ELECTRONIC RESEARCH ARCHIVE doi:10.3934/era.2020116
Volume **29**, Number **3**, August **2021** pp. **2269–2291**

PROPAGATION DYNAMICS OF NONLOCAL DISPERSAL EQUATIONS WITH INHOMOGENEOUS BISTABLE NONLINEARITY

Shao-Xia Qiao*

School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

Li-Jun Du

School of Science, Chang'an University
Xi'an, Shaanxi 710064, China

(Communicated by Yihong Du)

Abstract. This paper is concerned with the nonlocal dispersal equations with inhomogeneous bistable nonlinearity in one dimension. The varying nonlinearity consists of two spatially independent bistable nonlinearities, which are connected by a compact transition region. We establish the existence of a unique entire solution connecting two traveling wave solutions pertaining to the different nonlinearities. In particular, we use a "squeezing" technique to show that the traveling wave of the equation with one nonlinearity approaching from infinity, after going through the transition region, converges to the other traveling wave prescribed by the nonlinearity on the other side. Furthermore, we also prove that such an entire solution is Lyapunov stable.

1. **Introduction.**

In the present paper, we focus on the nonlocal dispersal equation with spatially dependent bistable nonlinearity

$$u_t(x,t) = J * u(x,t) - u(x,t) + f(x,u), \quad x,\ t \in \mathbb{R}, \tag{1}$$

where the nonlinearity $f(x,u)$ and kernel function $J(x)$ satisfy the following assumptions.

(F) The nonlinearity f satisfies that

$$\begin{cases} f(x,u) = f_1(u) \text{ for } x \geq 0,\ u \in [0,1], \\ f(x,u) = f_2(u) \text{ for } x \leq -L,\ u \in [0,1], \\ f_2(u) \leq f(x,u) \leq f_1(u),\ f_u(x,u) < 1 \text{ for } x \in [-L,0],\ u \in [0,1], \end{cases} \tag{2}$$

where $L > 0$, f_1 and f_2 are two given bistable nonlinearities satisfying that

$$f_i \in C^{1,1}([0,1]),\ f_i(0) = f_i(1) = 0, f_i'(0),\ f_i'(1) < 0,$$

$$f_i < 0 \text{ in } (0,\theta_i), f_i > 0 \text{ in } (\theta_i,1), 0 < \theta_1 < \theta_2 < 1, \text{ and } \int_0^1 f_i(s)ds > 0,\ i = 1,2.$$

2020 *Mathematics Subject Classification.* Primary: 35K57, 35R20; Secondary: 92D25.

Key words and phrases. Traveling wave, entire solution, nonlocal dispersal, comparison principle, squeezing technique.

The program was partially supported by NSF of China (11671180).

* Corresponding author: Shao-Xia Qiao.

(J) The kernel function $J \in C^1(\mathbb{R})$ satisfies that

$$\begin{cases} J(x) = J(-x), \ J(x) \geq 0, \ \displaystyle\int_{\mathbb{R}} J(y)dy = 1, \\ \displaystyle\int_{\mathbb{R}} |J'(y)|dy < \infty, \ \int_{\mathbb{R}} J(y)e^{-\lambda y}dy < \infty \text{ for all } \lambda > 0. \end{cases} \tag{3}$$

A typical example is $f(x,u) = u(1-u)(a(x)-u)$.

The nonlocal dispersal equation has attracted so much attention because of its extensive applications to account for diffusion phenomena involving jumps in biology, physics and chemistry [1, 12]. Traveling waves as one kind of special solutions with invariant profile and fixed speed as well as entire solutions have been adequately investigated. It is well-known that the importance of the study of entire solutions of reaction-diffusion equations (nonlocal dispersal equations) is frequently recalled in the literature. Since the pioneering works of Hamel and Nadirashvili [13, 14], there have been tremendous advances in studying the existence of entire solutions for various models. In particular, when the nonlinearity is homogeneous (i.e. $L = 0$ and $f_1 = f_2$), the theories of traveling waves and entire solutions for nonlocal dispersal equation 1 with various types of nonlinearities have been well established, the related results refer to [2, 6, 7, 8, 9, 15, 18, 23, 24, 26, 27] and references therein. Specifically, when the kernel $J(x)$ is compactly supported, Sun et al. [23] constructed a two-dimensional manifold of entire solutions which behave as two traveling wave solutions coming from both directions for bistable nonlocal dispersal equation.

However, for the inhomogeneous nonlinearity, several works are devoted to transition fronts (see [3, 19]) of nonlocal dispersal equations [17, 20] and forced waves in shifting habitats [25]. Traveling waves and spreading speed of monostable nonlocal dispersal equations with space periodic nonlinearity were studied in [21]. Li et al. [16] further obtained the existence of entire solutions for space periodic nonlinearity. Particularly, Eberle [10, 11] constructed a heteroclinic orbit connecting two traveling waves for bistable local dispersal equation 1 in cylinders. Meanwhile, Berestycki and Rodríguez [5] considered a bistable nonlocal dispersal equation with a gap in one dimension. As far as we know, there is no result on entire solutions for the nonlocal dispersal equation 1 with inhomogeneous nonlinearity $f(x,u)$ satisfying (F).

In this paper, we aim to construct an entire solution connecting traveling waves with the two different nonlinearities, motivated by [10, 11]. By constructing suitable sub- and super-solutions, we establish the existence and uniqueness of the entire solution behaving as the traveling wave coming from one side and eventually going to the other side, which is different with the one constructed in [23] for the case $L = 0$ and $f_2 = f_1$. It should be pointed out that there are many differences to [4], though the similar method is applied. More precisely, since the nonlocal operator is not a compact operator as the Laplacian operator in [4], we have to establish the Lipschitz continuity of entire solutions in order to prove its uniqueness as in [23]. Compared to [23], we drop the assumption the kernel $J(x)$ is compactly supported.

The crucial part of this paper is to figure out the long time asymptotic behavior of the entire solution. Since the lack of compactness of the nonlocal operator, we can not use the Lyapunov function argument as [10, 11] to show that the entire solution converges to a translation of the other traveling wave as time goes to positive infinity. However, inspired by the idea in [7], we use a "squeezing" technique to address this issue. Furthermore, we also apply sub- and super-solutions method with comparison principle to establish Lyapunov stability of the entire solution. This can be done because that we can obtain a positive estimate on the derivative of the entire solution with respect to t when it is not so close to 0 as well as 1 and t is large enough.

Now we state the main results of this paper as follows.

Theorem 1.1. *Let assumptions* (F) *and* (J) *hold. Then there exists a unique entire solution* $u(x,t)$ *of 1 with* $0 < u(x,t) < 1$, $u_t(x,t) > 0$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}$ *such that*

$$u(x,t) - \phi_1(x + c_1 t) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \mathbb{R}$$

and

$$u(x,t) - \phi_2(x + c_2 t + \beta) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}$$

for some $\beta \in \mathbb{R}$, *where* $(\phi_i, c_i)(i = 1, 2)$ *are the traveling wave solutions solving*

$$\begin{cases} J * \phi_i - \phi_i + f_i(\phi_i) - c_i\phi_i' = 0, \\ \phi_i(-\infty) = 0, \ \phi_i(+\infty) = 1, \\ 0 < \phi_i < 1. \end{cases} \tag{4}$$

Theorem 1.2. *The entire solution* $u(x,t)$ *constructed in Theorem 1.1 is Lyapunov stable in the following sense: For any given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any uniformly continuous* $v_0(x) \in [0,1]$ *with* $\sup_{x\in\mathbb{R}} |v_0(x) - u(x + a, t_0)| < \delta$, *the solution* $v(x, t, v_0(x))$ *of 1 with initial value* $v_0(x)$ *satisfies*

$$|v(x, t, v_0) - u(x + a, t + t_0)| < \epsilon$$

for $x \in \mathbb{R}$ *and* $t \geq 0$, *where* $a,\ t_0 \in \mathbb{R}$ *are two constants.*

The rest of this paper is organized as follows. In Section 2, we recall some results of the bistable traveling waves for homogeneous nonlinearities and prove the comparison principle for 1. Section 3 is devoted to constructing the unique entire solution. In Section 4, we study the asymptotic profile as time goes to positive infinity. Finally, we establish Lyapunov stability of the entire solution in Section 5.

2. Preliminaries.

In this section, some known results on the traveling waves of 4 are outlined and the comparison principle is established.

2.1. Traveling waves.

It follows from Theorem 2.7 in [2] and Theorem 2.7 in [23] that 4 admits a solution $\phi_1(z)$ satisfying

$$\begin{cases} \beta_0 e^{\lambda_0 z} \leq \phi_1(z) \leq \alpha_0 e^{\lambda_0 z}, \ z \leq 0, \\ \beta_1 e^{-\lambda_1 z} \leq 1 - \phi_1(z) \leq \alpha_1 e^{-\lambda_1 z}, \ z > 0, \end{cases} \tag{5}$$

where $\alpha_0,\ \alpha_1,\ \beta_0$ and β_1 are positive constants, λ_0 and λ_1 are the positive roots of

$$c\lambda_0 = \int_{\mathbb{R}} J_1(y) e^{-\lambda_0 y} dy - 1 + f_1'(0) \text{ and } c\lambda_1 = \int_{\mathbb{R}} J_1(y) e^{-\lambda_1 y} dy - 1 + f_1'(1),$$

respectively. Moreover, we have

$$\begin{cases} \tilde\beta_0 e^{\lambda_0 z} \leq \phi_1'(z) \leq \tilde\alpha_0 e^{\lambda_0 z}, \ z \leq 0, \\ \tilde\beta_1 e^{-\lambda_1 z} \leq \phi_1'(z) \leq \tilde\alpha_1 e^{-\lambda_1 z}, \ z > 0 \end{cases} \tag{6}$$

for some constants $\tilde\alpha_0,\ \tilde\alpha_1,\ \tilde\beta_0$ and $\tilde\beta_1 > 0$. At last, note that $f_1 \in C^{1,1}([0,1])$, there exists some $L_f > 0$ such that

$$|f_1(u + v) - f_1(u) - f_1(v)| \leq L_f uv \text{ for } 0 \leq u, v \leq 1.$$

2.2. **Comparison principle.** We show that the following comparison principle holds by a contradiction argument.

Proposition 1. *Suppose that assumptions (F) and (J) hold. Furthermore, let $u(x,t)$, $v(x,t)$ be continuous and bounded functions on $\mathbb{R}\times[0,T]$ for some $T>0$ satisfy*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \left(\displaystyle\int_{\mathbb{R}} J(x-y)[u(y,t)-u(x,t)]dy\right) - f(x,u(x,t)) \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (x,t)\in\mathbb{R}\times(0,T], \\ u(x,0)\geq 0,\ x\in\mathbb{R}, \end{cases}$$

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} - \left(\displaystyle\int_{\mathbb{R}} J(x-y)[v(y,t)-v(x,t)]dy\right) - f(x,v(x,t)) \leq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (x,t)\in\mathbb{R}\times(0,T], \\ v(x,0)\leq 0,\ x\in\mathbb{R}, \end{cases}$$

respectively. Then,

$$u(x,t)\geq v(x,t) \text{ in } \mathbb{R}\times[0,T].$$

Furthermore, if $u(x,0)\not\equiv v(x,0)$ for $x\in\mathbb{R}$, then $u(x,t)>v(x,t)$ for $x\in\mathbb{R}$, $t\in[0,T]$.

Proof of Proposition 1. Let $\bar{w}(x,t)=u(x,t)-v(x,t)$. In fact, it is sufficient to show $\bar{w}(x,t)\geq 0$ for $(x,t)\in\mathbb{R}\times[0,\epsilon_0 T]$ with $\epsilon_0\in(0,1)$. Otherwise, suppose that $\inf_{\mathbb{R}\times[0,\epsilon_0 T]}\bar{w}(x,t)<0$. Denote $\breve{w}(x,t)=e^{Zt}\bar{w}(x,t)$, where $Z=\|f_u(x,u)\|_\infty+1$. It follows that $\inf_{\mathbb{R}\times[0,\epsilon_0 T]}\breve{w}(x,t)<0$. Then there exists a sequence $(x_n,t_n)\in\mathbb{R}\times(0,\epsilon_0 T]$ such that

$$\lim_{n\to+\infty}\breve{w}(x_n,t_n)=\inf_{\mathbb{R}\times[0,\epsilon_0 T]}\breve{w}(x,t)<0.$$

Observe that

$$\begin{aligned} &\breve{w}_t(x,t)\\ =&Z\breve{w}(x,t)+e^{Zt}\bar{w}_t(x,t)\\ \geq&Z\breve{w}(x,t)+e^{Zt}\int_{\mathbb{R}}J(x-y)[\bar{w}(y,t)-\bar{w}(x,t)]dy+e^{Zt}[f(x,u(x,t))-f(x,v(x,t))]\\ =&Z\breve{w}(x,t)+\int_{\mathbb{R}}J(x-y)[\breve{w}(y,t)-\breve{w}(x,t)]dy+f_u(x,u_\theta(x,t))\breve{w}(x,t). \end{aligned}$$

where $u_\theta(x,t)$ is between $v(x,t)$ and $u(x,t)$. This implies that

$$\begin{aligned} &\breve{w}(x_n,t_n)-\breve{w}(x_n,0)\\ \geq&\int_0^{t_n}\left[J*\breve{w}(x_n,s)-\breve{w}(x_n,s)+Z\breve{w}(x_n,s)+f_u(x,u_\theta(x,s))\breve{w}(x_n,s)\right]ds\\ \geq&\int_0^{t_n}\left[J*\breve{w}(x_n,s)+(\|f_u(x,u)\|_\infty-f_u(x,u_\theta(x,s)))\inf_{\mathbb{R}\times[0,\epsilon_0 T]}\breve{w}(x,t)\right]ds. \end{aligned}$$

Letting n converge to infinity, we have

$$\inf_{\mathbb{R}\times[0,\epsilon_0 T]}\breve{w}(x,t)\geq(1+\|f_u(x,u)\|_\infty-\min_{\mathbb{R}\times[0,1]}|f_u(x,u_\theta(x,s))|)\epsilon_0 T\inf_{\mathbb{R}\times[0,\epsilon_0 T]}\breve{w}(x,t).$$

Since $\inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t) < 0$, we can choose ϵ_0 sufficiently small such that

$$\begin{aligned}\inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t) \geq &(1+\|f_u(x,u)\|_\infty - \min_{\mathbb{R}\times[0,1]} |f_u(x,u_\theta(x,s))|)\epsilon_0 T \inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t)\\ &> \inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t).\end{aligned}$$

Thus, we get a contradiction. Therefore, we obtain $u(x,t) \geq v(x,t)$ for $x \in \mathbb{R}$ and $t \in [0,T]$. The remaining part of this proposition can be proved similarly by replacing the auxiliary function $\breve{w}(x,t) = e^{Zt}\bar{w}(x,t)$ with $\breve{w}(x,t) = e^{Zt}\bar{w}(x,t) - \epsilon t$ for small $\epsilon > 0$. In fact, we can similarly obtain

$$\begin{aligned}\inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t) \geq &\Bigg(1+\|f_u(x,u)\|_\infty - \min_{\mathbb{R}\times[0,1]} |f_u(x,u_\theta(x,s))|\\ &- \frac{\epsilon}{\inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t)}\Bigg)\epsilon_0 T \inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t).\end{aligned}$$

Here, we can also choose ϵ_0 such that

$$\left(1+\|f_u(x,u)\|_\infty - \min_{\mathbb{R}\times[0,1]} |f_u(x,u_\theta(x,s))| - \frac{\epsilon}{\inf_{\mathbb{R}\times[0,\epsilon_0 T]} \breve{w}(x,t)}\right)\epsilon_0 T < 1.$$

This finishes the proof. □

3. **Construction of the entire solution.**

In this section, we focus on the construction of the entire solution which behaves like a traveling wave approaching from infinity. The main idea is to establish suitable sub- and super-solutions, which are defined as follows.

$$W^-(x,t) = \begin{cases}\phi_1(x+c_1 t-\xi(t)) - \phi_1(-x+c_1 t-\xi(t)),\ x \geq 0,\\ 0,\ x < 0,\end{cases}$$

and

$$W^+(x,t) = \begin{cases}\phi_1(x+c_1 t+\xi(t)) + \phi_1(-x+c_1 t+\xi(t)),\ x \geq 0,\\ 2\phi_1(c_1 t+\xi(t)),\ x < 0,\end{cases}$$

here $\xi(t)$ is the solution of the following equation

$$\dot{\xi}(t) = Me^{\lambda(c_1 t+\xi)},\ t < -T,\ \xi(-\infty) = 0,$$

where $M,\ \lambda$ and T are positive constants to be specified later. A direct calculation yields that

$$\xi(t) = \frac{1}{\lambda}\ln\frac{1}{1-c_1^{-1}Me^{\lambda c_1 t}}.$$

For the function $\xi(t)$ to be defined, one must have $1 - c_1^{-1}Me^{\lambda c_1 t} > 0$. Besides, there is

$$c_1 t + \xi(t) \leq 0 \text{ for } -\infty < t \leq -T,$$

where $T := \frac{1}{\lambda c_1}\ln\frac{c_1+M}{c_1}$.

Now we verify that $W^{\mp}$ are sub- and super-solutions of 1 for $t \leq -T$. Define

$$\mathcal{L}u = u_t - (J*u-u) - f(x,u).$$

3.1. **Sub-solution.** We first deal with the sub-solution $W^-(x,t)$. Since it is obvious for $x<0$, we only consider the case $x\geq 0$. A straightforward calculation yields that

$$\begin{aligned}\mathcal{L}W^-(x,t)=&(c_1-\dot{\xi}(t))[\phi_1'(x+c_1t-\xi(t))-\phi_1'(-x+c_1t-\xi(t))]\\&-\int_0^{+\infty}J(x-y)[\phi_1(y+c_1t-\xi(t))-\phi_1(-y+c_1t-\xi(t))]dy\\&+[\phi_1(x+c_1t-\xi(t))-\phi_1(-x+c_1t-\xi(t))]-f(x,W^-)\\=&-\dot{\xi}(t)[\phi_1'(x+c_1t-\xi(t))-\phi_1'(-x+c_1t-\xi(t))]\\&+\int_{-\infty}^0 J(x-y)[\phi_1(y+c_1t-\xi(t))-\phi_1(-y+c_1t-\xi(t))]dy\\&+f_1(\phi_1(x+c_1t-\xi(t)))-f_1(\phi_1(-x+c_1t-\xi(t)))-f(x,W^-).\end{aligned}$$

Recall that $f(x,u)=f_1(u)$ for $x\geq 0$ and $\phi_1'>0$, it follows that

$$\begin{aligned}&\mathcal{L}W^-(x,t)\\\leq&-\dot{\xi}(t)[\phi_1'(x+c_1t-\xi(t))-\phi_1'(-x+c_1t-\xi(t))]+f_1(\phi_1(x+c_1t-\xi(t)))\\&-f_1(\phi_1(-x+c_1t-\xi(t)))-f_1(\phi_1(x+c_1t-\xi(t))-\phi_1(-x+c_1t-\xi(t))).\end{aligned}$$

Then we continue to show $\mathcal{L}W^-(x,t)\leq 0$ in two cases $0<x<-c_1t+\xi(t)$ and $x\geq -c_1t+\xi(t)$.

Case 1. For $0<x<-c_1t+\xi(t)$, similar to [4], we have

$$\phi_1'(x+c_1t-\xi(t))-\phi_1'(-x+c_1t-\xi(t))>m[\phi_1(x+c_1t-\xi(t))-\phi_1(-x+c_1t-\xi(t))]$$

for some $m>0$. Consequently,

$$\begin{aligned}\mathcal{L}W^-(x,t)\leq&-\dot{\xi}(t)[\phi_1'(x+c_1t-\xi(t))-\phi_1'(-x+c_1t-\xi(t))]\\&+L_f\phi_1(-x+c_1t-\xi(t)))[\phi_1(x+c_1t-\xi(t))-\phi_1(-x+c_1t-\xi(t))]\\\leq&[\phi_1(x+c_1t-\xi(t))-\phi_1(-x+c_1t-\xi(t))]\big[-mMe^{\lambda(c_1t+\xi(t))}\\&+L_f\alpha_0e^{\lambda_0(-x+c_1t-\xi(t))}\big]\\=&[\phi_1(x+c_1t-\xi(t))-\phi_1(-x+c_1t-\xi(t))]e^{\lambda(c_1t+\xi(t))}\big[-mM\\&+L_f\alpha_0e^{\lambda_0(-x-2\xi(t))}\big].\end{aligned}$$

Therefore, if we choose $M\geq\frac{L_f\alpha_0}{m}$ then $\mathcal{L}W^-(x,t)\leq 0$.

Case 2. For $x\geq -c_1t+\xi(t)$, if $\lambda_0\geq\lambda_1$, then we obtain

$$\begin{aligned}\mathcal{L}W^-(x,t)\leq&-Me^{\lambda(c_1t+\xi(t))}\left[\tilde{\beta}_1e^{-\lambda_1(x+c_1t-\xi(t))}-\tilde{\alpha}e^{\lambda_0(-x+c_1t-\xi(t))}\right]\\&+L_f\alpha_0e^{\lambda_0(-x+c_1t-\xi(t))}\left[\alpha_1e^{-\lambda_1(x+c_1t-\xi(t))}-\beta_0e^{\lambda_0(-x+c_1t-\xi(t))}\right]\\\leq&-e^{\lambda_0(-x+c_1t-\xi(t))}\big[M\tilde{\beta}_1e^{(\lambda_0-\lambda_1)x}e^{-(\lambda_0+\lambda_1-\lambda)c_1t+(\lambda+\lambda_0+\lambda_1)\xi(t))}\\&-M\tilde{\alpha}e^{\lambda(c_1t+\xi(t))}-L_f\alpha_0\big].\end{aligned}$$

Since $c_1t+\xi(t)<0$ for $t<-T$ and $\lambda<\min\{\lambda_0,\lambda_1\}$, $\mathcal{L}W^-(x,t)\leq 0$ provided $M\geq\frac{2L_f\alpha_0}{\tilde{\beta}_1}$.

For $\lambda_0 < \lambda_1$, which means $f_1'(0) > f_1'(1)$, we have

$$\begin{aligned}&\mathcal{L}W^-(x,t)\\ \le& Me^{\lambda(c_1t+\xi(t))}\phi_1'(-x+c_1t-\xi(t)) + f_1(\phi_1(x+c_1t-\xi(t)))\\ &- f_1(\phi_1(-x+c_1t-\xi(t))) - f_1(\phi_1(x+c_1t-\xi(t)) - \phi_1(-x+c_1t-\xi(t))).\end{aligned}$$

Moreover, if $x+c_1t-\xi(t) > L$ for some $L \gg 1$ such that $\phi_1(x+c_1t-\xi(t)) \in [1-\sigma, 1]$ and $\phi_1(-x+c_1t-\xi(t)) \in [0,\sigma]$. It follows that

$$\begin{aligned}\mathcal{L}W^-(x,t) \le& Me^{\lambda(c_1t+\xi(t))}\phi_1'(-x+c_1t-\xi(t)) + [f_1'(1) - f'(0)]\phi_1(-x+c_1t-\xi(t))\\ &+ o(\phi_1(-x+c_1t-\xi(t)))\\ \le& e^{\lambda_0(-x+c_1t-\xi(t))}\left[M\tilde{\alpha}_0 e^{\lambda(c_1t+\xi(t))} + (f_1'(1) - f'(0)) - o(1)\right].\end{aligned}$$

Since $t < -T$ such that $c_1t + \xi(t) \ll -1$, we have $\mathcal{L}W^-(x,t) \le 0$. Finally, if $0 < x + c_1t - \xi(t) \le L$, then, from the above case $\lambda_0 \ge \lambda_1$, there holds

$$\begin{aligned}\mathcal{L}W^-(x,t) \le& -e^{\lambda_0(-x+c_1t-\xi(t))}\Big[M\tilde{\beta}_1 e^{(\lambda_0-\lambda_1)x} e^{-(\lambda_0+\lambda_1-\lambda)c_1t+(\lambda+\lambda_0+\lambda_1)\xi(t))}\\ &- M\tilde{\alpha}e^{\lambda(c_1t+\xi(t))} - L_f\alpha_0\Big]\\ \le& -e^{\lambda_0(-x+c_1t-\xi(t))}\big[M\tilde{\beta}_1 e^{(\lambda_0-\lambda_1)L} e^{(\lambda-2\lambda_0)c_1t+(\lambda+2\lambda_0)\xi(t))}\\ &- M\tilde{\alpha}e^{\lambda(c_1t+\xi(t))} - L_f\alpha_0\big].\end{aligned}$$

Thanks to $t < -T$, $\lambda < \min\{\lambda_0, \lambda_1\}$ and $M > \frac{2L_f\alpha_0}{\tilde{\beta}_1}e^{(\lambda_1-\lambda_0)L}$, we obtain

$$\mathcal{L}W^-(x,t) \le 0.$$

3.2. **Super-solution.** We intend to testify the super-solution $W^+(x,t)$ in two steps.

Step 1. $W^+(x,t) = 2\phi_1(c_1t + \xi(t))$ for $x < 0$. Following a direct calculation, we have

$$\begin{aligned}\mathcal{L}W^+(x,t) =& 2(c_1 + \dot{\xi}(t))\phi_1'(c_1t+\xi(t)) - \int_{\mathbb{R}} J(x-y)W^+(y,t)dy\\ &+ 2\phi_1(c_1t+\xi(t)) - f(x, 2\phi_1(c_1t+\xi(t)))\\ \ge& 2(c_1 + \dot{\xi}(t))\phi_1'(c_1t+\xi(t)) - f_1(2\phi_1(c_1t+\xi(t)))\\ &- \int_0^{+\infty} J(x-y)[\phi_1(y+c_1t+\xi(t)) + \phi_1(-y+c_1t+\xi(t))]dy.\end{aligned}$$

Denote

$$I = \int_0^{+\infty} J(x-y)[\phi_1(y+c_1t+\xi(t)) + \phi_1(-y+c_1t+\xi(t))]dy.$$

Then, it follows that

$$\begin{aligned}I =& \int_0^{-c_1t-\xi(t)} J(x-y)[\phi_1(y+c_1t+\xi(t)) + \phi_1(-y+c_1t+\xi(t))]dy\\ &+ \int_{-c_1t-\xi(t)}^{+\infty} J(x-y)[\phi_1(y+c_1t+\xi(t)) + \phi_1(-y+c_1t+\xi(t))]dy\\ \le& \alpha_0 e^{\lambda_0(c_1t+\xi(t))}\int_0^{+\infty} J(x-y)\left[e^{\lambda_0 y} + e^{-\lambda_0 y}\right]dy + 2\int_{-c_1t-\xi(t)}^{+\infty} J(x-y)dy.\end{aligned}$$

Let

$$J_{\lambda_0} = \int_0^{+\infty} J(y) \left[e^{\lambda_0 y} + e^{-\lambda_0 y} \right] dy.$$

It follows from assumption (J) that there exists $K_J > 0$ such that

$$J(x) \leq K_J e^{2\lambda_0 x} \text{ for } x \geq 0.$$

Furthermore, there hold

$$\begin{aligned} I \leq & J_{\lambda_0} \alpha_0 e^{\lambda_0 (c_1 t + \xi(t))} + 2K_J \int_{-c_1 t - \xi(t)}^{+\infty} e^{2\lambda_0 (x-y)} dy \\ \leq & J_{\lambda_0} \alpha_0 e^{\lambda_0 (c_1 t + \xi(t))} + \frac{2K_J}{\lambda_0} e^{2\lambda_0 (c_1 t + \xi(t))}. \end{aligned}$$

Therefore, according to $t < -T$ such that $\phi_1(c_1 t + \xi(t)) \ll 1$, we get

$$\begin{aligned} \mathcal{L}W^+(x,t) \geq & 2(c_1 + \dot{\xi}(t)) \phi_1'(c_1 t + \xi(t)) - J_{\lambda_0} \alpha_0 e^{\lambda_0 (c_1 t + \xi(t))} \\ & - \frac{2K_J}{\lambda_0} e^{2\lambda_0 (c_1 t + \xi(t))} - f_1(2\phi_1(c_1 t + \xi(t))) \\ \geq & \left(2c_1 \tilde{\beta}_0 - \alpha_0 J_{\lambda_0} - 2f_1'(0)\beta_0 - o(1) \right) e^{\lambda_0 (c_1 t + \xi(t))} \\ & + \left(2M\tilde{\beta}_0 - \frac{2K_J}{\lambda_0} \right) e^{2\lambda_0 (c_1 t + \xi(t))}. \end{aligned}$$

Since

$$M > \frac{K_J}{\tilde{\beta}_0 \lambda_0} \text{ and } J_{\lambda_0} < \frac{2c_1 \tilde{\beta}_0 + 2|f_1'(0)|\beta_0}{\alpha_0},$$

consequently, we have $\mathcal{L}W^+(x,t) \geq 0$ for $t < -T$.

Step 2. For $x \geq 0$, $W^+(x,t) = \phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t))$. Then we can obtain

$$\begin{aligned} & \mathcal{L}W^+(x,t) \\ = & (c_1 + \dot{\xi}(t))[\phi_1'(x + c_1 t + \xi(t)) + \phi_1'(-x + c_1 t + \xi(t))] - \int_{\mathbb{R}} J(x-y) W^+(y,t) dy \\ & + [\phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t))] - f_1(W^+) \\ = & \dot{\xi}(t)[\phi_1'(x + c_1 t + \xi(t)) + \phi_1'(-x + c_1 t + \xi(t))] \\ & + \int_{-\infty}^0 J(x-y)[\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t)) - 2\phi_1(c_1 t + \xi(t))] dy \\ & + f_1(\phi_1(x + c_1 t + \xi(t))) + f_1(\phi_1(-x + c_1 t + \xi(t))) \\ & - f_1(\phi_1(x + c_1 t + \xi(t)) + \phi_1(-x + c_1 t + \xi(t))). \end{aligned}$$

Denote

$$II = \int_{-\infty}^0 J(x-y)[\phi_1(y + c_1 t + \xi(t)) + \phi_1(-y + c_1 t + \xi(t)) - 2\phi_1(c_1 t + \xi(t))] dy.$$

We consider two cases.

Case 1. $0 \le x \le -c_1 t - \xi(t)$. Without loss of generality, let $\phi_1(0) = \theta_1$. Since $\phi_1' > 0$ and $t < -T$ such that $\phi_1(c_1 t + \xi(t)) < \frac{1}{2}\theta_1$, we have

$$\begin{aligned}
II =& \int_{c_1t+\xi(t)}^{0} J(x-y)[\phi_1(y+c_1t+\xi(t)) + \phi_1(-y+c_1t+\xi(t)) \\
&- 2\phi_1(c_1t+\xi(t))]dy + \int_{-\infty}^{c_1t+\xi(t)} J(x-y)[\phi_1(y+c_1t+\xi(t)) \\
&+ \phi_1(-y+c_1t+\xi(t)) - 2\phi_1(c_1t+\xi(t))]dy \\
\ge& \int_{c_1t+\xi(t)}^{0} J(x-y)C_{\lambda_0}e^{\lambda_0(c_1t+\xi(t))}\Big[e^{\lambda_0 y} + e^{-\lambda_0 y} - 2 \\
&- \left(2 + e^{(\lambda_0+\eta)y} + e^{-(\lambda_0+\eta)y}\right) C_\eta e^{\eta(c_1t+\xi(t))}\Big]dy \\
\ge& -(2+J_\eta)C_{\lambda_0}C_\eta e^{(\lambda_0+\eta)(c_1t+\xi(t))}.
\end{aligned}$$

The second inequality follows from

$$\left|\phi_1(x) - C_{\lambda_0}e^{\lambda_0 x}\right| \le C_\eta e^{(\lambda_0+\eta)x} \text{ for } x \le 0 \text{ and some } 0 < \eta < \lambda_0,$$

which can be easily obtained by 5 and 6. As a consequence,

$$\begin{aligned}
\mathcal{L}W^+(x,t) \ge& M e^{(\lambda+\lambda_0)(c_1t+\xi(t))}\tilde{\beta}_0\left(e^{\lambda_0 x} + e^{-\lambda_0 x}\right) - (2+J_\eta)C_{\lambda_0}C_\eta e^{(\lambda_0+\eta)(c_1t+\xi(t))} \\
&- L_f\alpha_0^2 e^{2\lambda_0(c_1t+\xi(t)))} \\
\ge& e^{(\lambda+\lambda_0)(c_1t+\xi(t))}\big[2M\tilde{\beta}_0 - (2+J_\eta)C_{\lambda_0}C_\eta e^{(\eta-\lambda)(c_1t+\xi(t))} \\
&- L_f\alpha_0^2 e^{(\lambda_0-\lambda)(c_1t+\xi(t))}\big] \\
\ge& e^{(\lambda+\lambda_0)(c_1t+\xi(t))}\left[2M\tilde{\beta}_0 - (2+J_\eta)C_{\lambda_0}C_\eta - L_f\alpha_0^2\right].
\end{aligned}$$

The last inequality holds since $\lambda < \min\{\lambda_0, \lambda_1, \eta\}$. Thus $\mathcal{L}W^+(x,t) \ge 0$, provided that

$$M \ge \frac{(2+J_\eta)C_{\lambda_0}C_\eta + L_f\alpha_0^2}{2\tilde{\beta}_0}.$$

Case 2. Here $x > -c_1t - \xi(t)$. From Case 1, we know

$$\begin{aligned}
II \ge& -\int_{c_1t+\xi(t)}^{0} J(x-y)C_{\lambda_0}e^{\lambda_0(c_1t+\xi(t))} \\
&\left(2 + e^{(\lambda_0+\eta)y} + e^{-(\lambda_0+\eta)y}\right) C_\eta e^{\eta(c_1t+\xi(t))}dy \\
\ge& -C_{\lambda_0}C_\eta K_J e^{(\lambda_0+\eta)(c_1t+\xi(t))}\int_{c_1t+\xi(t)}^{0} e^{-2\lambda_0(x-y)}\left(3 + e^{-(\lambda_0+\eta)y}\right)dy \\
\ge& -C_{\lambda_0}C_\eta K_J e^{\lambda_0(-x+c_1t+\xi(t))-\lambda_0 x}\left(\frac{3}{\lambda_0}e^{\eta(c_1t+\xi(t))} + \frac{1}{\eta}\right).
\end{aligned}$$

Moreover, if $\lambda_1 \le \lambda_0$, then

$$
\begin{aligned}
&\mathcal{L}W^+(x,t)\\
\ge& Me^{\lambda(c_1t+\xi(t))}\left(\tilde{\beta}_1 e^{-\lambda_1(x+c_1t+\xi(t))}+\tilde{\beta}_0 e^{\lambda_0(-x+c_1t+\xi(t))}\right)\\
&-L_f\alpha_0 e^{\lambda_0(-x+c_1t+\xi(t))}-C_{\lambda_0}C_\eta K_J e^{\lambda_0(-x+c_1t+\xi(t))}\left(\frac{3}{\lambda_0}e^{\eta(c_1t+\xi(t))}+\frac{1}{\eta}\right)\\
\ge& e^{\lambda(c_1t+\xi(t))+\lambda_0(-x+c_1t+\xi(t))}\Bigg[M\tilde{\beta}_1 e^{(\lambda_0-\lambda_1)x-(\lambda_0+\lambda_1)(c_1t+\xi(t))}\\
&-\frac{3C_{\lambda_0}C_\eta K_J}{\lambda_0}e^{(\eta-\lambda)(c_1t+\xi(t))}-\left(L_f\alpha_0+\frac{C_{\lambda_0}C_\eta K_J}{\eta}\right)e^{-\lambda(c_1t+\xi(t))}\Bigg].
\end{aligned}
$$

Remember that $\lambda < \min\{\lambda_0,\lambda_1,\eta\}$ and $t < -T$. As a matter of fact, take M sufficiently large such that

$$
M\tilde{\beta}_1-\frac{3C_{\lambda_0}C_\eta K_J}{\lambda_0}-\left(L_f\alpha_0+\frac{C_{\lambda_0}C_\eta K_J}{\eta}\right)\ge 0. \tag{7}
$$

This yields that $\mathcal{L}W^+(x,t)\ge 0$.

If $\lambda_0<\lambda_1$, we have $f_1'(0)>f_1'(1)$. In addition, for $x>-c_1t-\xi(t)+\mathcal{M}$ with $\mathcal{M}\gg 1$ such that $\phi_1(x+c_1t+\xi(t))\in[1-\sigma,1]$, we have

$$
\begin{aligned}
&f_1(\phi_1(x+c_1t+\xi(t)))+f_1(\phi_1(-x+c_1t+\xi(t)))-f_1(W^+)\\
=&[f_1'(0)-f_1'(1)]\phi_1(-x+c_1t+\xi(t))+o(\phi_1(-x+c_1t+\xi(t))).
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
\mathcal{L}W^+(x,t)\ge& e^{\lambda(c_1t+\xi(t))+\lambda_0(-x+c_1t+\xi(t))}\Bigg(M\tilde{\beta}_1 e^{(\lambda_0-\lambda_1)x-(\lambda_0+\lambda_1)(c_1t+\xi(t))}\\
&-\frac{3C_{\lambda_0}C_\eta K_J}{\lambda_0}e^{(\eta-\lambda)(c_1t+\xi(t))}\Bigg)+\Bigg([f_1'(0)-f_1'(1)]\beta_0\\
&-\frac{C_{\lambda_0}C_\eta K_J}{\eta}e^{-\lambda_0x}\Bigg)e^{\lambda_0(-x+c_1t+\xi(t))}.
\end{aligned}
$$

Then $\mathcal{L}W^+(x,t)\ge 0$ by 7 and $-x<c_1t+\xi(t)-\mathcal{M}\ll -1$ such that

$$
[f_1'(0)-f_1'(1)]\beta_0-\frac{C_{\lambda_0}C_\eta K_J}{\eta}e^{-\lambda_0x}\ge 0.
$$

When $-c_1t-\xi(t)<x\le -c_1t-\xi(t)+\mathcal{M}$, it follows from the case $\lambda_0\ge\lambda_1$ that

$$
\begin{aligned}
\mathcal{L}W^+(x,t)\ge& e^{\lambda(c_1t+\xi(t))+\lambda_0(-x+c_1t+\xi(t))}\Bigg[M\tilde{\beta}_1 e^{(\lambda_0-\lambda_1)x-(\lambda_0+\lambda_1)(c_1t+\xi(t))}\\
&-\frac{3C_{\lambda_0}C_\eta K_J}{\lambda_0}e^{(\eta-\lambda)(c_1t+\xi(t))}-\left(L_f\alpha_0+\frac{C_{\lambda_0}C_\eta K_J}{\eta}\right)e^{-\lambda(c_1t+\xi(t))}\Bigg]\\
\ge& e^{\lambda(c_1t+\xi(t))+\lambda_0(-x+c_1t+\xi(t))}\Bigg[M\tilde{\beta}_1 e^{(\lambda_0-\lambda_1)M-2\lambda_0(c_1t+\xi(t))}\\
&-\frac{3C_{\lambda_0}C_\eta K_J}{\lambda_0}e^{(\eta-\lambda)(c_1t+\xi(t))}-\left(L_f\alpha_0+\frac{C_{\lambda_0}C_\eta K_J}{\eta}\right)e^{-\lambda(c_1t+\xi(t))}\Bigg].
\end{aligned}
$$

Similar to the case $\lambda_1\le\lambda_0$, we can obtain $\mathcal{L}W^+(x,t)\ge 0$ for $t<-T$ by letting M be sufficiently large.

Proof of Theorem 1.1. Let $u_n(x,t)$ be the unique solution of the Cauchy problem

$$\begin{cases}(u_n)_t(x,t) = J*u_n(x,t)-u_n(x,t)+f(x,u_n),\ x\in\mathbb{R},\ t>-n,\\ u_n(x,-n) = W^-(x,-n),\ x\in\mathbb{R}.\end{cases}$$

Since $W^-(x,-n) = u_n(x,-n) \le W^+(x,-n)$, the comparison principle yields for $n > T$

$$W^-(x,t) \le u_n(x,t) \le W^+(x,t) \text{ for } x\in\mathbb{R},\ t\in[-n,-T].$$

Moreover, since $m := \min_{\mathbb{R}\times[0,1]}(1-f_u(x,u)) > 0$ by the assumption $f_u(x,u) < 1$, it is not difficult to see that $\delta u_n(x,t) = u_n(x+\eta,t) - u_n(x,t)$ with $\eta\in\mathbb{R}$ is a sub-solution of

$$\begin{cases}v'(t) = L\eta - mv(t),\ t>-n,\\ v(-n) = M\eta,\end{cases}$$

where $L \ge \int_{\mathbb{R}} |J'(y)|dy$ and M is some positive constant. It is then similar to the proof of Proposition 2.4 in [23] that $u_n(x,t)$ and $(u_n)_t(x,t)$ are Lipschiz continuous with respect to x. Besides, a direct calculation gives that $|u_n(x,t)| < C$ and $|(u_n)_t(x,t)| < C$ for some positive constant C. Then applying Arzela-Ascoli Theorem, there exists a subsequence of $\{u_n\}_{n=1}^{+\infty}$ and $\{(u_n)_t\}_{n=1}^{+\infty}$, still denoted by $\{u_n\}_{n=1}^{+\infty}$ and $\{(u_n)_t\}_{n=1}^{+\infty}$, such that $\{u_n\}_{n=1}^{+\infty}$ and $\{(u_n)_t\}_{n=1}^{+\infty}$ converge to a function $u(x,t)$ and $u_t(x,t)$. Now we have obtained the entire solution. In fact, the regularity of this entire solution is not so well. But we can show the entire solution is Lipschitz continuous in $x\in\mathbb{R}$, which is important to show the uniqueness of the entire solution.

Proposition 2. *Let $u(x,t)$ be the entire solution in Theorem 1.1. Then $u(x,t)$ satisfies*

$$|u(x+\eta,t)-u(x,t)| \le M'\eta,$$

and

$$\left|\frac{\partial u(x+\eta,t)}{\partial t} - \frac{\partial u(x,t)}{\partial t}\right| \le M''\eta$$

for $M',\ M'' > 0$.

The proof is similar to that of Proposition 3.1 in [23] by virtue of the fact $f_u(x,u) < 1$ for $x\in\mathbb{R}$ and $u\in[0,1]$, so we omit it here.

As for the uniqueness, since it is easy to see $u(x,t)\to\phi_1(x+c_1t)$ as $t\to-\infty$ by the sub- and super-solutions established above, we can similarly show that the conclusion of Lemma 3.1 in [4] also holds true here. Then referring to the process of Section 3 in [4], we can obtain the uniqueness of the entire solution by constructing similar sub- and super-solutions. Thus we have shown the first part of Theorem 1.1. □

4. **Long time asymptotic profile.** In this section, we are going to show the entire solution established in previous section converges to a shift of $\phi_2(x+c_2t)$ as $t\to+\infty$. The main idea here is first to construct suitable sub-and super-solutions taking advantage of the traveling wave $\phi_2(x+c_2t)$ to get proper lower and upper bounds, which is a classical and regular method in study of traveling waves. Then we apply a "squeezing" technique which is used to prove the asymptotic stability of

the traveling waves for nonlocal dispersal equation in [7] to obtain the convergence of the entire solution. Here, we assume that for some $X > 0,\ \omega,\ \sigma > 0$ and

$$\omega < \min\left\{\frac{|f_1'(0)|}{4}, \frac{|f_2'(0)|}{4}, \frac{|f_1'(1)|}{4}, \frac{|f_2'(1)|}{4}\right\},$$

there hold

$$\phi_1(x),\ \phi_2(x) \le \frac{\sigma}{2} \text{ for } x \le -X \text{ and } \phi_1(x),\ \phi_2(x) \ge 1 - \frac{\sigma}{2} \text{ for } x \ge X,$$

besides,

$$f_1'(s), f_2'(s) \le -\omega \text{ for } s \in [0,\sigma] \cup [1-\sigma, 1].$$

Theorem 4.1. *Let assumptions (F) and (J) hold, and $u(x,t)$ be the unique entire solution of 1 with $0 < u(x,t) < 1$, $u_t(x,t) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$ such that*

$$u(x,t) - \phi_1(x + c_1 t) \to 0 \textit{ as } t \to -\infty \textit{ uniformly in } x \in \mathbb{R}. \tag{8}$$

Then

$$u(x,t) - \phi_2(x + c_2 t + \beta) \to 0 \textit{ as } t \to +\infty \textit{ uniformly in } x \in \mathbb{R}$$

for some $\beta \in \mathbb{R}$.

Lemma 4.2. *Suppose assumptions (F) and (J) hold. Then there exist $T_-,\ \beta_-,\ \delta_-,\ \omega > 0$ such that*

$$u(x,t) \ge \max\left\{\phi_2(x + c_2 t - \beta_-) - \delta_- e^{-\omega t}, 0\right\}$$

for $x \in \mathbb{R}$ and $t \ge T_-$.

Proof of Lemma 4.2. In fact, it follows from 8 that there exists $t_- \in \mathbb{R}$ such that

$$|u(x,t_-) - \phi_1(x + c_1 t_-)| \le \frac{\epsilon_-}{2}$$

for any $\epsilon_- > 0$ and $x \in \mathbb{R}$. Denote

$$\tilde{u}(x,t) = u(x, t + t_-) \text{ and } \underline{u}(x,t) = \max\left\{\phi_2(\xi_-(x,t)) - C_v e^{-\omega t}, 0\right\},$$

where

$$\xi_-(x,t) = x + c_2(t + t_-) - \beta^0 + C^v e^{-\omega t} - C^v \text{ for } t \ge 0,\ x \in \mathbb{R}.$$

We first show $\tilde{u}(x,0) \ge \underline{u}(x,0)$ for $x \in \mathbb{R}$. Then there exists $x_1 > 0$ such that

$$\phi_1(x + c_1 t_-) \ge 1 - \frac{\epsilon_-}{2} \text{ and } \phi_2(x + c_2 t_-) \ge 1 - \frac{\epsilon_-}{2} \text{ for } x \ge x_1.$$

As $u_t > 0$ and $\phi_i' > 0$ $(i = 1, 2)$, we have

$$1 - \frac{\epsilon_-}{2} \le u(x,t) \le 1 \text{ and } 1 - \frac{\epsilon_-}{2} \le \phi_i \le 1,\ i = 1, 2$$

for $t \ge t_-,\ x \ge x_1$. Then,

$$|\phi_i - u(x,t)| \le \epsilon_-,\ i = 1, 2 \text{ for } t \ge t_-,\ x \ge x_1.$$

Meanwhile, there exists $x_2 \in \mathbb{R}$ such that

$$0 < \phi_i(x + c_i t_-) \le \frac{\epsilon_-}{2},\ i = 1, 2 \text{ for } x \le x_2.$$

Particularly, there holds

$$|u(x,t_-) - \phi_2(x + c_2 t_-)| \le \epsilon_- \text{ for } x \in \mathbb{R}\backslash(x_1, x_2).$$

In addition, $\min\limits_{x \in [x_1, x_2]} u(x,t) > 0$ since $0 < u(x,t) < 1$. Then we can choose $\beta^0 > 0$ such that

$$\phi_2(x + c_2 t_- - \beta^0) \le u(x, t_-) \text{ for } x \in [x_1, x_2].$$

From all the discussion above we obtain $\tilde{u}(x,0) \geq \underline{u}(x,0)$. Next, we prove

$$\mathcal{L}\underline{u}(x,t) = \underline{u}_t(x,t) - (J * \underline{u}(x,t) - \underline{u}(x,t)) - f(x,\underline{u}) \leq 0 \text{ for } t \geq 0,\ x \in \mathbb{R}.$$

In fact, we only need to consider $\underline{u}(x,t) = \phi_2(\xi_-(x,t)) - C_v e^{-\omega t}$ because 0 is obviously a sub-solution of 1. Hence,

$$\begin{aligned}\mathcal{L}\underline{u}(x,t) = &\left(c_2 - C^v \omega e^{-\omega t}\right)\phi_2'(\xi_-(x,t)) + C_v \omega e^{-\omega t} - \int_{\mathbb{R}} J(x-y)\phi_2(\xi_-(y,t))dy \\ &+ \phi_2(\xi_-(x,t)) - f(x,\underline{u}(x,t)) \\ \leq &- C^v \omega e^{-\omega t}\phi_2'(\xi_-(x,t)) + f_2(\phi_2(\xi_-(x,t))) \\ &- f_2(\phi_2(\xi_-(x,t)) - C_v e^{-\omega t}) + C_v \omega e^{-\omega t}.\end{aligned}$$

In the following, we continue the proof in three cases.

Case $\xi_-(x,t) \leq -X$. In this case,

$$\phi_2(\xi_-) - C_v e^{-\omega t} \leq \phi_2(\xi_-) \leq \sigma.$$

Then, by $\omega < \frac{|f_2'(0)|}{4}$,

$$\mathcal{L}\underline{u}(x,t) \leq (f_2'(0) + 2\omega)C_v e^{-\omega t} \leq 0.$$

Case $\xi_-(x,t) \geq X$. For this case, by $2C_v < \sigma$,

$$1 - \sigma \leq 1 - \frac{\sigma}{2} - C_v \leq \phi_2(\xi_-) - C_v e^{-\omega t} \leq \phi_2(\xi_-) < 1.$$

Then, since $\omega < \frac{|f_2'(1)|}{4}$,

$$\mathcal{L}\underline{u}(x,t) \leq (f_2'(1) + 2\omega)C_v e^{-\omega t} \leq 0.$$

Case $\xi_- \in [-X, X]$. Let $\tau_0 = \min\limits_{\xi \in [-X,X]} \phi_2'(\xi)$. Then,

$$\begin{aligned}\mathcal{L}\underline{u}(x,t) \leq &- \tau_0 C^v \omega e^{-\omega t} + \|f_2'\|_\infty C_v e^{-\omega t} + C_v \omega e^{-\omega t} \\ \leq &(\|f_2'\|_\infty C_v + + C_v \omega - \tau_0 C^v \omega)e^{-\omega t}.\end{aligned}$$

Choose $C^v = \tau_0^{-1}\omega_0^{-1}(\|f_2'\|_\infty C_v + + C_v\omega)$. This gives $\mathcal{L}\underline{u}(x,t) \leq 0$. We finish the proof by letting $\beta_- = \beta^0 + C^v$ and $\delta_- = C_v$. □

Similarly, we can prove the following lemma.

Lemma 4.3. *Under the assumptions of Lemma 4.2, there exist $T_+ > T_-$, β_+, C_+, $\omega > 0$ such that*

$$\min\left\{\phi_1(x + c_1 t + \beta_+) + C_+ e^{-\omega t}, 1\right\} \geq u(x,t)$$

for $x \in \mathbb{R}$, $t \geq T_+$, where ω, T_- are defined in Lemma 4.2.

Now we are in a position to establish an important lemma.

Lemma 4.4. *Under the assumptions of Lemma 4.2, there exist $T > \max\{T_+, T_-\}$, β, C, $\omega > 0$ such that*

$$\min\left\{\phi_2(x + c_2 t + \beta) + Ce^{-\omega t}, 1\right\} \geq u(x,t)$$

for $x \in \mathbb{R}$, $t \geq T$, where ω, T_-, T_+ are defined in Lemmas 4.2 and 4.3.

Proof of Lemma 4.4. Define $\hat{u}(x,t) = u(x, t+T)$ and $\overline{u}(x,t) = \min\{\phi_2(\xi_+(x,t)) + v(t), 1\}$, where

$$\xi_+(x,t) = x + c_2(t+T) + \beta^+ + V(t),\ V(t) = K_V \int_0^t v(s)ds,$$

and

$$v(t) = (\epsilon_1 + C_3)e^{-\omega t} - C_3 e^{-\lambda^1 c_2 t}$$

for $0 < \epsilon_1,\ C_3 < \frac{\sigma}{4}$ and $t \geq 0$. Here, λ^1 is the positive root of

$$c_2\lambda^1 = \int_{\mathbb{R}} J(x-y)e^{-\lambda^1 y}dy - 1 + f_2'(1).$$

Then

$$v(0) = \epsilon_1,\ 0 < v(t) < \frac{\sigma}{2} \text{ for } t \geq 0$$

and

$$\dot{v}(t) = -\omega v(t) - (w - \lambda^1 c_2)C_3 e^{\lambda^1 c_2 t} \text{ for } t \geq 0.$$

Now we intend to show $\hat{u}(x,0) \leq \overline{u}(x,0)$. Recall the fact that $|1 - \phi_2(x)| \leq C^2 e^{-\lambda^1 x}$ for $x \in \mathbb{R},\ C^2 > 0$. Meanwhile, we assume that $0 < \omega < \lambda^1 c_2$. Additionally, by the assumptions of $f(x,u)$, there exists $C_0 > 0$ such that

$$|f_1(s) - f_2(s)| \leq C_0|1-s| \text{ for } s \in [0,1].$$

Since $\phi_i(-\infty) = 0$ and $\phi_i(+\infty) = 1$, there exists $x_3 < 0$ such that

$$\phi_1(x + c_1 T + \beta_+) < \frac{\epsilon_1}{2} \text{ and } \phi_2(x + c_2 T) < \epsilon_1 \text{ for } x < x_3$$

as well as $x_4 > 0$ such that

$$\phi_2(x + c_2 T - \beta_-) \geq 1 - \epsilon_1 \text{ and } \phi_2(x + c_2 T) \geq 1 - \epsilon_1$$

for $x > x_4$. Note that we can take $T > 0$ sufficiently large such that

$$\max\{C_+, \delta_-\}e^{-\omega T} < \frac{\epsilon_1}{2} \text{ and } X - c_2 T \leq -x_0.$$

Therefore, it is obvious that

$$u(x,T) \leq \min\{\phi_2(x + c_2 T) + \epsilon_1, 1\} \leq \overline{u}(x,0) \text{ for } x \in \mathbb{R} \setminus (x_3, x_4).$$

Furthermore, since $\max_{x \in [x_3, x_4]} u(x,T) < 1$, one can choose $\beta^+ > 0$ such that

$$\phi_2(x + c_2 T + \beta^+) \geq u(x,T) \text{ for } x \in [x_3, x_4].$$

This yields that $\hat{u}(x,0) \leq \overline{u}(x,0)$.

In the following, we are going to prove

$$\mathcal{L}\overline{u}(x,t) = \overline{u}_t(x,t) - (J * \overline{u}(x,t) - \overline{u}(x,t)) - f(x, \overline{u}) \geq 0 \text{ for } t \geq 0,\ x \in \mathbb{R}.$$

Here we only need to consider $\overline{u}(x,t) = \phi_2(\xi_+(x,t)) + v(t)$ since 1 is obviously a super-solution of 1. A direct calculation gives that

$$\begin{aligned}
&\mathcal{L}\overline{u}(x,t)\\
=&\dot{\xi}_+(x,t)\phi_2'(\xi_+(x,t)) + \dot{v}(t) - \int_{\mathbb{R}} J(x-y)\phi_2(\xi_+(y,t))dy\\
&+ \phi_2(\xi_+(x,t)) - f(x, \phi_2(\xi_+(x,t)) + v(t))
\end{aligned}$$

$$
\begin{aligned}
&=\dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)+f_2(\phi_2(\xi_+(x,t)))-f(x,\phi_2(\xi_+(x,t))+v(t))\\
&\geq\begin{cases}\dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)+f_2(\phi_2(\xi_+(x,t)))-f_2(\phi_2(\xi_+(x,t))+v(t)) \text{ for } x<-L,\\ \dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)+f_2(\phi_2(\xi_+(x,t)))-f_1(\phi_2(\xi_+(x,t))+v(t)) \text{ for } x\geq -L.\end{cases}
\end{aligned}
$$

We continue to prove $\mathcal{L}\overline{u}(x,t)\geq 0$ in three cases.

Case 1. If $\xi_+>X$ and $x\geq -L$, then

$$
\begin{aligned}
\mathcal{L}\overline{u}(x,t)\geq&\dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)+f_1(\phi_2(\xi_+(x,t)))\\
&-f_1(\phi_2(\xi_+(x,t))+v(t))-C_0|1-\phi_2(\xi_+(x,t))|\\
\geq&\dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)-(f_1'(1)+\omega)\dot v(t)-C^2C_0e^{-\lambda^1(x+c_2(t+T)+\beta^++V(t))}\\
\geq&\dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)-(f_1'(1)+\omega)\dot v(t)-C^2C_0e^{-\lambda^1c_2t}e^{\lambda^1(L-c_2T)}.
\end{aligned}
$$

Particularly, choose C_3 such that

$$\dot v(t)=-\omega v(t)+C^2C_0e^{-\lambda^1c_2t}e^{\lambda^1(L-c_2T)},$$

which means

$$C_3=\frac{C^2C_0e^{\lambda^1(L-c_2T)}}{\lambda^1c_2-\omega}.$$

As a consequence,

$$\mathcal{L}\overline{u}(x,t)\geq-(f_1'(1)+2\omega)v(t)\geq 0.$$

In addition, if $\xi_+>X$ and $x<-L$, then

$$
\begin{aligned}
\mathcal{L}\overline{u}(x,t)\geq&-\omega v(t)-(w-\lambda^1c_2)C_3e^{\lambda^1c_2t}-(f_2'(1)+\omega)v(t)\\
\geq&-(f_2'(1)+2\omega)v(t)\\
\geq&0.
\end{aligned}
$$

Case 2. If $\xi_+<-X$, then $x<-L$. Similar to Case 1 with $x<-L$, we can obtain $\mathcal{L}\overline{u}(x,t)\geq 0$.

Case 3. For $\xi_+\in[-X,X]$, we know $x<-L$. Therefore,

$$
\begin{aligned}
\mathcal{L}\overline{u}(x,t)\geq&\dot V(t)\phi_2'(\xi_+(x,t))+\dot v(t)-\|f_2'\|_\infty v(t)\\
\geq&\dot V(t)\tau_0\dot V(t)-\omega v(t)-\|f_2'\|_\infty v(t)\\
\geq&(K_V\tau_0-\omega-\|f_2'\|_\infty)v(t),
\end{aligned}
$$

where τ_0 is defined as in Lemma 4.2. It follows from

$$K_V\geq\frac{\omega+\|f_2'\|_\infty}{\tau_0}$$

that $\mathcal{L}\overline{u}(x,t)\geq 0$. This finishes the proof by letting $\beta=\beta^++\epsilon_1+C_3$ and $C=C_3+\epsilon_1$. □

Now we are ready to show that the entire solution converges to a shift of $\phi_2(x+c_2t)$ using a "squeezing " technique [7]. We should declare that assumption (**A**) (see **(B2)** in [7]) holds.

(**A**) There exists a positive nonincreasing function $\zeta(n)$ defined on $[1,+\infty)$ such that for any $u_1(x,t)$, $u_2(x,t)$ satisfying $-1\leq u_1(x,t)$, $u_2(x,t)\leq 2$, $\mathcal{L}u_1(x,t)\geq 0$, $\mathcal{L}u_2(x,t)\leq 0$ and $u_1(x,0)\geq u_2(x,0)$, there holds

$$\min_{x\in[-n,n]}\{u_1(x,1)-u_2(x,1)\}\geq\zeta(n)\int_0^1[u_1(y,0)-u_2(y,0)]dy \text{ for } m\geq 1.$$

In fact, it follows from the comparison principle that $u(x,t) \geq v(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$. Furthermore, denote $\breve{w}(x,t) = e^{Kt}[u(x,t) - v(x,t)] \geq 0$ with $K \geq 1 + \max_{s\in[-1,2]} |f_s(x,s)|$. Then

$$\begin{aligned}\breve{w}_t(x,t) \geq& K\breve{w}(x,t) + J * \breve{w}(x,t) - \breve{w}(x,t) + f(x,u(x,t)) - f(x,v(x,t))\\ \geq& J * \breve{w}(x,t) + [K - (1 + \max_{s\in[-1,2]} |f_s(x,s)|)]\breve{w}(x,t)\\ \geq& J * \breve{w}(x,t),\end{aligned}$$

which implies that $\breve{w}(x,t) \geq \breve{w}(x,0)$ for $t \geq 0$. Therefore, $\breve{w}_t(x,t) \geq J * \breve{w}(x,0)$ and $\breve{w}(x,t_*) \geq t_* J * \breve{w}(x,0)$. Repeating the same progress on $[t_*, 2t_*]$, ..., $[(N-1), Nt_*]$, we have $\breve{w}(x,Nt_*) \geq t_*^N J * \breve{w}(x,0)$ for $N \geq 1$. In addition, since $J(x) \geq 0$ and $\int_{\mathbb{R}} J(y)dy = 1$, there exists $N_0 = N_0(M) \geq 1$ such that by letting $t_* = \frac{T}{N_0}$ we have

$$\breve{w}(x,T) \geq \left(\frac{T}{N_0}\right)^N J * \breve{w}(x,0) \geq \left(\frac{T}{N_0}\right)^N c(M) \int_0^1 \breve{w}(y,0)dy,$$

where

$$c(M) = \min_{x\in[-M-1,M+1]} J(x) > 0.$$

Thus, there exists a positive function $\eta(x,t) \in C([0,\infty),[0,\infty))$ such that

$$u(x,t) - v(x,t) \geq \eta(|x|,t) \int_0^1 [u(y,0) - v(y,0)]dy \text{ for } x \in \mathbb{R}, t > 0,$$

which implies the assumption (**A**) holds.

Now, we start to prove the following lemma, which plays an important role in the proof of Theorem 4.1.

Lemma 4.5. *Suppose that assumptions* (F) *and* (J) *hold. Then there exists a small* ϵ_0 *such that if for some* $\tau \geq 0$, $\xi \in \mathbb{R}$, $\delta \in (0, \frac{\sigma}{2}]$, *and* $h > 0$, *there holds*

$$\phi_2(x + c_2\tau + \xi) - \delta \leq u(x,t) \leq \phi_2(x + c_2\tau + \xi + h) + \delta \text{ for } x \in \mathbb{R}, \tag{9}$$

then for every $t > \tau + 1$, *there exist* $\tilde{\xi}(t)$, $\tilde{\delta}(t)$ *and* $\tilde{h}(t)$ *satisfying*

$$\begin{aligned}&\tilde{\xi}(t) \in \left[\xi - \delta\frac{\gamma}{C_m}, \xi + h + \delta\frac{\gamma}{C_m}\right],\\ &\tilde{\delta}(t) \leq e^{-\omega(t-\tau-1)}\left[\delta + \epsilon_0 \min\{h,1\}\right],\\ &\tilde{h}(t) \leq \tau h - \frac{\gamma\epsilon_0}{C_m}\min\{h,1\} + 2\delta\frac{\gamma}{C_m}\end{aligned}$$

such that 9 holds with (τ,ξ,δ,h) *replaced by* $(t,\tilde{\xi}(t),\tilde{\delta}(t),\tilde{h}(t))$. *Here,* $C_m := \max\{\delta_-, C\}$ *and* $\gamma := \max\left\{C^v, \frac{K_V}{\omega}\right\}$, *where the parameters are defined in Lemmas 4.2 and 4.4.*

Proof of Lemma 4.5. In view of Lemmas 4.2 and 4.4, it is easy to see that

$$\phi_2(x + c_2t - \beta_-) - \delta_- e^{-\omega t} \leq u(x,t) \leq \phi_2(x + c_2t + \beta) + Ce^{-\omega t}.$$

Furthermore, as in the proofs of Lemmas 4.2 and 4.4,

$$\phi_2\left(x + c_2t - \beta^0 + C^v e^{-\omega t} - C^v\right) - \delta_- e^{-\omega t} \leq u(x,t)$$

and

$$\phi_2\left(x + c_2t + \beta^+ + \frac{K_V}{\omega} - \frac{K_V}{\omega}e^{-\omega t}\right) + Ce^{-\omega t} \geq u(x,t).$$

Denote

$$\kappa = \max\{-\beta^0 - \gamma\},\ h = \beta^+ + \frac{K_V}{\omega} - \kappa.$$

In addition, by the definition of C_m, γ and letting $\check{u}(x,t) = u(x-\kappa,t)$, we have

$$\phi_2\left(x + c_2 t + \gamma e^{-\omega t}\right) - C_m e^{-\omega t} \le \check{u}(x,t) \le \phi_2\left(x + c_2 t + h + \gamma e^{-\omega t}\right) + C_m e^{-\omega t}.$$

Let $\overline{h} = \min\{h, 1\}$ and $\vartheta_0 = \frac{1}{2}\min_{[0,2]} \phi_2'(x)$. Then

$$\int_0^1 \left[\phi_2(y+\overline{h}) - \phi_2(y)\right] dy \ge 2\vartheta_0 \overline{h}.$$

Therefore, at least one of the following two inequalities is true

$$\text{(i)}\ \int_0^1 [\check{u}(y,0) - \phi_2(y)]dy \ge \vartheta_0\overline{h}, \qquad \text{(ii)}\ \int_0^1 [\phi_2(y+\overline{h}) - \check{u}(y,0)]dy \ge \vartheta_0\overline{h}.$$

Next, we consider the case (i) since the case (ii) is similar. According to the assumption (**A**), for $\zeta = \zeta(M + c_2 + 2)$ with $M \gg 1$ such that $\frac{\gamma}{C_m}\phi_2'(x) < 1$ for $|x| \ge M$, and for every $x \in [-M - c_2 - \gamma, M + c_2 + \gamma]$, there holds

$$\check{u}(x,1) - \left[\phi_2\left(x + \gamma e^{-\omega}\right) - C_m e^{-\omega}\right] \ge \zeta \int_0^1 [\check{u}(y,0) - (\phi_2(y) - C_m)]dy \ge \zeta\vartheta_0\overline{h}.$$

Now define

$$\epsilon_0 = \min\left\{\frac{\sigma}{2}, \frac{\gamma}{2C_m}, \min_{x\in[-M-c_2-\gamma, M+c_2+\gamma]} \frac{\zeta\vartheta_0\gamma}{2C_m\phi_2'(x)}\right\}.$$

Accordingly, there exists $\tilde{\theta} \in (0,1)$ such that

$$\begin{aligned}\phi_2\left(x + \gamma e^{-\omega} + 2\epsilon_0\overline{h}\frac{\gamma}{C_m}\right) - \phi_2\left(x + \gamma e^{-\omega}\right) =&\phi_2'\left(x + \gamma e^{-\omega} + 2\epsilon_0\overline{h}\tilde{\theta}\frac{\gamma}{C_m}\right)2\epsilon_0\overline{h}\frac{\gamma}{C_m}\\ \le&\zeta\vartheta_0\overline{h}\end{aligned}$$

for all $x \in [-M - c_2 - (\gamma - 1), M + c_2 + (\gamma - 1)]$. Hence,

$$\check{u}(x,1) \ge \phi_2\left(x + \gamma e^{-\omega} + 2\epsilon_0\overline{h}\frac{\gamma}{C_m}\right) - C_m e^{-\omega}$$

for $x \in [-M - c_2 - (\gamma - 1), M + c_2 + (\gamma - 1)]$.

For $|x| \ge M + c_2 + (\gamma - 1)$, by the choice of M, we know that

$$\phi_2\left(x + \gamma e^{-\omega}\right) \ge \phi_2\left(x + \gamma e^{-\omega} + 2\epsilon_0\overline{h}\frac{\gamma}{C_m}\right) - \epsilon_0\overline{h}.$$

Then, it follows that

$$u(x,1) \ge \phi_2\left(x + \kappa + \gamma e^{-\omega} + 2\epsilon_0\overline{h}\frac{\gamma}{C_m}\right) - \epsilon_0\overline{h} - C_m e^{-\omega} \text{ for } x \in \mathbb{R}.$$

Note that $p := \delta e^{-\omega} + \epsilon_0\overline{h} \le \sigma$, repeat the operation above with 1 replaced by $1 + t'$, which implies that

$$\begin{aligned}u(x,1+t') \ge&\phi_2\left(x + c_2 t + \kappa + \gamma e^{-\omega} + 2\epsilon_0\overline{h}\frac{\gamma}{C_m} + \gamma e^{-\omega t'}\right) - p e^{-\omega t'}\\ \ge&\phi_2\left(x + c_2(t'+1) + \kappa + \epsilon_0\overline{h}\frac{\gamma}{C_m} - \frac{\delta\gamma}{C_m}\right) - (\delta + \epsilon_0\overline{h})e^{-\omega t'}.\end{aligned}$$

Thus we finish the proof by setting

$$t = 1 + t',\ \tilde{\xi}(t) = \frac{\gamma\epsilon_0\overline{h}}{C_m},\ \tilde{\delta} = (\delta + \epsilon_0\overline{h})e^{-\omega(t-1)}$$
$$\tilde{h} = \left[h + \delta\frac{\gamma}{C_m}e^{-\omega t}\right] - \tilde{\xi}(t) = h - \frac{\epsilon_0\gamma}{C_m}\overline{h} + \delta\frac{\gamma}{C_m}\left[2 - e^{-\omega t}\right].$$

□

Now we shall prove Theorem 4.1.

Proof of Theorem 4.1. We shall divide the proof in three steps.

Step 1. Following from Lemmas 4.2 and 4.4, there exist $T^*,\ M^* > 0$ such that

$$\phi_2(x + c_2T^* - M^*) - C_m \le u(x, T^*) \le \phi_2(x + c_2T^* + M^*) + C_m \text{ for } x \in \mathbb{R}. \quad (10)$$

Here, C_m is defined as in the proof of Lemma 4.4. Define

$$\epsilon^* = \min\left\{\frac{\sigma}{2}, \frac{\epsilon_0}{4}\right\} \text{ and } k_0 = \epsilon_0\frac{\gamma}{C_m} - 2\epsilon^*\frac{\gamma}{C_m} \ge \frac{\gamma\epsilon_0}{2C_m} > 0.$$

Meanwhile, fix $t^* \ge 2$ such that

$$e^{-\omega(t^*-1)}\left[1 + \frac{\epsilon_0}{\epsilon^*}\right] \le 1 - k^*.$$

Then, replace C_m with ϵ^* and denote $M^*,\ T^*$ by $\frac{h_0}{2},\ T_0$. Assume that $h_0 \ge 1$, otherwise, we directly go to Step 2.

By 10, applying Lemma 4.5 with $\tau = T_0,\ \xi = -\frac{h_0}{2},\ h = h_0,\ \delta = \epsilon^*$, 9 holds with $\tau = T_0 + t^*$, some $\xi \in \left[-\frac{h_0}{2} - \frac{\gamma\epsilon^*}{C_m}, \frac{h_0}{2} + \frac{\gamma\epsilon^*}{C_m}\right]$, $\delta = \epsilon^*$ and $h = h_0 - k^*$, by the definition of $k^*,\ t^*$, it follows that

$$\hat{\delta}(T_0 + t^*) \le [\epsilon^* + \epsilon_0]e^{-\omega t^*} \le \epsilon^* \text{ and } \hat{h}(T_0 + t^*) \le h_0 - \frac{\gamma}{C_m}\epsilon_0 + 2\epsilon^*\frac{\gamma}{C_m} \le h_0 - k^*.$$

Repeat the same process, it yields that 9 holds for $\tau = T_0 + Nt^*$, $\delta = \epsilon^*$ and $h = h_0 - Nk^*$ with N such that $h_0 - (N-1)t^* \ge 1$. Thus, there exists $T_1 > T_0$ such that 9 holds for $\tau = T_1$, $\delta = \epsilon^*$ and $h = 1$ and some $\xi \in \mathbb{R}$.

Step 2. In this step, we use a mathematical induction to show that for every nonnegative integer k, 9 holds for $\xi = \xi^k \in \mathbb{R}$ and

$$\tau = T^k := T_1 + kt^*,\ \delta = \delta^k := (1 - k^*)^k\epsilon^*,\ h = h^k := (1 - k^*)^k.$$

It is obvious that the assertion holds for $k = 0$ by Step 2. Then suppose that the assertion is true for $k = l \ge 0$. We show that it is true for $k = l + 1$. In fact, as can be seen in Lemma 4.5 with $\tau = T^l$ and $t = T^{l+1}$, one can obtain that 9 holds with (τ, ξ, δ, h) replaced by $(\hat{\tau}, \hat{\xi}, \hat{\delta}, \hat{h})$ satisfying

$$\hat{\xi} \in \left[\xi^l - \delta^l\frac{\gamma}{C_m}, \xi^l + \delta^l\frac{\gamma}{C_m}\right],$$
$$\hat{\delta} \le \left(\delta^l + \epsilon_0 h^l\right) = [1 - k^*]^l\epsilon^*\left(1 + \frac{\epsilon_0}{\epsilon^*}\right)e^{-\omega(t^*-1)} \le (1 - k^*)^{l+1}\epsilon^*,$$
$$\hat{h} \le h^l - h^l\epsilon_0\frac{\gamma}{C_m} + 2\delta^l\frac{\gamma}{C_m} = [1 - k^*]^l\left[1 - \epsilon_0\frac{\gamma}{C_m} + 2\epsilon^*\frac{\gamma}{C_m}\right] = [1 - k^*]^{l+1}$$

by the definition of ϵ^*, k^* and t^*. This means that 9 holds for $\tau = T^{l+1}$, some $\xi = \xi^{l+1} \in \left[\xi^l - \delta^l \frac{\gamma}{C_m}, \xi^l + \delta^l \frac{\gamma}{C_m}\right]$, $\delta = [1-k^*]^{l+1}\epsilon^*$ and $h = [1-k^*]^{l+1}$. Now we finish the mathematical induction.

Step 3. So far, we have known that 9 holds for $(\tau, \xi, \delta, h) = (T^k, \xi^k, \delta^k, h^k)$ for all $k = 0, 1....$ Furthermore, 9 holds with $\tau \in [T^k, \infty)$, $\delta = \delta^k$, $h = h^k + 2\delta^k \frac{\gamma}{C_m}$ and $\xi = \xi^k - \delta^k \frac{\gamma}{C_m}$, $k = 0, 1....$

Now we define

$$\delta(t) = \delta^k,\ \xi(t) = \xi^k - \delta^k \frac{\gamma}{C_m},\ h(t) = h^k + 2\delta^k \frac{\gamma}{C_m}$$

for $t \in \left[T^k, T^{k+1}\right]$, $k = 0, 1....$ Then,

$$\phi_2(x + c_2 t + \xi(t)) - \delta(t) \le u(x,t) \le \phi_2(x + c_2 t + \xi(t + h(t)) + \delta(t) \text{ for } t \ge T_1,\ x \in \mathbb{R}.$$

It follows from the definition of $\delta(t)$ and $h(t)$ that

$$\delta(t) = \delta^k = [1-k^*]^k \epsilon^* \le \epsilon^* \exp\left[\left(\frac{t-T_1}{t^*} - 1\right)\ln(1-k^*)\right] \text{ for } t \ge T_1,$$

$$h(t) = h^k + 2\delta^k \frac{\gamma}{C_m} \le \left[1 + 2\epsilon^* \frac{\gamma}{C_m}\right] \exp\left[\left(\frac{t-T_1}{t^*} - 1\right)\ln(1-k^*)\right] \text{ for } t \ge T_1.$$

Moreover, since for any $t \ge \tau \ge T_1$,

$$\xi(t) \in \left[\xi(\tau) - \delta(\tau)\frac{\gamma}{C_m}, \xi(\tau) + h(\tau) + \delta(\tau)\frac{\gamma}{C_m}\right],$$

there holds

$$|\xi(t) - \xi(\tau)| \le h(\tau) + 2\delta(\tau)\frac{\gamma}{C_m},$$

which implies that $\xi(\infty) := \lim_{t\to+\infty} \xi(t)$ exists and

$$|\xi(\infty) - \xi(\tau)| \le h(\tau) + 2\delta(\tau)\frac{\gamma}{C_m} \le \left[1 + 4\epsilon^* 2\delta(\tau)\frac{\gamma}{C_m}\right] e^{\left[\left(\frac{t-T_1}{t^*} - 1\right)\ln(1-k^*)\right]} \text{ for } t \ge T_1.$$

Therefore, we have that

$$|u(x,t) - \phi_2(x + c_2 t + \xi(\infty))| \to 0 \text{ as } t \to +\infty.$$

Furthermore, the convergence is exponential. Then we complete the proof. □

5. **Lyapunov stability of the entire solution.** We investigate the Lyapunov stability of the entire solution obtained in Theorem 1.1 in this section. That is, the aim here is to prove Theorem 1.2. The following lemma plays an important role in proving Theorem 1.2.

Lemma 5.1. *Let $u(x,t)$ be the unique entire solution in Theorem 1.1. Then for any $\varphi \in (0, \frac{1}{2}]$, there exist constants $T_\varphi = T_\varphi(\varphi) > 1$ and $K_\varphi = K_\varphi(\varphi) > 0$ such that*

$$u_t(x,t) \ge K_\varphi \text{ for any } t \ge T_\varphi \text{ and } x \in \Omega_\varphi(t),$$

where

$$\Omega_\varphi(t) = \{x \in \mathbb{R} :\ \varphi \le u(x,t) \le 1 - \varphi\}.$$

Proof of Lemma 5.1. It is easy to choose T_φ and M_φ such that

$$\Omega_\varphi(t) \subset \{x \in \mathbb{R} : |x + c_2 t| \le M_\varphi\} \subset \{x \in \mathbb{R} : x \le -1\}.$$

Now suppose there exist sequences $t_k \in [T_\varphi, +\infty)$ and $x^k \in \Omega_\varphi(t)$ such that

$$u_t(t_k, x^k) \to 0 \text{ as } k \to +\infty.$$

Here only two cases happen, $t_k \to +\infty$ or $t_k \to t_*$ for some $t_* \in [T_\varphi, +\infty)$ as $k \to +\infty$.

For the former case, denote

$$u_k(x,t) = u(x + x^k, t + t_k).$$

By Proposition 2, $\{u_k(x,t)\}_{k=1}^\infty$ is equicontinuous in $x \in \mathbb{R}$ and $t \in \mathbb{R}$. Furthermore, there exists a subsequence still denoted by $\{u_k(x,t)\}_{k=1}^\infty$ such that

$$u_k \to u_* \text{ as } k \to +\infty$$

for some function u_* satisfying 1 with $\frac{\partial u_*(0,0)}{\partial t} = 0$ and $(u_*(x,t))_t \ge 0$. By the comparison principle, we further have

$$\frac{\partial u_*(x,t)}{\partial t} \equiv 0 \text{ for } t \ge 0.$$

However, this is impossible because by Theorem 4.1

$$u_*(x,t) = \phi_2(x + c_2 t + \beta + a) \text{ for some } a \in [-M_\eta, M_\eta].$$

For the second case, x^k remains bounded by the definition of $\Omega_\varphi(t)$. Therefore, we assume that $x^k \to x^*$ as $k \to +\infty$ and let

$$u_k(x,t) := u(x + x^k, t).$$

Then, each $u_k(x,t)$ is defined for all $(x,t) \in \{x \in \mathbb{R} \mid x \le -1\} \times [T_\varphi, +\infty)$ by the definition of $\Omega_\varphi(t)$. Similarly, there exists a subsequence, again denoted by $\{u_k\}_{k=1}^\infty$, such that

$$u_k \to u^* \text{ as } k \to +\infty$$

for some function u^* satisfying 1 with $\frac{\partial u^*}{\partial t}(0, t_*) = 0$, $(u^*(x,t))_t \ge 0$. Then by the comparison principle, we obtain $\frac{\partial u^*}{\partial t}(x,t) \equiv 0$ for $t \ge t_*$, but this is impossible since by Theorem 4.1,

$$u^*(x,t) - \phi_2(x + \beta + x^* + c_2 t) \to 0 \text{ as } t \to +\infty.$$

This ends the proof. □

Proof of Theorem 1.2. We first define a pair of sub- and super-solutions as follows.

$$U^\pm(x,t) = u(x + a, t + t_0 \pm \tilde{\delta}\varpi(1 - e^{-\omega t})) \pm \varpi e^{-\omega t},$$

where ω is defined as in Section 4 and $t_0 \ge T_\varphi$ and $\tilde{\delta}$, $\varpi > 0$ are constants. Besides, we claim that $U^\pm$ are super- and sub-solutions of 1. Then, by the comparison principle, we have

$$U^-(x,t) \le v(x,t) \le U^+(x,t) \text{ for } x \in \mathbb{R},\ t \ge 0. \tag{11}$$

In view of that for all $|\tau| \le \varpi$ with $\varpi =: \varpi(\epsilon) = \frac{\epsilon}{2\tilde{M}}$ and $\tilde{M} = 2 + \max\limits_{u \in [0,1]} (f_L(x,u))_u$, we have

$$\sup_{x \in \mathbb{R},\ t \in \mathbb{R}} |u(x,t) - u(x, t + \tau)| \le \sup_{x \in \mathbb{R},\ t \in \mathbb{R}} |u_t(x,t)|\varpi \le \frac{\epsilon}{2}. \tag{12}$$

It then follows from 11 and 12 that

$$|v(x,t,v_0) - u(x + a, t + t_0)| < \epsilon$$

for $x \in \mathbb{R}$ and $t \geq 0$.

Now we prove the claim. We show $U^+(x,t)$ is a super-solution of 1 for $t \geq 0$. Then it can be similarly shown that $U^-(x,t)$ is a sub-solution of 1 for $t \geq 0$. Since it is easy to see that $U^+(x,0) \geq v_0(x)$ for $x \in \mathbb{R}$ and $\delta < \frac{\varpi}{2}$, we only need to show that

$$\begin{aligned}\mathcal{L}U^+(x,t) =& U_t^+(x,t) - (J * U^+(x,t) - U^+(x,t)) - f(x,U^+)\\ =& \tilde{\delta}\varpi\omega e^{-\omega t} - \varpi\omega e^{-\omega t} + f(x,u) - f(x,U^+)\\ \geq& 0.\end{aligned}$$

We go further to show $\mathcal{L}U^+(x,t) \geq 0$ in two cases.

Case 1. For any $u = u(x+a, t+\gamma+\tilde{\delta}\varpi(1-e^{-\omega t})) \in [0,\sigma] \cup [1-\sigma, 1]$ with σ defined as in Section 4,

$$\begin{aligned}\mathcal{L}U^+(x,t) \geq& -\min\left\{\frac{|f_u(x,0)|}{2}, \frac{|(f_u(x,1)|}{2}\right\}\varpi e^{-\omega t} - \omega\varpi e^{-\omega t}\\ \geq& \varpi e^{-\omega t}(2\omega - \omega)\\ \geq& 0.\end{aligned}$$

Case 2. For $u \in [\sigma, 1-\sigma]$, it follows from Lemma 5.1 that there exists K_σ such that $u_t \geq K_\sigma$. Therefore,

$$\begin{aligned}\mathcal{L}U^+(x,t) \geq& \tilde{\delta}\varpi\omega K_\sigma e^{-\omega t} - \varpi\omega e^{-\omega t} - \|f_u(x,u)\|_\infty \varpi e^{-\omega t}\\ \geq& \varpi e^{-\omega t}\left(\tilde{\delta}\omega K_\sigma - \omega - \|f_u(x,u))\|_\infty\right).\end{aligned}$$

Let $\tilde{\delta}$ be sufficiently large such that

$$\tilde{\delta} \geq \frac{\omega K_\sigma}{\omega + \|f_u(x,u)\|_\infty}.$$

Thus, we have $\mathcal{L}U^+(x,t) \geq 0$. Now, we obtain the claim. The proof of Theorem 1.2 is completed. □

Remark 1. In this paper, we have considered the existence, uniqueness, asymptotic behavior and Lyapunov stability of entire solutions of the nonlocal dispersal equation 1 under the assumption $J(x) = J(-x)$, $x \in \mathbb{R}$. It is well-known from [8, 22, 28, 29] that the asymmetry of J has a great influence on the profile of the traveling waves and the sign of the wave speeds, which further makes the properties of the entire solution more diverse. Naturally, an interesting problem is to consider entire solutions of 1 under asymmetric conditions.

Remark 2. The method used here can be also applied to consider the bistable lattice differential equations with f_i satisfying (F)

$$\dot{u}_i(t) = u_{i+1}(t) + u_{i-1}(t) - 2u_i(t) + f_i(u_i(t)),\ i \in \mathbb{Z},\ t \in \mathbb{R}. \tag{13}$$

The existence, uniqueness, asymptotic behavior and the Lyapunov stability of entire solutions to 13 can be similarly obtained.

While for bistable random diffusion equations with $f(x,u)$ satisfying (F)

$$u_t = u_{xx} + f(x,u),\ x \in \mathbb{R},\ t \in \mathbb{R},$$

the construction of the entire solution behaving as the traveling wave pertaining to f_1 coming from infinity is similar. In fact, since regularity of the entire solution is known, the Lyapunov function method can be used to show the approach of the entire solution to the traveling wave pertaining to f_2, one can refer to [10, 11].

REFERENCES

[1] P. W. Bates, On some nonlocal evolution equations arising in materials science, *Nonlinear Dynamics and Evolution Equations*, in: Fields Inst. Commun., Amer. Math. Soc., Providence, RI, **48** (2006), 13–52.

[2] P. W. Bates, P. C. Fife, X. Ren and X. Wang, Traveling waves in a convolution model for phase transitions, *Arch. Rational Mech. Anal.*, **138** (1997), 105–136.

[3] H. Berestycki and F. Hamel, Generalized transition waves and their properties, *Comm. Pure Appl. Math.*, **65** (2012), 592–648.

[4] H. Berestycki, F. Hamel and H. Matano, Bistable traveling waves around an obstacle, *Comm. Pure Appl. Math.*, **62** (2009), 729–788.

[5] H. Berestycki and N. Rodríguez, A non-local bistable reaction-diffusion equaiton with a gap, *Discrete Contin. Dyn. Syst.*, **37** (2017), 685–723.

[6] J. Carr and A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, *Proc. Amer. Math. Soc.*, **132** (2004), 2433–2439.

[7] X. Chen, Existence, uniqueness and asymptotic stablility of traveling waves in nonlocal evolution equations, *Adv. Differential Equations*, **2** (1997), 125–160.

[8] J. Coville, J. Dávila and S. Martínez, Nonlocal anisotropic dispersal with monostable nonlinearity, *J. Differential Equations*, **244** (2008), 3080–3118.

[9] J. Coville and L. Dupaigne, On a nonlocal reaction diffusion equation arising in population dynamics, *Proc. Roy. Soc. Edinburgh Sect.*, **137** (2007), 727–755.

[10] S. Eberle, A heteroclinic orbit connecting traveling waves pertaining to different nonlinearities, *J. Differential Equations*, **265** (2018), 804–829.

[11] S. Eberle, A heteroclinic orbit connecting traveling waves pertaining to different nonlinearities in a channel with decreasing cross section, *Nonlinear Anal.*, **172** (2018), 99–114.

[12] P. Fife, Some nonclassical trends in parabolic and parabolic–like evolutions, in: *Trends in Nonlinear Analysis*, Springer, Berlin (2003), 153–191.

[13] F. Hamel and N. Nadirashvili, Entire solutions of the KPP equation, *Comm. Pure Appl. Math.*, **52** (1999), 1255–1276.

[14] F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.*, **157** (2001), 91–163.

[15] W.-T. Li, Y.-J. Sun and Z.-C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, *Nonlinear Anal. Real World Appl.*, **11** (2010), 2302–2313.

[16] W.-T. Li, J.-B. Wang and L. Zhang, Entire solutions of nonlocal dispersal equations with monostable nonlinearity in space periodic habitats, *J. Differential Equations*, **261** (2016), 2472–2501.

[17] T. S. Lim and A. Zlatoš, Transition fronts for inhomogeneous Fisher-KPP reactions and non-local diffusion, *Trans. Amer. Math. Soc.*, **368** (2016), 8615–8631.

[18] S. Pan, W.-T. Li and G. Lin, Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay, *Nonlinear Anal.*, **72** (2010), 3150–3158.

[19] W. Shen, Traveling waves in diffusive random media, *J. Dynam. Differential Equations,* **16** (2004), 1011–1060.

[20] W. Shen, Stability of transition waves and positive entire solutions of Fisher-KPP equations with time and space dependence, *Nonlinearity*, **30** (2017), 3466–3491.

[21] W. Shen and A. Zhang, Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats, *J. Differential Equations*, **249** (2010), 747–795.

[22] Y.-J. Sun, W.-T. Li and Z.-C. Wang, Traveling waves for a nonlocal anisotropic dispersal equation with monostable nonnlinearity, *Nonlinear Anal.*, **74** (2011), 814–826.

[23] Y.-J. Sun, W.-T. Li and Z.-C. Wang, Entire solutions for nonlocal dispersal equations with bistable nonlineartiy, *J. Differential Equations*, **251** (2011), 551–581.

[24] M. Wang and G. Lv, Entire solutions of a diffusion and competitive Lotka-Volterra type system with nonlocal delayes, *Nonlinearity*, **23** (2010), 1609–1630.

[25] J.-B. Wang and C. Wu, Forced waves and gap formations for a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats, *Nonlinear Anal. Real World Appl.*, **58** (2021), 103208.

[26] S.-L. Wu and S. Ruan, Entire solutions for nonlocal dispersal equations with spatio-temporal delay: Monostable case, *J. Differential Equations*, **258** (2015), 2435–2470.

[27] H. Yagisita, Existence of traveling wave solutions for a nonlocal bistable equation: An abstract approach, *Publ. Res. Inst. Math. Sci.*, **45** (2009), 955–979.

[28] L. Zhang, W.-T. Li and Z.-C. Wang, Entire solution in an ignition nonlocal dispersal equation: Asymmetric kernel, *Sci. China Math.*, **60** (2017), 1791–1804.

[29] L. Zhang, W.-T. Li, Z.-C. Wang and Y.-J. Sun, Entire solutions in nonlocal bistable equations: Asymmetric case, *Acta Math. Sin. (Engl. Ser.)*, **35** (2019), 1771–1794.

Received November 2019; revised July 2020.

E-mail address: qiaoshx14@lzu.edu.cn
E-mail address: dulj13@lzu.edu.cn