents the rate that the treatment for the infectious reduce their contacts with others.

$$\begin{cases} \dfrac{\partial S_1(x,t)}{\partial t} = d_1\Delta S_1(x,\ t) + \Lambda - \beta S_1(x,\ t)\left[\theta I_1(x,\ t) + I_2(x,\ t)\right] - (\rho+\mu)S_1(x,\ t), \\ \dfrac{\partial S_2(x,\ t)}{\partial t} = d_2\Delta S_2(x,\ t) + \rho S_1(x,\ t) - \beta\sigma S_2(x,\ t)\left[\theta I_1(x,\ t) + I_2(x,\ t)\right] - \mu S_2(x,\ t), \\ \dfrac{\partial E(x,\ t)}{\partial t} = D_0\Delta E(x,\ t) + \beta\left[S_1(x,\ t) + \sigma S_2(x,\ t)\right]\left[\theta I_1(x,\ t) + I_2(x,\ t)\right] - (\mu+\gamma)E(x,\ t), \\ \dfrac{\partial I_1(x,\ t)}{\partial t} = D_1\Delta I_1(x,\ t) + q_1\gamma E(x,\ t) - (\mu+\alpha_1)I_1(x,\ t), \\ \dfrac{\partial I_2(x,\ t)}{\partial t} = D_2\Delta I_2(x,\ t) + q_2\gamma E(x,\ t) - (\mu+\alpha_2)I_2(x,\ t), \\ \dfrac{\partial R(x,\ t)}{\partial t} = D\Delta R(x,\ t) + \alpha_1 I_1(x,\ t) + \alpha_2 I_2(x,\ t) - \mu R(x,\ t). \end{cases} \tag{1}$$

Throughout this aritcle, we make the following assumption:

(A) $d_1 = d_2 := d,\ d > D_0 > D_i > 0,\ \Lambda,\ \beta,\ \theta,\ \rho,\ \mu,\ \sigma,\ \gamma,\ q_1,\ q_2,\ \alpha_i > 0,\ q_1+q_2 = 1, \theta \in (0,\ 1],\ \sigma \in (0,1]$ for $i = 1,\ 2$.

In fact, for making a living, the susceptible has to move, and their diffusion is almost same without special cases. Especially, the susceptible with self-protection may take some methods, such as wearing masks, avoiding crowed public places and so on, to decrease the likelihood of connects with the infectious. The infectious could reduce going out owing to discomfort by disease. Thus, the assumption can better close to reality. In mathematical analysis of this model, the assumption (A) also can help us get an upper bound for the different susceptible and infected components to research the asymptotic behavior as $x \to +\infty$.

Since the equations $S_i(x,\ t),\ E(x,\ t)$ and $I_i(x,\ t)$ $(i = 1,\ 2)$ are fully decoupled from $R(x,\ t)$, we only need to study the sub-system:

$$\begin{cases} \dfrac{\partial S_1(x,\ t)}{\partial t} = d\Delta S_1(x,\ t) + \Lambda - \beta S_1(x,\ t)\left[\theta I_1(x,\ t) + I_2(x,\ t)\right] \\ \qquad\qquad - (\rho+\mu)S_1(x,\ t), \\ \dfrac{\partial S_2(x,\ t)}{\partial t} = d\Delta S_2(x,\ t) + \rho S_1(x,\ t) - \beta\sigma S_2(x,\ t)\left[\theta I_1(x,\ t) + I_2(x,\ t)\right] \\ \qquad\qquad - \mu S_2(x,\ t), \\ \dfrac{\partial E(x,\ t)}{\partial t} = D_0\Delta E(x,\ t) + \beta\left[S_1(x,\ t) + \sigma S_2(x,\ t)\right]\left[\theta I_1(x,\ t) + I_2(x,\ t)\right] \\ \qquad\qquad - (\mu+\gamma)E(x,\ t), \\ \dfrac{\partial I_1(x,\ t)}{\partial t} = D_1\Delta I_1(x,\ t) + q_1\gamma E(x,\ t) - (\mu+\alpha_1)I_1(x,\ t), \\ \dfrac{\partial I_2(x,\ t)}{\partial t} = D_2\Delta I_2(x,\ t) + q_2\gamma E(x,\ t) - (\mu+\alpha_2)I_2(x,\ t), \end{cases} \tag{2}$$

where $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

The structure of this article is as follows. In Section 2, we firstly obtain the basic reproduction number R_0 and two non-negative constant equilibriums for the corresponding reaction system, and these equilibriums are also two non-negative constant equilibriums for system (2). Then, we find a $c^* > 0$ called as the minimal spread speed. Next, as $R_0 > 1$ and the spread speed $c > c^*$, we prove that system (2) admits non-trivial and non-negative traveling wave solution by applying the Schauder fixed point theorem. Meanwhile, a suitable Lyapunov functional is used to prove the asymptotic behavior of traveling wave solution at $x \to +\infty$. At last, we obtain the existence of traveling wave solution connecting two non-negative constant equilibriums when $R_0 > 1$ and $c = c^*$. In Section 3, we firstly prove the non-existence of non-trivial and non-negative traveling wave solution connecting two non-negative constant equilibriums as $R_0 \leq 1$. In particular, when $R_0 > 1$ and $c \in (0,\ c^*)$, we also study the non-existence via the two-sided Laplace transform. In Section 4, we take advantage of numerical simulations to display the existence of traveling wave solution connecting two non-negative constant equilibriums. Simultaneously, we also conclude that self-protection and treatment can decrease the spread speed for an epidemic via numerical simulations.

2. **Existence of traveling wave solution.**

In order to research traveling wave solution of system (2), the first step is to find what kind of constant equilibriums of system (2) exist. It is well known that there always exists an equilibrium

$(S_1^0,\ S_2^0,\ 0,\ 0,\ 0) = (\frac{\Lambda}{\rho+\mu},\ \frac{\rho\Lambda}{\mu(\mu+\rho)},\ 0,\ 0,\ 0)$ named as the disease-free equilibrium of system (2). We can also find a positive equilibrium by the following system,

$$\begin{cases} \dfrac{\mathrm{d}S_1(t)}{\mathrm{d}t} = \Lambda - \beta S_1(t)\left[\theta I_1(t) + I_2(t)\right] - (\rho+\mu)S_1(t), \\ \dfrac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \rho S_1(t) - \beta\sigma S_2(t)\left[\theta I_1(t) + I_2(t)\right] - \mu S_2(t), \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = \beta\left[S_1(t) + \sigma S_2(t)\right]\left[\theta I_1(t) + I_2(t)\right] - (\mu+\gamma)E(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = q_1\gamma E(t) - (\mu+\alpha_1)I_1(t), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = q_2\gamma E(t) - (\mu+\alpha_2)I_2(t). \end{cases} \tag{3}$$

The point $(S_1^0,\ S_2^0,\ 0,\ 0,\ 0)$ is also the disease-free equilibrium of system (3). Via the next generation matrix method formulated in [24], the basic reproduction number for system (3) denoted by R_0 can be expressed as

$$R_0 = \left[\frac{q_1\gamma\theta\beta}{(\mu+\gamma)(\mu+\alpha_1)} + \frac{q_2\gamma\beta}{(\mu+\gamma)(\mu+\alpha_2)}\right](S_1^0 + \sigma S_2^0).$$

Furthermore, taking advantage of the direct Lyapunov functional method, which is similar in [15], can claim the below theorem.

Theorem 2.1. *If $R_0 \le 1$, then there exists a unique constant equilibrium of system (3) which is the disease-free equilibrium $(S_1^0,\ S_2^0,\ 0,\ 0,\ 0)$, and it is globally asymptotically stable. If $R_0 > 1$, system (3) admits a positive equilibrium which is the endemic equilibrium $(S_1^*,\ S_2^*,\ E^*,\ I_1^*,\ I_2^*)$, and it is globally asymptotically stable.*

In the rest of Section 2 , we always assume $R_0 > 1$. Then system (2) exists two constant equilibriums: the disease-free equilibrium $(S_1^0,\ S_2^0,\ 0,\ 0,\ 0)$ and the endemic equilibrium $(S_1^*,\ S_2^*,\ E^*,\ I_1^*,\ I_2^*)$. So we research the existence of traveling wave solution satisfying system (2) and connecting $(S_1^0,\ S_2^0,\ 0,\ 0,\ 0)$ and $(S_1^*,\ S_2^*,\ E^*,\ I_1^*,\ I_2^*)$. Now, we show that traveling wave solution of system (2) is special solution with the form as

$$(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi)), \qquad \xi = x + ct \in \mathbb{R}. \tag{4}$$

Substitute formula (4) into systmen (2), and we can obtain the wave form equations as follows:

$$\begin{cases} dS_1^{''}(\xi) - cS_1^{'}(\xi) + \Lambda - \beta S_1(\xi)[\theta I_1(\xi) + I_2(\xi)] - (\rho+\mu)S_1(\xi) = 0, \\ dS_2^{''}(\xi) - cS_2^{'}(\xi) + \rho S_1(\xi) - \beta\sigma S_2(\xi)[\theta I_1(\xi) + I_2(\xi)] - \mu S_2(\xi) = 0, \\ D_0E^{''}(\xi) - cE^{'}(\xi) + \beta\left[\theta I_1(\xi) + I_2(\xi)\right]\left[S_1(\xi) + \sigma S_2(\xi)\right] - (\mu+\gamma)E(\xi) = 0, \\ D_1I_1^{''}(\xi) - cI_1^{'}(\xi) + q_1\gamma E(\xi) - (\mu+\alpha_1)I_1(\xi) = 0, \\ D_2I_2^{''}(\xi) - cI_2^{'}(\xi) + q_2\gamma E(\xi) - (\mu+\alpha_2)I_2(\xi) = 0, \end{cases} \tag{5}$$

for $\xi \in \mathbb{R}$, where $'$ and $''$ represent the first and the second derivative with respect to ξ, respectively. Since we would like to find positive traveling wave solution connecting two equilibria, we need the positive solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ of system (5) with following boundary conditions

$$\begin{aligned} &S_i(-\infty) = S_i^0,\ E(-\infty) = 0,\ I_i(-\infty) = 0, \\ &S_i(+\infty) = S_i^*,\ E(+\infty) = E^*,\ I_i(+\infty) = I_i^*, \quad i = 1,\ 2. \end{aligned} \tag{6}$$

Linearizing the E-th and I_i-th equations of (5) at the disease-free equilibrium $(S_1^0,\ S_2^0,\ 0,\ 0,\ 0)$ yields

$$\begin{cases} D_0E^{''}(\xi) - cE^{'}(\xi) + \beta\left[\theta I_1(\xi) + I_2(\xi)\right](S_1^0 + \sigma S_2^0) - (\mu+\gamma)E(\xi) = 0, \\ D_1I_1^{''}(\xi) - cI_1^{'}(\xi) + q_1\gamma E(\xi) - (\mu+\alpha_1)I_1(\xi) = 0, \\ D_2I_2^{''}(\xi) - cI_2^{'}(\xi) + q_2\gamma E(\xi) - (\mu+\alpha_2)I_2(\xi) = 0. \end{cases}$$

Set $(E(\xi),\ I_1(\xi),\ I_2(\xi)) = (\eta_0,\ \eta_1,\ \eta_2)e^{\lambda\xi}$ and plug it into above equations, then we can obtain characteristic equations

$$\begin{cases} D_0\eta_0\lambda^2 - c\eta_0\lambda + \beta(\theta\eta_1 + \eta_2)(S_1^0 + \sigma S_2^0) - (\mu+\gamma)\eta_0 = 0, \\ D_1\eta_1\lambda^2 - c\eta_1\lambda + q_1\gamma\eta_0 - (\mu+\alpha_1)\eta_1 = 0, \\ D_2\eta_2\lambda^2 - c\eta_2\lambda + q_2\gamma\eta_0 - (\mu+\alpha_2)\eta_2 = 0. \end{cases} \tag{7}$$

Let

$$\tilde{A} = \begin{pmatrix} D_0 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix},$$

$$\tilde{V} = \begin{pmatrix} \mu+\gamma & 0 & 0 \\ 0 & \mu+\alpha_1 & 0 \\ 0 & 0 & \mu+\alpha_2 \end{pmatrix},$$

$$\tilde{F} = \begin{pmatrix} 0 & \beta\theta(S_1^0+\sigma S_2^0) & \beta(S_1^0+\sigma S_2^0) \\ q_1\gamma & 0 & 0 \\ q_2\gamma & 0 & 0 \end{pmatrix}.$$

Denote $\Theta(\lambda,\ c) = \lambda^2\tilde{A} - \mu\tilde{B} - \tilde{V} + \tilde{F}$. Then system (7) can reduce to

$$\Theta(\lambda,\ c)(\eta_0,\ \eta_1,\ \eta_2)^{\mathrm{T}} = 0.$$

And define $A = \tilde{V}^{-1}\tilde{A},\ B = \tilde{V}^{-1}\tilde{B},\ F = \tilde{V}^{-1}\tilde{F}$, then we yield that

$$(-A\lambda^2 + B\lambda + I)^{-1}F\eta = \eta, \tag{8}$$

and

$$(-A\lambda^2 + B\lambda + I)^{-1}F = \begin{pmatrix} 0 & \frac{\beta\theta(S_1^0+\sigma S_2^0)}{m_0(\lambda,\ c)} & \frac{\beta(S_1^0+\sigma S_2^0)}{m_0(\lambda,\ c)} \\ \frac{q_1\gamma}{m_1(\lambda,\ c)} & 0 & 0 \\ \frac{q_2\gamma}{m_2(\lambda,\ c)} & 0 & 0 \end{pmatrix},$$

where $\eta = (\eta_0,\ \eta_1,\ \eta_2)^{\mathrm{T}}$, $m_0(\lambda,\ c) = -D_0\lambda^2 + c\lambda + \mu + \gamma$, $m_1(\lambda,\ c) = -D_1\lambda^2 + c\lambda + \mu + \alpha_1$, $m_2(\lambda,\ c) = -D_2\lambda^2 + c\lambda + \mu + \alpha_2$. Let $\mathcal{M}(\lambda,\ c) = (-A\lambda^2 + B\lambda + I)^{-1}F$, then equation (8) becomes

$$\mathcal{M}(\lambda,\ c)\eta = \eta.$$

Let $\rho(\lambda,\ c)$ be the principal eigenvalue of $\mathcal{M}(\lambda,\ c)$ and define

$$\lambda_c = \min\left\{\frac{c + \sqrt{c^2 - 4D_0(\mu+\gamma)}}{2D_0},\ \frac{c + \sqrt{c^2 - 4D_1(\mu+\alpha_1)}}{2D_1},\ \frac{c + \sqrt{c^2 - 4D_2(\mu+\alpha_2)}}{2D_2}\right\}.$$

For $c \geq 0$ and $\lambda \in [0,\ \lambda_c)$, a direct calculation indicates that

$$\rho(\lambda,\ c) = \left[\left(\frac{\beta\theta q_1\gamma}{m_0(\lambda,\ c)m_1(\lambda,\ c)} + \frac{\beta q_2\gamma}{m_0(\lambda,\ c)m_2(\lambda,\ c)}\right)(S_1^0 + \sigma S_2^0)\right]^{\frac{1}{2}}. \tag{9}$$

Moreover, some properties of $\rho(\lambda,\ c)$ can be described as the following lemma:

Lemma 2.2. *Below statements hold:*

(i) λ_c *is strictly increasing in* $c \in [0, +\infty)$, *and* $\lim_{c\to+\infty} \lambda_c = +\infty$,

(ii) $\rho(0,\ c) = \sqrt{R_0}$ *for any* $c \in [0,\ +\infty)$, $\rho(\lambda,\ 0)$ *is strictly increasing in* $[0,\ \lambda_c)$, *and* $\rho(\lambda,\ c) \to +\infty$ *as* $\lambda \to \lambda_c - 0$ *for any* $c \geq 0$, *where* R_0 *is the basic reproduction number for system (3),*

(iii) *for* $\forall\lambda \in (0, \lambda_c)$, $\frac{\partial}{\partial c}\rho(\lambda,\ c) < 0$.

Proof. It is well known that (i) is established, so we only prove (ii) and (iii). It follows from the definitions of both R_0 and $\rho(\lambda,\ c)$ that $\rho(0,\ c) = \sqrt{R_0}$ for any $c \in [0,\ +\infty)$. Due to $\rho(\lambda,\ 0) > 0$ and $m_i(\lambda,\ 0) > 0$ for $\lambda \in [0,\ \lambda_c)$ with $i = 0,\ 1,\ 2,$ it is the fact that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\rho(\lambda,\ 0) \quad &= \quad \rho^{-\frac{1}{2}}(\lambda,\ 0)\left(\frac{\beta\theta q_1\gamma(S_1^0+\sigma S_2^0)(D_0\lambda m_1(\lambda,\ 0)+D_1\lambda m_0(\lambda,\ 0))}{m_0^2(\lambda,\ 0)m_1^2(\lambda,\ 0)}\right.\\ &\quad \left.+\frac{\beta q_2\gamma(S_1^0+\sigma S_2^0)(D_0\lambda m_2(\lambda,\ 0)+D_2\lambda m_0(\lambda,\ 0))}{m_0^2(\lambda,\ 0)m_2^2(\lambda,\ 0)}\right)\\ &> \quad 0\end{aligned}$$

for $\lambda \in [0,\ \lambda_c)$. In addition, we can indicate $\lim_{\lambda\to\lambda_c-0} m_i(\lambda,\ c) = 0$ $(i = 0,\ 1,\ 2)$, respectively, so the principal eigenvalue $\rho(\lambda,\ c)$ tends to $+\infty$ as $\lambda \to \lambda_c - 0$. The direct derivative of the spread speed c can show that

$$\frac{\partial}{\partial c}\rho(\lambda,\ c) = -\frac{1}{2}\lambda\rho^{-\frac{1}{2}}(\lambda,\ c)\left(\frac{\beta\theta q_1\gamma(S_1^0+\sigma S_2^0)}{m_0^2(\lambda,\ c)m_1^2(\lambda,\ c)}+\frac{\beta q_2\gamma(S_1^0+\sigma S_2^0)}{m_0^2(\lambda,\ c)m_2^2(\lambda,\ c)}\right) < 0,$$

for $\forall\lambda \in [0,\ \lambda_c)$. The proof is complete. □

According to Lemma 2.2, we define

$$\hat{\lambda}(c) = \min_{\lambda\in[0,\ \lambda_c)} \rho(\lambda,\ c), \quad \forall c \geq 0.$$

Then we conclude $\hat{\lambda}(0) = \sqrt{R_0}$ and $\lim_{c\to+\infty} \hat{\lambda}(c) = 0$. Meanwhile, $\hat{\lambda}(c)$ is continuous and strictly decreasing in $c \geq 0$. Assume that $R_0 > 1$, then there exists a $c^* > 0$ such that $\hat{\lambda}(c^*) = 1,\ \hat{\lambda}(c) > 1$ for $c \in [0,\ c^*)$ and $\hat{\lambda}(c) < 1$ for $c \in (c^*,\ +\infty)$. Let

$$\lambda_* = \inf\{\lambda \in [0,\ \lambda_{c^*}) :\ \rho(\lambda,\ c^*) = 1\},$$

it indicates that $\rho(\lambda_*,\ c^*) = 1$ and $\rho(\lambda_*,\ c) < 1$ for any $c > c^*$. Define

$$\lambda_1(c) = \sup\{\lambda \in (0,\ \lambda_*) :\ \rho(\lambda,\ c) = 1,\ \rho(\lambda',\ c) \geq 1,\ \forall\lambda' \in (0,\ \lambda)\}.$$

Because of $\rho(\lambda_*,\ c) < 1$ for $c > c^*$, we claim that the following lemma is established.

Lemma 2.3. *Assume that* $R_0 > 1$, *then there exist* $c^* > 0$, *named as the minimal spread speed, and* $\lambda_* \in (0,\ \lambda_{c^*})$ *such that*

(i) $\rho(\lambda,\ c) > 1$ *for any* $c \in [0,\ c^*)$ *and* $\lambda \in (0,\ \lambda_c)$,

(ii) $\rho(\lambda_*,\ c^*) = 1,\ \rho(\lambda,\ c^*) > 1$ *for any* $\lambda \in (0,\ \lambda_*)$ *and* $\rho(\lambda,\ c^*) \geq 1$ *for any* $\lambda \in (0,\ \lambda_{c^*})$,

(iii) *as* $c > c^*$, *there exists* $\lambda_1(c) \in (0,\ \lambda_*)$ *such that* $\rho(\lambda_1(c),\ c) = 1,\ \rho(\lambda,\ c) \geq 1$ *for* $\lambda \in (0,\ \lambda_1(c))$ *and* $\rho(\lambda_1(c) + \varepsilon_n(c),\ c) < 1$ *for some decreasing sequence* $\{\varepsilon_n(c)\}$ *satisfying* $\lim_{n\to+\infty} \varepsilon_n = 0$ *and* $\lambda_1(c)+\varepsilon_n(c) < \lambda_*$ $(n \in \mathbb{Z})$. *Especially,* $\lambda_1(c)$ *is strictly decreasing in* $c > c^*$.

Since the $\mathcal{M}(\lambda,\ c)$ is non-negative and irreducible for $\lambda \in [0,\ \lambda_c)$, the bleow lemma holds with utilizing the Perron-Frobenius theorem.

Lemma 2.4. *Assume that $R_0 > 1$, for $c > c^*$, there are positive unit vectors $\eta(c) = (\eta_0(c),\ \eta_1(c), \eta_2(c))^{\mathrm{T}}$ and $\zeta^n(c) = (\zeta_0^n(c),\ \zeta_1^n(c),\ \zeta_2^n(c))^{\mathrm{T}}$ for $n \in \mathbb{Z}$ such that*

$$\mathcal{M}(\lambda_1(c),\ c)\eta(c) = \eta(c),$$
$$\mathcal{M}(\lambda_1(c) + \varepsilon_n(c),\ c)\zeta^n(c) = \rho(\lambda_1(c) + \varepsilon_n(c),\ c)\zeta^n(c).$$

Fix $c > c^*$ and let $\lambda_1(c),\ \eta(c) = (\eta_0(c),\ \eta_1(c),\ \eta_2(c))^{\mathrm{T}}$ and $\zeta^n(c) = (\zeta_0^n(c),\ \zeta_1^n(c),\ \zeta_2^n(c))^{\mathrm{T}}$ $(n \in \mathbb{Z})$ own same definitions in Lemma 2.3 and Lemma 2.4. For convenience, we use $\lambda_1,\ \varepsilon_n,\ \eta = (\eta_0,\ \eta_1,\ \eta_2)^{\mathrm{T}}$ and $\zeta^n = (\zeta_0^n,\ \zeta_1^n,\ \zeta_2^n)^{\mathrm{T}}$ $(n \in \mathbb{Z})$ instant of them, respectively. It follows from Lemma 2.4 and $\rho(\lambda_1 + \varepsilon_n,\ c) < 1$ that following equations and inequalities are established for $n \in \mathbb{Z}$

$$\begin{cases} -m_0(\lambda_1,\ c)\eta_0 + (S_1^0 + \sigma S_2^0)(\theta\eta_1 + \eta_2) = 0, \\ -m_1(\lambda_1,\ c)\eta_1 + q_1\gamma\eta_0 = 0, \\ -m_2(\lambda_1,\ c)\eta_2 + q_2\gamma\eta_0 = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} -m_0(\lambda_1 + \varepsilon_n,\ c)\zeta_0^n + (S_1^0 + \sigma S_2^0)(\theta\zeta_1^n + \zeta_2^n) < 0, \\ -m_1(\lambda_1 + \varepsilon_n,\ c)\zeta_1^n + q_1\gamma\zeta_0^n < 0, \\ -m_2(\lambda_1 + \varepsilon_n,\ c)\zeta_2^n + q_2\gamma\zeta_0^n < 0. \end{cases} \tag{11}$$

According to the above argument, we can construct suitable sub- and super-solutions which are defined in below lemmas. And then the local existence of solution for system (5) is proved via the Schauder fixed point theorem.

Lemma 2.5. *Let the vector function $\mathcal{P}(\xi) = (p_0(\xi),\ p_1(\xi),\ p_2(\xi))^{\mathrm{T}}$ with $p_i(\xi) = \eta_i(\xi)e^{\lambda_1\xi}$ for $\xi \in \mathbb{R}$ and $i = 0,\ 1,\ 2,$ and it satisfies*

$$\begin{cases} D_0 p_0''(\xi) - cp_0'(\xi) + \beta(S_1^0 + \sigma S_2^0)(\theta p_1(\xi) + p_2(\xi)) - (\mu + \gamma)p_0(\xi) = 0, \\ D_1 p_1''(\xi) - cp_1'(\xi) + q_1\gamma p_0(\xi) - (\mu + \alpha_1)p_1(\xi) = 0, \\ D_2 p_2''(\xi) - cp_2'(\xi) + q_2\gamma p_0(\xi) - (\mu + \alpha_2)p_2(\xi) = 0. \end{cases}$$

Lemma 2.6. *For each $\omega > 0$ small enough with $\omega < \min\{\lambda_1, \frac{c}{d}\}$ and $M > 1$ sufficiently large, the vector function $S^-(\xi) = (S_1^-(\xi), S_2^-(\xi))^{\mathrm{T}}$ defined by $S_i^-(\xi) = \max\{S_i^0(1 - Me^{\omega\xi}),\ 0\}$ $(i = 1,\ 2)$ satisfies*

$$cS_1^{-'}(\xi) \leq dS_1^{-''}(\xi) + \Lambda - (\mu + \rho)S_1^-(\xi) - \beta S_1^-(\xi)(\theta p_1(\xi) + p_2(\xi)), \tag{12}$$
$$cS_2^{-'}(\xi) \leq dS_2^{-''}(\xi) + \rho S_1^-(\xi) - \mu S_2^-(\xi) - \beta\sigma S_2^-(\xi)(\theta p_1(\xi) + p_2(\xi)), \tag{13}$$

for $\xi \in \mathbb{R}$ and $\xi \neq -\frac{1}{\omega}\ln M$.

Proof. When $\xi > -\frac{1}{\omega}\ln M$, $S_i^-(\xi) = 0$ for $i = 1,\ 2$, and inequalities (12)-(13) hold. When $\xi < -\frac{1}{\omega}\ln M$, it implies that $S_i^-(\xi) = S_i^0(1 - Me^{\omega\xi}$ and $p_i(\xi) = \eta_i e^{\lambda_1\xi}$. Due to $\omega < \frac{c}{d}$ and $e^{-\frac{\lambda_1 - \omega}{\omega}\ln M} \to 0$ as $M \to +\infty$, it is shown that

$$\begin{aligned} &(-d\omega + c)S_1^0\omega Me^{\omega\xi} + \Lambda - (\mu + \rho)S_1^0(1 - Me^{\omega\xi}) - \beta(\theta\eta_1 + \eta_2)S_1^0(1 - Me^{\omega\xi})e^{\lambda_1\xi} \\ &= (-d\omega + c)S_1^0\omega Me^{\omega\xi} + (\mu + \rho)S_1^0 Me^{\omega\xi} - \beta(\theta\eta_1 + \eta_2)S_1^0(1 - Me^{\omega\xi})e^{\lambda_1\xi} \\ &\geq \left[(-d\omega + c)S_1^0\omega M + (\mu + \rho)S_1^0 M - \beta(\theta\eta_1 + \eta_2)S_1^0 e^{-\frac{\lambda_1 - \omega}{\omega}\ln M}\right]e^{\omega\xi} \\ &\geq 0, \end{aligned}$$

so the inequality (12) is established. Furthermore,

$$\begin{aligned}
&(-d\omega+c)S_2^0\omega Me^{\omega\xi}+\rho S_1^0(1-Me^{\omega\xi})-\beta\sigma(\theta\eta_1+\eta_2)S_2^0(1-Me^{\omega\xi})e^{\lambda_1\xi}\\
&-\mu S_2^0(1-Me^{\omega\xi})\\
&=(-d\omega+c)S_2^0\omega Me^{\omega\xi}-\beta\sigma(\theta\eta_1+\eta_2)S_2^0(1-Me^{\omega\xi})e^{\lambda_1\xi}\\
&\geq\left[(-d\omega+c)S_1^0\omega M-\beta\sigma(\theta\eta_1+\eta_2)S_1^0e^{-\frac{\lambda_1-\omega}{\omega}\ln M}\right]e^{\omega\xi}\\
&\geq 0,
\end{aligned}$$

and the inequality (13) is set up. The lemma is completely proved. □

Lemma 2.7. *Let $0<\epsilon<\frac{\omega}{2}$ with $\epsilon=\varepsilon_{n_0}$ for some $n_0\in\mathbb{Z}$, and the eigenvector $\zeta^{n_0}=(\zeta_0^{n_0},\ \zeta_1^{n_0},\ \zeta_2^{n_0})^{\mathrm{T}}$ is defined by $\zeta=(\zeta_0,\ \zeta_1,\ \zeta_2)^{\mathrm{T}}$. We define the vector map $\mathcal{H}(\xi)=(h_0(\xi),\ h_1(\xi),\ h_2(\xi))^{\mathrm{T}}$ with $h_i(\xi)=\max\{\eta_ie^{\lambda_1\xi}-K\zeta_ie^{(\lambda_1+\epsilon)\xi},\ 0\}$, where η_i and ζ_i are defined in Lemma 2.4 with $i=0,\ 1,\ 2$, respectively. For $K>0$ large enough such that*

$$\min\left\{\frac{1}{\epsilon}\ln\frac{K\zeta_0}{\eta_0},\ \frac{1}{\epsilon}\ln\frac{K\zeta_1}{\eta_1},\ \frac{1}{\epsilon}\ln\frac{K\zeta_2}{\eta_2}\right\}>\frac{1}{\omega}\ln M,$$

then the vector map $\mathcal{H}(\xi)$ satisfies

$$\begin{aligned}
&ch_0^{'}(\xi)\leq D_0h_0^{''}(\xi)-(\mu+\gamma)h_0(\xi)+\beta(S_1^-(\xi)+\sigma S_2^-(\xi))(\theta h_1(\xi)+h_2(\xi)),\\
&\qquad\xi<\frac{1}{\epsilon}\ln\frac{\eta_0}{K\zeta_0},
\end{aligned}\tag{14}$$

$$ch_1^{'}(\xi)\leq D_1h_1^{''}(\xi)+q_1\gamma h_0(\xi)-(\mu+\alpha_1)h_1(\xi),\ \xi<\frac{1}{\epsilon}\ln\frac{\eta_1}{K\zeta_1},\tag{15}$$

$$ch_2^{'}(\xi)\leq D_2h_2^{''}(\xi)+q_2\gamma h_0(\xi)-(\mu+\alpha_2)h_2(\xi),\ \xi<\frac{1}{\epsilon}\ln\frac{\eta_2}{K\zeta_2}.\tag{16}$$

Proof. Firstly, we prove the inequality (14). For $\xi<\frac{1}{\epsilon}\ln\frac{\eta_0}{K\zeta_0}$, we yield $h_0(\xi)=\eta_0e^{\lambda_1\xi}-K\zeta_0e^{(\lambda_1+\epsilon)\xi}$, $S_1^-(\xi)=S_1^0(1-Me^{\omega\xi})$, and $S_2^-(\xi)=S_2^0(1-Me^{\omega\xi})$. To prove the inequality (14), it is sufficient to show the following inequality:

$$\begin{aligned}
&K\left[-D_0\zeta_0(\lambda_1+\epsilon)^2+c\zeta_0(\lambda_1+\epsilon)+(\mu+\gamma)\zeta_0\right]e^{(\lambda_1+\epsilon)\xi}+\beta(S_1^-+\sigma S_2^-)\left[\theta h_1(\xi)\right.\\
&\left.+h_2(\xi)\right]+D_0\eta_0\lambda_1^2e^{\lambda_1\xi}-c\eta_0\lambda_1e^{\lambda_1\xi}-(\mu+\gamma)\eta_0e^{\lambda_1\xi}\\
&\geq 0,
\end{aligned}\tag{17}$$

which is

$$-K\zeta_0m_0(\lambda_1+\epsilon,\ c)e^{(\lambda_1+\epsilon)\xi}+\eta_0m_0(\lambda_1,\ c)e^{\lambda_1\xi}-\beta(S_1^-+\sigma S_2^-)\left[\theta h_1(\xi)+h_2(\xi)\right]\leq 0.$$

According to equations (10), the proof of the inequality (17) can be replaced by proving

$$\begin{aligned}
&-K\zeta_0m_0(\lambda_1+\epsilon,\ c)e^{(\lambda_1+\epsilon)\xi}+\beta(S_1^0+\sigma S_2^0)(\theta\eta_1+\eta_2)e^{\lambda_1\xi}-\beta(S_1^-\\
&+\sigma S_2^-)\left[\theta h_1(\xi)+h_2(\xi)\right]\leq 0.
\end{aligned}\tag{18}$$

Because $S_i^0 - S_i^- \le S_i^0 M e^{\omega\xi}$ and $\eta_i e^{\lambda_1 \xi} - h_i(\xi) \le K\zeta_i e^{(\lambda_1+\epsilon)\xi}$ for $i = 1,\ 2$, it is obtained that

$$\begin{aligned}
&\beta(S_1^0 + \sigma S_2^0)(\theta\eta_1 + \eta_2)e^{\lambda_1\xi} - \beta(S_1^- + \sigma S_2^-)[\theta h_1(\xi) + h_2(\xi)]\\
&= \beta(S_1^0 + \sigma S_2^0)(\theta\eta_1 + \eta_2)e^{\lambda_1\xi} - \beta(S_1^0 + \sigma S_2^0)[\theta h_1(\xi) + h_2(\xi)]\\
&\quad + \beta(S_1^0 + \sigma S_2^0)[\theta h_1(\xi) + h_2(\xi)] - \beta(S_1^- + \sigma S_2^-)[\theta h_1(\xi) + h_2(\xi)]\\
&= \beta(S_1^0 + \sigma S_2^0)[\theta(\eta_1 e^{\lambda_1\xi} - h_1(\xi)) + (\eta_2 e^{\lambda_1\xi}) - h_2(\xi)]\\
&\quad + \beta[\theta h_1(\xi) + h_2(\xi)][(S_1^0 - S_1^-) + \sigma(S_2^0 - S_2^-)]\\
&\le \beta(S_1^0 + \sigma S_2^0)(\theta K\zeta_1 e^{(\lambda_1+\omega)\xi} + K\zeta_2 e^{(\lambda_1+\omega)\xi}) + \beta[\theta h_1(\xi) + h_2(\xi)](S_1^0 M e^{\omega\xi}\\
&\quad + \sigma S_2^0 M e^{\omega\xi}).
\end{aligned}$$

Thus, for the proof of the inequality (18), we only need to prove

$$\begin{aligned}
&Ke^{(\lambda_1+\epsilon)\xi}[-m_0(\lambda_1+\epsilon)\zeta_0 + \beta(S_1^0 + \sigma S_2^0)(\theta\zeta_1 + \zeta_2)] + M\beta(S_1^0 + \sigma S_2^0)[\theta h_1(\xi)\\
&+ h_2(\xi)]e^{\omega\xi} \le 0,
\end{aligned}$$

which is

$$\begin{aligned}
&K[-m_0(\lambda_1+\epsilon)\zeta_0 + \beta(S_1^0 + \sigma S_2^0)(\theta\zeta_1 + \zeta_2)] + M\beta(S_1^0 + \sigma S_2^0)[\theta\eta_1(\xi)\\
&+ \eta_2(\xi)]e^{(\omega-\epsilon)\xi} \le 0. \qquad (19)
\end{aligned}$$

For $\xi < \frac{1}{\epsilon}\ln\frac{\eta_0}{K\zeta_0}$, it is the fact that $e^{(\omega-\epsilon)\xi} \to 0$ as $K \to +\infty$. Thus, combining inequalities (11), the inequality (19) is obtained, and the inequality (14) is established. Finally, we can prove inequalities (15) and (16) via the similar way in the proof of the inequality (14). The proof is complete. □

Now, we set $X > \max\left\{\frac{1}{\epsilon}\ln\frac{\eta_0}{K\zeta_0},\ \frac{1}{\epsilon}\ln\frac{\eta_1}{K\zeta_1},\ \frac{1}{\epsilon}\ln\frac{\eta_2}{K\zeta_2}\right\}$. Define

$$\Gamma_X = \left\{ \begin{array}{l|l} & \chi_i(\pm X) = S_i^-(\pm X),\\ \chi_1(\cdot),\ \chi_2(\cdot),\ \varphi_0(\cdot),\ \varphi_1(\cdot),\ \varphi_2(\cdot) & \varphi_j(\pm X) = h_j(\pm X),\\ \in C([-X,\ X],\ \mathbb{R}^5) & S_i^-(\xi) \le \chi_i(\xi) \le S_i^0,\\ & h_j^-(\xi) \le \varphi_j(\xi) \le p_j(\xi), \end{array} \right\}$$

where $i = 1,\ 2$ and $j = 0,\ 1,\ 2$. And it is well known that Γ_X is closed and convex.

For any

$$(\chi_1(\cdot),\ \chi_2(\cdot),\ \varphi_0(\cdot),\ \varphi_1(\cdot),\ \varphi_2(\cdot)) \in \Gamma_X,$$

we consider the following boundary-value problem for $\xi \in (-X,\ X)$,

$$\begin{cases}
-dS_{1,X}''(\xi) + cS_{1,X}'(\xi) - \Lambda + (\mu+\rho)S_{1,X}(\xi) + \beta S_{1,X}(\xi)[\theta\varphi_1(\xi) + \varphi_2(\xi)] = 0,\\
-dS_{2,X}''(\xi) + cS_{2,X}'(\xi) - \rho\chi_1(\xi) + \mu S_{2,X}(\xi) + \beta\sigma S_{2,X}(\xi)[\theta\varphi_1(\xi) + \varphi_2(\xi)] = 0,\\
-D_0E_X''(\xi) + cE_X'(\xi) - \beta[\chi_1(\xi) + \sigma\chi_2(\xi)][\theta\varphi_1(\xi) + \varphi_2(\xi)] + (\mu+\gamma)E_X(\xi) = 0,\\
-D_1I_{1,X}''(\xi) + cI_{1,X}'(\xi) - q_1\gamma\varphi_0(\xi) + (\mu+\alpha_1)I_{1,X}(\xi) = 0,\\
-D_2I_{2,X}''(\xi) + cI_{2,X}'(\xi) - q_2\gamma\varphi_0(\xi) + (\mu+\alpha_2)I_{2,X}(\xi) = 0,
\end{cases} \qquad (20)$$

satisfying the below boundary condition:

$$\begin{aligned}
S_{1,X}(\pm X) = S_1^-(\pm X),\ S_{2,X}(\pm X) = S_2^-(\pm X),\\
E_X(\pm X) = h_0(\pm X),\ I_{1,X}(\pm X) = h_1(\pm X),\ I_{2,X}(\pm X) = h_2(\pm X).
\end{aligned} \qquad (21)$$

Applying the Gilbarg and Trudinger's Corollary 9.18 in [6], we can claim that there exists a unique solution

$$(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X}),$$

satisfying the problems (20)-(21), where

$$(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X}) \in W^{2,p}((-X,\ X),\ \mathbb{R}^5) \cap C([-X,\ X],\ \mathbb{R}^5),$$

for $p > 1$. And it is shown that $(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X}) \in W^{2,p}((-X,\ X),\ \mathbb{R}^5) \hookrightarrow C^{1,\alpha}([-X,\ X],\ \mathbb{R}^5)$ for some $\alpha \in (0.\ 1)$ via the embedding theorem (see Gilbarg and Trudinger's Theorem 7.26 in [6]). Define the operator $\mathcal{T} = (\mathcal{T}_1,\ \mathcal{T}_2,\ \mathcal{T}_3,\ \mathcal{T}_4,\ \mathcal{T}_5)$ on Γ_X such that

$$\begin{aligned}
S_{1,X} &= \mathcal{T}_1(\chi_1,\ \chi_2,\ \varphi_0,\ \varphi_1,\ \varphi_2),\\
S_{2,X} &= \mathcal{T}_2(\chi_1,\ \chi_2,\ \varphi_0,\ \varphi_1,\ \varphi_2),\\
E_X &= \mathcal{T}_3(\chi_1,\ \chi_2,\ \varphi_0,\ \varphi_1,\ \varphi_2),\\
I_{1,X} &= \mathcal{T}_4(\chi_1,\ \chi_2,\ \varphi_0,\ \varphi_1,\ \varphi_2),\\
I_{1,X} &= \mathcal{T}_5(\chi_1,\ \chi_2,\ \varphi_0,\ \varphi_1,\ \varphi_2),
\end{aligned}$$

for any $(\chi_1,\ \chi_2,\ \varphi_0,\ \varphi_1,\ \varphi_2) \in \Gamma_X$.

Lemma 2.8. *The operator $\mathcal{T}$ maps Γ_X into Γ_X.*

Proof. Firstly, we consider $S_i(\xi)$ $(i = 1,\ 2)$. It is well known that 0 is the sub-solution of the S_i-th equation in equation (20), respectively. S_1^0 and S_2^0 are super-solutions of S_i-th equations in equation (20), respectively. According to $0 < S_{i,X}(X) = S_i^-(X) < S_i^0$ and $0 = S_{i,X}(-X) = S_i^-(-X) < S_i^0$, while combining the maximum principle in [19], it is the fact that $0 \le S_{i,X}(\xi) \le S_i^0$ for $\forall \xi \in [-X,\ X]$. Due to Lemma 2.6, it yields that $S_i^-(\xi)$ satisfies

$$\begin{aligned}
0 &\ge -dS_1^{-''}(\xi) + cS_1^{-'}(\xi) - \Lambda + (\mu+\rho)S_1^-(\xi) + \beta S_1^-(\xi)(\theta p_1(\xi) + p_2(\xi))\\
&\ge -dS_1^{-''}(\xi) + cS_1^{-'}(\xi) - \Lambda + (\mu+\rho)S_1^-(\xi) + \beta S_1^-(\xi)(\theta \varphi_1(\xi) + \varphi_2(\xi)),
\end{aligned}$$

and

$$\begin{aligned}
0 &\ge -dS_2^{-''}(\xi) + cS_2^{-'}(\xi) - \rho S_1^-(\xi) + \mu S_2^-(\xi) + \beta\sigma S_2^-(\xi)(\theta p_1(\xi) + p_2(\xi))\\
&\ge -dS_2^{-''}(\xi) + cS_2^{-'}(\xi) - \rho \chi_1(\xi) + \mu S_2^-(\xi) + \beta\sigma S_2^-(\xi)(\theta \varphi_1(\xi) + \varphi_2(\xi)).
\end{aligned}$$

for any $\xi \in [-X,\ X']$ with $X' = -\frac{1}{\omega}\ln M$. Because of $S_{i,X}(-X) = S_i^-(-X)$ and $S_{i,X}(X') \ge S_i^-(X') = 0$, it is concluded that $S_i^-(\xi) \le S_{i,X}(\xi)$ for $\xi \in [-X,\ X']$ by the maximum principle. Therefore, we claim that $S_i^-(\xi) \le S_{i,X}(\xi) \le S_i^0$ for $\xi \in [-X,\ X]$.

Secondly, we consider about $E_X,\ I_{1,X}$ and $I_{2,X}$. It follows from the maximum principle that $E_X \ge 0,\ I_{i,X} \ge 0$ for any $\xi \in [-X,\ X]$. According to Lemma 2.5, it is obtained that

$$\begin{aligned}
0 &= -D_0 p_0^{''}(\xi) + cp_0^{'}(\xi) - \beta(S_1^0 + \sigma S_2^0)(\theta p_1(\xi) + p_2(\xi)) + (\mu+\gamma)p_0(\xi)\\
&\le -D_0 p_0^{''}(\xi) + cp_0^{'}(\xi) - \beta(S_1^0 + \sigma S_2^0)(\theta \varphi_1(\xi) + \varphi_2(\xi)) + (\mu+\gamma)p_0(\xi),\\
0 &= -D_1 p_1^{''}(\xi) + cp_1^{'}(\xi) - q_1\gamma p_0(\xi) + (\mu+\alpha_1)p_1(\xi)\\
&\le -D_1 p_1^{''}(\xi) + cp_1^{'}(\xi) - q_1\gamma \varphi_0(\xi) + (\mu+\alpha_1)p_1(\xi),
\end{aligned}$$

and

$$\begin{aligned}0 &= -D_2 p_2^{''}(\xi) + c p_2^{'}(\xi) - q_2\gamma p_0(\xi) + (\mu+\alpha_2)p_2(\xi)\\ &\le -D_2 p_2^{''}(\xi) + c p_2^{'}(\xi) - q_2\gamma \varphi_0(\xi) + (\mu+\alpha_2)p_2(\xi),\end{aligned}$$

for any $\xi \in [-X,\ X]$. So the maximum principle shows that $E_X(\xi) \le p_0(\xi),\ I_{1,X}(\xi) \le p_1(\xi)$ and $I_{2,X}(\xi) \le p_2(\xi)$. for $\xi \in [-X,\ X]$. Furthermore, Lemma 2.7 implies that

$$\begin{aligned}0 &\ge -D_0 h_0^{''}(\xi) + c h_0^{'}(\xi) + (\mu+\gamma)h_0(\xi) - \beta(S_1^-(\xi) + \sigma S_2^-)(\xi)(\theta h_1(\xi) + h_2(\xi))\\ &\ge -D_0 h_0^{''}(\xi) + c h_0^{'}(\xi) + (\mu+\gamma)h_0(\xi) - \beta(\chi_1^-(\xi) + \sigma \chi_2^-)(\xi)(\theta \varphi_1(\xi) + \varphi_2(\xi)),\end{aligned}$$

for $\xi \in [-X, X_0^{'}]$ with $X_0^{'} = \frac{1}{\epsilon}\ln\frac{\eta_0}{K\zeta_0}$,

$$\begin{aligned}0 &\ge -D_1 h_1^{''}(\xi) + c h_1^{'}(\xi) - q_1\gamma h_0(\xi) + (\mu+\alpha_1)h_1(\xi)\\ &\ge -D_1 h_1^{''}(\xi) + c h_1^{'}(\xi) - q_1\gamma \varphi_0(\xi) + (\mu+\alpha_1)h_1(\xi),\end{aligned}$$

for $\xi \in [-X, X_1^{'}]$ with $X_1^{'} = \frac{1}{\epsilon}\ln\frac{\eta_1}{K\zeta_1}$, and

$$\begin{aligned}0 &\ge -D_2 h_2^{''}(\xi) + c h_2^{'}(\xi) - q_2\gamma h_0(\xi) + (\mu+\alpha_2)h_2(\xi)\\ &\ge -D_2 h_2^{''}(\xi) + c h_2^{'}(\xi) - q_2\gamma \varphi_0(\xi) + (\mu+\alpha_2)h_2(\xi),\end{aligned}$$

for $\xi \in [-X, X_2^{'}]$ with $X_2^{'} = \frac{1}{\epsilon}\ln\frac{\eta_2}{K\zeta_2}$. Owing to $E_X(-X) = h_0(-X),\ I_{1,X}(-X) = h_1(-X)$ and $I_{2,X}(-X) = h_2(-X)$, while combining the fact $E_X(X_0^{'}) \ge h_0(X_0^{'}) = 0,\ I_{1,X}(X_1^{'}) \ge h_1(X_1^{'}) = 0$ and $I_{2,X}(X_2^{'}) \ge h_2(X_2^{'}) = 0$, we can implies that $h_0(\xi) \le E_X(\xi),\ h_1(\xi) \le I_{1,X}(\xi)$ and $h_2(\xi) \le I_{2,X}(\xi)$ for $\xi \in [-X,\ X_i^{'}]$ $(i = 0,\ 1,\ 2)$ by the maximum principle. Thus, one yields $h_0(\xi) \le E_X(\xi) \le p_0(\xi),\ h_1(\xi) \le I_{1,X}(\xi) \le p_1(\xi)$ and $h_2(\xi) \le I_{2,X}(\xi) \le p_2(\xi)$ for $\xi \in [-X,\ X]$. This completes the proof. □

By taking advantage of the classic embedding theorem, $\mathcal{T}$ is a compact operator from Γ_X to Γ_X. In fact, $\mathcal{T}:\ \Gamma_X \to \Gamma_X$ is also a completely continuous operator (see [27]). Above all, the Schauder fixed point theorem implies that there exists a vector function $(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X}) \in \Gamma_X$ satisfying

$$(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X}) = \mathcal{T}(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X}),$$

for $\xi \in [-X,\ X]$, which includes that $(S_{1,X},\ S_{2,X},\ E_X,\ I_{1,X},\ I_{2,X})$ satisfies following equations for $i = 1,\ 2$,

$$\begin{cases} -dS_{1,X}^{''}(\xi) + cS_{1,X}^{'}(\xi) - \Lambda + (\mu+\rho)S_{1,X}(\xi) + \beta S_{1,X}(\xi)[\theta I_{1,X}(\xi) + I_{2,X}(\xi)] = 0,\\ -dS_{2,X}^{''}(\xi) + cS_{2,X}^{'}(\xi) - \rho S_{1,X}(\xi) + \mu S_{2,X}(\xi) + \beta\sigma S_{2,X}(\xi)[\theta I_{1,X}(\xi) + I_{2,X}(\xi)]\\ \quad = 0,\\ -D_0 E_X^{''}(\xi) + cE_X^{'}(\xi) - \beta[S_{1,X}(\xi) + \sigma S_{2,X}(\xi)][\theta I_{1,X}(\xi) + I_{2,X}(\xi)]\\ \quad + (\mu+\gamma)E_X(\xi) = 0,\\ -D_1 I_{1,X}^{''}(\xi) + cI_{1,X}^{'}(\xi) - q_1\gamma E_X(\xi) + (\mu+\alpha_1)I_{1,X}(\xi) = 0,\\ -D_2 I_{2,X}^{''}(\xi) + cI_{2,X}^{'}(\xi) - q_2\gamma E_X(\xi) + (\mu+\alpha_2)I_{2,X}(\xi) = 0,\\ S_{i,X}(\pm X) = S_i^-(\pm X),\ E_X(\pm X) = h_0(\pm X),\ I_{i,X}(\pm X) = h_i(\pm X). \end{cases} \tag{22}$$

We have proved the local existence of solution for system (5). In order to obtain the global existence, we need the following estimate.

Lemma 2.9. *For a given $Y > 0$, there exist some positive constants $N_{S_i}(Y), N_E(Y)$ and $N_{I_i}(Y)$ such that*

$$\| S_{i,X} \|_{C^3[-Y,\ Y]} \leq N_{S_i}(Y),\ \| E_X \|_{C^3[-Y,\ Y]} \leq N_E(Y),\ \| I_{i,X} \|_{C^3[-Y,\ Y]} \leq N_{I_i}(Y), \tag{23}$$

with $i = 1,\ 2$, and these positive constants are independent of

$$X > \max\left\{ Y,\ \frac{1}{\omega}\ln\frac{1}{M},\ \frac{1}{\epsilon}\ln\frac{\eta_0}{K\zeta_0},\ \frac{1}{\epsilon}\ln\frac{\eta_1}{K\zeta_1},\ \frac{1}{\epsilon}\ln\frac{\eta_2}{K\zeta_2} \right\}.$$

Proof. We always set $i = 1,\ 2$. The equation (22) can imply that $S_{i,X}(\xi) \leq S_i^0,\ E_X(\xi) \leq \eta_0 e^{\lambda_1 Y} \leq \widehat{N_E}(Y)$, and $I_{i,X}(\xi) \leq \eta_i e^{\lambda_1 Y} \leq \widehat{N_{I_i}}(Y)$ for any $\xi \in [-Y,\ Y]$. And applying L^p $(p \geq 2)$ estimates of linear elliptic differential equations to $S_{i,X}$, we can claim that

$$\| S_{1,X} \|_{W^{2,p}(-Y,Y)} \leq \Omega_1 \left(\Lambda + \beta S_1^0 [\theta \widehat{N_{I_1}}(Y) + \widehat{N_{I_2}}(Y)] + \| \phi_1 \|_{W^{2,p}(-Y,Y)} \right),$$

and

$$\| S_{2,X} \|_{W^{2,p}(-Y,Y)} \leq \Omega_2 \left(\rho S_1^0 + \beta\sigma S_2^0 [\theta \widehat{N_{I_1}}(Y) + \widehat{N_{I_2}}(Y)] + \| \phi_2 \|_{W^{2,p}(-Y,Y)} \right),$$

where Ω_i is a constant depending on Y. ϕ_i can be considered as a linear function connecting points $(-Y,\ S_{i,X}(-Y))$ and $(Y,\ S_{i,X}(Y))$, respectively. Therefore, there exists a constant $\widehat{Q_{S_i}}(Y)$ such that $\| S_{i,X} \|_{W^{2,p}(-Y,Y)} \leq \widehat{Q_{S_i}}(Y)$ for any $X > Y$, respectively. Owing to $W^{2,p}(-Y,\ Y) \hookrightarrow C^{1,\alpha}[-Y,\ Y]$ for $\alpha = 1 - \frac{1}{p}$, it is shown that $\| S_{i,X} \|_{C^{1,\alpha}[-Y,\ Y]} \leq \widetilde{N_{S_i}}(Y) \| S_{i,X} \|_{W^{p,\alpha}(-Y,\ Y)}$ where $\widetilde{N_{S_i}}(Y)$ is a constant depending on Y, which can lead to $\| S_{i,X} \|_{C^{1,\alpha}[-Y,\ Y]} \leq \widehat{N_{S_i}}(Y)$ for $\widehat{N_{S_i}}(Y) = \widehat{Q_{S_i}}(Y)\widetilde{N_{S_i}}(Y) > 0$. According to the first and second equations in equations (22), we also obtain $\| S_{i,X} \|_{C^2[-Y,\ Y]} \leq \widehat{N_{S_i}}(Y)$ for $\widehat{N_{S_i}}(Y) > 0$, respectively. Via the similar way, we further yield that $\| E_X \|_{C^2[-Y,\ Y]} \leq \widehat{N_E}(Y)$ and $\| I_{i,X} \|_{C^2[-Y,\ Y]} \leq \widehat{N_{I_i}}(Y)$. Finally, the estimate (23) is established with differentiating two sides of the first five equations in equations (22). The proof is finished. □

Set a sequence of positive numbers $\{X_m\}_{m>0}$ satisfying $X_m \to +\infty$ as $m \to +\infty$. Thus, by Lemma 2.9, there exists a solution $(S_1,\ S_2,\ E,\ I_1,\ I_2) \in C^2(\mathbb{R},\ \mathbb{R}^5)$ of system (5) such that

$$\begin{aligned} S_1^-(\xi) \leq S_1(\xi) \leq S_1^0,\ S_2^-(\xi) \leq S_2(\xi) \leq S_2^0, \\ h_0(\xi) \leq E(\xi) \leq p_0(\xi),\ h_1(\xi) \leq I_1(\xi) \leq p_1(\xi),\ h_2(\xi) \leq I_2(\xi) \leq p_2(\xi) \end{aligned} \tag{24}$$

with $\forall\xi \in \mathbb{R}$. According to inequalities (24), we can gain

$$\lim_{\xi\to-\infty} S_1(\xi) = S_1^0,\ \lim_{\xi\to-\infty} S_2(\xi) = S_2^0,\ \lim_{\xi\to-\infty} E(\xi) = 0,$$

$$\lim_{\xi\to-\infty} I_1(\xi) = 0,\ \lim_{\xi\to-\infty} I_2(\xi) = 0.$$

Now, we need to show some estimates about solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ in order to research the asymptotic behavior as $\xi \to +\infty$.

Lemma 2.10. *Let $r \le \min\{\mu,\ \mu+\alpha_1,\ \mu+\alpha_2\}$, then we have*

$$0 \le E(\xi) + I_1(\xi) + I_2(\xi) \le \sqrt{\frac{d}{D_{min}}\frac{\Lambda}{r}} \tag{25}$$

and

$$\begin{aligned} \frac{\Lambda}{\mu+\rho+\beta\sqrt{\frac{d}{D_{min}}\frac{\Lambda}{r}}} &\le S_1(\xi) \le S_1^0, \\ \frac{\rho\Lambda}{\left(\mu+\rho+\beta\sqrt{\frac{d}{D_{min}}\frac{\Lambda}{r}}\right)\left(\mu+\beta\sigma\sqrt{\frac{d}{D_{min}}\frac{\Lambda}{r}}\right)} &\le S_2(\xi) \le S_2^0, \end{aligned} \tag{26}$$

where $D_{min} = \min\{D_0,\ D_1,\ D_2\}$ and $\forall \xi \in \mathbb{R}$.

Proof. In this proof, we still set $i = 1,\ 2$. As $E(\xi),\ I_1(\xi),\ I_2(\xi)$ are non-negative and not identically zero, the strong maximum principle implies that $E(\xi),\ I_1(\xi),\ I_2(\xi) > 0$ for any $\xi \in \mathbb{R}$ and $E(\xi) + I_1(\xi) + I_2(\xi) > 0$. Define

$$m_1(\xi) = \rho S_1(\xi) + \beta S_1(\xi)[\theta I_1(\xi) + I_2(\xi)],\ m_2(\xi) = \rho S_1(\xi) - \beta\sigma S_2(\xi)[\theta I_1(\xi) + I_2(\xi)],$$

and

$$n_1(\xi) = q_1\gamma E(\xi),\ n_2(\xi) = q_1\gamma E(\xi).$$

Due to the define of r, it then follows that

$$\begin{cases} -dS_1^{''}(\xi) + cS_1^{'}(\xi) + rS_1(\xi) \le \Lambda - m_1(\xi), \\ -dS_2^{''}(\xi) + cS_2^{'}(\xi) + rS_2(\xi) \le m_2(xi), \\ -D_0E^{''}(\xi) + cE^{'}(\xi) + rE(\xi) \le m_1(\xi) - m_2(\xi) - n_1(\xi) - n_2(\xi),\ \forall \xi \in \mathbb{R}, \\ -D_1I_1^{''}(\xi) + cI_1^{'}(\xi) + rI_1(\xi) \le n_1(\xi), \\ -D_2I_2^{''}(\xi) + cI_2^{'}(\xi) + rI_2(\xi) \le n_2(\xi). \end{cases} \tag{27}$$

So we need to consider the following Cauchy problems

$$\begin{cases} \frac{\partial}{\partial t}u_1(t,\ \xi) - d\frac{\partial}{\partial \xi^2}u_1(t,\ \xi) + c\frac{\partial}{\partial \xi}u_1(t,\ \xi) + ru_1(t,\ \xi) = \Lambda - m_1(\xi), \\ u_1(0,\ \xi) = S_1(\xi), \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}u_2(t,\ \xi) - d\frac{\partial}{\partial \xi^2}u_2(t,\ \xi) + c\frac{\partial}{\partial \xi}u_2(t,\ \xi) + ru_2(t,\ \xi) = m_2(\xi), \\ u_2(0,\ \xi) = S_2(\xi), \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}v_0(t,\ \xi) - d\frac{\partial}{\partial \xi^2}v_0(t,\ \xi) + c\frac{\partial}{\partial \xi}v_0(t,\ \xi) + rv_0(t,\ \xi) = m_1(\xi) - m_2(\xi) \\ -n_1(\xi) - n_2(\xi), \\ v_0(0,\ \xi) = E(\xi), \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}v_i(t,\ \xi) - d\frac{\partial}{\partial \xi^2}v_i(t,\ \xi) + c\frac{\partial}{\partial \xi}v_i(t,\ \xi) + rv_i(t,\ \xi) = n_i(\xi), \\ v_i(0,\ \xi) = I_i(\xi), \end{cases}$$

for $\forall t > 0,\ \forall \xi \in \mathbb{R}$. Then, via the Theorem 12 and the Theorem 16 in [5], we can indicate

$$\begin{aligned}
u_1(t,\ \xi) = e^{-rt} &\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi dt}} e^{-\frac{(\xi-ct-y)^2}{4dt}} S_1(y)\mathrm{d}y \\
&+ \int_0^t \int_{\mathbb{R}} \frac{e^{-rs}}{\sqrt{4\pi ds}} e^{-\frac{(\xi-ct-y)^2}{4ds}} (\Lambda - m_1(y))\mathrm{d}y\mathrm{d}s, \\
u_2(t,\ \xi) = e^{-rt} &\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi dt}} e^{-\frac{(\xi-ct-y)^2}{4dt}} S_2(y)\mathrm{d}y \\
&+ \int_0^t \int_{\mathbb{R}} \frac{e^{-rs}}{\sqrt{4\pi ds}} e^{-\frac{(\xi-ct-y)^2}{4ds}} m_2(y)\mathrm{d}y\mathrm{d}s, \\
v_0(t,\ \xi) = e^{-rt} &\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi D_0 t}} e^{-\frac{(\xi-ct-y)^2}{4D_0 t}} E(y)\mathrm{d}y \\
&+ \int_0^t \int_{\mathbb{R}} \frac{e^{-rs}}{\sqrt{4\pi D_0 s}} e^{-\frac{(\xi-ct-y)^2}{4D_0 s}} (m_1(y) - m_2(y) - n(y) - n(y))\mathrm{d}y\mathrm{d}s, \\
v_i(t,\ \xi) =& e^{-rt} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi D_i t}} e^{-\frac{(\xi-ct-y)^2}{4D_i t}} I_i(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} \frac{e^{-rs}}{\sqrt{4\pi D_i s}} e^{-\frac{(\xi-ct-y)^2}{4D_i s}} n_i(y)\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

for $\forall t > 0,\ \forall \xi \in \mathbb{R}$. Applying the comparison principle in [19] concludes

$$S_i(\xi) \le u_i(t,\ \xi),\ E(\xi) \le v_0(t,\ \xi),\ I_i(\xi) \le v_i(t,\ \xi),\ \forall t > 0,\ \forall \xi \in \mathbb{R}.$$

Let $t \to +\infty$, then one yields

$$S_1(\xi) \le \frac{\Lambda}{r} - f_1(\xi),\ S_2(\xi) \le f_2(\xi),\ E(\xi) \le f_0(\xi) - g_0(\xi),\ I_i(\xi) \le g_i(\xi),$$

for $\forall t > 0,\ \forall \xi \in \mathbb{R}$, where

$$\begin{aligned}
f_i(\xi) &= \int_0^{+\infty} e^{-\frac{-rt}{\sqrt{4\pi dt}}} \int_{-\infty}^{+\infty} m_i(\xi - y - ct) e^{-\frac{y^2}{4dt}} \mathrm{d}y\mathrm{d}t, \\
f_0(\xi) &= \int_0^{+\infty} e^{-\frac{-rt}{\sqrt{4\pi D_0 t}}} \int_{-\infty}^{+\infty} [m_1(\xi - y - ct) - m_2(\xi - y - ct)] e^{-\frac{y^2}{4D_0 t}} \mathrm{d}y\mathrm{d}t, \\
g_0(\xi) &= \int_0^{+\infty} e^{-\frac{-rt}{\sqrt{4\pi D_0 t}}} \int_{-\infty}^{+\infty} [n_1(\xi - y - ct) + n_2(\xi - y - ct)] e^{-\frac{y^2}{4D_0 t}} \mathrm{d}y\mathrm{d}t, \\
g_i(\xi) &= \int_0^{+\infty} e^{-\frac{-rt}{\sqrt{4\pi D_i t}}} \int_{-\infty}^{+\infty} n_i(\xi - y - ct) e^{-\frac{y^2}{4D_i t}} \mathrm{d}y\mathrm{d}t.
\end{aligned}$$

Owing to $d \ge D_0 \ge D_i$, we yield

$$\sqrt{D_0} g_0(\xi) \ge \sqrt{D_1} g_1(\xi) + \sqrt{D_2} g_2(\xi)$$

and

$$\sqrt{d}[f_1(\xi) - f_2(\xi)] \ge \sqrt{D_0} f_0(\xi)$$

for any $\xi \in \mathbb{R}$. We also have $\sqrt{D_i} I_i(\xi) \le \sqrt{D_i} g_i(\xi)$. Consequently, it is shown that

$$\begin{aligned}
&E(\xi) + \sqrt{\frac{D_1}{D_0}} I_1(\xi) + \sqrt{\frac{D_2}{D_0}} I_2(\xi) \le f_0(\xi) \le \sqrt{\frac{d}{D_0}} f_1(\xi) - \sqrt{\frac{d}{D_0}} f_2(\xi) \\
&\le \sqrt{\frac{d}{D_0}} [\frac{\lambda}{r} - S_1(\xi)] - \sqrt{\frac{d}{D_0}} S_2(\xi) \\
&\le \sqrt{\frac{d}{D_0}} \frac{\Lambda}{r},
\end{aligned}$$

which implies

$$E(\xi)+I_1(\xi)+I_2(\xi)\le\sqrt{\frac{D_0}{D_{min}}}E(\xi)+\sqrt{\frac{D_1}{D_{min}}}I_1(\xi)+\sqrt{\frac{D_2}{D_{min}}}I_2(\xi)$$
$$\le\sqrt{\frac{d}{D_{min}}}\frac{\Lambda}{r},$$

for $D_{min}=\min\{D_0,\ D_1,\ D_2\}$ and any $\xi\in\mathbb{R}$.

On the other hand, one has

$$dS_1^{''}(\xi)-cS_1^{'}(\xi)+\Lambda-\beta S_1(\xi)\sqrt{\frac{d}{D_{min}}}\frac{\Lambda}{r}-(\mu+\rho)S_1(\xi).$$

for any $\xi\in\mathbb{R}$. Then the maximum principle implies that

$$\frac{\Lambda}{\mu+\rho+\beta\sqrt{\frac{d}{D_{min}}}\frac{\Lambda}{r}}\le S_1(\xi),\ \forall\xi\in\mathbb{R}.$$

Via the similar argument on $S_1(\xi)$, we can indicate

$$\frac{\rho\Lambda}{\left(\mu+\rho+\beta\sqrt{\frac{d}{D_{min}}}\frac{\Lambda}{r}\right)\left(\mu+\beta\sigma\sqrt{\frac{d}{D_{min}}}\frac{\Lambda}{r}\right)}\le S_2(\xi),\ \forall\xi\in\mathbb{R}.$$

This lemma is completely proved. □

Since the system consisted by E-th and I_i-th equations in system (5) is cooperation and irreducible, the Theorem 2.2 in [2] and Lemma 2.10 can claim that there exists a positive constant M_1 such that

$$\max\left\{\max_{[\xi-1,\xi+1]}E,\ \max_{[\xi-1,\xi+1]}I_1,\ \max_{[\xi-1,\xi+1]}I_2\right\}$$
$$\le M_1\min\left\{\min_{[\xi-1,\xi+1]}E,\ \min_{[\xi-1,\xi+1]}I_1,\ \min_{[\xi-1,\xi+1]}I_2\right\}. \tag{28}$$

Furthermore, there exists a constant $\hat{M}>0$ such that

$$\left|\frac{E^{'}(\xi)}{E(\xi)}\right|+\left|\frac{I_1^{'}(\xi)}{I_1(\xi)}\right|+\left|\frac{I_2^{'}(\xi)}{I_2(\xi)}\right|\le\hat{M}, \tag{29}$$

for $\forall\xi\in\mathbb{R}$. Actually, L^p interior estimate shows there exists a positive constant $M_2>0$ satisfying

$$\max\left\{\parallel E\parallel_{W^{2,p}(\xi-\frac{1}{2},\xi+\frac{1}{2})},\ \parallel I_1\parallel_{W^{2,p}(\xi-\frac{1}{2},\xi+\frac{1}{2})},\ \parallel I_2\parallel_{W^{2,p}(\xi-\frac{1}{2},\xi+\frac{1}{2})}\right\}$$
$$\le M_2\left\{\parallel E\parallel_{L^p(\xi-1,\xi+1)}+\parallel I_1\parallel_{L^p(\xi-1,\xi+1)}+\parallel I_2\parallel_{L^p(\xi-1,\xi+1)}\right\}$$
$$\le 6M_2\max\left\{\max_{[\xi-1,\xi+1]}E,\ \max_{[\xi-1,\xi+1]}I_1,\ \max_{[\xi-1,\xi+1]}I_2\right\},$$

for any $\xi\in\mathbb{R}$ and $p>1$. The above inequality, combining the embedding theorem, can conclude that there exists a positive costant M_3 satisfying

$$\max\left\{\parallel E^{'}\parallel_{C[x-\frac{1}{2},x+\frac{1}{2}]},\ \parallel I_1^{'}\parallel_{C[x-\frac{1}{2},x+\frac{1}{2}]},\ \parallel I_2^{'}\parallel_{C[x-\frac{1}{2},x+\frac{1}{2}]}\right\}$$
$$\le M_3\max\left\{\max_{[x-1,x+1]}E,\ \max_{[x-1,x+1]}I_1,\ \max_{[x-1,x+1]}I_2\right\}.\quad\forall\xi\in\mathbb{R}. \tag{30}$$

Set $\hat{M}=M_1M_3$, then the inequality (29) is established via the estimate (30).

In order to take advantage of a suitable Lyapunov functional to research the asymptotic behavior of $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ as $\xi \to +\infty$, we define

$$\hat{E} = \left\{ \begin{array}{l} S_1(\cdot),\ S_2(\cdot),\ E(\cdot),\ I_1(\cdot),\ I_2(\cdot) \in C^1(\mathbb{R},\ (0,\ +\infty)) \times \cdots \\ \times\, C^1(\mathbb{R},\ (0,\ +\infty)), \\ S_1(\cdot) > 0,\ S_2(\cdot) > 0,\ E(\cdot) > 0,\ I_1(\cdot) > 0,\ I_2(\cdot) > 0, \\ \exists \hat{M} > 0,\ \left|\dfrac{E'(\xi)}{E(\xi)}\right| + \left|\dfrac{I_1'(\xi)}{I_1(\xi)}\right| + \left|\dfrac{I_2'(\xi)}{I_2(\xi)}\right| \le \hat{M}. \end{array} \right\},$$

Let $g(x) = x - 1 - \ln x$ and define a suitable Lyapunov functional

$$\begin{aligned} V(S_1,\ S_2,\ E,\ I_1,\ I_2)(\xi) =& S_1^* \left[dS_1' \left(\frac{1}{S_1(\xi)} - \frac{1}{S_1^*} \right) + cg\left(\frac{S_1(\xi)}{S_1^*} \right) \right] \\ &+ S_2^* \left[dS_2' \left(\frac{1}{S_2(\xi)} - \frac{1}{S_2^*} \right) + cg\left(\frac{S_2(\xi)}{S_2^*} \right) \right] \\ &+ E^* \left[D_0 E' \left(\frac{1}{E(\xi)} - \frac{1}{E^*} \right) + cg\left(\frac{E(\xi)}{E^*} \right) \right] \\ &+ C_1 I_1^* \left[D_1 I_1' \left(\frac{1}{I_1(\xi)} - \frac{1}{I_1^*} \right) + cg\left(\frac{I_1(\xi)}{I_1^*} \right) \right] \\ &+ C_2 I_2^* \left[D_2 I_2' \left(\frac{1}{I_2(\xi)} - \frac{1}{I_2^*} \right) + cg\left(\frac{I_2(\xi)}{I_2^*} \right) \right], \end{aligned} \tag{31}$$

where

$$C_1 = \frac{\beta\theta I_1^*(S_1^* + \sigma S_2^*)}{q_1 \gamma E^*}, \quad C_2 = \frac{\beta I_2^*(S_1^* + \sigma S_2^*)}{q_2 \gamma E^*},$$

for each $(S_1(\cdot),\ S_2(\cdot),\ E(\cdot),\ I_1(\cdot),\ I_2(\cdot)) \in \hat{E}$. Then we claim the below lemma.

Lemma 2.11. *Let (A) be satisfied and $(S_1(\cdot),\ S_2(\cdot),\ E(\cdot),\ I_1(\cdot),\ I_2(\cdot))$ be a positive solution of system (5) satisfying*

$$\frac{1}{\mathcal{N}} \le S_i(\xi) \le S_i^*, \tag{32}$$

$$E(\xi) \le \mathcal{N} E^*, \tag{33}$$

$$I_i(\xi) \le \mathcal{N} I_i^*, \tag{34}$$

and

$$\left|\frac{E'(\xi)}{E(\xi)}\right| + \left|\frac{I_1'(\xi)}{I_1(\xi)}\right| + \left|\frac{I_2'(\xi)}{I_2(\xi)}\right| \le \mathcal{N} \tag{35}$$

for any $\xi \in \mathbb{R}$ and $i = 1,\ 2$, where $\mathcal{N}$ is a positive constant. Then there exists a constant $m > 0$, depending on $\mathcal{N}$, such that

$$-m \le V(\xi) < +\infty, \quad \forall \xi \in \mathbb{R}, \tag{36}$$

where the map $V(\xi)$ is defined as formula (31). Moreover, the map $V(\xi)$ is not increasing. In particular, $S_i(\xi) = S_i^*,\ E(\xi) = E^*,\ I_i(\xi) = I_i^*$ as the map $V(\xi)$ is a constant.

Proof. The previous description has shown S_1 and S_2 are bounded in $C^2(\mathbb{R})$. Via inequalities (32)-(35), we can conclude that for any $\xi \in \mathbb{R}$

$$\begin{aligned}
&\Bigg| d\sum_{i=1}^{2} S_i^* S_i^{'}(\xi)\left(\frac{1}{S_i(\xi)}-\frac{1}{S_i^*}\right)+D_0E^*E^{'}(\xi)\left(\frac{1}{E(\xi)}-\frac{1}{E^*}\right)\\
&\qquad+\sum_{i=1}^{2} C_iI_i^*D_iI_i^{'}(xi)\left(\frac{1}{I_i(\xi)}-\frac{1}{I_i^*}\right)\Bigg|\\
&\le d\sum_{i=1}^{2} S_i^* \parallel S_i\parallel_\infty (\mathcal{N}+\frac{1}{S_i^*})+\hat{D}\hat{C}\mathcal{N}+\hat{D}\hat{C}\left(\left|\frac{E^{'}(\xi)}{E^*}\right|+\left|\frac{I_1^{'}(\xi)}{I_1^*}\right|+\left|\frac{I_2^{'}(\xi)}{I_2^*}\right|\right)\\
&\le d\sum_{i=1}^{2} S_i^* \parallel S_i\parallel_\infty (\mathcal{N}+\frac{1}{S_i^*})+\hat{D}\hat{C}\mathcal{N}+\hat{D}\hat{C}\mathcal{N}^2, \qquad (37)
\end{aligned}$$

where $\hat{D}=\max\{D_0E^*,\ D_1I_1^*\ D_2I_2^*\}$, and $\hat{C}=\max\{1,\ C_1,\ C_2\}$. Let

$$\Phi(\xi)=cg\left(\frac{S_1(\xi)}{S_1^*}\right)+cg\left(\frac{S_2(\xi)}{S_2^*}\right)+cg\left(\frac{E(\xi)}{E^*}\right)+cC_1g\left(\frac{I_1(\xi)}{I_1^*}\right)+cC_2g\left(\frac{I_2(\xi)}{I_2^*}\right). \qquad (38)$$

According to both the definition of $g(\cdot)$ and (32)-(34), it is shown that $0 \le \Phi(\xi) < +\infty, \forall \xi \in \mathbb{R}$. Then, by combining the inequality (37), the inequality (36) holds.

Since a direct calculation with letting $x_1=\frac{S_1(\xi)}{S_1^*},\ x_2=\frac{S_2(\xi)}{S_2^*},\ x_3=\frac{E(\xi)}{E^*},\ x_4=\frac{I_1(\xi)}{I_1^*},\ x_5=\frac{I_2(\xi)}{I_2^*}$ leads to

$$\begin{aligned}
\frac{\mathrm{d}V(\xi)}{\mathrm{d}\xi}=&(\Lambda-\beta\theta S_1^*I_1^*-\beta S_1^*I_2^*-\rho S_1^*)\left(2-x_1-\frac{1}{x_1}\right)\\
&+(\rho S_1^*-\beta\sigma\theta S_2^*I_1^*-\beta\sigma S_2^*I_2^*)\left(3-\frac{1}{x_1}-x_2-\frac{x_1}{x_2}\right)\\
&+\beta\theta S_1^*I_1^*\left(3-\frac{1}{x_1}-\frac{x_1x_4}{x_3}-\frac{x_3}{x_4}\right)+\beta S_1^*I_2^*\left(3-\frac{1}{x_1}-\frac{x_1x_5}{x_3}-\frac{x_3}{x_5}\right)\\
&+\beta\sigma\theta S_2^*I_1^*\left(4-\frac{1}{x_1}-\frac{x_1}{x_2}-\frac{x_2x_4}{x_3}-\frac{x_3}{x_4}\right)\\
&+\beta\sigma S_2^*I_2^*\left(4-\frac{1}{x_1}-\frac{x_1}{x_2}-\frac{x_2x_5}{x_3}-\frac{x_3}{x_5}\right).
\end{aligned}$$

Therefore, via the mean inequality, we conclude that

$$\frac{\mathrm{d}V(\xi)}{\mathrm{d}\xi}\le 0, \quad \forall \xi \in \mathbb{R},$$

which implies that the map $V(\xi)$ is non-increasing. Especially, when

$$\frac{\mathrm{d}V(\xi)}{\mathrm{d}\xi}= 0, \quad \forall \xi \in \mathbb{R},$$

the map $V(\xi)$ is a constant, which indicates that

$$S_1(\xi)\equiv S_1^*,\ S_2(\xi)\equiv S_2^*,\ E(\xi)\equiv E^*,\ I_1(\xi)\equiv I_1^*,\ I_2(\xi)\equiv I_2^*, \quad \forall \xi \in \mathbb{R}.$$

This completes the proof. □

Now, we gain the first theorem for existence of traveling wave solution for system (2) as below:

Theorem 2.12. *If (A) and $R_0 > 1$ hold, system (2) admits a non-trivial and non-negative traveling wave solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ satisfying the boundary condition (6) for each $c > c^*$, where c^* is the minimal spread speed and R_0 is the basic reproduction number for system (3).*

Proof. We still set $i = 1,\ 2$ in this proof. It follows from the previous argument that there exists a vector function

$$(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$$

satisfying system (5), and

$$\lim_{\xi\to-\infty} S_1(\xi) = S_1^0,\ \lim_{\xi\to-\infty} S_2(\xi) = S_2^0,\ \lim_{\xi\to-\infty} E(\xi) = 0,$$
$$\lim_{\xi\to-\infty} I_1(\xi) = 0,\ \lim_{\xi\to-\infty} I_2(\xi) = 0,$$

for any $\xi \in \mathbb{R}$. Therefore we only need to show that

$$\lim_{\xi\to+\infty} S_1(\xi) = S_1^*,\ \lim_{\xi\to+\infty} S_2(\xi) = S_2^*,\ \lim_{\xi\to+\infty} E(\xi) = E^*,$$
$$\lim_{\xi\to+\infty} I_1(\xi) = I_1^*,\ \lim_{\xi\to+\infty} I_2(\xi) = I_2^*,$$

Take an arbitrary increasing sequence $\{\xi_m\}$ with $\xi_m > 0$ and $m \geq 0$ such that $\xi_m \to +\infty$ as $m \to +\infty$. Define

$$S_{i,m}(\xi) = S_i(\xi + \xi_m),\ E_m(\xi) = E(\xi + \xi_m),\ I_{i,m}(\xi) = I_i(\xi + \xi_m).$$

Via the elliptic estimate, it may assume that the sequence $(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m},\ I_{2,m})$ converges towards $(S_{1,\infty},\ S_{2,\infty},\ E_\infty,\ I_{1,\infty},\ I_{2,\infty})$ in $C^1_{loc}(\mathbb{R})\times\cdots\times C^1_{loc}(\mathbb{R})$. As a result, $(S_{1,\infty},\ S_{2,\infty},\ E_\infty,\ I_{1,\infty}, I_{2,\infty})$ is also a solution of system (5). In addition, the map $V(\xi)$ defined as the formula (31) is not increasing, then we yield

$$V(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m},\ I_{2,m})(\xi) \leq V(S_1,\ S_2,\ E_m,\ I_1,\ I_2)(\xi)$$

for any $\xi \in \mathbb{R}$. Since the map $V(\xi)$ is bounded via Lemma 2.11, there exists a constant $\hat{G} \in \mathbb{R}$ such that

$$\lim_{m\to+\infty} V(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m},\ I_{2,m})(\xi) = \hat{G}, \quad \forall \xi \in \mathbb{R},$$

which implies that

$$V(S_{1,\infty},\ S_{2,\infty},\ E_\infty,\ I_{1,\infty},\ I_{2,\infty})(\xi) \equiv \hat{G}$$

in $C^1_{loc}(\mathbb{R})$. Combining Lemma 2.11, we can claim that

$$S_{i,\infty} = S_i^*,\ E_\infty = E^*,\ I_{i,\infty} = I_i^*,$$
$$S'_{i,\infty} = 0,\ E'_\infty = 0,\ I'_{i,\infty} = 0.$$

Via the arbitrariness of the sequence $\{\xi_m\}$, we finally indicate

$$\lim_{\xi\to+\infty} S_i(\xi) = S_i^*,\ \lim_{\xi\to+\infty} E(\xi) = E^*,\ \lim_{\xi\to+\infty} I_i(\xi) = I_i^*.$$

The proof is completed. □

Furthermore, the second theorem of existence for traveling wave solution for system (2) is stated as below:

Theorem 2.13. *Assume that (A) is satisfied and $R_0 > 1$. Then for $c = c^*$, system (2) also admits a non-trivial and non-negative traveling wave solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ satisfying the boundary condition (6), where c^* is the minimal spread speed and R_0 is the basic reproduction number for system (3).*

Proof. **Step 1.** Take a decreasing sequence $\{c_m\} \in (c^*,\ c^*+1)$ with $\lim_{m\to+\infty} c_m = c^*$. It follows from Theorem 2.12 that there exists a solution $(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m},\ I_{2,m})$ of system (5) for each c_m satisfying conditions (6), (25), (26), (28) and (29). Since $(S_{1,m}(\cdot+a),\ S_{2,m}(\cdot+a),\ E_m(\cdot+a),\ I_{1,m}(\cdot+a),\ I_{2,m}(\cdot+a))$ is also the solution of system (5) satisfying equations (6) for any $a \in \mathbb{R}$, we can let

$$S_{1,m}(0) = \frac{S_1^0 + S_1^*}{2}.$$

The interior elliptic estimates, Arzela-Ascoli theorem and a diagonalization argument can indicate a subsequence of $\{(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m}.\ I_{2,m})\}$ defined again by $\{(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m}.\ I_{2,m})\}$ satisfies

$$(S_{1,m},\ S_{2,m},\ E_m,\ I_{1,m}.\ I_{2,m}) \to (S_1,\ S_2,\ E,\ I_1.\ I_2)$$

as $m \to +\infty$ in $C^2_{loc}(\mathbb{R},\ \mathbb{R}^5)$. And it is well known that $(S_1,\ S_2,\ E,\ I_1.\ I_2)$ satisfies system (5) and

$$S_1(0) = \frac{S_1^0 + S_1^*}{2}, \tag{39}$$

which implies that

$$(S_1,\ S_2,\ E,\ I_1.\ I_2) \not\equiv (S_1^0,\ S_2^0,\ 0,\ 0,\ 0).$$

Moreover, we yield $S_i > 0,\ E > 0,\ I_i > 0$. Above all, it is concluded that $(S_1,\ S_2,\ E,\ I_1.\ I_2)$ satisfies estimates (25), (26), (28) and (29). Via Lemma 2.11, one gains

$$S_i(+\infty) = S_i^*,\ E(+\infty) = E^*,\ I_i(+\infty) = I_i^*,\ i = 1,\ 2.$$

Next, we need to prove what the solution converges to as $\xi \to -\infty$. Since Lemma 2.11 implies $V(\xi)$ defined as the formula (31) is non-increasing, then we can obtain either

$$\lim_{\xi\to-\infty} V(\xi) = L < +\infty, \tag{40}$$

or

$$\lim_{\xi\to-\infty} V(\xi) = +\infty. \tag{41}$$

If the formula (40) holds, via the similar way in Lemma 2.11, it can claim that

$$S_i(-\infty) = S_i^*, E(-\infty) = E^*,\ I_i(-\infty) = I^*,\ i = 1,\ 2,$$

and then $L = 0$, which implies that $V \equiv 0$ for any $\xi \in \mathbb{R}$. Consequently, applying Lemma 2.11 concludes that

$$S_i(\xi) \equiv S_i^*,\ E(\xi) \equiv E^*,\ I_i(\xi) \equiv I_i^*,\ \forall \xi \in \mathbb{R},\ i = 1,\ 2,$$

which contradicts the equation (39). Thus, the equation (41) must be only workable. Due to the inequality (37), it is shown that

$$\lim_{\xi\to-\infty} \Phi(\xi) = +\infty, \tag{42}$$

where $\Phi(\xi)$ is defined as the formula (38) in the proof of Lemma 2.11. Now, we firstly show that

$$\lim_{\xi\to-\infty} \inf E(\xi) = 0. \tag{43}$$

On the contrary, if $E(\xi) > \delta$ in $\xi \in \mathbb{R}$ for some $\delta > 0$. According to $E(+\infty) = E^*$, then it follows from the estimate (28) that there exists a constant $\hat{\delta} > 0$ such that

$$I_i(\xi) > \hat{\delta}, \quad \forall \xi \in \mathbb{R},$$

which indicates that

$$\lim_{\xi\to-\infty}\Phi(\xi) < +\infty,$$

and contradicts the equation (42). Therefore, the equation (43) holds. Secondly, we prove that

$$\lim_{\xi\to-\infty} E(\xi) = 0.$$

If

$$\lim_{\xi\to-\infty}\sup E(\xi) = \delta > 0,$$

there exists a sequence $\{\xi_j\}$ satisfying

$$\lim_{j\to\infty} E(\xi_j) = \delta,$$

Via the estimate (28), we can have

$$I_i(\xi_j) \le \frac{1}{2M_1}\delta, \quad i = 1,\ 2.$$

Then, it is implied that

$$\lim_{j\to\infty}\sup V(\xi_j) < +\infty,$$

which contradicts the equation (41). Therefore,

$$\lim_{\xi\to-\infty} E(\xi) = 0.$$

In the similar way, we can also claim that

$$\lim_{\xi\to-\infty} I_i(\xi) = 0.$$

Finally, set $i = 1,\ 2$ and we show that

$$\lim_{\xi\to-\infty} S_i(\xi) = S_i^0$$

In order to hit the target, we firstly prove the existence of $\lim_{\xi\to-\infty} S_i(\xi)$. On the contrary, we assume that one of $\lim_{\xi\to-\infty} S_i(\xi)$ does not exist. Since $S_i(\xi)$ satisfies (26), we obtain

$$\lim_{\xi\to-\infty}\inf\{S_1(\xi)+S_2(\xi)\} < \lim_{\xi\to-\infty}\sup\{S_1(\xi)+S_2(\xi)\} \le \frac{\Lambda}{\mu}.$$

Take a sequence $\{\xi_n\}$ satisfies $\xi_n \to -\infty$ as $n \to +\infty$ and

$$\begin{aligned}&\lim_{n\to+\infty}[S_1(\xi_n)+S_2(\xi_n)] = \lim_{\xi\to-\infty}\inf\{S_1(\xi)+S_2(\xi)\} < \frac{\Lambda}{\mu},\\ &\frac{\mathrm{d}}{\mathrm{d}\xi_n}[S_1(\xi_n)+S_2(\xi_n)] = 0,\ \frac{\mathrm{d}^2}{\mathrm{d}\xi_n^2}[S_1(\xi_n)+S_2(\xi_n)] \le 0.\end{aligned} \tag{44}$$

Since $\lim_{n\to+\infty} I_i(\xi_n) = 0$, adding the S_1-th equation and S_2-th equation implies that

$$\lim_{n\to+\infty}[S_1(\xi_n)+S_2(\xi_n)] \ge \frac{\Lambda}{\mu},$$

which leads to a contradiction to inequalities (44). Thus, $\lim_{\xi\to-\infty} S_i(\xi)$ exists, respectively.

Step 2. This step is to prove $\lim_{\xi\to-\infty} S_i(\xi) = S_i^0$ Let $\lim_{\xi\to-\infty} S_i(\xi) = k_i$. Utilizing the equation

$$-dS_1^{''}(\xi) + cS_1^{'}(\xi) + (\mu+\rho)S_1(\xi) = \Lambda - \beta S_1(\xi)[\theta I_1(\xi) + I_2(\xi)], \quad \xi \in \mathbb{R},$$

we can obtain that

$$\begin{aligned} S_1(\xi) =& \frac{1}{z}\int_{-\infty}^{\xi} e^{A_1(\xi-x)}[\lambda-\beta S_1(x)(\theta I_1(x)+I_2(x))]\mathrm{d}x \\ &+\frac{1}{z}\int_{\xi}^{+\infty} e^{A_2(\xi-x)}[\lambda-\beta S_1(x)(\theta I_1(x)+I_2(x))]\mathrm{d}x \\ =&\frac{1}{z}\int_{-\infty}^{0} e^{A_1 x}[\lambda-\beta S_1(\xi-x)(\theta I_1(\xi-x)+I_2(\xi-x))]\mathrm{d}x \\ &+\frac{1}{z}\int_{0}^{+\infty} e^{A_2 x}[\lambda-\beta S_1(\xi-x)(\theta I_1(\xi-x)+I_2(\xi-x))]\mathrm{d}x, \end{aligned}$$

where

$$z=d(A_1-A_2),\ A_1=\frac{c-\sqrt{c^2+4d(\mu+\rho)}}{2},\ A_2=\frac{c-\sqrt{c^2+4d(\mu+\rho)}}{2}.$$

As $\xi\to-\infty$, it follows from the Lebesgue dominated convergence theorem that

$$k_1=\frac{\Lambda}{z}\left(\int_{-\infty}^{0} s^{A_1 x}\mathrm{d}x+\int_{0}^{+\infty} s^{A_2 x}\mathrm{d}x\right),$$

which leads to

$$k_1=\frac{\Lambda}{\mu+\rho}=S_1^0.$$

Via the similar argument on $S_2(\xi)$, we can get

$$k_2=\frac{\rho k_1}{\mu}=S_2^0.$$

This completes the proof. □

3. **Non-existence of traveling wave solution.**

The previous section has stated the existence of traveling wave solution connecting disease-free equilibrium and endemic equilibrium when $R_0>1$ and $c\geq c^*$. In this section, we aim to prove the non-existence of this solution in three cases: (1) $R_0<1$, (2) $R_0=1$, (3) $R_0>1$ and $c\in(0,\ c^*)$.

3.1. **Case(1):** $R_0<1$.

Theorem 3.1. *If $R_0<1$, there does not exist a non-negative and non-trivial traveling wave solution of system (5) satisfying the boundary condition (6), where R_0 is the basic reproduction number for system (3).*

Proof. On the contrary, assume that there exists a solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),$ $I_1(\xi),\ I_2(\xi))$ satisfying system (5) and the boundary condition (6). Let $E_{sup}=\sup_{\xi\in\mathbb{R}}E(\xi)$ and take $i=1,\ 2$. We can yield from the I_i-th equation that

$$D_i I_i^{''}(\xi)-cI_i^{'}(\xi)+q_i\gamma E_{\text{sup}}-(\mu+\alpha_i)I_i(\xi)\geq 0,\quad \forall\xi\in\mathbb{R}.$$

It follows from the comparison principle that

$$I_i(\xi)\leq\frac{q_i\gamma E_{\text{sup}}}{\mu+\alpha_i},\quad \forall\xi\in\mathbb{R}.$$

Next we research the following equation

$$D_0\bar{E}^{''}(\xi)-c\bar{E}^{'}(\xi)+\bar{\Lambda}\beta\left(\theta\frac{q_1\gamma E_{\text{sup}}}{\mu+\alpha_1}+\frac{q_2\gamma E_{\text{sup}}}{\mu+\alpha_2}\right)-(\mu+\gamma)\bar{E}(\xi)=0,$$

where $\bar{\Lambda} = S_1^0 + \sigma S_2^0$. Via the comparison principle, we can show that

$$E(\xi) \le \bar{\Lambda}\left(\theta\frac{q_1\gamma}{\mu+\alpha_1} + \frac{q_2\gamma}{\mu+\alpha_2}\right)E_{\sup},$$

which indicates

$$\bar{\Lambda}\left(\theta\frac{q_1\gamma}{\mu+\alpha_1} + \frac{q_2\gamma}{\mu+\alpha_2}\right) \ge 1.$$

and contradicts the fact $R_0 < 1$. Above all, there exists no non-negative and non-trivial traveling wave solution as $R_0 < 1$. □

3.2. **Case(2):** $R_0 = 1$**.**

Theorem 3.2. *As* $R_0 = 1$, *there does not exist a non-negative and non-trivial traveling wave solution of system (5) satisfying the boundary condition (6), where* R_0 *is the basic reproduction number for system (3).*

Proof. Set a new sequence $\{\xi_m\} \subset \mathbb{R}$ such that

$$\lim_{m\to+\infty} E(\xi_m) = \hat{B} = \sup_{\xi\in\mathbb{R}} E(\xi).$$

Now, we need to show $\hat{B} = 0$ by contradiction. Assume $\hat{B} > 0$, and we consider the function sequence $(S_{1,m}(\xi),\ S_{2,m}(\xi),\ E_m(\xi),\ I_{1,m}(\xi),\ I_{2,m}(\xi)) = (S_1(\xi + \xi_m),\ S_2(\xi+\xi_m),\ E_m(\xi+\xi_m),\ I_1(\xi+\xi_m),\ I_2(\xi+\xi_m))$. We take a function subsequence denoted again by $(S_{1,m}(\xi),\ S_{2,m}(\xi),\ E_m(\xi),\ I_{1,m}(\xi),$
$I_{2,m}(\xi))$, and utilizing elliptic estimates shows that

$$(S_{1,m}(\xi),\ S_{2,m}(\xi),\ E_m(\xi),\ I_{1,m}(\xi),\ I_{2,m}(\xi)) \to (\tilde{S}_1(\xi),\ \tilde{S}_2(\xi),\ \tilde{E}(\xi),\ \tilde{I}_1(\xi),\ \tilde{I}_2(\xi)),$$

as $m \to +\infty$ in $C^2_{loc}(\mathbb{E})$ with $(\tilde{S}_1(\xi),\ \tilde{S}_2(\xi),\ \tilde{E}(\xi),\ \tilde{I}_1(\xi),\ \tilde{I}_2(\xi))$ satisfying

$$\begin{cases} d\tilde{S}_1^{''}(\xi) - c\tilde{S}_1^{'}(\xi) + \Lambda - \beta\tilde{S}_1(\xi)(\theta\tilde{I}_1(\xi) + \tilde{I}_2(\xi)) - (\rho+\mu)\tilde{S}_1(\xi) = 0,\\ d\tilde{S}_2^{''}(\xi) - c\tilde{S}_2^{'}(\xi) + \rho\tilde{S}_1(\xi) - \beta\sigma\tilde{S}_2(\xi)(\theta\tilde{I}_1(\xi) + \tilde{I}_2(\xi)) - \mu\tilde{S}_2(\xi) = 0,\\ D_0\tilde{E}^{''}(\xi) - c\tilde{E}^{'}(\xi) + \beta\left[\theta\tilde{I}_1(\xi) + \tilde{I}_2(\xi)\right]\left[\tilde{S}_1(\xi) + \sigma\tilde{S}_2(\xi)\right] - (\mu+\gamma)\tilde{E}(\xi) = 0,\\ D_1\tilde{I}_1^{''}(\xi) - c\tilde{I}_1^{'}(\xi) + q_1\gamma\tilde{E}(\xi) - (\mu+\alpha_1)\tilde{I}_1(\xi) = 0,\\ D_2\tilde{I}_2^{''}(\xi) - c\tilde{I}_2^{'}(\xi) + q_2\gamma\tilde{E}(\xi) - (\mu+\alpha_2)\tilde{I}_2(\xi) = 0,\\ \tilde{E}(0) = \hat{B},\ \tilde{E}(\xi) \le \hat{B},\\ 0 \le \tilde{S}_i(\xi) \le S_i^0,\ i = 1,\ 2,\ \forall\xi\in\mathbb{R}. \end{cases}$$

The maximum principle implies that

$$\tilde{I}_i(\xi) \le \frac{q_i\gamma\hat{B}}{\mu+\alpha_i}.$$

Moreover, we can yields

$$0 \le D_0\tilde{E}^{''}(0) + \beta[\tilde{S}_1(0) + \sigma\tilde{S}_1(0)]\left(\theta\frac{q_1\gamma}{\mu+\alpha_1} + \frac{q_2\gamma}{\mu+\alpha_2}\right)\tilde{B} - (\mu+\gamma)\tilde{B}.$$

Owing to $\tilde{E}^{''}(0) \le 0$, $\beta[\tilde{S}_1(0) + \sigma\tilde{S}_1(0)]\left(\theta\frac{q_1\gamma}{\mu+\alpha_1} + \frac{q_2\gamma}{\mu+\alpha_2}\right)\hat{B} - (\mu+\gamma)\hat{B} \ge 0$ and $\tilde{S}_i \le S_i^0$, we must claim $\beta[\tilde{S}_1(0)+\sigma\tilde{S}_1(0)]\left(\theta\frac{q_1\gamma}{\mu+\alpha_1} + \frac{q_2\gamma}{\mu+\alpha_2}\right)\hat{B} - (\mu+\gamma)\hat{B} = 0$, which implies that $\tilde{S}_i(0) = S_i^0$. The minimal principle with combining the $\tilde{S}_i$-th equation can lead to $\tilde{S}_i(\xi) \equiv S_i^0,\ \tilde{E}(\xi) \equiv 0,\ \tilde{I}_i(\xi) \equiv 0,\ \forall \xi \in \mathbb{R}$. Consequently, we have $\hat{B} = 0$ that contradicts $\hat{B} > 0$. This case is completely proved. □

3.3. **Case(3): $R_0 > 1$ and $c \in (0,\ c^*)$.** In this case, we attempt to utilize the two-sides Laplace transform to gain the non-existence of traveling wave solution for system (2). Therefore, we firstly need to imply the exponential boundedness for traveling wave solution via next two lemmas.

Lemma 3.3. *Assume that $R_0 > 1$, where R_0 is the basic reproduction number for system (3). For any $c > 0$, if system (5) admits a non-trivial and non-negative solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi), I_2(\xi))$ satisfying the boundary condition (6), then there exist two constants $\mathcal{J} > 0$ and $G > 0$ large enough such that*

$$\int_{-\infty}^{\xi} E(x)\mathrm{d}x < \mathcal{J},\ \int_{-\infty}^{\xi} I_1(x)\mathrm{d}x < \mathcal{J},\ \int_{-\infty}^{\xi} I_2(x)\mathrm{d}x < \mathcal{J},$$

for $\xi < -2G$.

Proof. Set $i = 1,\ 2$. Let $c > 0$ and $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ be traveling wave solution of system (5) satisfying the boundary condition (6). Thus, we can claim there exist $G > 0$ large enough and small $\varepsilon \in (0,\ 1)$ such that

$$S_i(\xi) > S_i^0(1-\varepsilon), \quad \xi \in (-\infty,\ -2G).$$

For any $\xi < -2G$, we can yield

$$\begin{aligned}\beta(S_1(\xi) + \sigma S_2(\xi))(\theta I_1(\xi) + I_2(\xi)) - (\mu+\gamma)E(\xi) \geq & \beta(S_1^0 + \sigma S_2^0)(1-\varepsilon)[\theta I_1(\xi) \\ & + I_2(\xi)] - (\mu+\gamma)E(\xi) \end{aligned} \tag{45}$$

For $y < \xi < -2G$, let

$$\bar{J}_0(\xi,\ y) = \int_y^{\xi} E(x)\mathrm{d}x,\ \bar{J}_1(\xi,\ y) = \int_y^{\xi} I_1(x)\mathrm{d}x,\ \bar{J}_2(\xi,\ y) = \int_y^{\xi} I_2(x)\mathrm{d}x.$$

Integrate both sides of inequalities (45) from y to ξ, with $y < \xi < -2G$, and we can yield

$$\begin{aligned}&\beta(S_1^0 + \sigma S_2^0)(1-\varepsilon)(\theta \bar{J}_1(\xi,\ y) + \bar{J}_2(\xi,\ y)) - (\mu+\gamma)\bar{J}_0(\xi,\ y) \\ &\leq \int_y^{\xi} [\beta(S_1(x) + \sigma S_2(x))(\theta I_1(x) + I_2(x)) - (\mu+\gamma)E(x)]\mathrm{d}x.\end{aligned} \tag{46}$$

It follows from Lemma 2.10 and inequality (29) that

$$\| E(\cdot) \|_{C^2(\mathbb{R})} \leq \hat{P}, \| I_i(\cdot) \|_{C^2(\mathbb{R})} \leq \hat{P}, \tag{47}$$

and

$$\lim_{\xi \to -\infty} E^{'}(\xi) = \lim_{\xi \to -\infty} I_i^{'}(\xi) = 0, \tag{48}$$

where $\hat{P}$ is a positive constant. Due to inequalities (47)-(48), we have

$$\int_{-\infty}^{\xi}[\beta(S_1(x)+\sigma S_2(x))(\theta I_1(x)+I_2(x))-(\mu+\gamma)E(x)]\mathrm{d}x \tag{49}$$
$$=\lim_{y\to-\infty}\int_y^{\xi}[-D_0E^{''}(x)+cE^{'}(x)]\mathrm{d}x=-D_0E^{'}(\xi)+cE(\xi),$$

$$\int_{-\infty}^{\xi}[q_i\gamma E(x)-(\mu+\alpha_i)I_i(x)]\mathrm{d}x=\lim_{y\to-\infty}\int_y^{\xi}[-D_iI_i^{''}(x)+cI_i^{'}(x)]\mathrm{d}x$$
$$=-D_iI_i^{'}(\xi)+cI_i(\xi). \tag{50}$$

In the following, we aim to prove that there exists a constant $\mathcal{J}>0$ such that

$$\int_{-\infty}^{\xi}E(x)\mathrm{d}x<\mathcal{J},\quad \int_{-\infty}^{\xi}I_i(x)\mathrm{d}x<\mathcal{J},\quad \forall\xi<-2G.$$

Let

$$A=\begin{pmatrix}\mu+\gamma & \beta\theta(S_1^0+\sigma S_2^0)(1-\varepsilon) & \beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\\ q_1\gamma & \mu+\alpha_1 & 0\\ q_2\gamma & \mu+\alpha_2 & 0\end{pmatrix},$$

and we conclude $|A|<0$ for $\varepsilon>0$ small enough on the basis of assumption $R_0>1$. Add

$$\bar{J}(\xi,\ y)=-(\mu+\alpha_2)\beta\theta(S_1^0+\sigma S_2^0)(1-\varepsilon)(q_1\gamma\bar{J}_0(\xi,\ y)-(\mu+\alpha_1)\bar{J}_1(\xi,\ y))$$
$$-(\mu+\alpha_1)\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)(q_2\gamma\bar{J}_0(\xi,\ y)-(\mu+\alpha_2)\bar{J}_2(\xi,\ y))$$

to both sides of the inequality (46) multiplied by $-(\mu+\alpha_1)(\mu+\alpha_2)$, and it is obtained that

$$-|A|\bar{J}_0(\xi,\ y)\le(\mu+\alpha_1)(\mu+\alpha_2)\int_y^{\xi}[\beta(S_1(x)+\sigma S_2(x))(\theta I_1(x)+I_2(x))$$
$$-(\mu+\gamma)E(x)]\mathrm{d}x-\bar{J}(\xi,\ y), \tag{51}$$

for $y<\xi<-2G$. As $y\to-\infty$, the inequality (51), combining equations (49) and (50), leads to

$$\int_{-\infty}^{\xi}E(x)\mathrm{d}x<\mathcal{J},\quad \forall\xi<-2G.$$

In the similar way, we also can claim that

$$\int_{-\infty}^{\xi}I_i(x)\mathrm{d}x<\mathcal{J},\quad \forall\xi<-2G.$$

The proof is completed. □

Lemma 3.4. *Assume that $R_0>1$, where R_0 is the basic reproduction number for system (3). For any $c>0$, if there exists a non-trivial and non-negative traveling wave solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),I_1(\xi),\ I_2(\xi))$ of system (5) satisfying the boundary condition (6), a positive constant μ_0 can be found and satisfies*

$$\sup_{\xi\in\mathbb{R}}E(\xi)e^{-\mu_0\xi}<\infty,\ \sup_{\xi\in\mathbb{R}}|E^{'}(\xi)|e^{-\mu_0\xi}<\infty,\ \sup_{\xi\in\mathbb{R}}|E^{''}(\xi)|e^{-\mu_0\xi}<\infty,$$
$$\sup_{\xi\in\mathbb{R}}I_1(\xi)e^{-\mu_0\xi}<\infty,\ \sup_{\xi\in\mathbb{R}}|I_1^{'}(\xi)|e^{-\mu_0\xi}<\infty,\ \sup_{\xi\in\mathbb{R}}|I_1^{''}(\xi)|e^{-\mu_0\xi}<\infty, \tag{52}$$
$$\sup_{\xi\in\mathbb{R}}I_2(\xi)e^{-\mu_0\xi}<\infty,\ \sup_{\xi\in\mathbb{R}}|I_2^{'}(\xi)|e^{-\mu_0\xi}<\infty,\ \sup_{\xi\in\mathbb{R}}|I_2^{''}(\xi)|e^{-\mu_0\xi}<\infty.$$

Proof. Fix $c > 0$. According to $\varepsilon > 0$ and $G > 0$ defined in Lemma 3.3, it is also shown that

$$S_1(\xi) > S_1^0(1-\varepsilon), \quad S_2(\xi) > S_2^0(1-\varepsilon), \quad \forall \xi < -2G.$$

And $R_0 > 1$ implies that

$$\frac{[q_1\gamma\beta\theta(\mu+\alpha_2) + q_2\gamma\beta(\mu+\alpha_1)](S_1^0 + \sigma S_0^2)(1-\varepsilon)}{(\mu+\gamma)(\mu+\alpha_1)(\mu+\alpha_2)} > 1. \tag{53}$$

Thus, for any $\xi < -2G$, we yield that

$$cE^{'}(\xi) \geq D_0 E^{''}(\xi) + \beta(S_1^0 + S_2^0)(1-\varepsilon)[\theta I_1(\xi) + I_2(\xi)] - (\mu+\gamma)E(\xi), \tag{54}$$

$$cI_1^{'}(\xi) = D_1 I_1^{''}(\xi) + q_1\gamma E(\xi) - (\mu+\alpha_1)I_1(\xi), \tag{55}$$

$$cI_2^{'}(\xi) = D_2 I_2^{''}(\xi) + q_2\gamma E(\xi) - (\mu+\alpha_2)I_2(\xi). \tag{56}$$

Set $i = 1,\ 2$ and remain in what follows. According to Lemma 3.3, it is indicated that

$$\bar{J}_0(\xi) = \int_{-\infty}^{\xi} E(x)\mathrm{d}x < \mathcal{J},\ \ \bar{J}_i(\xi) = \int_{-\infty}^{\xi} I_i(x)\mathrm{d}x < \mathcal{J},$$

for any $\xi < -2G$. Moreover, we integrate tow sides of the inequality (54) from $-\infty$ to ξ and can obtain that

$$cE(\xi) \geq D_0 E^{'}(\xi) + \beta(S_1^0 + \sigma S_2^0)(1-\varepsilon)[\theta \bar{J}_1(\xi) + \bar{J}_2(\xi)] - (\mu+\gamma)\bar{J}_0(\xi). \tag{57}$$

Integrating two sides of the inequality (57) from $-\infty$ to ξ with $\xi < -2G$ indicates that

$$\begin{aligned}\beta(S_1^0 + \sigma S_2^0)(1-\varepsilon)[\theta\int_{-\infty}^{\xi} \bar{J}_1(x)\mathrm{d}x + \int_{-\infty}^{\xi} \bar{J}_2(x)\mathrm{d}x] - (\mu+\gamma)\int_{-\infty}^{\xi} \bar{J}_0(x)\mathrm{d}x \\ + D_0 E(\xi) \geq c\bar{J}_0(\xi).\end{aligned} \tag{58}$$

Via the similar way in equations (55) and (56), we can yield

$$q_1\gamma \int_{-\infty}^{\xi} \bar{J}_0(x)\mathrm{d}x - (\mu+\alpha_1)\int_{-\infty}^{\xi} \bar{J}_1(x)\mathrm{d}x + D_1 I_1(\xi) = c\bar{J}_1(\xi), \tag{59}$$

and

$$q_2\gamma \int_{-\infty}^{\xi} \bar{J}_0(x)\mathrm{d}x - (\mu+\alpha_2)\int_{-\infty}^{\xi} \bar{J}_2(x)\mathrm{d}x + D_2 I_2(\xi) = c\bar{J}_2(\xi), \tag{60}$$

which reduces to

$$\int_{-\infty}^{\xi} \bar{J}_1(x)\mathrm{d}x = \frac{1}{\mu+\alpha_1}\left(q_1\gamma\int_{-\infty}^{\xi} \bar{J}_0(x)\mathrm{d}x + D_1 I_1(\xi) - c\bar{J}_1(\xi)\right), \tag{61}$$

and

$$\int_{-\infty}^{\xi} \bar{J}_2(x)\mathrm{d}x = \frac{1}{\mu+\alpha_2}\left(q_2\gamma\int_{-\infty}^{\xi} \bar{J}_0(x)\mathrm{d}x + D_2 I_2(\xi) - c\bar{J}_2(\xi)\right). \tag{62}$$

Next, we need to claim there exist two positive constants a and b such that

$$a\sum_{j=0}^{2}\int_{-\infty}^{\xi} \bar{J}_j(x)\mathrm{d}x \leq b\sum_{j=0}^{2} \bar{J}_j(x)\mathrm{d}x. \tag{63}$$

Substituting equations (61) and (62) into the equation (59) has

$$\begin{aligned}
c\bar{J}_0(\xi) \geq& \beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left[\frac{\theta}{\mu+\alpha_1}\left(q_1\gamma\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x+D_1I_1(\xi)-c\bar{J}_1(\xi)\right)\right.\\
&\left.+\frac{1}{\mu+\alpha_2}\left(q_2\gamma\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x+D_2I_2(\xi)-c\bar{J}_2(\xi)\right)\right]\\
&+D_0E(\xi)-(\mu+\gamma)\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x\\
=&\left[\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{q_1\gamma\theta}{\mu+\alpha_1}+\frac{q_2\gamma}{\mu+\alpha_2}\right)-(\mu+\gamma)\right]\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x\\
&+\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{\theta D_1}{\mu+\alpha_1}I_1(\xi)+\frac{D_2}{\mu+\alpha_2}I_2(\xi)\right)+D_0E(\xi)\\
&-c\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{\theta}{\mu+\alpha_1}\bar{J}_1(\xi)+\frac{1}{\mu+\alpha_2}\bar{J}_2(\xi)\right),
\end{aligned}$$

which implies

$$\begin{aligned}
&c\left[\bar{J}_0(\xi)+\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{\theta}{\mu+\alpha_1}\bar{J}_1(\xi)+\frac{1}{\mu+\alpha_2}\bar{J}_2(\xi)\right)\right]\\
&\geq\left[\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{q_1\gamma\theta}{\mu+\alpha_1}+\frac{q_2\gamma}{\mu+\alpha_2}\right)-(\mu+\gamma)\right]\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x\\
&\quad+\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{\theta D_1}{\mu+\alpha_1}I_1(\xi)+\frac{D_2}{\mu+\alpha_2}I_2(\xi)\right)+D_0E(\xi).
\end{aligned}$$

Because $E(\xi)$ and $I_i(\xi)$ are non-negative, the above inequality reduces to

$$\begin{aligned}
&\left[\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{q_1\gamma\theta}{\mu+\alpha_1}+\frac{q_2\gamma}{\mu+\alpha_2}\right)-(\mu+\gamma)\right]\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x\\
&\leq c\left[\bar{J}_0(\xi)+\beta(S_1^0+\sigma S_2^0)(1-\varepsilon)\left(\frac{\theta}{\mu+\alpha_1}\bar{J}_1(\xi)+\frac{1}{\mu+\alpha_2}\bar{J}_2(\xi)\right)\right].
\end{aligned}$$

According to the inequality (53), we can claim that there exist two positive constants $\bar{a_0}$ and $\bar{b_0}$ such that

$$\bar{a_0}\int_{-\infty}^{\xi}\bar{J}_0(x)\mathrm{d}x\leq\bar{b_0}(\bar{J}_0(\xi)+\bar{J}_1(\xi)+\bar{J}_2(\xi)). \tag{64}$$

Plug the inequality (64) into the inequality (58), and it is shown that there are two constants $\bar{a}$ and $\bar{b}$ satisfying

$$\bar{a}\left(\int_{-\infty}^{\xi}\bar{J}_1(x)\mathrm{d}x+\int_{-\infty}^{\xi}\bar{J}_2(x)\mathrm{d}x\right)\leq\bar{b}(\bar{J}_0(\xi)+\bar{J}_1(\xi)+\bar{J}_2(\xi)). \tag{65}$$

Hence, for any $\xi<-2G$, the inequality (3.3) is established by adding the inequality (64) and the inequality (65). Let

$$\mathcal{J}(\xi)=\bar{J}_0(\xi)+\bar{J}_1(\xi)+\bar{J}_2(\xi),$$

then we gain that

$$a\int_{-\infty}^{\xi}\mathcal{J}(x)\mathrm{d}x\leq b\mathcal{J}(\xi),\quad\forall\xi<-2G,$$

that is

$$a\int_{-\infty}^{0}\mathcal{J}(x+\xi)\mathrm{d}x \le b\mathcal{J}(\xi), \quad \forall \xi < -2G.$$

Because $\mathcal{J}(\cdot)$ is increasing, it is implied that $a\eta\mathcal{J}(\xi-\eta) \le b\mathcal{J}(\xi)$ for any $\xi < -2G$ and $\eta > 0$. Therefore, there exist a large enough $\xi_0 > 0$ and a small $\omega_0 \in (0,\ 1)$ such that

$$\mathcal{J}(\xi-\xi_0) \le \omega_0\mathcal{J}(\xi) \quad \forall \xi < -2G.$$

Let $\Psi(\xi) = \mathcal{J}(\xi)e^{-\mu_0\xi}$ with $0 < \mu_0 = \frac{1}{\xi_0}\ln\frac{1}{\omega_0} < \lambda_1$, and then it is concluded that

$$\Psi(\xi-\xi_0) = \mathcal{J}(\xi-\xi_0)e^{-\mu_0(\xi-\xi_0)} \le \omega_0\mathcal{J}(\xi)e^{-\mu_0(\xi-\xi_0)} = \Psi(\xi),$$

for any $\xi < -2G$. According to $\mathcal{J}(\xi) < \infty$ for any $\xi < -2G$, we can find a constant $k_0 > 0$ such that $\Psi(\xi) \le k_0$ for $\forall \xi < -2G$, that is $\mathcal{J}(\xi) \le k_0 e^{\mu_0\xi}$ for any $\xi < -2G$. Therefore, there exists a constant $q_0 > 0$ such that $\int_{-\infty}^{\xi}\bar{J}(x)\mathrm{d}x \le q_0 e^{\mu_0\xi}$ in $\xi \in (-\infty,\ -2G)$. Combining equations (59)-(60), we can also find a constant $p_0 > 0$ satisfying

$$E(\xi) \le p_0 e^{\mu_0\xi}, \quad I_i(\xi) \le p_0 e^{\mu_0\xi}, \quad \forall \xi < -2G.$$

Since $E(\xi)$ and $I_i(\xi)$ are bounded in $\mathbb{R}$, we obtain that

$$E(\xi) \le p_0 e^{\mu_0\xi}, \quad I_i(\xi) \le p_0 e^{\mu_0\xi}, \quad \forall \xi \in \mathbb{R}.$$

According to estimates (29) and (57), it yields

$$\sup_{\xi\in\mathbb{R}}|E^{'}(\xi)|e^{-\mu_0\xi} < \infty.$$

Via the E-th equation in system (5) and the inequality (54), we can conclude that

$$\sup_{\xi\in\mathbb{R}}|E^{''}(\xi)|e^{-\mu_0\xi} < \infty.$$

Finally, applying the similar argument on $I_i(\xi)$ shows that

$$\sup_{\xi\in\mathbb{R}} I_i(\xi)e^{-\mu_0\xi} < \infty, \quad \sup_{\xi\in\mathbb{R}}|I_i^{'}(\xi)|e^{-\mu_0\xi} < \infty, \quad \sup_{\xi\in\mathbb{R}}|I_i^{''}(\xi)|e^{-\mu_0\xi} < \infty.$$

This completes the proof. □

According to above two theorems, we can obtain the following non-existence theorem.

Theorem 3.5. *If $R_0 > 1$, for $c \in (0,\ c^*)$, there exists no non-trivial and non-negative traveling wave solution $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ of system (5) satisfying the boundary condition (6), where c^* is the minimal spread speed and R_0 is the basic reproduction number for system (3).*

Proof. We intend to prove this theorem by contradiction. Fix $c \in (0,\ c^*)$. On the contrary, we assume there exists a non-trivial and non-negative $(S_1(\xi),\ S_2(\xi),\ E(\xi),\ I_1(\xi),\ I_2(\xi))$ of system (5) satisfying the boundary condition (6). Via Lemma 3.4, then there exists a positive constant μ_0 defined in the proof of Lemma 3.4 such that inequalities (52) are established. Set $K_1(\xi) = S_1^0 - S_1(\xi)$ and $K_2(\xi) = S_2^0 - S_2(\xi)$ in $\mathbb{R}$. Plugging $K_1(\xi)$ and $K_2(\xi)$ into S_1-th and S_2-th equations yields

$$cK_1^{'}(\xi) - dK_1^{''}(\xi) + (\mu+\rho)K_1^{'}(\xi) - \beta S_1(\xi)[\theta I_1(\xi) + I_2(\xi)] = 0, \tag{66}$$

$$cK_2^{'}(\xi) - dK_2^{''}(\xi) - \rho K_2^{''}(\xi) + \mu K_2^{'}(\xi) - \beta\sigma S_2(\xi)[\theta I_1(\xi) + I_2(\xi)] = 0. \tag{67}$$

According to the inequality

$$\| K_i^{'} \|_{C((\infty,0],\mathbb{R})} \leq 2 \| K_i^{''} \|_{C((\infty,0],\mathbb{R})}^{\frac{1}{2}} \| K_i \|_{C((\infty,0],\mathbb{R})}^{\frac{1}{2}},$$

and the fact that

$$\lim_{\xi \to -\infty} K_i(\xi) = 0, \tag{68}$$

we can claim

$$\lim_{\xi \to -\infty} K_i^{'}(\xi) = 0, \tag{69}$$

for $i = 1,\ 2$. In addition, since $K_i(\xi)$ is bounded and satisfies equations (68)-(69), integrating both sides of the sum of equations (66) and (67) from $-\infty$ to $\xi \leq 0$ can show that

$$c[K_1(\xi) + K_2(\xi)] - d[K_1^{'}(\xi) + K_2^{'}(\xi)] + \mu \int_{-\infty}^{\xi} [K_1(x) + K_2(x)]\mathrm{d}x$$
$$- \int_{-\infty}^{\xi} \beta[S_1(x) + \sigma S_2(x)][\theta I_1(x) + I_2(x)]\mathrm{d}x = 0.$$

Let $f(\xi) = \int_{-\infty}^{\xi} \beta[S_1(x) + \sigma S_2(x)][\theta I_1(x) + I_2(x)]\mathrm{d}x$, and $K_\mu(\xi) = \mu \int_{-\infty}^{\xi} [K_1(x) + K_2(x)]\mathrm{d}x$ for any $\xi \leq 0$. It follows from Lemma 3.4 that $f(\xi) \leq C_f e^{\mu_0 \xi}$ on $\xi \in \mathbb{R}$, where C_f is a positive constant. Hence, a direct calculation indicates that

$$\begin{aligned} K_1(\xi) + K_2(\xi) &= [K_1(0) + K_2(0)]e^{\frac{c}{d}\xi} + \frac{1}{d} e^{\frac{c}{d}\xi} \int_{\xi}^{0} e^{-\frac{c}{d}x}[-K_\mu(x) + f(x)]\mathrm{d}x \\ &\leq [K_1(0) + K_2(0)]e^{\frac{c}{d}\xi} + \frac{1}{d} e^{\frac{c}{d}\xi} \int_{\xi}^{0} e^{-\frac{c}{d}x} f(x)\mathrm{d}x, \end{aligned}$$

for $\xi \leq 0$. Owing to $f(\xi) = O(e^{\mu_0 \xi})$ as $\xi \to -\infty$, we can claim that $K_1(\xi) + K_2(\xi) = O(e^{\mu_0^{'} \xi})$ as $\xi \to -\infty$, where $\mu_0^{'} = \min\left\{\mu_0,\ \frac{c}{d}\right\}$. Combining $0 \leq K_1(\xi) + K(\xi) \leq \frac{\lambda}{\mu}$, we can conclude that

$$\sup_{\xi \in \mathbb{R}} [K_1(\xi) + K_2(\xi)]e^{-\mu_0^{'} \xi} < \infty,$$

which implies

$$\sup_{\xi \in \mathbb{R}} K_1(\xi)e^{-\mu_0^{'} \xi} < \infty, \quad \sup_{\xi \in \mathbb{R}} K_2(\xi)e^{-\mu_0^{'} \xi} < \infty. \tag{70}$$

On the basis of the above discussion, we define the one-sided Laplace transforms for $E(\xi)$, $I_1(\xi)$ and $I_2(\xi)$ by

$$L_0(\hat{\lambda}) = \int_{-\infty}^{0} e^{-\hat{\lambda}\xi} E(\xi)\mathrm{d}\xi,\ L_1(\hat{\lambda}) = \int_{-\infty}^{0} e^{-\hat{\lambda}\xi} I_1(\xi)\mathrm{d}\xi,\ L_2(\hat{\lambda}) = \int_{-\infty}^{0} e^{-\hat{\lambda}\xi} I_2(\xi)\mathrm{d}\xi. \tag{71}$$

Next, we only consider $\hat{\lambda} \in \mathbb{R}_+$. Since $E(\xi) > 0$, $I_1(\xi) > 0$, $I_2(\xi) > 0$ for any $\xi \in \mathbb{R}$, and $L_i(\cdot)$ is increasing for $\hat{\lambda} \in \mathbb{R}_+$, respectively, there exist two possibilities: (i) a positive constant $v_i > \mu_0$ can be found such that $L_i(\cdot) < \infty$ for any $0 \leq \hat{\lambda} < v_i$ and $\lim_{\hat{\lambda} \to v_i - 0} L_i(\hat{\lambda}) = \infty$. (ii) $L_i(\cdot) < \infty$ for any $\hat{\lambda} \geq 0$, where $i = 0,\ 1,\ 2$. Therefore, we define two-sided Laplace transforms for $E(\cdot)$, $I_1(\cdot)$ and $I_2(\cdot)$ by

$$\begin{aligned} \mathcal{L}_0(\hat{\lambda}) = \int_{-\infty}^{+\infty} e^{-\hat{\lambda}\xi} E(\xi)\mathrm{d}\xi, \quad \mathcal{L}_1(\hat{\lambda}) = \int_{-\infty}^{+\infty} e^{-\hat{\lambda}\xi} I_1(\xi)\mathrm{d}\xi, \quad \mathcal{L}_2(\hat{\lambda}) \\ = \int_{-\infty}^{+\infty} e^{-\hat{\lambda}\xi} I_2(\xi)\mathrm{d}\xi, \end{aligned}$$

and we also only research $\hat{\lambda} > 0$. Because $E(\cdot)$, $I_1(\cdot)$ and $I_2(\cdot)$ are bounded in $\mathbb{R}$, respectively, it is the fact that

$$\int_0^{+\infty} e^{-\hat{\lambda}\xi} E(\xi)\mathrm{d}\xi < \infty, \quad \int_0^{+\infty} e^{-\hat{\lambda}\xi} I_1(\xi)\mathrm{d}\xi < \infty, \quad \int_0^{+\infty} e^{-\hat{\lambda}\xi} I_2(\xi)\mathrm{d}\xi < \infty,$$

for any $\hat{\lambda} > 0$, which implies $\mathcal{L}_I(\cdot)$ owns the same property as $L_i(\cdot)$ in $\mathbb{R}_+$ with $i = 0,\ 1,\ 2$, respectively.

In the following, we indicate that there indeed exists a $v_i = +\infty$ such that $\mathcal{L}_i(\hat{\lambda}) < \infty$ for any $\hat{\lambda} \geq 0$ with $i = 0,\ 1,\ 2$, respectively. The first step is to prove $v_0 = v_1 = v_2$. According to

$$\begin{aligned} &D_0 E^{''}(\xi) - cE^{'}(\xi) + \beta(S_1^0 + \sigma S_2^0)[\theta I_1(\xi) + I_2(\xi)] - (\mu + \gamma)E(\xi) \\ &\quad = \beta[S_1^0 - S_1(\xi) + \sigma S_2^0 - S_2(\xi)][\theta I_1(\xi) + I_2(\xi)], \qquad \forall \xi \in \mathbb{R}, \end{aligned}$$

it is implied that

$$\begin{aligned} &[D_0\hat{\lambda}^2 - c\hat{\lambda} - (\mu + \gamma)]L_0(\hat{\lambda}) + \beta(S_1^0 + \sigma S_2^0)[\theta \mathcal{L}_1(\hat{\lambda}) + \mathcal{L}_2(\hat{\lambda})] \\ &\quad = \beta \int_{-\infty}^{+\infty} [S_1^0 - S_1(\xi) + \sigma(S_2^0 - S_2(\xi))][\theta I_1(\xi) + I_2(\xi)]e^{-\hat{\lambda}\xi}\mathrm{d}\xi, \qquad \forall \xi \in \mathbb{R}. \end{aligned} \tag{72}$$

In the same way, we can obtain

$$[D_1\hat{\lambda}^2 - c\hat{\lambda} - (\mu + \alpha_1)]\mathcal{L}_1(\hat{\lambda}) + q_1\gamma\mathcal{L}_0(\hat{\lambda}) = 0, \tag{73}$$

and

$$[D_2\hat{\lambda}^2 - c\hat{\lambda} - (\mu + \alpha_2)]\mathcal{L}_2(\hat{\lambda}) + q_2\gamma\mathcal{L}_0(\hat{\lambda}) = 0. \tag{74}$$

Thus, it follows from equations (73) and (74) that $v_0 = v_1 = v_2$. Let $v = v_0 = v_1 = v_2$. The second step is to prove $v = +\infty$ via a contradiction. We assume that $v < +\infty$ and $\mathcal{L}_i(\hat{\lambda}) < \infty$ $(i = 0,\ 1,\ 2)$ for any $\hat{\lambda} \in [0,\ v)$ and $\lim_{\hat{\lambda} \to v-0} \mathcal{L}_i(\hat{\lambda}) = \infty$. Equations (72)-(74) can show that $-D_1v^2 + cv + (\mu + \alpha_1) > 0$ and $-D_2v^2 + cv + (\mu + \alpha_2) > 0$. If $D_0v^2 - cv - (\mu + \gamma) \geq 0$ and let $\hat{\lambda} \to v - 0$ in both sides of (72), the right side shows

$$\begin{aligned} &\beta \int_{-\infty}^{+\infty} [S_1^0 - S_1(\xi) + \sigma(S_2^0 - S_2(\xi))][\theta I_1(\xi) + I_2(\xi)]e^{-v\xi}\mathrm{d}\xi \\ &\quad \leq \beta \sup_{\xi \in \mathbb{R}} \left\{ [S_1^0 - S_1(\xi) + \sigma(S_2^0 - S_2(\xi))]e^{-\mu_0^{'}\xi} \right\} \left[\int_{-\infty}^{-\infty} \theta I_1(\xi) e^{-(v-\mu_0^{'})\xi}\mathrm{d}\xi \right. \\ &\qquad \left. + \int_{-\infty}^{-\infty} I_2(\xi)e^{-(v-\mu_0^{'})\xi}\mathrm{d}\xi \right] \\ &\quad \leq \infty. \end{aligned}$$

However, the left side tends to ∞, which leads to a contraction. If $D_0v^2 - cv - (\mu + \gamma) < 0$, plugging equations (73) and (74) into the equation (72) can obtain that

$$\begin{aligned} m_0(\hat{\lambda},\ c)\mathcal{L}_0(\hat{\lambda})[\rho^2(\hat{\lambda},\ c) - 1] =& \beta \int_{-\infty}^{+\infty} [S_1^0 - S_1(\xi) + \sigma(S_2^0 - S_2(\xi))][\theta I_1(\xi) \\ &+ I_2(\xi)]e^{-\hat{\lambda}\xi}\mathrm{d}\xi, \end{aligned} \tag{75}$$

and $0 < v < \lambda_c$. As $\hat{\lambda} \to v - 0$, the right side is bounded, but the left side tends to ∞. As a result, the case (i) is impossible.

If the case (ii) is established, similarly, let $\hat{\lambda} \to +\infty$ in both sides of the equation (72). The right side is bounded but the left side tends to ∞ owing to $D_0\hat{\lambda}^2 - c\hat{\lambda} - (\mu+\gamma) \to \infty$, which also leads to a contraction. Above all, the non-existence is completely proved. □

4. **Numerical simulations and discussion.** Section 2 and Section 3 have proved the existence and non-existence of traveling wave solution for system (2) satisfying the boundary condition (6). In this section, we aim to visually display the existence of traveling wave solution for system (2) connecting disease-free equilibrium and endemic equilibrium. Now, we firstly take a set of parameters for system (2) as follows:

$$\begin{aligned}
&d = 0.02,\ D_0 = 0.015,\ D_1 = 0.005,\ D_2 = 0.01,\ D = 0.02,\\
&\Lambda = 8,\ \mu = 0.3,\ \beta = 0.375,\ \rho = 0.7,\\
&\sigma = 0.5,\ \gamma = 0.75,\ \alpha_1 = 0.65,\ \alpha_2 = 0.5,\\
&\theta = 0.5,\ q_1 = 0.6,\ q_2 = 0.4.
\end{aligned}$$

As a result, we can obtain the disease-free equilibrium $(S_1^0,\ S_2^0,\ E^0,\ I_1^0,\ I_2^0,\ R^0) = (8,\ 18.6667,\ 0,\ 0,\ 0,\ 0)$, the endemic equilibrium $(S_1^*,\ S_2^*,\ E^*,\ I_1^*,\ I_2^*,\ R^*) \approx (3.3762,\ 2.3999,\ 5.9687,\ 2.8274,\ 2.2383,\ 9.8565)$ and the basic reproduction number $R_0 \approx 3.2821 > 1$. For simulations, we further intercept $[-200,\ 200]$ from spatial domain and $[0,\ 200]$ from time domain. Moreover, we take the Neumann boundary condition and the below piecewise functions as initial conditions for system (2):

$$S_i(x,\ t) = \begin{cases} S_i^*, & 0 < x \le 200, \quad t = 0, \quad i = 1,\ 2,\\ S_i^0, & -200 \le x \le 0, \quad t = 0, \quad i = 1,\ 2,\end{cases}$$

$$E(x,\ t) = \begin{cases} E^*, & 0 < x \le 200, \quad t = 0,\\ 0, & -200 \le x \le 0, \quad t = 0,\end{cases}$$

$$I_i(x,\ t) = \begin{cases} I_i^*, & 0 < x \le 200, \quad t = 0, \quad i = 1,\ 2,\\ 0, & -200 \le x \le 0, \quad t = 0, \quad i = 1,\ 2,\end{cases}$$

$$R(x,\ t) = \begin{cases} R^*, & 0 < x \le 200, \quad t = 0,\\ 0, & -200 \le x \le 0, \quad t = 0.\end{cases}$$

The figure 1, simulations with applying above conditions, indicate that there exists a traveling wave solution of system (2) connecting disease-free equilibrium and endemic equilibrium. Meanwhile, we cross section curves of traveling wave solution in figure 1 as $t = 200$ (see figure 2). And we can find that traveling waves for system (2) are not monotonic in figure 2.

Since we pay more attention to influences of self-protection and treatment in the spatial spread for an epidemic, it is critical to research the change of the minimal spread speed c^* while self-protection σ and treatment θ changing. The direct derivations of $\rho(\lambda,\ c)$ with respect to σ and θ show that

$$\frac{\partial \rho}{\partial \theta} > 0, \quad \frac{\partial \rho}{\partial \sigma} > 0, \tag{76}$$

respectively, where $\rho(\lambda,\ c)$ is defined as the formula (9). The inequality (76) means that the numerical increase in σ and θ could lead to increasing of the minimal spread speed c^* with applying Lemma 2.3. The figures (a) and (b) in figure 3 also display that the minimal spread speed c^* is increasing with respect to σ and θ for $\sigma,\ \theta \in [0.05,\ 1]$, respectively. In fact, measures about the enhancement of

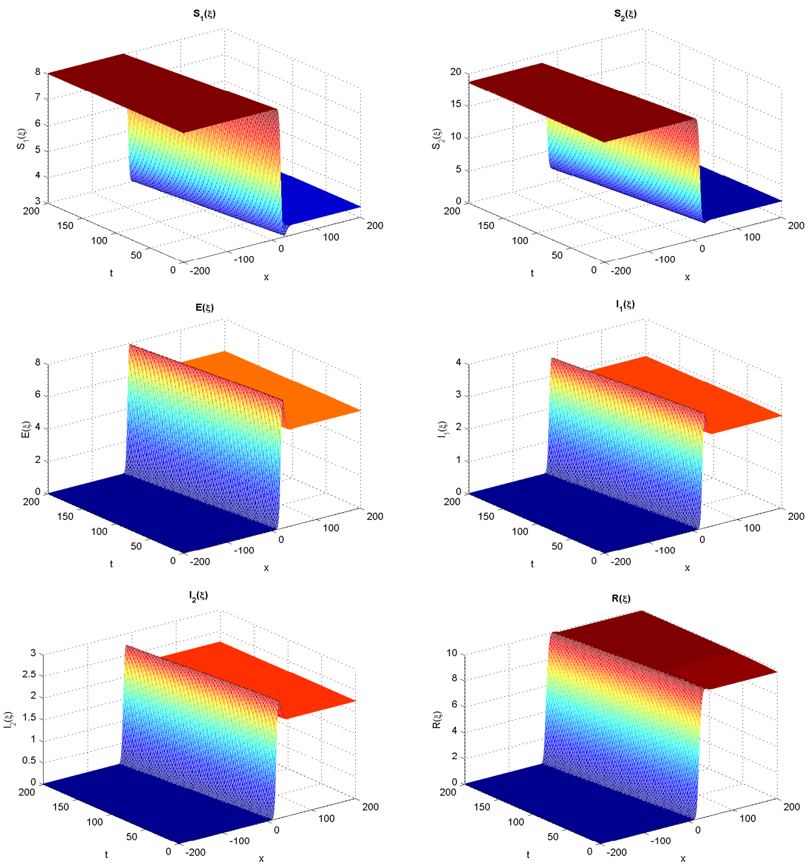

FIGURE 1. The numerical simulations of existence for traveling wave solution of system (2)

self-protection and treatment can lead to numerical reductions about both σ and θ, which implies the minimal spread speed c^* will decrease via the inequality (76) and figure 3. Above all, the effective self-protection and treatment can reduce the spread speed for an epidemic.

In this paper, we mainly construct a non-monotonic reaction diffusion SEIR model with effects of self-protection and treatment in incident rate, and determine the existence and non-existence of traveling wave solution connecting disease-free equilibrium and endemic equilibrium. We prove the existence as $R_0 > 1$ and $c \geq c^*$. And when $R_0 \leq 1$ or $R_0 > 1$ with $c \in (0,\ c^*)$, there exists no non-trivial and non-negative traveling wave solution satisfying the boundary condition (6). Finally, the numerical simulations show the existence and indicate that self-protection and treatment can reduce the spread speed of an epidemic. However, self-protection and treatment may affect the movement of different individuals in various times and spaces. Some individuals may even choose or be forcibly quarantined. These factors are worthy being researched in the future.

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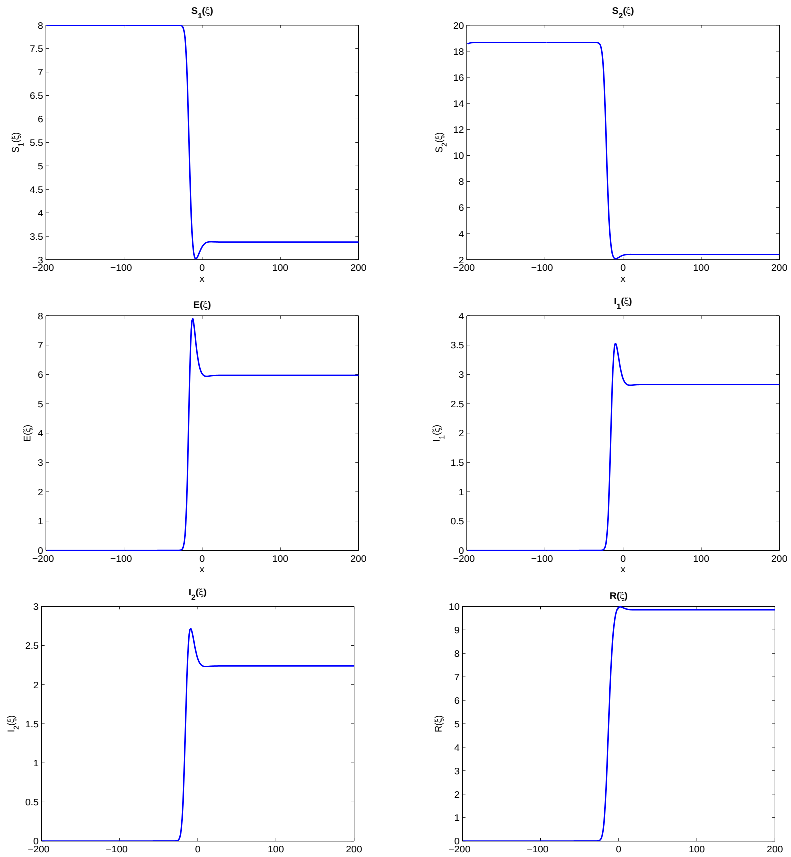
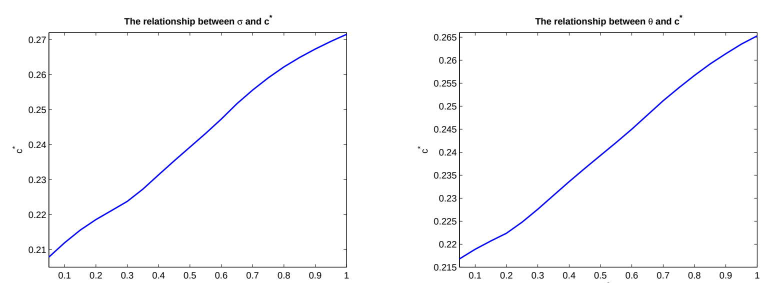


FIGURE 2. Cross section curve of traveling wave solution for system (2) as $t = 200$.



(a) (b)

FIGURE 3. Show the effects of self-protection σ and treatment θ on minimal spread speed c^*, where σ and θ are taken from 0.05 to 1.

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