

PULLBACK ATTRACTORS FOR STOCHASTIC RECURRENT NEURAL NETWORKS WITH DISCRETE AND DISTRIBUTED DELAYS

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ABSTRACT. In this paper, we investigate a class of stochastic recurrent neural networks with discrete and distributed delays for both biological and mathematical interests. We do not assume any Lipschitz condition on the nonlinear term, just a continuity assumption together with growth conditions so that the uniqueness of the Cauchy problem fails to be true. Moreover, the existence of pullback attractors with or without periodicity is presented for the multi-valued noncompact random dynamical system. In particular, a new method for checking the asymptotical compactness of solutions to the class of nonautonomous stochastic lattice systems with infinite delay is used.

1. Introduction. Recurrent Neural Networks arise in a wide range of applications such as classification, combinatorial optimization, parallel computing, signal processing and pattern recognition, (see, e.g. [7, 9, 14, 16, 22, 23, 28]). Due to the finite switching speed of neurons and amplifiers, time delays commonly occurred in neural networks. Since time delays will affect the stability of the neural system and may lead to some complex dynamic behavior, it is critical to study delayed recurrent neural networks. In particular, signal propagation is not instantaneous and may not be suitably modeled with discrete delay, so it is more appropriate to incorporate continuously distributed delays in neural network models.

Random effects arise naturally in neural network models to take into account the uncertainty. Given $\tau \in \mathbb{R}$ and $t > \tau$, in this paper, we will consider the following general class of stochastic neural networks with discrete and distributed delays:

$$\dot{x}_i(t) = f_i(x_i(t)) + \sum_{j=i-N}^{i+N} a_{ij}(t)g_{1j}(\theta_t\omega, x_j(t)) + \sum_{j=i-N}^{i+N} b_{ij}(t)g_{2j}(\theta_t\omega, x_j(t - \hat{h}(t)))$$

2020 *Mathematics Subject Classification.* Primary: 34A33; Secondary: 34D45, 34F05.

Key words and phrases. Pullback attractor, multi-valued dynamical system, stochastic recurrent neural networks, variable delays, unbounded delays.

This work was supported by NSF of China (Grant Nos. 41875084, 11571153), the Fundamental Research Funds for the Central Universities under Grant Nos. lzujbky-2018-it58 and lzujbky-2018-ot03.

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$$+ \sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, x_j(t+r)) dr + J_i(t), \quad i \in \mathbb{Z}, \quad (1.1)$$

with the initial condition

$$x_i(t) = \phi_i(t - \tau), \quad t \in (-\infty, \tau], \quad i \in \mathbb{Z}, \quad (1.2)$$

where \mathbb{Z} denotes the integer set; $x_i(t)$ represents the state variable of the potential for the i -th neuron at time t ; f_i denotes the behaved function; g_{kj} ($k = 1, 2, 3$) are activation functions of the neuron; $a_{ij}(t)$, $b_{ij}(t)$ and $c_{ij}(t)$ denote the connection weight, discretely delayed connection weight and distributively delayed connection weight, respectively, between the j -th and i -th neurons, $\hat{a}_{ij}(t)$, $\hat{b}_{ij}(t)$ and $\hat{c}_{ij}(t)$ belong to $C(\mathbb{R}; \mathbb{R}^+)$; $\hat{h}(t)$ stands for discrete time varying delay and $\tilde{h}(t)$ belongs to $C(\mathbb{R}; [0, h])$ with constant $h > 0$; $J_i(t)$ represents the external force.

Robust analysis for stochastic neural networks with time-varying delay can be found in [20, 35]. Exponential stability of stochastic neural networks with constant or time-varying delays has been studied in [8, 15, 16, 19, 21, 30]. Exponential stability of stochastic recurrent neural networks with time-varying delays was investigated in [25]. Asymptotic stability of stochastic neural networks with discrete and distributed delays has been developed, e.g., Markovian jumping parameters [26, 27, 29], Brownian motion [12], impulsive effects [23], and infinite delay [2, 18]. There has, however, been little mention of pullback attractors for stochastic neural networks.

The long-time behavior of multi-valued non-autonomous and random dynamical systems has been extensively developed over the last one and a half decades; see, e.g. [3, 4, 10, 11, 13, 17, 24] etc. The theory of pullback attractors for single-valued noncompact random dynamical systems has been established in [31]. The existence of pullback attractors has been studied in [33] for reaction-diffusion equations on an unbounded domain with non-autonomous deterministic as well as stochastic forcing terms for which the uniqueness of solutions need not hold (see also [34] for unbounded delay case). Based on the previous work, our main goal in this paper is to develop new theory of multi-valued noncompact random dynamical systems in a biological context to analyze the dynamics of a class of stochastic recurrent neural networks with discrete and distributed delays. It is worthy mentioning that we do not assume any Lipschitz condition on the nonlinear term, just a continuity assumption together with growth conditions.

The paper is organized as follows. Section 2 gives some preliminary definitions and results regarding pullback attractors of multi-valued noncompact random dynamical systems, while in Section 3 the existence of solutions for the multi-valued noncompact random dynamical systems is considered. Sections 4-6 are devoted to the existence of pullback attractors and periodic attractors for stochastic recurrent neural networks with discrete and distributed delays.

2. Multi-valued noncompact random dynamical systems. We now recall some basic definitions for multi-valued noncompact random dynamical systems and some results ensuring the existence of a pullback attractor for these systems.

Let Q be a nonempty set, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and (X, d) be a Polish space with Borel σ -algebra $\mathcal{B}(X)$. Denote by $P(X)$ and $C(X)$ the sets of all nonempty and nonempty closed subsets of X , respectively. Let also denote by

$dist(A, B)$ the Hausdorff semidistance, i.e., for given subsets A and B of X we have

$$dist(A, B) = \sup \{d(x, B) : x \in A\},$$

where $d(x, B) = \inf \{d(x, y) : y \in B\}$. Finally, denote by $\mathcal{N}_r(A)$ the open r -neighborhood $\{y \in X : d(y, A) < r\}$ of radius $r > 0$ of a subset A of X .

Assume that there are two groups $\{\sigma_t\}_{t \in \mathbb{R}}$ and $\{\theta_t\}_{t \in \mathbb{R}}$ acting on Q and Ω , respectively. Specifically, $\sigma : \mathbb{R} \times Q \rightarrow Q$ is a mapping such that σ_0 is the identity on Q , $\sigma_{t+\tau} = \sigma_t \circ \sigma_\tau$ for all $t, \tau \in \mathbb{R}$. Similarly, $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ is a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ -measurable mapping such that θ_0 is the identity on Ω , $\theta_{t+\tau} = \theta_t \circ \theta_\tau$ for all $t, \tau \in \mathbb{R}$ and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$. In the sequel, we will call both $(Q, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ parametric dynamical systems.

Definition 2.1. Let $(Q, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ be parametric dynamical systems. A multi-valued mapping $\Phi : \mathbb{R}^+ \times Q \times \Omega \times X \rightarrow P(X)$ is called a multi-valued cocycle on X over $(Q, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ if for all $q \in Q, \omega \in \Omega$ and $t, \tau \in \mathbb{R}^+$, the following conditions are satisfied:

- (1) $\Phi(0, q, \omega, \cdot)$ is the identity on X ;
- (2) $\Phi(t + \tau, q, \omega, \cdot) = \Phi(t, \sigma_\tau q, \theta_\tau \omega, \Phi(\tau, q, \omega, \cdot))$.

For the above composition of multi-valued mappings, we use that for any nonempty set $V \subset X$, $\Phi(t, q, \omega, V)$ is defined by

$$\Phi(t, q, \omega, V) = \bigcup_{x_0 \in V} \Phi(t, q, \omega, x_0).$$

Definition 2.2. (See [3, 31, 33].) A set-valued mapping $K : Q \times \Omega \rightarrow P(X)$ is called measurable with respect to \mathcal{F} in Ω if the mapping $\omega \in \Omega \rightarrow d(x, K(q, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable for every fixed $x \in X$ and $q \in Q$.

In what follows denote by \mathcal{D} be a collection of some families of nonempty subsets of X parametrized by $q \in Q$ and $\omega \in \Omega$.

Definition 2.3. Let \mathcal{D} be a collection of some families of nonempty subsets of X parametrized by $q \in Q$ and $\omega \in \Omega$. \mathcal{D} is said to be neighborhood closed if for each $D = \{D(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$, there exists a positive number ε depending on D such that the family

$$\{B(q, \omega) : B(q, \omega) \text{ is a nonempty subset of } \mathcal{N}_\varepsilon(D(q, \omega)), \forall q \in Q, \forall \omega \in \Omega\} \quad (2.1)$$

also belongs to \mathcal{D} .

Note that the neighborhood closedness of \mathcal{D} implies for each $D \in \mathcal{D}$,

$$\{\tilde{D}(q, \omega) : \tilde{D}(q, \omega) \text{ is a nonempty subset of } D(q, \omega), \forall q \in Q, \forall \omega \in \Omega\} \in \mathcal{D}. \quad (2.2)$$

A collection \mathcal{D} satisfying (2.2) is said to be inclusion-closed in the literature, see, e.g., [11].

Definition 2.4. (See [3, 31, 33].) Let \mathcal{D} be a collection of some families of nonempty subsets of X and $K = \{K(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$. Then K is called a \mathcal{D} -pullback absorbing set for Φ if for all $q \in Q, \omega \in \Omega$ and for every $B = \{B(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(B, q, \omega) > 0$ such that

$$\Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega)) \subseteq K(q, \omega), \quad \text{for all } t \geq T.$$

In addition, if K is measurable with respect to the \mathbb{P} -completion of \mathcal{F} , then K is said to be a measurable \mathcal{D} -pullback absorbing set for Φ .

Definition 2.5. Let \mathcal{D} be a collection of some families of nonempty subsets of X . Then Φ is said to be \mathcal{D} -pullback asymptotically upper semicompact in X if for all $q \in Q$ and $\omega \in \Omega$, any sequence $y_n \in \Phi(t_n, \sigma_{-t_n}q, \theta_{-t_n}\omega, x_n)$ has a convergent subsequence in X whenever $t_n \rightarrow +\infty$ ($n \rightarrow \infty$), $x_n \in B(\sigma_{-t_n}q, \theta_{-t_n}\omega)$ with $\{B(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.6. Let \mathcal{D} be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$. Then \mathcal{A} is said to be a \mathcal{D} -pullback attractor for Φ if it satisfies:

- (1) $\mathcal{A}(q, \omega)$ is compact for all $q \in Q$ and $\omega \in \Omega$.
- (2) \mathcal{A} is invariant, that is, for every $q \in Q$ and $\omega \in \Omega$,

$$\Phi(t, q, \omega, \mathcal{A}(q, \omega)) = \mathcal{A}(\sigma_tq, \theta_t\omega), \quad \forall t \geq 0.$$

- (3) \mathcal{A} attracts every member of \mathcal{D} , that is, for every $B = \{B(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathcal{D}$ and for every $q \in Q$ and $\omega \in \Omega$,

$$\lim_{t \rightarrow +\infty} \text{dist}(\Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega)), \mathcal{A}(q, \omega)) = 0.$$

The following result shows a sufficient and necessary criterion for the existence and uniqueness of pullback attractors associated to multi-valued cocycles [33], see also [31] for the single-valued case.

Theorem 2.7. *Let \mathcal{D} be a neighborhood closed collection of some families of nonempty subsets of X , and let Φ be a multi-valued cocycle on X over $(Q, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ possessing the norm-to-weak upper semicontinuity on X , i.e., if $x_n \rightarrow x$ in X , then for any $y_n \in \Phi(t, q, \omega, x_n)$, there exist a subsequence y_{n_k} and a $y \in \Phi(t, q, \omega, x)$ such that $y_{n_k} \rightarrow y$ (weak convergence). Then Φ has a \mathcal{D} -pullback attractor \mathcal{A} in \mathcal{D} if and only if Φ is \mathcal{D} -pullback asymptotically upper semicompact in X and Φ has a closed \mathcal{D} -pullback absorbing set K in \mathcal{D} . The \mathcal{D} -pullback attractor \mathcal{A} is unique and is given by, for each $q \in Q$ and $\omega \in \Omega$,*

$$\mathcal{A}(q, \omega) = \Theta(K, q, \omega) = \bigcup_{B \in \mathcal{D}} \Theta(B, q, \omega), \tag{2.3}$$

where the family $\{\Theta(B, q, \omega) : q \in Q, \omega \in \Omega\}$ is called the Θ -limit set of B defined by

$$\Theta(B, q, \omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega))}.$$

By the similar arguments of Theorem 2.25 in [31], we have a sufficient and necessary criterion for the periodicity of pullback attractors of multi-valued cocycles.

Theorem 2.8. *Let \mathcal{D} be a neighborhood closed collection of some families of nonempty subsets of X , and let Φ be a norm-to-weak upper semicontinuous periodic multi-valued cocycle with period $T > 0$ on X over $(Q, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$, i.e., for every $t \geq 0$, $q \in Q$ and $\omega \in \Omega$, there holds*

$$\Phi(t, \sigma_Tq, \omega, \cdot) = \Phi(t, q, \omega, \cdot).$$

Suppose Φ has a \mathcal{D} -pullback attractor $\mathcal{A} \in \mathcal{D}$. Then \mathcal{A} is periodic with period T , i.e., $\mathcal{A}(\sigma_Tq, \omega) = \mathcal{A}(q, \omega)$ for all $q \in Q$ and $\omega \in \Omega$ if and only if Φ has a closed \mathcal{D} -pullback absorbing set $K \in \mathcal{D}$ with K being periodic with period T .

3. The existence of solutions for random recurrent neural networks with discrete and distributed delays without uniqueness. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. On this probability space we consider a measurable non-autonomous group θ :

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \rightarrow (\Omega, \mathcal{F}).$$

In addition, we assume that \mathbb{P} is ergodic with respect to θ , which means that every θ_t -invariant set has measure zero or one, $t \in \mathbb{R}$. Therefore \mathbb{P} is invariant with respect to θ_t . Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$, which is the model for a noise, is a parametric dynamical system. Suppose $Q = \mathbb{R}$. Define a family $\{\sigma_t\}_{t \in \mathbb{R}}$ of shift operators by $\sigma_t(s) = s + t$ for all $t, s \in \mathbb{R}$.

We also recall the following well-known ergodic theorem.

Theorem 3.1. *Suppose Y is a real random variable in L^1 . Then*

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t Y(\theta_s \omega) ds = \mathbb{E}Y$$

on a $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set of measure one.

Outside this set of measure one we will replace the values of Y by $\mathbb{E}Y$ so that this version of Y has the above limit for all $\omega \in \Omega$.

Let

$$l^2 = \{x = (x_i)_{i \in \mathbb{Z}}, x_i \in \mathbb{R} : \sum_{i \in \mathbb{Z}} x_i^2 < +\infty\},$$

and equip it with the inner product and norm as

$$(x, y) = \sum_{i \in \mathbb{Z}} x_i y_i, \|x\|^2 = (x, x), \quad \forall x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in l^2.$$

We denote by C_{γ, l^2} the space

$$C_{\gamma, l^2} = \left\{ \varphi \in C((-\infty, 0]; l^2) : \lim_{s \rightarrow -\infty} \varphi(s) e^{\gamma s} \text{ exists} \right\},$$

where the parameter $\gamma > 0$ will be determined later on. If we define

$$\|\varphi\|_{C_{\gamma, l^2}} = \left(\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |\varphi_i(s)|^2 \right)^{\frac{1}{2}}, \quad \forall \varphi \in C_{\gamma, l^2},$$

then $(C_{\gamma, l^2}, \|\cdot\|_{C_{\gamma, l^2}})$ is a Banach space. Given $\tau \in \mathbb{R}$, $T > \tau$ and a function $x : (-\infty, T] \rightarrow l^2$, for each $t \in [\tau, T]$ we denote by x_t the function defined on $(-\infty, 0]$ by the relation $x_t(s) = x(t + s)$, $s \in (-\infty, 0]$. In the following sections, C denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

We consider the following conditions:

- (C1) The mappings $\omega \rightarrow g_{kj}(\omega, x)$ are \mathcal{F} -measurable for any fixed $x \in \mathbb{R}$, and the mappings $(t, x) \rightarrow g_{kj}(\theta_t \omega, x)$ are continuous from $\mathbb{R} \times \mathbb{R}$ into \mathbb{R} for any fixed $\omega \in \mathbb{R}$, where $k = 1, 2$ and $j \in \mathbb{Z}$. Similarly, the mappings $\omega \rightarrow g_{3j}(\omega, r, x)$ are \mathcal{F} -measurable for any fixed $(r, x) \in \mathbb{R} \times \mathbb{R}$, and the mappings $(t, r, x) \rightarrow g_{3j}(\theta_t \omega, r, x)$ are continuous from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into \mathbb{R} for any fixed $\omega \in \Omega$, where $j \in \mathbb{Z}$.

- (C2) For each $i \in \mathbb{Z}$, f_i is continuous, and there exist a positive constant $h_1 > 0$ and $h_2 = (h_{2i})_{i \in \mathbb{Z}} \in l^2$ such that

$$f_i(x)x \leq -h_1x^2 + h_{2i}^2, \quad \forall i \in \mathbb{Z}, x \in \mathbb{R}.$$

Besides, there exist a positive constant $l_1 > 0$ and $l_2 = (l_{2i})_{i \in \mathbb{Z}} \in l^2$ such that

$$|f_i(x)| \leq l_1|x| + l_{2i}, \quad \forall i \in \mathbb{Z}, x \in \mathbb{R}.$$

- (C3) For $k = 1, 2$ and $j \in \mathbb{Z}$, there exist nonnegative functions $p_{kj}, q_{kj} : \Omega \rightarrow \mathbb{R}$, which are measurable with respect to \mathcal{F} , such that

$$|g_{kj}(\omega, x)|^2 \leq p_{kj}^2(\omega)|x|^2 + q_{kj}^2(\omega), \quad \forall \omega \in \Omega, x \in \mathbb{R},$$

where the mappings $t \rightarrow p_{kj}(\theta_t\omega)$ and $t \rightarrow q_{kj}(\theta_t\omega)$ are continuous from \mathbb{R} into \mathbb{R} for any fixed $\omega \in \Omega$.

Besides, there exist nonnegative functions $\hat{p}_{3j}, \hat{q}_{3j} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|g_{3j}(\omega, r, x)| \leq \hat{p}_{3j}(\omega, r)|x| + \hat{q}_{3j}(\omega, r), \quad \forall \omega \in \Omega, r \in \mathbb{R}, x \in \mathbb{R},$$

where the mappings $\omega \rightarrow \hat{p}_{3j}(\omega, r)$ and $\omega \rightarrow \hat{q}_{3j}(\omega, r)$ are \mathcal{F} -measurable for any fixed $r \in \mathbb{R}$, and the mappings $r \rightarrow \hat{p}_{3j}(\omega, r)$ and $r \rightarrow \hat{q}_{3j}(\omega, r)$ are continuous from \mathbb{R} into \mathbb{R} for any fixed $\omega \in \Omega$.

Also, for any $\omega \in \Omega$ and $t \in \mathbb{R}$, we define

$$\tilde{p}_{3j}(\theta_t\omega) := \int_{-\infty}^0 e^{-\gamma r} \hat{p}_{3j}(\theta_t\omega, r) dr, \quad \tilde{q}_{3j}(\theta_t\omega) := \int_{-\infty}^0 \hat{q}_{3j}(\theta_t\omega, r) dr,$$

where the mappings $t \rightarrow \hat{p}_{3j}(\theta_t\omega, r)$ and $t \rightarrow \hat{q}_{3j}(\theta_t\omega, r)$ are continuous from \mathbb{R} into \mathbb{R} for any fixed $\omega \in \Omega$ and $r \in \mathbb{R}$. In addition, there exists a measurable mapping $\Lambda : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that the mapping $t \rightarrow \Lambda(\theta_t\omega)$ is continuous from \mathbb{R} into \mathbb{R} for any fixed $\omega \in \Omega$, and

$$p_{kj}(\omega) \leq \Lambda(\omega), q_{kj}(\omega) \leq \Lambda(\omega), \tilde{p}_{3j} \leq \Lambda(\omega), \tilde{q}_{3j} \leq \Lambda(\omega), j \in \mathbb{Z}, k = 1, 2, \omega \in \Omega.$$

- (C4) For $i, j \in \mathbb{Z}$, a_{ij}, b_{ij} and c_{ij} belong to $C(\mathbb{R}; \mathbb{R}^+)$, and

$$a_{ij}(t) \leq \tilde{a}_{ij}, b_{ij}(t) \leq \tilde{b}_{ij}, c_{ij}(t) \leq \tilde{c}_{ij}, \quad \forall t \in \mathbb{R}.$$

Moreover,

$$\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} (\tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2) < \infty.$$

- (C5) The external force $(J_i)_{i \in \mathbb{Z}}$ belongs to $C(\mathbb{R}; l^2)$, and

$$\int_{-\infty}^{\tau} \sum_{i \in \mathbb{Z}} e^{\frac{h_1}{3}r} |J_i(r)|^2 dr < \infty, \quad \forall \tau \in \mathbb{R},$$

which implies that

$$\lim_{k \rightarrow +\infty} \int_{-\infty}^{\tau} \sum_{|i| \geq k} e^{\frac{h_1}{3}r} |J_i(r)|^2 dr = 0, \quad \forall \tau \in \mathbb{R},$$

where the constant h_1 is the same as that of Assumption (C2).

- (C6) The mappings $t \rightarrow p_{1j}^2(\theta_t\omega)$, $t \rightarrow p_{2j}^2(\theta_t\omega)$ and $t \rightarrow \tilde{p}_{3j}^2(\theta_t\omega)$ are sub-exponentially growing as $t \rightarrow \pm\infty$ for any fixed $\omega \in \Omega$, where $j \in \mathbb{Z}$. In other words, for $\varepsilon > 0$ and $\omega \in \Omega$, there exists a $t_0(\varepsilon, \omega) > 0$ such that for $|t| \geq t_0(\varepsilon, \omega)$ it holds that

$$p_{1j}^2(\theta_t\omega) \leq e^{\varepsilon|t|}, \quad p_{2j}^2(\theta_t\omega) \leq e^{\varepsilon|t|}, \quad \tilde{p}_{3j}^2(\theta_t\omega) \leq e^{\varepsilon|t|}.$$

Similarly, the mappings $t \rightarrow q_{1j}^2(\theta_t\omega)$, $t \rightarrow q_{2j}^2(\theta_t\omega)$ and $t \rightarrow \tilde{q}_{3j}^2(\theta_t\omega)$ are sub-exponentially growing as $t \rightarrow \pm\infty$ for any fixed $\omega \in \Omega$, where $j \in \mathbb{Z}$, which means that for $\varepsilon > 0$ and $\omega \in \Omega$, there exists a $t'_0(\varepsilon, \omega) > 0$ such that for $|t| \geq t'_0(\varepsilon, \omega)$ it holds that

$$q_{1j}^2(\theta_t\omega) \leq e^{\varepsilon|t|}, \quad q_{2j}^2(\theta_t\omega) \leq e^{\varepsilon|t|}, \quad \tilde{q}_{3j}^2(\theta_t\omega) \leq e^{\varepsilon|t|}.$$

(C7) We suppose that $\mathbb{E}\Lambda^2 < \infty$, and also that

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t \Lambda^2(\theta_r\omega) dr = \bar{\Lambda}.$$

By the ergodicity assumption and Theorem 3.1 we obtain that

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t \Lambda^2(\theta_r\omega) dr = \mathbb{E}\Lambda^2 =: \bar{\Lambda},$$

on a $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant set of full measure. Let us replace outside this set (which has measure zero) the values of $\Lambda^2(\omega)$ by $\bar{\Lambda}$.

Remark 1. Let us define

$$\alpha_{ij}(\theta_t\omega) := \tilde{a}_{ij}^2 p_{1j}^2(\theta_t\omega) + \tilde{b}_{ij}^2 p_{2j}^2(\theta_t\omega) + \tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_t\omega)$$

and

$$\beta_{ij}(\theta_t\omega) := \tilde{a}_{ij}^2 q_{1j}^2(\theta_t\omega) + \tilde{b}_{ij}^2 q_{2j}^2(\theta_t\omega) + \tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_t\omega),$$

where $i, j \in \mathbb{Z}$. Then from Assumption (C6) we obtain that $\alpha_{ij}(\theta_t\omega)$ and $\beta_{ij}(\theta_t\omega)$ are sub-exponentially growing as $t \rightarrow \pm\infty$ for any fixed $\omega \in \Omega$.

Lemma 3.2. *Let (C1)-(C5) hold. Then for any fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$ and each $M > 0$, there exists $T(M, \omega) > 0$ such that if $\phi \in C_{\gamma, l^2}$ and $\|\phi\|_{C_{\gamma, l^2}} \leq M$, then problem (1.1) and (1.2) admits at least a solution $x(t) = x(t; \tau, \omega, \phi)$ defined on $[\tau, \tau + T(M, \omega)]$, and x belongs to the space $C^1([\tau, \tau + T(M, \omega)]; l^2)$.*

The proof of Lemma 3.2 is given in the Appendix.

By slightly modifying the proof of Lemmas 4.1 and 4.2, we see that every solution can be globally defined. Hence we now define a multi-valued mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{\gamma, l^2} \rightarrow P(C_{\gamma, l^2})$ by

$$\begin{aligned} \Phi(t, \tau, \omega, \phi) &= \{x_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \phi) \mid x(\cdot) \text{ is a solution of Eqs. (1.1)-(1.2)} \\ &\quad \text{with } \phi \in C_{\gamma, l^2}\}. \end{aligned}$$

Lemma 3.3. *The mapping Φ is a multi-valued cocycle on C_{γ, l^2} over $(\mathbb{R}, \{\sigma_t\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.*

Proof. We only need to check condition (2) in Definition 2.1, since condition (1) follows immediately. Let $z \in \Phi(t+s, \tau, \omega, \phi)$. Then there exists a solution x of (1.1)-(1.2) such that $z = x_{t+\tau+s}(\cdot, \tau, \theta_{-\tau}\omega, \phi)$. We define a function u by $u_{t+\tau} = x_{t+\tau+s}$ for $t \geq 0$ and $\tau \in \mathbb{R}$. It is clear that $u_\tau = x_{\tau+s}$, and the function u solves (1.1) with ω , $a_{ij}(t)$, $b_{ij}(t)$, $c_{ij}(t)$, $J_i(t)$, $\hat{h}(t)$ replaced by $\theta_s\omega$, $a_{ij}(t+s)$, $b_{ij}(t+s)$, $c_{ij}(t+s)$, $J_i(t+s)$ and $\hat{h}(t+s)$, respectively, and $\phi = x_{\tau+s}$. Indeed, for $r \in [-t, 0]$, we obtain

$$\begin{aligned} u_{t+\tau}(r) &= x_{t+\tau+s}(r) = \phi(0) + \int_\tau^{t+\tau+s+r} \hat{f}(\theta_{r'}\omega, r', x_{r'}) dr' \\ &= \phi(0) + \int_\tau^{\tau+s} \hat{f}(\theta_{r'}\omega, r', x_{r'}) dr' + \int_{\tau+s}^{t+\tau+s+r} \hat{f}(\theta_{r'}\omega, r', x_{r'}) dr' \end{aligned}$$

$$\begin{aligned}
 &= x_{\tau+s}(0) + \int_{\tau+s}^{t+\tau+s+r} \hat{f}(\theta_{r'}\omega, r', x_{r'}) dr' \\
 &= u_{\tau}(0) + \int_{\tau}^{t+\tau+r} \hat{f}(\theta_{r''+s}\omega, r'' + s, u_{r''}) dr'',
 \end{aligned}$$

where $\hat{f}(\theta_{r'}\omega, r', x_{r'}) = \left(\hat{f}_i(\theta_{r'}\omega, r', x_{ir'}) \right)_{i \in \mathbb{Z}}$ and $\hat{f}_i(\theta_{r'}\omega, r', x_{ir'}) = f_i(x_i(r')) + \sum_{j=i-N}^{i+N} a_{ij}(r') g_{1j}(\theta_{r'}\omega, x_j(r')) + \sum_{j=i-N}^{i+N} b_{ij}(r') g_{2j}(\theta_{r'}\omega, x_j(r' - \hat{h}(r')))$
 $+ \sum_{j=i-N}^{i+N} c_{ij}(r') \int_{-\infty}^0 g_{3j}(\theta_{r'}\omega, r''', x_j(r' + r''')) dr''' + J_i(r')$.

Note that $u_{t+\tau}(r) = x_s(t + \tau + r)$ for $r < -t$. Therefore

$$z \in \Phi(t, \tau + s, \theta_s\omega, x_{\tau+s}) \subset \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \phi)).$$

Since z is arbitrary, we have $\Phi(t + s, \tau, \omega, \phi) \subset \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \phi))$.

On the other hand, let $z \in \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \phi))$. Then there exist x solving (1.1) and y solving (1.1)(with $\omega, a_{ij}(t), b_{ij}(t), c_{ij}(t), J_i(t)$ and $\hat{h}(t)$ replaced by $\theta_s\omega, a_{ij}(t + s), b_{ij}(t + s), c_{ij}(t + s), J_i(t + s)$ and $\hat{h}(t + s)$, respectively) and such that $y_{\tau} = x_{s+\tau}$ and $y_{t+\tau} = z$. Define the function

$$w_{t'} = \begin{cases} x_{t'}, & \text{if } \tau \leq t' \leq s + \tau, \\ y_{t'-s}, & \text{if } s + \tau \leq t', \end{cases}$$

which is a solution to (1.1). Indeed, for $t' \leq s + \tau$ the equality $w_{t'} = x_{t'}$ ensures that $w(\cdot)$ is a solution. If $t' \geq s + \tau$, then for $r \in [\tau + s - t', 0]$ we have

$$\begin{aligned}
 w_{t'}(r) &= y_{t'-s}(r) = y_{\tau}(0) + \int_{\tau}^{t'-s+r} \hat{f}(\theta_{r'+s}\omega, r' + s, y_{r'}) dr' \\
 &= x_{s+\tau}(0) + \int_{\tau+s}^{t'+r} \hat{f}(\theta_{r''}\omega, r'', w_{r''}) dr'' = x_{\tau}(0) + \int_{\tau}^{t'+r} \hat{f}(\theta_{r''}\omega, r'', w_{r''}) dr''.
 \end{aligned}$$

Also, for $r \in [\tau - t', \tau + s - t']$ we find that

$$\begin{aligned}
 w_{t'}(r) &= y_{t'-s}(r) = x_s(t' + r - s) \\
 &= x_{\tau}(0) + \int_{\tau}^{t'+r} \hat{f}(\theta_{r'}\omega, r', x_{r'}) dr' = x_{\tau}(0) + \int_{\tau}^{t'+r} \hat{f}(\theta_{r'}\omega, r', w_{r'}) dr'.
 \end{aligned}$$

Finally, for $r < \tau - t'$ it is easy to see that

$$w_{t'}(r) = x_s(t' + r - s) = \phi(t' + r - \tau).$$

Therefore, $z = y_{t+\tau} = w_{t+\tau+s} \in \Phi(t + s, \tau, \omega, \phi)$. Since z is arbitrary, we obtain that

$$\Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \phi)) \subset \Phi(t + s, \tau, \omega, \phi). \quad \square$$

4. Uniform estimates of solutions. In this section, we establish uniform estimates of solutions of problem (1.1)-(1.2) which are needed for proving the existence of pullback absorbing sets of the system.

Let B be a bounded nonempty subset of C_{γ,l^2} , and denote by $\|B\|_{C_{\gamma,l^2}} = \sup_{\varphi \in B} \|\varphi\|_{C_{\gamma,l^2}}$. Assume $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a family of bounded nonempty subsets of C_{γ,l^2} satisfying, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{r \rightarrow -\infty} e^{\frac{h_1}{2}r} \|D(\tau + r, \theta_r\omega)\|_{C_{\gamma,l^2}}^2 = 0, \tag{4.1}$$

where the constant h_1 is the same as that of Assumption (C2). Denote by \mathcal{D} the collection of all families of bounded nonempty subsets of C_{γ,l^2} which fulfill condition (4.1), i.e.,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (4.1)}\}.$$

Obviously, \mathcal{D} is neighborhood closed.

Lemma 4.1. *Suppose (C1)-(C5) hold and assume that*

$$\frac{5}{8}h_1 < \gamma. \tag{4.2}$$

Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, any solution x of Eqs. (1.1)-(1.2) with ω replaced by $\theta_{-\tau}\omega$ satisfies for all $t \geq 0$,

$$\begin{aligned} \|x_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \phi)\|_{C_{\gamma,l^2}}^2 &\leq C e^{-h_1 t + \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \int_{-\tau}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s \omega) ds} \|\phi\|_{C_{\gamma,l^2}}^2 \\ &+ C \int_{-\tau}^0 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s \omega) + \|J(\tau + s)\|^2 + \|h_2\|^2 \right) \\ &\times e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'} \omega) ds'} ds, \end{aligned}$$

where $\alpha_{ij}(\theta_s \omega)$ and $\beta_{ij}(\theta_s \omega)$ are the same as those of Remark 1.

The proof of Lemma 4.1 is given in the Appendix.

Lemma 4.2. *Let (C1)-(C7) and (4.2) hold. Also, assume that*

$$8(2N + 1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} (\tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2) \mathbb{E} \Lambda^2 < \frac{1}{2} h_1^2. \tag{4.3}$$

Then the closed ball $K(\tau, \omega)$ in C_{γ,l^2} with center zero and random radius $R(\tau, \omega)$ where

$$\begin{aligned} (R(\tau, \omega))^2 &= C \int_{-\infty}^0 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s \omega) + \|J(\tau + s)\|^2 + \|h_2\|^2 \right) \\ &\times e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'} \omega) ds'} ds \end{aligned}$$

is contained in \mathcal{D} , and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a measurable \mathcal{D} -pullback absorbing set for Φ .

Proof. It follows from Remark 1 that for any fixed $\omega \in \Omega$, the mappings $t \rightarrow \beta_{ij}(\theta_t \omega)$ are sub-exponentially growing for $t \rightarrow \pm\infty$, where $i \in \mathbb{Z}$ and $j = i - N, \dots, i + N$. Hence for $0 < \varepsilon < \frac{1}{12}h_1$ and $\omega \in \Omega$, there exists a $t'_0(\varepsilon, \omega)$ such that for $|t| \geq t'_0(\varepsilon, \omega)$,

$$\beta_{ij}(\theta_t \omega) \leq (\tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2) e^{\varepsilon|t|},$$

where $i \in \mathbb{Z}$ and $j = i - N, \dots, i + N$.

Thanks to Assumptions (C4)-(C7), in view of Remark 1, we deduce that

$$\begin{aligned} &C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{s+r} \omega) e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'+r} \omega) ds'} ds \\ &\leq C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 e^{\varepsilon|s+r|} e^{h_1 s} \\ &\times e^{\frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \left((\int_{s+r}^0 - \int_r^0) (\Lambda^2(\theta_{s''} \omega) - \bar{\Lambda}) ds'' - \bar{\Lambda} s \right)} ds \end{aligned}$$

$$\begin{aligned} &\leq C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 e^{3\varepsilon|s+r|} e^{h_1 s - \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda} s} ds \\ &\leq C e^{(\frac{1}{2}h_1 - 3\varepsilon)r} \int_{-\infty}^0 e^{(\frac{1}{2}h_1 - 3\varepsilon)s} ds \leq C e^{(\frac{1}{2}h_1 - 3\varepsilon)r} \rightarrow 0 \end{aligned} \tag{4.4}$$

as $r \rightarrow -\infty$, where we have used the notations $\bar{\Lambda} = \mathbb{E}\Lambda^2$ and

$$M_{ij} := \tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2,$$

and in the similar way, we have

$$\begin{aligned} &C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 (\|J(\tau + r + s)\|^2 + \|h_2\|^2) \\ &\quad \times e^{h_1 s + \int_s^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'+r}\omega) ds'} ds \\ &\leq C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 (\|J(\tau + r + s)\|^2 + \|h_2\|^2) \\ &\quad \times e^{h_1 s + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \left(\int_{s+r}^0 - \int_r^0 (\Lambda^2(\theta_{s''}\omega) - \bar{\Lambda}) ds'' - \bar{\Lambda} s \right)} ds \\ &\leq C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 (\|J(\tau + r + s)\|^2 + \|h_2\|^2) \\ &\quad \times e^{h_1 s + 2\varepsilon|s+r| - \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda} s} ds \\ &\leq C e^{(\frac{1}{2}h_1 - 2\varepsilon)r} \int_{-\infty}^0 (\|J(\tau + r + s)\|^2 + \|h_2\|^2) e^{(\frac{1}{2}h_1 - 2\varepsilon)s} ds \\ &\leq C e^{-(\frac{1}{2}h_1 - 2\varepsilon)\tau} \int_{-\infty}^{\tau+r} e^{(\frac{1}{2}h_1 - 2\varepsilon)s'} (\|J(s')\|^2 + \|h_2\|^2) ds' \rightarrow 0 \end{aligned} \tag{4.5}$$

as $r \rightarrow -\infty$. Therefore,

$$\begin{aligned} &e^{\frac{h_1 r}{2}} (R(\tau + r, \theta_r \omega))^2 \\ &= C e^{\frac{h_1 r}{2}} \int_{-\infty}^0 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{s+r}\omega) + \|J(\tau + r + s)\|^2 + \|h_2\|^2 \right) \\ &\quad \times e^{h_1 s + \int_s^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'+r}\omega) ds'} ds \rightarrow 0 \text{ as } r \rightarrow -\infty. \end{aligned}$$

This implies that

$$\lim_{r \rightarrow -\infty} e^{\frac{h_1 r}{2}} \|K(\tau + r, \theta_r \omega)\|_{C_{\gamma, t^2}}^2 = 0, \tag{4.6}$$

and thus $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to \mathcal{D} . By (C7), (4.3) and the ergodic Theorem 3.1, we obtain that

$$\begin{aligned} &e^{-h_1 t + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \int_{-t}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s \omega) ds} \|B(\tau - t, \theta_{-t} \omega)\|_{C_{\gamma, t^2}}^2 \\ &\leq e^{-\frac{1}{2}h_1 t} \|B(\tau - t, \theta_{-t} \omega)\|_{C_{\gamma, t^2}}^2 \rightarrow 0 \end{aligned} \tag{4.7}$$

as $t \rightarrow +\infty$, where $\phi \in B(\tau - t, \theta_{-t} \omega)$ and $B \in \mathcal{D}$. Note that for each $\tau \in \mathbb{R}$, $(R(\tau, \omega))^2 : \Omega \rightarrow \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable. Then it follows from Lemma 4.1, (4.6) and (4.7) that $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable \mathcal{D} -pullback absorbing set in \mathcal{D} for Φ . This completes the proof of the lemma. \square

5. Estimate of the tails. In order to prove the asymptotically upper semicom-
pactness for the multi-valued cocycle Φ , we need the following lemma.

Lemma 5.1. *Suppose (C1)-(C7), (4.2) and (4.3) hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for every $\varepsilon > 0$, there exist $T = T(\tau, \omega, B, \varepsilon) > 0$ and $N = N(\tau, \omega, B, \varepsilon) > 0$ such that any solution $x(\cdot)$ of Eqs. (1.1)-(1.2), given by x with $x_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, \phi) \in \Phi(t, \tau - t, \theta_{-\tau}\omega, \phi)$ and $\phi \in B(\tau - t, \theta_{-t}\omega)$, satisfies*

$$\sum_{|i| \geq N} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{i\tau}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \leq \varepsilon, \quad \text{for all } t \geq T. \tag{5.1}$$

Proof. Choose a smooth function ρ such that $0 \leq \rho(r) \leq 1$ for $r \in \mathbb{R}^+$, and

$$\rho(r) = 0 \quad \text{for } 0 \leq r \leq 1, \quad \rho(r) = 1 \quad \text{for } r \geq 2.$$

Then there exists a constant C_0 such that $|\rho'(r)| \leq C_0$ for $r \in \mathbb{R}^+$. Define $\rho_M(|i|) := \rho\left(\frac{|i|}{M}\right)$. Multiplying (1.1) by $\rho_M(|i|)x_i$ we have that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \rho_M(|i|) |x_i(t)|^2 &= \rho_M(|i|) f_i(x_i(t)) x_i(t) \\ &+ \sum_{j=i-N}^{i+N} \rho_M(|i|) a_{ij}(t) g_{1j}(\theta_t \omega, x_j(t)) x_i(t) \\ &+ \sum_{j=i-N}^{i+N} \rho_M(|i|) b_{ij}(t) g_{2j}(\theta_t \omega, x_j(t - \hat{h}(t))) x_i(t) \\ &+ \sum_{j=i-N}^{i+N} \rho_M(|i|) c_{ij}(t) x_i(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, x_j(t+r)) dr + \rho_M(|i|) J_i(t) x_i(t). \end{aligned}$$

In a similar way as in Lemma 4.1, by Assumptions (C2)-(C4), Young's inequality, and $\left(\sum_{j=1}^N u_j\right)^2 \leq N \sum_{j=1}^N u_j^2$, we obtain that

$$\rho_M(|i|) f_i(x_i(t)) x_i(t) \leq -\rho_M(|i|) h_1 |x_i(t)|^2 + \rho_M(|i|) h_{2i}^2, \tag{5.2}$$

$$\begin{aligned} &\sum_{j=i-N}^{i+N} \rho_M(|i|) a_{ij}(t) g_{1j}(\theta_t \omega, x_j(t)) x_i(t) \\ &\leq \frac{4}{h_1} (2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) a_{ij}^2(t) g_{1j}^2(\theta_t \omega, x_j(t)) + \frac{1}{16} h_1 \rho_M(|i|) |x_i(t)|^2 \\ &\leq \frac{4}{h_1} (2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{a}_{ij}^2 p_{1j}^2(\theta_t \omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\ &\quad + \frac{1}{16} h_1 \rho_M(|i|) |x_i(t)|^2 + \frac{4}{h_1} (2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{a}_{ij}^2 q_{1j}^2(\theta_t \omega), \end{aligned} \tag{5.3}$$

$$\sum_{j=i-N}^{i+N} \rho_M(|i|) b_{ij}(t) g_{2j}(\theta_t \omega, x_j(t - \hat{h}(t))) x_i(t)$$

$$\begin{aligned}
 &\leq \frac{4}{h_1}(2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) b_{ij}^2(t) g_{2j}^2(\theta_t \omega, x_j(t - \hat{h}(t))) + \frac{1}{16} h_1 \rho_M(|i|) |x_i(t)|^2 \\
 &\leq \frac{4}{h_1}(2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{b}_{ij}^2 p_{2j}^2(\theta_t \omega) e^{2\gamma h} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\
 &\quad + \frac{1}{16} h_1 \rho_M(|i|) |x_i(t)|^2 + \frac{4}{h_1}(2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{b}_{ij}^2 q_{2j}^2(\theta_t \omega), \tag{5.4}
 \end{aligned}$$

$$\begin{aligned}
 &\left| \sum_{j=i-N}^{i+N} \rho_M(|i|) c_{ij}(t) x_i(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, x_j(t+r)) dr \right| \\
 &\leq \sum_{j=i-N}^{i+N} \rho_M(|i|) c_{ij}(t) |x_i(t)| \int_{-\infty}^0 (\hat{p}_{3j}(\theta_t \omega, r) |x_j(t+r)| + \hat{q}_{3j}(\theta_t \omega, r)) dr \\
 &\leq \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{c}_{ij} \left(\tilde{p}_{3j}(\theta_t \omega) \sup_{s \in (-\infty, 0]} e^{\gamma s} |x_{jt}(s)| + \tilde{q}_{3j}(\theta_t \omega) \right) |x_i(t)| \\
 &\leq \frac{4}{h_1}(2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_t \omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\
 &\quad + \frac{1}{8} h_1 \rho_M(|i|) |x_i(t)|^2 + \frac{4}{h_1}(2N+1) \sum_{j=i-N}^{i+N} \rho_M(|i|) \tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_t \omega), \tag{5.5}
 \end{aligned}$$

and

$$\rho_M(|i|) J_i(t) x_i(t) \leq \frac{1}{16} h_1 \rho_M(|i|) |x_i(t)|^2 + \frac{4}{h_1} \rho_M(|i|) |J_i(t)|^2. \tag{5.6}$$

Note that $e^{2\gamma h} > 1$. Then, from (5.2)-(5.6) it follows that

$$\begin{aligned}
 &\frac{d}{dt} (\rho_M(|i|) |x_i(t)|^2) \leq -\frac{11}{8} h_1 \rho_M(|i|) |x_i(t)|^2 \\
 &\quad + \frac{8}{h_1} (2N+1) \rho_M(|i|) \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_t \omega) e^{2\gamma h} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\
 &\quad + \frac{8}{h_1} \rho_M(|i|) |J_i(t)|^2 + \frac{8}{h_1} (2N+1) \rho_M(|i|) \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_t \omega) + 2\rho_M(|i|) h_{2i}^2, \tag{5.7}
 \end{aligned}$$

where $\alpha_{ij}(\theta_t \omega)$ and $\beta_{ij}(\theta_t \omega)$ are given in Remark 1. And consequently,

$$\begin{aligned}
 &\frac{d}{dt} \left(e^{\frac{5}{4} h_1 t} \rho_M(|i|) |x_i(t)|^2 \right) = \frac{5}{4} h_1 e^{\frac{5}{4} h_1 t} \rho_M(|i|) |x_i(t)|^2 + e^{\frac{5}{4} h_1 t} \frac{d}{dt} (\rho_M(|i|) |x_i(t)|^2) \\
 &\leq -\frac{1}{8} h_1 e^{\frac{5}{4} h_1 t} \rho_M(|i|) |x_i(t)|^2 + \frac{8}{h_1} (2N+1) \rho_M(|i|) e^{\frac{5}{4} h_1 t} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_t \omega) \\
 &\quad + \frac{8}{h_1} \rho_M(|i|) e^{\frac{5}{4} h_1 t} |J_i(t)|^2 + 2\rho_M(|i|) e^{\frac{5}{4} h_1 t} h_{2i}^2 \\
 &\quad + \frac{8}{h_1} (2N+1) e^{2\gamma h} \rho_M(|i|) \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_t \omega) e^{\frac{5}{4} h_1 t} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2. \tag{5.8}
 \end{aligned}$$

Integrating (5.8) over $[\tau - t, t^*]$ with $t \geq 0$ and $t^* \geq \tau$, we find that for every $\omega \in \Omega$,

$$\begin{aligned}
 & e^{\frac{5}{4}h_1 t^*} \rho_M(|i|) |x_i(t^*, \tau - t, \omega, \phi)|^2 \leq e^{\frac{5}{4}h_1(\tau - t)} \rho_M(|i|) |x_i(\tau - t, \tau - t, \omega, \phi)|^2 \\
 & - \frac{1}{8} h_1 \int_{\tau - t}^{t^*} e^{\frac{5}{4}h_1 r} \rho_M(|i|) |x_i(r, \tau - t, \omega, \phi)|^2 dr \\
 & + \frac{8}{h_1} (2N + 1) e^{2\gamma h} \rho_M(|i|) \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_r \omega) e^{\frac{5}{4}h_1 r} \\
 & \times \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \omega, \phi)|^2 dr \\
 & + \frac{8}{h_1} (2N + 1) \rho_M(|i|) \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_r \omega) e^{\frac{5}{4}h_1 r} dr \\
 & + \rho_M(|i|) \int_{\tau - t}^{t^*} e^{\frac{5}{4}h_1 r} \left(\frac{8}{h_1} |J_i(r)|^2 + 2h_{2i}^2 \right) dr. \tag{5.9}
 \end{aligned}$$

Neglecting the second term on the right-hand side of (5.9). Note that $\frac{5}{8}h_1 < \gamma$, so $e^{(2\gamma - \frac{5}{4}h_1)s} \leq 1$ for $s \leq 0$. Setting $t^* + s$ instead of t^* , multiplying (5.9) by $e^{-\frac{5}{4}h_1(t^* + s)} e^{2\gamma s}$ and replacing ω by $\theta_{-\tau}\omega$, we have that for all $s \in [\tau - t - t^*, 0]$,

$$\begin{aligned}
 & \rho_M(|i|) e^{2\gamma s} |x_{it^*}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & \leq e^{-\frac{5}{4}h_1(t^* + t - \tau)} \rho_M(|i|) |x_i(\tau - t, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & + \frac{8}{h_1} (2N + 1) e^{2\gamma h} e^{-\frac{5}{4}h_1 t^*} \rho_M(|i|) \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) e^{\frac{5}{4}h_1 r} \\
 & \times \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 dr \\
 & + \frac{8}{h_1} (2N + 1) e^{-\frac{5}{4}h_1 t^*} \rho_M(|i|) \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) e^{\frac{5}{4}h_1 r} dr \\
 & + e^{-\frac{5}{4}h_1 t^*} \rho_M(|i|) \int_{\tau - t}^{t^*} e^{\frac{5}{4}h_1 r} \left(\frac{8}{h_1} |J_i(r)|^2 + 2h_{2i}^2 \right) dr. \tag{5.10}
 \end{aligned}$$

Note that for all $s \in (-\infty, \tau - t - t^*]$,

$$\begin{aligned}
 & \sum_{i \in \mathbb{Z}} \rho_M(|i|) e^{2\gamma s} |x_{it^*}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & = \sum_{i \in \mathbb{Z}} \rho_M(|i|) e^{-2\gamma(t^* + t - \tau)} e^{2\gamma(s + t^* - \tau + t)} |x_i(t^* + s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & \leq e^{-\frac{5}{4}h_1(t^* + t - \tau)} \sum_{i \in \mathbb{Z}} e^{2\gamma(s + t^* - \tau + t)} |x_i(t^* + s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & \leq e^{-\frac{5}{4}h_1(t^* + t - \tau)} \|\phi\|_{C_{\gamma, t^2}}^2,
 \end{aligned}$$

and

$$\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2$$

$$\begin{aligned}
 &\leq \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \right) \\
 &\quad \times \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) \right) \\
 &\leq (2N + 1) \left(\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{ir}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \right) \\
 &\quad \times \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) \right). \tag{5.11}
 \end{aligned}$$

Let $t^* = \tau$. Then it follows that for all $t \geq 0$,

$$\begin{aligned}
 &\sum_{i \in \mathbb{Z}} \rho_M(|i|) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{i\tau}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \leq C e^{-\frac{5}{4}h_1 t} \|\phi\|_{C_{\gamma, t^2}}^2 \\
 &\quad + C e^{-\frac{5}{4}h_1 \tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \beta_{ij}(\theta_{r-\tau}\omega) e^{\frac{5}{4}h_1 r} dr \\
 &\quad + C e^{-\frac{5}{4}h_1 \tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \rho_M(|i|) e^{\frac{5}{4}h_1 r} (|J_i(r)|^2 + h_{2i}^2) dr \\
 &\quad + C e^{-\frac{5}{4}h_1 \tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) e^{\frac{5}{4}h_1 r} \|x_r\|_{C_{\gamma, t^2}}^2 dr. \tag{5.12}
 \end{aligned}$$

Now we estimate each term on the right-hand side of (5.12). For the first term, since $\phi \in B(\tau - t, \theta_{-t}\omega)$ and $B \in \mathcal{D}$, we see that

$$\limsup_{t \rightarrow +\infty} C e^{-\frac{5}{4}h_1 t} \|\phi\|_{C_{\gamma, t^2}}^2 \leq \limsup_{t \rightarrow +\infty} C e^{-\frac{5}{4}h_1 t} \|B(\tau - t, \theta_{-t}\omega)\|_{C_{\gamma, t^2}}^2 = 0. \tag{5.13}$$

For the third term, Assumption (C5) ensures that we can find N'' large enough such that for all $t \geq 0$,

$$\begin{aligned}
 &C e^{-\frac{5}{4}h_1 \tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \rho_M(|i|) e^{\frac{5}{4}h_1 r} (|J_i(r)|^2 + h_{2i}^2) dr \\
 &\leq C e^{-\frac{5}{4}h_1 \tau} \int_{-\infty}^{\tau} \sum_{i \in \mathbb{Z}} \rho_M(|i|) e^{\frac{5}{4}h_1 r} (|J_i(r)|^2 + h_{2i}^2) dr \leq C\varepsilon, \text{ if } M \geq N''. \tag{5.14}
 \end{aligned}$$

Let $\varepsilon > 0$ be given arbitrarily. Then there is $N' = N'(\varepsilon)$ such that for all $M \geq N'$,

$$\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) M_{ij} \leq \varepsilon. \tag{5.15}$$

where $M_{ij} = \tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2$ is given in Lemma 4.2. By Assumption (C6) and Remark 1, we see that for $0 < \eta < \min \left\{ \frac{h_1}{8}, \frac{\frac{1}{2}h_1 - \frac{8}{h_1}(2N+1)^2 e^{2\gamma h}}{2 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda}} \right\}$ and $\omega \in \Omega$, there exists a $t_0'' = t_0''(\eta, \omega)$ such that for $|t| \geq t_0''$,

$$\alpha_{ij}(\theta_t\omega) \leq M_{ij} e^{\eta|t|}, \quad \beta_{ij}(\theta_t\omega) \leq M_{ij} e^{\eta|t|}, \tag{5.16}$$

where $i \in \mathbb{Z}$ and $j = i - N, \dots, i + N$. Hence, for the second term, using Assumption (C6) and (5.16), we have that for all $M \geq N'$ and $t \geq 0$,

$$\begin{aligned} & C e^{-\frac{5}{4}h_1\tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \beta_{ij}(\theta_{r-\tau}\omega) e^{\frac{5}{4}h_1r} dr \\ &= C \int_{-t}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \beta_{ij}(\theta_{r'}\omega) e^{\frac{5}{4}h_1r'} dr' \\ &\leq C \int_{-\infty}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) M_{ij} e^{-\eta r'} e^{\frac{5}{4}h_1r'} dr' \\ &\leq C \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) M_{ij} \leq C\varepsilon. \end{aligned} \tag{5.17}$$

Now we estimate the last term in (5.12). Similar to (5.17), we find that for all $M \geq N'$ and $t \geq 0$,

$$\begin{aligned} & C e^{-\frac{1}{4}h_1\tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) e^{\frac{1}{4}h_1r} dr \\ &= C \int_{-t}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r'}\omega) e^{\frac{1}{4}h_1r'} dr' \\ &\leq C \int_{-\infty}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) M_{ij} e^{-\eta r'} e^{\frac{1}{4}h_1r'} dr' \\ &\leq C \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) M_{ij} \leq C\varepsilon. \end{aligned} \tag{5.18}$$

Note that $\phi \in B(\tau - t, \theta_{-t}\omega)$ and $B \in \mathcal{D}$, using Assumptions (C4), (C7) and the ergodic Theorem 3.1, in view of (4.3), we deduce that

$$\begin{aligned} & e^{-h_1t + \int_{-t}^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s\omega) ds} \|\phi\|_{C_{\gamma,t^2}}^2 \\ &\leq e^{-h_1t + \int_{-t}^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \Lambda^2(\theta_s\omega) ds} \|\phi\|_{C_{\gamma,t^2}}^2 \\ &= e^{-h_1t + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} (\int_{-t}^0 (\Lambda^2(\theta_s\omega) - \bar{\Lambda}) ds + \bar{\Lambda}t)} \|\phi\|_{C_{\gamma,t^2}}^2 \\ &\leq e^{-\frac{1}{2}h_1t} \|B(\tau - t, \theta_{-t}\omega)\|_{C_{\gamma,t^2}}^2 \\ &\quad \times e^{-\frac{1}{2}h_1t + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} (\int_{-t}^0 (\Lambda^2(\theta_s\omega) - \bar{\Lambda}) ds + \bar{\Lambda}t)} \rightarrow 0 \end{aligned} \tag{5.19}$$

as $t \rightarrow +\infty$. In a similar way as in (5.19), by Assumptions (C4) and (C7), (4.3), (5.16) and Theorem 4.1, we have

$$\begin{aligned} & \frac{8}{h_1} (2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s\omega) e^{h_1s + \int_s^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} \\ &\leq C e^{-\eta s + h_1s + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} (\int_s^0 (\Lambda^2(\theta_{s'}\omega) - \bar{\Lambda}) ds' - \bar{\Lambda}s)} \\ &\leq C e^{h_1s - 2\eta s - \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda}s}, \end{aligned} \tag{5.20}$$

where we have used

$$\frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \int_s^0 (\Lambda^2(\theta_{s'}\omega) - \bar{\Lambda}) ds' \leq -\eta s$$

for sufficiently large $|s|$. This and (5.20) ensure that for all $t \geq 0$,

$$\begin{aligned} & \int_{-t}^0 \frac{8}{h_1}(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s\omega) e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} ds \\ & \leq C, \end{aligned} \quad (5.21)$$

thanks to $\eta < \frac{\frac{1}{2}h_1 - \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda}}{2}$. From (4.3) we see that $\frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda} < \frac{1}{2}h_1$. Then similar to (5.20), we obtain that

$$\begin{aligned} & e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} \\ & \leq e^{h_1 s - \eta s - \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda} s} \end{aligned}$$

for sufficiently large $|s|$ and $h_1 - \eta - \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda} > \frac{1}{2}h_1$. Hence it follows from Assumption (C5) that for all $t \geq 0$,

$$\int_{-t}^0 \left(\frac{8}{h_1} \|J(\tau+s)\|^2 + 2\|h_2\|^2 \right) e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} ds \leq C. \quad (5.22)$$

Hence for the last term in (5.12), by (F.43), (5.18)-(5.19) and (5.21)-(5.22), we can choose M and t sufficiently large such that

$$\begin{aligned} & C e^{-\frac{5}{4}h_1\tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) e^{\frac{5}{4}h_1 r} \|x_r\|_{C_{\gamma, i_2}}^2 dr \\ & \leq C e^{-\frac{1}{4}h_1\tau} e^{-h_1 t + \int_{-t}^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s\omega) ds} \|\phi\|_{C_{\gamma, i_2}}^2 \\ & \quad \times \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) e^{\frac{1}{4}h_1 r} dr \\ & \quad + C e^{-\frac{1}{4}h_1\tau} \int_{\tau-t}^{\tau} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_M(|i|) \alpha_{ij}(\theta_{r-\tau}\omega) e^{\frac{1}{4}h_1 r} dr \\ & \quad \times \int_{-t}^0 \left(\frac{8}{h_1}(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s\omega) + \frac{8}{h_1} \|J(\tau+s)\|^2 + 2\|h_2\|^2 \right) \\ & \quad \times e^{h_1 s + \int_s^0 \frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} ds \leq C\varepsilon. \end{aligned} \quad (5.23)$$

Finally, if we take M and t sufficiently large, then we deduce from (5.12)-(5.13), (5.14), (5.17) and (5.23) that for all $\phi \in B(\tau-t, \theta_{-t}\omega)$,

$$\begin{aligned} & \sum_{|i| > 2M} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{i\tau}(s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \\ & \leq \sum_{i \in \mathbb{Z}} \rho_M(|i|) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{i\tau}(s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \leq C\varepsilon. \end{aligned} \quad (5.24)$$

Thus the proof of this lemma is complete. \square

6. Existence of pullback attractors. First, let us prove some properties of the multi-valued cocycle Φ .

Lemma 6.1. *Suppose (C1)-(C5), (4.2) and (4.3) hold. Let ϕ^n be a sequence converging to ϕ in C_{γ,l^2} and fix $T > 0$. Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon > 0$, there exist $K^*(\varepsilon)$ and $N^*(\varepsilon, \omega)$ such that for any solution $x^n(\cdot) = (x_i^n(\cdot, \tau, \theta_{-\tau}\omega, \phi))_{i \in \mathbb{Z}}$ of problem (1.1) with ω replaced by $\theta_{-\tau}\omega$ and $n \geq K^*(\varepsilon)$ it follows*

$$\sum_{|i| \geq 2N^*(\varepsilon, \omega)} |x_i^n(r)|^2 \leq \varepsilon, \quad \forall r \in [\tau, \tau + T]. \tag{6.1}$$

Moreover, there exist $x_{t+\tau}(\cdot) \in \Phi(t, \tau, \omega, \phi)$ and a subsequence x^{n_k} satisfying

$$x^{n_k} \rightarrow x \text{ in } C([\tau, \tau + T]; l^2) \text{ as } k \rightarrow \infty. \tag{6.2}$$

Proof. For any $\varepsilon > 0$, there exist $\widetilde{K}'(\varepsilon)$ and $\widetilde{N}'(\varepsilon)$ such that

$$\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |\phi_i^n(s) - \phi_i(s)|^2 < \frac{\varepsilon}{8}, \quad \forall n \geq \widetilde{K}'(\varepsilon),$$

and

$$\sum_{i \in \mathbb{Z}} \rho_M(|i|) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |\phi_i(s)|^2 < \frac{\varepsilon}{8}, \quad \forall M \geq \widetilde{N}'(\varepsilon).$$

Hence,

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} \rho_M(|i|) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |\phi_i^n(s)|^2 \\ & \leq 2 \left(\sum_{i \in \mathbb{Z}} \rho_M(|i|) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |\phi_i^n(s) - \phi_i(s)|^2 + \sum_{i \in \mathbb{Z}} \rho_M(|i|) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |\phi_i(s)|^2 \right) \\ & \leq \frac{\varepsilon}{2}, \end{aligned} \tag{6.3}$$

if $n \geq \widetilde{K}'$ and $M \geq \widetilde{N}'$. On the other hand, by slightly modifying the proof of Lemma 4.1, in view of $\phi^n \rightarrow \phi$ in C_{γ,l^2} and Assumptions (C1)-(C5), there exists $\widetilde{R}'(\tau, \omega) > 0$ such that

$$\sum_{i \in \mathbb{Z}} |x_i^n(r)|^2 \leq \widetilde{R}'(\tau, \omega), \quad \forall r \in [\tau, \tau + T], \quad \forall n \in \mathbb{N}. \tag{6.4}$$

Integrating (5.8) over $[\tau, \tau + t]$ with $t \in [0, T]$, by (6.3)-(6.4), the continuity of $\Lambda(\theta_r\omega)$, we can choose n and M sufficiently large such that for all $t \in [0, T]$,

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} \rho_M(|i|) |x_i^n(\tau + t)|^2 \\ & \leq e^{-\frac{5}{4}h_1 t} \sum_{i \in \mathbb{Z}} \rho_M(|i|) |\phi_i^n(0)|^2 + \frac{8}{h_1} (2N + 1) e^{2\gamma h} e^{-\frac{5}{4}h_1(\tau+t)} \sum_{i \in \mathbb{Z}} \rho_M(|i|) \\ & \times \int_{\tau}^{\tau+t} \sum_{j=i-N}^{i+N} \left(\widetilde{a}_{ij}^2 + \widetilde{b}_{ij}^2 + \widetilde{c}_{ij}^2 \right) \Lambda^2(\theta_r\omega) e^{\frac{5}{4}h_1 r} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}^n(s)|^2 dr \\ & + \frac{8}{h_1} (2N + 1) e^{-\frac{5}{4}h_1(\tau+t)} \sum_{i \in \mathbb{Z}} \rho_M(|i|) \int_{\tau}^{\tau+t} \sum_{j=i-N}^{i+N} \left(\widetilde{a}_{ij}^2 + \widetilde{b}_{ij}^2 + \widetilde{c}_{ij}^2 \right) \Lambda^2(\theta_r\omega) e^{\frac{5}{4}h_1 r} dr \\ & + e^{-\frac{5}{4}h_1(\tau+t)} \sum_{i \in \mathbb{Z}} \rho_M(|i|) \int_{\tau}^{\tau+t} e^{\frac{5}{4}h_1 r} \left(\frac{8}{h_1} |J_i(r)|^2 + 2h_2^2 \right) dr \end{aligned}$$

$$\begin{aligned} &\leq \frac{\varepsilon}{2} + C \sum_{i \in \mathbb{Z}} \rho_M(|i|) \sum_{j=i-N}^{i+N} \left(\tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2 \right) \\ &+ C \int_{\tau}^{\tau+t} \sum_{i \in \mathbb{Z}} \rho_M(|i|) e^{\frac{1}{3}h_1 r} (|J_i(r)|^2 + h_{2i}^2) dr \leq \varepsilon, \end{aligned} \tag{6.5}$$

thanks to Assumptions (C4) and (C5). From (6.3) and (6.5), the conclusion (6.1) follows immediately.

Now it only remains to prove (6.2). Fix now $r \in [\tau, \tau + T]$. Taking into account (6.4), passing to a subsequence, we can state that $x^n(r) \rightarrow y$ weakly in l^2 . This and (6.1) imply that for any $\eta > 0$, there exist $K^{**}(\eta)$ and $N^{**}(\eta)$ such that

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |x_i^n(r) - y_i|^2 &\leq \sum_{|i| \leq N^{**}} |x_i^n(r) - y_i|^2 + \sum_{|i| > N^{**}} |x_i^n(r) - y_i|^2 \\ &\leq \sum_{|i| \leq N^{**}} |x_i^n(r) - y_i|^2 + 2 \sum_{|i| > N^{**}} |x_i^n(r)|^2 + 2 \sum_{|i| > N^{**}} |y_i|^2 < \eta, \end{aligned} \tag{6.6}$$

if $n \geq K^{**}(\eta)$. Therefore, $x^n(r) \rightarrow y$ strongly in l^2 , and consequently, $x^n(r)$ is precompact for any r .

On the other hand, in view of (6.4) and $\phi^n \rightarrow \phi$ in C_{γ, l^2} , by Assumptions (C1)-(C4) we deduce that there exists $R^{**}(\tau, \omega)$ such that for all $r \in [\tau, \tau + T]$ and $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \frac{d}{dr} x_i^n(r) \right\|^2 &= \sum_{i \in \mathbb{Z}} |\dot{x}_i^n(r)|^2 \leq 5 \sum_{i \in \mathbb{Z}} |f_i(x_i^n(r))|^2 \\ &+ 5 \sum_{i \in \mathbb{Z}} \left| \sum_{j=i-N}^{i+N} a_{ij}(r) g_{1j}(\theta_r \omega, x_j^n(r)) \right|^2 \\ &+ 5 \sum_{i \in \mathbb{Z}} \left| \sum_{j=i-N}^{i+N} b_{ij}(r) g_{2j}(\theta_r \omega, x_j^n(r - \hat{h}(r))) \right|^2 \\ &+ 5 \sum_{i \in \mathbb{Z}} \left| \sum_{j=1}^N c_{ij}(r) \int_{-\infty}^0 g_{3j}(\theta_r \omega, r', x_j^n(r + r')) dr' \right|^2 + 5 \sum_{i \in \mathbb{Z}} |J_i(r)|^2 \\ &\leq 10l_1^2 \sum_{i \in \mathbb{Z}} |x_i^n(r)|^2 + 5(2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 \Lambda^2(\theta_r \omega) |x_{jr}^n(0)|^2 \\ &+ 5(2N + 1) e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \Lambda^2(\theta_r \omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}^n(s)|^2 \\ &+ 10(2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \Lambda^2(\theta_r \omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}^n(s)|^2 \\ &+ 5(2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 \Lambda^2(\theta_r \omega) + 5(2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \Lambda^2(\theta_r \omega) \end{aligned}$$

$$+ 10(2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \Lambda^2(\theta_r \omega) + 5\|J(r)\|^2 + 10\|l_2\|^2 \leq R^{**}(\tau, \omega), \tag{6.7}$$

thanks to (6.4), $J \in C(\mathbb{R}; l^2)$ and the continuity of $\Lambda(\theta_r \omega)$. This implies that the sequence x^n is equi-continuous. Then the Arzelà-Ascoli theorem ensures the existence of a subsequence x^{n_k} converging in $C([\tau, \tau + T]; l^2)$ to some function $x(\cdot)$. It is easy to show that x is a solution of (1.1). Also, it is clear that $x(\tau) = \phi(0)$. \square

From Lemma 6.1 we have the following two results.

Corollary 1. *Suppose (C1)-(C5), (4.2) and (4.3) hold. Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, the map $\Phi(t, \tau, \omega, \cdot)$ has compact values.*

Corollary 2. *Suppose (C1)-(C5), (4.2) and (4.3) hold. Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, the map $\phi \rightarrow \Phi(t, \tau, \omega, \phi)$ is upper semi-continuous, i.e., if $\phi^n \rightarrow \phi$ in C_{γ, l^2} , then for any $x_{t+\tau}^n(\cdot) \in \Phi(t, \tau, \omega, \phi^n)$, there exists a subsequence x^{n_k} and a $x_{t+\tau}(\cdot) \in \Phi(t, \tau, \omega, \phi)$ such that $x_{t+\tau}^{n_k}(\cdot) \rightarrow x_{t+\tau}(\cdot)$ in C_{γ, l^2} .*

We are now ready to show the existence of pullback attractors for Φ .

Theorem 6.2. *Suppose (C1)-(C7), (4.2) and (4.3) hold. Then the multi-valued cocycle Φ associated with problem (1.1)-(1.2) has a unique \mathcal{D} -pullback attractor $\mathcal{A} \in \mathcal{D}$ in C_{γ, l^2} .*

Proof. Note that by Lemma 4.2, Corollary 2 and Theorem 2.7, it only remains to prove the asymptotically upper semicompactness for Φ .

In order to prove the asymptotically upper semicompactness for Φ , arguing as in Theorem 2.5 in [34], we only need to show that for any fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$, every $B \in \mathcal{D}$ and any $\varepsilon > 0$, there exist $T_0^* = T_0^*(\tau, \omega, B, \varepsilon) > 0$, $T^* = T^*(\tau, \omega, B, \varepsilon) > 0$, a $m > 0$ and a $\delta > 0$ such that

- (1) for all $t \geq T^*$, $x_r(\cdot) \in \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))$,

$$\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} e^{2\gamma s} |x_i(\tau + s)|^2 < \varepsilon;$$

- (2) for each fixed $s \in [-T_0^*, 0]$,

$$\left\| \bigcup_{t \geq T^*} \bigcup_{x_r(\cdot) \in \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))} (x_i(\tau + s))_{|i| \leq m} \right\|_{\mathbb{R}^{2m+1}} \text{ is bounded};$$

- (3) for all $t \geq T^*$, $x_r(\cdot) \in \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))$, $s_1, s_2 \in [-T_0^*, 0]$ with $|s_2 - s_1| < \delta$,

$$\|(x_i(\tau + s_1) - x_i(\tau + s_2))_{|i| \leq m}\|_{\mathbb{R}^{2m+1}}^2 < \varepsilon;$$

- (4) for all $t \geq T^*$, $x_r(\cdot) \in \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))$,

$$\sum_{|i| > m} \left(\sup_{s \in [-T_0^*, 0]} e^{2\gamma s} |x_i(\tau + s)|^2 \right) < \varepsilon.$$

We divide the proof into two steps.

Step 1. For (1), by making use of (F.41) with t^* replaced by $\tau + s$, we deduce that for all $t \geq 0$ and s with $-t \leq s \leq 0$,

$$e^{2\gamma s} |x_i(\tau + s, \tau - t, \theta_{-t}\omega, \phi)|^2 \leq e^{2\gamma s} \sup_{r \in (-\infty, 0]} e^{2\gamma r} |x_{i(\tau+s)}(r, \tau - t, \theta_{-t}\omega, \phi)|^2$$

$$\begin{aligned} &\leq e^{(2\gamma-h_1)s} e^{-h_1 t} |x_i(\tau-t, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \\ &\quad + C e^{(2\gamma-h_1)s} e^{-h_1 \tau} \int_{\tau-t}^{\tau+s} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) e^{h_1 r} \\ &\quad \times \sup_{s' \in (-\infty, 0]} e^{2\gamma s'} |x_{jr}(s', \tau-t, \theta_{-\tau}\omega, \phi)|^2 dr \\ &\quad + C e^{(2\gamma-h_1)s} e^{-h_1 \tau} \int_{\tau-t}^{\tau+s} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) e^{h_1 r} dr \\ &\quad + C e^{(2\gamma-h_1)s} e^{-h_1 \tau} \int_{\tau-t}^{\tau+s} e^{h_1 r} (|J_i(r)|^2 + h_{2i}^2) dr, \end{aligned}$$

and further by (F.43) with t^* replaced by r , we obtain

$$\begin{aligned} &e^{2\gamma s} |x_i(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \leq e^{(2\gamma-h_1)s} e^{-h_1 t} |x_i(\tau-t, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \\ &\quad + C e^{(2\gamma-h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_r\omega) \\ &\quad \times \left(e^{-h_1 t} e^{\int_{-t}^r \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} \|\phi\|_{C_{\gamma, t^2}}^2 \right. \\ &\quad \left. + \int_{-t}^r \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{s'}\omega) + \|J(\tau+s')\|^2 + \|h_2\|^2 \right) \right. \\ &\quad \left. \times e^{h_1 s'} e^{\int_{s'}^r \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s''}\omega) ds'' ds' \right) dr \\ &\quad + C e^{(2\gamma-h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_r\omega) e^{h_1 r} dr \\ &\quad + C e^{(2\gamma-h_1)s} \int_{-t}^s e^{h_1 r} (|J_i(r+\tau)|^2 + h_{2i}^2) dr. \tag{6.8} \end{aligned}$$

Note that for all $s \leq -t$,

$$\begin{aligned} &e^{2\gamma s} |x_i(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 = e^{-2\gamma t} e^{2\gamma(s+t)} |x_i(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \\ &\leq e^{-h_1 t} e^{2\gamma(s+t)} |x_i(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2, \tag{6.9} \end{aligned}$$

thanks to $h_1 < 2\gamma$. Note that $\phi \in B(\tau-t, \theta_{-t}\omega)$ and $B \in \mathcal{D}$, by Assumption (C4), (C7), and the ergodic Theorem 3.1, in view of (4.2), (4.3) and (5.16), we find that there exists a $T^* > 0$ and then we can choose T_0^* large enough such that for all $t \geq T^*$,

$$\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} e^{-h_1 t} e^{2\gamma(s+t)} |x_i(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \leq e^{-h_1 t} \|\phi\|_{C_{\gamma, t^2}}^2 < \frac{\varepsilon}{8}, \tag{6.10}$$

$$\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} e^{(2\gamma-h_1)s} e^{-h_1 t} |x_i(\tau-t, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \leq e^{-h_1 t} \|\phi\|_{C_{\gamma, t^2}}^2 < \frac{\varepsilon}{8}, \tag{6.11}$$

$$\begin{aligned}
 & \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_r \omega) e^{-h_1 t} \\
 & \quad \times e^{\int_{-t}^r \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'} \omega) ds'} \|\phi\|_{C_{\gamma, t^2}}^2 dr \\
 & \leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{-\eta r} e^{-h_1 t} \\
 & \quad \times e^{\frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} ((\int_{-t}^0 - \int_r^0) (\Lambda^2(\theta_{s'} \omega) - \bar{\Lambda}) ds' + \bar{\Lambda}(r+t))} \|\phi\|_{C_{\gamma, t^2}}^2 dr \\
 & \leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \\
 & \quad \times \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{-2\eta r - h_1 t + \eta t + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda}(r+t)} \|\phi\|_{C_{\gamma, t^2}}^2 dr \\
 & \leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \sum_{j=i-N}^{i+N} M_{ij} e^{(-\eta - \frac{1}{2} h_1)t} \int_{-t}^s e^{(\frac{1}{2} h_1 - 4\eta)r} dr \|\phi\|_{C_{\gamma, t^2}}^2 \\
 & \leq C e^{-(\frac{1}{2} h_1 + \eta)t} \|\phi\|_{C_{\gamma, t^2}}^2 < \frac{\varepsilon}{8}, \tag{6.12}
 \end{aligned}$$

where $M_{ij} = \tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2$, and in a similar way, we have

$$\begin{aligned}
 & \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_r \omega) \\
 & \quad \times \int_{-t}^r \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{s'} \omega) e^{h_1 s' + \int_{s'}^r \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s''} \omega) ds''} ds' dr \\
 & \leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{-\eta r} \int_{-t}^r \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} e^{-\eta s' + h_1 s'} \\
 & \quad \times e^{\frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} ((\int_{s'}^0 - \int_r^0) (\Lambda^2(\theta_{s''} \omega) - \bar{\Lambda}) ds'' + \bar{\Lambda}(r-s'))} ds' dr \\
 & \leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{-\eta r} \\
 & \quad \times \int_{-t}^r e^{(h_1 - 2\eta)s' - \eta r + \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} \bar{\Lambda}(r-s')} ds' dr \\
 & \leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \sum_{j=i-N}^{i+N} M_{ij} \int_{-t}^s e^{(\frac{1}{2} h_1 - 2\eta)r} \int_{-t}^r e^{(\frac{1}{2} h_1 - 2\eta)s'} ds' dr \\
 & \leq C e^{-(2\gamma - 4\eta)T_0^*} < \frac{\varepsilon}{8}, \tag{6.13}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_r \omega) \int_{-t}^r (\|J(\tau + s')\|^2 + \|h_2\|^2) \\
 & \quad \times e^{h_1 s' + \int_{s'}^r \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s''} \omega) ds''} ds' dr
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{-\eta r} \int_{-t}^r (\|J(\tau + s')\|^2 + \|h_2\|^2) e^{h_1 s'} \\
 &\quad \times e^{\frac{8}{h_1}(2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} M_{ij} ((\int_{s'}^0 - \int_r^0)(\Lambda^2(\theta_s'' \omega) - \bar{\Lambda})) ds'' + \bar{\Lambda}(r-s'))} ds' dr \\
 &\leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{(\frac{1}{2}h_1 - 2\eta)r} \\
 &\quad \times \int_{-t}^r (\|J(\tau + s')\|^2 + \|h_2\|^2) e^{(\frac{1}{2}h_1 - \eta)s'} ds' dr \\
 &\leq C \left(\int_{-\infty}^0 e^{-\frac{h_1 s'}{3}} \|J(\tau + s')\|^2 + \|h_2\|^2 \right) e^{-(2\gamma - \frac{1}{2}h_1 - 2\eta)T_0^*} < \frac{\varepsilon}{8}, \tag{6.14}
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_r \omega) e^{h_1 r} dr \\
 &\leq \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s \sum_{j=i-N}^{i+N} M_{ij} e^{(h_1 - \eta)r} dr \\
 &\leq C e^{-(2\gamma - \eta)T_0^*} < \frac{\varepsilon}{8}, \tag{6.15}
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} C e^{(2\gamma - h_1)s} \int_{-t}^s e^{h_1 r} (|J_i(r + \tau)|^2 + h_{2i}^2) dr \\
 &\leq C e^{-(2\gamma - h_1)T_0^*} \left(\int_{-\infty}^0 e^{\frac{h_1 r}{3}} \|J(r + \tau)\|^2 + \|h_2\|^2 \right) < \frac{\varepsilon}{8}. \tag{6.16}
 \end{aligned}$$

Inserting (6.11)-(6.16) into (6.8), in view of (6.10), we deduce that for all $t \geq T^*$ and $x_\tau(\cdot) \in \Phi(\tau, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))$,

$$\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, -T_0^*]} e^{2\gamma s} |x_i(\tau + s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 < \varepsilon,$$

which implies that (1) holds true.

Step 2. Thanks to Lemmas 4.1-4.2 and 5.1, (2) and (4) follow immediately.

For (3), without loss of generality, we assume that $s_1, s_2 \in [-T_0^*, 0]$ with $0 < s_1 - s_2 < 1$, by (C1)-(C4) we have that for all $r \in [\tau + s_2, \tau + s_1]$,

$$\begin{aligned}
 &\left\| (\dot{x}_i(r, \tau - t, \theta_{-\tau}\omega, \phi))_{|i| \leq m} \right\|_{\mathbb{R}^{2m+1}}^2 = \sum_{|i| \leq m} |\dot{x}_i(r, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &\leq 5 \sum_{|i| \leq m} (f_i(x_i(r, \tau - t, \theta_{-\tau}\omega, \phi)))^2 \\
 &\quad + 5 \sum_{|i| \leq m} \left| \sum_{j=i-N}^{i+N} a_{ij}(r) g_{1j}(\theta_{r-\tau}\omega, x_j(r, \tau - t, \theta_{-\tau}\omega, \phi)) \right|^2 \\
 &\quad + 5 \sum_{|i| \leq m} \left| \sum_{j=i-N}^{i+N} b_{ij}(r) g_{2j}(\theta_{r-\tau}\omega, x_j(r - \hat{h}(r), \tau - t, \theta_{-\tau}\omega, \phi)) \right|^2
 \end{aligned}$$

$$\begin{aligned}
 & + 5 \sum_{|i| \leq m} \left| \sum_{j=i-N}^{i+N} c_{ij}(r) \int_{-\infty}^0 g_{3j}(\theta_{r-\tau}\omega, r', x_j(r+r')) dr' \right|^2 + 5 \sum_{|i| \leq m} |J_i(r)|^2 \\
 & \leq 10l_1^2 \sum_{|i| \leq m} |x_i(r, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & \quad + 10(2N + 1)e^{2\gamma h} \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 & \quad + 10(2N + 1) \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) + 5 \sum_{|i| \leq m} |J_i(r)|^2 + 10 \sum_{|i| \leq m} l_{2i}^2, \tag{6.17}
 \end{aligned}$$

where $\alpha_{ij}(\theta_{r-\tau}\omega)$ and $\beta_{ij}(\theta_{r-\tau}\omega)$ are given in Lemma 4.1. Using (F.43), (5.19), (5.21)-(5.22) and (6.17), we obtain that for t sufficiently large,

$$\begin{aligned}
 & \left\| (x_i(\tau + s_1, \tau - t, \theta_{-\tau}\omega, \phi) - x_i(\tau + s_2, \tau - t, \theta_{-\tau}\omega, \phi))_{|i| \leq m} \right\|_{\mathbb{R}^{2m+1}} \\
 & \leq \int_{\tau+s_2}^{\tau+s_1} \left\| (\dot{x}_i(r, \tau - t, \theta_{-\tau}\omega, \phi))_{|i| \leq m} \right\|_{\mathbb{R}^{2m+1}} dr \\
 & \leq C(s_1 - s_2) + C \int_{\tau+s_2}^{\tau+s_1} \sum_{|i| \leq m} |\dot{x}_i(r, \tau - t, \theta_{-\tau}\omega, \phi)|^2 dr \\
 & \leq C(s_1 - s_2) + C \int_{\tau+s_2}^{\tau+s_1} \sum_{|i| \leq m} (|x_i(r, \tau - t, \theta_{-\tau}\omega, \phi)|^2 + l_{2i}^2) dr \\
 & \quad + C \int_{\tau+s_2}^{\tau+s_1} \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 dr \\
 & \quad + C \int_{\tau+s_2}^{\tau+s_1} \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) dr + C \int_{\tau+s_2}^{\tau+s_1} \sum_{|i| \leq m} |J_i(r)|^2 dr \\
 & \leq C(s_1 - s_2) + C \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \sup_{r \in [\tau+s_2, \tau+s_1]} \beta_{ij}(\theta_{r-\tau}\omega) (s_1 - s_2) \\
 & \quad + C \sup_{r \in [\tau+s_2, \tau+s_1]} \|J(r)\|^2 (s_1 - s_2) \\
 & \quad + C \left(1 + \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \sup_{r \in [\tau+s_2, \tau+s_1]} \alpha_{ij}(\theta_{r-\tau}\omega) \right) \int_{\tau+s_2}^{\tau+s_1} \|x_r\|_{C_{\gamma, l^2}}^2 dr \\
 & \leq C(s_1 - s_2) + C \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \sup_{r \in [\tau+s_2, \tau+s_1]} \beta_{ij}(\theta_{r-\tau}\omega) (s_1 - s_2) \\
 & \quad + C \sup_{r \in [\tau+s_2, \tau+s_1]} \|J(r)\|^2 (s_1 - s_2) \\
 & \quad + C \left(1 + \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \sup_{r \in [\tau+s_2, \tau+s_1]} \alpha_{ij}(\theta_{r-\tau}\omega) \right) \|\phi\|_{C_{\gamma, l^2}}^2 (e^{-h_1 s_2} - e^{-h_1 s_1}) \\
 & \quad \times e^{-h_1 t} + \int_{-t}^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s \omega) ds
 \end{aligned}$$

$$\begin{aligned}
 & + C \left(1 + \sum_{|i| \leq m} \sum_{j=i-N}^{i+N} \sup_{r \in [\tau+s_2, \tau+s_1]} \alpha_{ij}(\theta_{r-\tau}\omega) \right) (e^{-h_1 s_2} - e^{-h_1 s_1}) \\
 & \times \int_{-t}^0 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s \omega) + \|J(\tau+s)\|^2 + \|h_2\|^2 \right) \\
 & \times e^{h_1 s + \int_s^0 \frac{s}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'} \omega) ds'} ds \\
 & \leq C(s_1 - s_2) + C(e^{-h_1 s_2} - e^{-h_1 s_1}), \tag{6.18}
 \end{aligned}$$

thanks to the continuity of $\alpha_{ij}(\theta_{r-\tau}\omega)$, $\beta_{ij}(\theta_{r-\tau}\omega)$ and $J \in C(\mathbb{R}; l^2)$, and thus (3) holds. The proof is complete. \square

By a similar argument as in [31], the following result can be obtained immediately by using Theorem 2.8.

Theorem 6.3. *Suppose (C1)-(C7), (4.2) and (4.3) hold. If there exists $T > 0$ such that for all $t \in \mathbb{R}$,*

$$a_{ij}(t+T) = a_{ij}(t), b_{ij}(t+T) = b_{ij}(t), c_{ij}(t+T) = c_{ij}(t), \tag{6.19}$$

$$J_i(t+T) = J_i(t), \hat{h}(t+T) = \hat{h}(t), \tag{6.20}$$

where $i \in \mathbb{Z}$ and $j = i - N, \dots, i + N$, then the multi-valued cocycle Φ associated with problem (1.1)-(1.2) has a unique periodic \mathcal{D} -pullback attractor $\mathcal{A} \in \mathcal{D}$ in C_{γ, l^2} .

Appendix.

Proof of Lemma 3.2. *Proof.* Let us fix some $\omega \in \Omega$. We can rewrite Eq. (1.1) as

$$\dot{x}(t) = \tilde{f}(t, x_t),$$

where $\tilde{f}(t, x_t) := \left(\tilde{f}_i(t, x_{it}) \right)_{i \in \mathbb{Z}}$ and

$$\begin{aligned}
 \tilde{f}_i(t, x_{it}) & = f_i(x_i(t)) + \sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_t \omega, x_j(t)) + J_i(t) \\
 & + \sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_t \omega, x_j(t - \hat{h}(t))) + \sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, x_j(t+r)) dr.
 \end{aligned}$$

We divide the proof into two steps.

Step 1. $\tilde{f} : \mathbb{R} \times C_{\gamma, l^2} \rightarrow l^2$ is well defined and bounded.

We note that $\tilde{f}(t, v) = \left(\tilde{f}_i(t, v_i) \right)_{i \in \mathbb{Z}}$ and

$$\begin{aligned}
 \tilde{f}_i(t, v_i) & = f_i(v_i(0)) + \sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_t \omega, v_j(0)) + \sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_t \omega, v_j(-\hat{h}(t))) \\
 & + \sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, v_j(r)) dr + J_i(t).
 \end{aligned}$$

In view of the Assumption (C2) and the trivial bound $\|v(r)\| \leq \|v_r\|_{C_{\gamma, l^2}}$, we can obtain that

$$\|f(v(0))\|^2 \leq 2l_1^2 \|v\|_{C_{\gamma, l^2}}^2 + 2\|l_2\|^2. \tag{F.21}$$

By (C3) and using the fact that $(\sum_{j=1}^N u_j)^2 \leq N \sum_{j=1}^N u_j^2$, we have

$$\begin{aligned} & \left\| \sum_{j=i-N}^{i+N} a_{ij}(t)g_{1j}(\theta_t\omega, v_j(0)) \right\|^2 \\ & \leq (2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} a_{ij}^2(t) (p_{1j}^2(\theta_t\omega)|v_j(0)|^2 + q_{1j}^2(\theta_t\omega)) \\ & \leq (2N+1)^2 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} a_{ij}^2(t)p_{1j}^2(\theta_t\omega) \right) \|v\|_{C_{\gamma,t^2}}^2 \\ & \quad + (2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} a_{ij}^2(t)q_{1j}^2(\theta_t\omega). \end{aligned} \tag{F.22}$$

In a similar way as above, by (C3) we deduce that

$$\begin{aligned} & \left\| \sum_{j=i-N}^{i+N} b_{ij}(t)g_{2j}(\theta_t\omega, v_j(-\hat{h}(t))) \right\|^2 \\ & \leq (2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} b_{ij}^2(t) (p_{2j}^2(\theta_t\omega)|v_j(-\hat{h}(t))|^2 + q_{2j}^2(\theta_t\omega)) \\ & \leq (2N+1)^2 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} b_{ij}^2(t)p_{2j}^2(\theta_t\omega) \right) e^{2\gamma h} \|v\|_{C_{\gamma,t^2}}^2 \\ & \quad + (2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} b_{ij}^2(t)q_{2j}^2(\theta_t\omega), \end{aligned} \tag{F.23}$$

and

$$\begin{aligned} & \left\| \sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t\omega, r, v_j(r))dr \right\|^2 \\ & \leq \sum_{i \in \mathbb{Z}} \left(\sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 (\hat{p}_{3j}(\theta_t\omega, r)|v_j(r)| + \hat{q}_{3j}(\theta_t\omega, r)) dr \right)^2 \\ & \leq \sum_{i \in \mathbb{Z}} \left(\sum_{j=i-N}^{i+N} c_{ij}(t) \left(\tilde{p}_{3j}(\theta_t\omega) \sup_{s \in (-\infty, 0]} e^{\gamma s} |v_j(s)| + \tilde{q}_{3j}(\theta_t\omega) \right) \right)^2 \\ & \leq 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} c_{ij}^2(t) \tilde{p}_{3j}^2(\theta_t\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |v_j(s)|^2 \\ & \quad + 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} c_{ij}^2(t) \tilde{q}_{3j}^2(\theta_t\omega) \end{aligned}$$

$$\begin{aligned}
&\leq 2(2N+1)^2 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} c_{ij}^2(t) \tilde{p}_{3j}^2(\theta_t \omega) \right) \|v\|_{C_{\gamma, l^2}}^2 \\
&\quad + 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} c_{ij}^2(t) \tilde{q}_{3j}^2(\theta_t \omega). \tag{F.24}
\end{aligned}$$

Then, using (F.21)-(F.24) and the assumption on Λ we obtain that

$$\begin{aligned}
\|\tilde{f}(t, v)\|^2 &\leq 5 \left(2l_1^2 \|v\|_{C_{\gamma, l^2}}^2 + 2\|l_2\|^2 \right) + 5 \sum_{i \in \mathbb{Z}} \left(\sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_t \omega, v_j(0)) \right)^2 \\
&\quad + 5 \sum_{i \in \mathbb{Z}} \left(\sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_t \omega, v_j(-\hat{h}(t))) \right)^2 \\
&\quad + 5 \sum_{i \in \mathbb{Z}} \left(\sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, v_j(r)) dr \right)^2 + 5 \sum_{i \in \mathbb{Z}} |J_i(t)|^2 \\
&\leq 10l_1^2 \|v\|_{C_{\gamma, l^2}}^2 + 5(2N+1)^2 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 \right) \Lambda^2(\theta_t \omega) \|v\|_{C_{\gamma, l^2}}^2 \\
&\quad + 5(2N+1)^2 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \right) \Lambda^2(\theta_t \omega) e^{2\gamma h} \|v\|_{C_{\gamma, l^2}}^2 \\
&\quad + 10(2N+1)^2 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \right) \Lambda^2(\theta_t \omega) \|v\|_{C_{\gamma, l^2}}^2 \\
&\quad + 5(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 \Lambda^2(\theta_t \omega) + 5(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \Lambda^2(\theta_t \omega) \\
&\quad + 10(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \Lambda^2(\theta_t \omega) + 5\|J(t)\|^2 + 10\|l_2\|^2. \tag{F.25}
\end{aligned}$$

Since $\Lambda^2(\theta_t \omega)$ belongs to $C(\mathbb{R}; \mathbb{R}^+)$ for any fixed $\omega \in \Omega$, in view of $J(t) \in C(\mathbb{R}; l^2)$, it follows from (F.25) and $\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} (\tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2) < \infty$ that \tilde{f} maps the bounded sets of $\mathbb{R} \times C_{\gamma, l^2}$ into the bounded set of l^2 .

Step 2. $\tilde{f} : \mathbb{R} \times C_{\gamma, l^2} \rightarrow l^2$ is continuous.

We consider $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ and $t \in \mathbb{R}$ such that $t_n \rightarrow t$, and $\{v^n\}_{n \in \mathbb{N}} \subset C_{\gamma, l^2}$ and $v^0 \in C_{\gamma, l^2}$ such that $v^n \rightarrow v^0$. Let $\varepsilon > 0$ be given arbitrarily. Then there exists $k = k(\varepsilon)$ such that for all $n \in \mathbb{N}$,

$$\sum_{|i| > k} l_{2i}^2 \leq \varepsilon, \quad \sum_{|i| > k} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |v_i^0(s)|^2 \leq \varepsilon, \quad \sum_{|i| > k} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |v_i^n(s)|^2 \leq \varepsilon, \tag{F.26}$$

$$\sum_{|i| > k} \sum_{j=i-N}^{i+N} \left(\tilde{a}_{ij}^2 + \tilde{b}_{ij}^2 + \tilde{c}_{ij}^2 \right) \leq \varepsilon. \tag{F.27}$$

Due to the continuity of f_i and $v^n \rightarrow v^0$, in view of the Assumption (C2), for any $\varepsilon > 0$ and sufficiently large n , we have

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} |f_i(v_i^n(0)) - f_i(v_i^0(0))|^2 \\ & \leq \sum_{|i| \leq k} |f_i(v_i^n(0)) - f_i(v_i^0(0))|^2 + 2 \sum_{|i| > k} |f_i(v_i^n(0))|^2 + 2 \sum_{|i| > k} |f_i(v_i^0(0))|^2 \\ & \leq \sum_{|i| \leq k} |f_i(v_i^n(0)) - f_i(v_i^0(0))|^2 + 4 \sum_{|i| > k} (l_1 |v_i^n(0)|^2 + l_{2i}^2) + 4 \sum_{|i| > k} (l_1 |v_i^0(0)|^2 + l_{2i}^2) \\ & \leq C\varepsilon. \end{aligned} \tag{F.28}$$

By (C1), (C3)-(C4) and (F.27), in view of the continuity of $a_{ij}(t)$, we find that for all n sufficiently large,

$$\begin{aligned} & \left\| \sum_{j=i-N}^{i+N} a_{ij}(t_n) g_{1j}(\theta_{t_n} \omega, v_j^n(0)) - \sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_t \omega, v_j^0(0)) \right\|^2 \\ & \leq 2 \left\| \sum_{j=i-N}^{i+N} a_{ij}(t_n) g_{1j}(\theta_{t_n} \omega, v_j^n(0)) - \sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_{t_n} \omega, v_j^n(0)) \right\|^2 \\ & \quad + 2 \left\| \sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_{t_n} \omega, v_j^n(0)) - \sum_{j=i-N}^{i+N} a_{ij}(t) g_{1j}(\theta_t \omega, v_j^0(0)) \right\|^2 \\ & \leq 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} (a_{ij}(t_n) - a_{ij}(t))^2 g_{1j}^2(\theta_{t_n} \omega, v_j^n(0)) \\ & \quad + 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} a_{ij}^2(t) (g_{1j}(\theta_{t_n} \omega, v_j^n(0)) - g_{1j}(\theta_t \omega, v_j^0(0)))^2 \\ & \leq 2(2N+1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} (a_{ij}(t_n) - a_{ij}(t))^2 (\Lambda^2(\theta_{t_n} \omega) |v_j^n(0)|^2 + \Lambda^2(\theta_{t_n} \omega)) \\ & \quad + 4(2N+1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} 2\tilde{a}_{ij}^2 \left(\Lambda^2(\theta_{t_n} \omega) (2N+1) \|v^n\|_{C_{\gamma, t_2}}^2 + \Lambda^2(\theta_{t_n} \omega) \right) \\ & \quad + 2(2N+1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 (g_{1j}(\theta_{t_n} \omega, v_j^n(0)) - g_{1j}(\theta_t \omega, v_j^0(0)))^2 \\ & \quad + 4(2N+1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 \left(\Lambda^2(\theta_{t_n} \omega) (2N+1) \|v^n\|_{C_{\gamma, t_2}}^2 + \Lambda^2(\theta_{t_n} \omega) \right. \\ & \quad \left. + \Lambda^2(\theta_t \omega) (2N+1) \|v^0\|_{C_{\gamma, t_2}}^2 + \Lambda^2(\theta_t \omega) \right) \leq C\varepsilon, \end{aligned} \tag{F.29}$$

Arguing in the similar way as above, we deduce from (C1), (C3)-(C4), (F.27) and the continuity of $b_{ij}(t)$, $c_{ij}(t)$, $\Lambda(\theta_t \omega)$ and $\hat{h}(t)$ that for all n sufficiently large,

$$\begin{aligned}
 & \left\| \sum_{j=i-N}^{i+N} b_{ij}(t_n) g_{2j}(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) - \sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_t \omega, v_j^0(-\hat{h}(t))) \right\|^2 \\
 & \leq 2 \left\| \sum_{j=i-N}^{i+N} b_{ij}(t_n) g_{2j}(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) - \sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) \right\|^2 \\
 & \quad + 2 \left\| \sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) - \sum_{j=i-N}^{i+N} b_{ij}(t) g_{2j}(\theta_t \omega, v_j^0(-\hat{h}(t))) \right\|^2 \\
 & \leq 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} (b_{ij}(t_n) - b_{ij}(t))^2 g_{1j}^2(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) \\
 & \quad + 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} b_{ij}^2(t) \left(g_{2j}(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) - g_{2j}(\theta_t \omega, v_j^0(-\hat{h}(t))) \right)^2 \\
 & \leq 2(2N+1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} (b_{ij}(t_n) - b_{ij}(t))^2 \left(\Lambda^2(\theta_{t_n} \omega) |v_j^n(-\hat{h}(t_n))|^2 + \Lambda^2(\theta_{t_n} \omega) \right) \\
 & \quad + 4(2N+1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} 2\tilde{b}_{ij}^2 \left(\Lambda^2(\theta_{t_n} \omega) e^{2\gamma h} (2N+1) \|v^n\|_{C_{\gamma, t^2}}^2 + \Lambda^2(\theta_{t_n} \omega) \right) \\
 & \quad + 4(2N+1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \left(g_{2j}(\theta_{t_n} \omega, v_j^n(-\hat{h}(t_n))) - g_{2j}(\theta_t \omega, v_j^0(-\hat{h}(t_n))) \right)^2 \\
 & \quad + 4(2N+1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \left(g_{2j}(\theta_t \omega, v_j^0(-\hat{h}(t_n))) - g_{2j}(\theta_t \omega, v_j^0(-\hat{h}(t))) \right)^2 \\
 & \quad + 4(2N+1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 \left(\Lambda^2(\theta_{t_n} \omega) e^{2\gamma h} (2N+1) \|v^n\|_{C_{\gamma, t^2}}^2 + \Lambda^2(\theta_{t_n} \omega) \right) \\
 & \quad + \Lambda^2(\theta_t \omega) e^{2\gamma h} (2N+1) \|v^0\|_{C_{\gamma, t^2}}^2 + \Lambda^2(\theta_t \omega) \leq C\varepsilon, \tag{F.30}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left\| \sum_{j=i-N}^{i+N} c_{ij}(t_n) \int_{-\infty}^0 g_{3j}(\theta_{t_n} \omega, r, v_j^n(r)) dr - \sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, v_j^0(r)) dr \right\|^2 \\
 & \leq 2 \left\| \sum_{j=i-N}^{i+N} (c_{ij}(t_n) - c_{ij}(t)) \int_{-\infty}^0 g_{3j}(\theta_{t_n} \omega, r, v_j^n(r)) dr \right\|^2 \\
 & \quad + 2 \left\| \sum_{j=i-N}^{i+N} c_{ij}(t) \left(\int_{-\infty}^0 g_{3j}(\theta_{t_n} \omega, r, v_j^n(r)) dr - \int_{-\infty}^0 g_{3j}(\theta_t \omega, r, v_j^0(r)) dr \right) \right\|^2 \\
 & \leq 2(2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} (c_{ij}(t_n) - c_{ij}(t))^2 \\
 & \quad \times \left(\int_{-\infty}^0 (\hat{p}_{3j}(\theta_{t_n} \omega, r) |v_j^n(r)| + \hat{q}_{3j}(\theta_{t_n} \omega, r)) dr \right)^2
 \end{aligned}$$

$$\begin{aligned}
 & + 2(2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} c_{ij}^2(t) \left(\int_{-\infty}^0 (g_{3j}(\theta_{t_n}\omega, r, v_j^n(r)) - g_{3j}(\theta_t\omega, r, v_j^0(r))) dr \right)^2 \\
 \leq & 4(2N + 1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} (c_{ij}(t_n) - c_{ij}(t))^2 \\
 & \times \left(\Lambda^2(\theta_{t_n}\omega)(2N + 1)\|v^n\|_{C_{\gamma, l^2}}^2 + \Lambda^2(\theta_{t_n}\omega) \right) \\
 & + 8(2N + 1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} 2\tilde{c}_{ij}^2 \left(\Lambda^2(\theta_{t_n}\omega)(2N + 1)\|v^n\|_{C_{\gamma, l^2}}^2 + \Lambda^2(\theta_{t_n}\omega) \right) \\
 & + 2(2N + 1) \sum_{|i| \leq k} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \left(\int_{-\infty}^0 (g_{3j}(\theta_{t_n}\omega, r, v_j^n(r)) - g_{3j}(\theta_t\omega, r, v_j^0(r))) dr \right)^2 \\
 & + 8(2N + 1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \left(\Lambda^2(\theta_{t_n}\omega)(2N + 1)\|v^n\|_{C_{\gamma, l^2}}^2 + \Lambda^2(\theta_{t_n}\omega) \right) \\
 & + 8(2N + 1) \sum_{|i| > k} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \left(\Lambda^2(\theta_t\omega)(2N + 1)\|v^0\|_{C_{\gamma, l^2}}^2 + \Lambda^2(\theta_t\omega) \right) \leq C\varepsilon, \quad (\text{F.31})
 \end{aligned}$$

thanks to Assumption (C1) and Lebesgue’s dominated convergence theorem.

Note that $J(t) \in C(\mathbb{R}; l^2)$, hence for sufficiently large n , we can deduce that

$$\sum_{i \in \mathbb{Z}} |J_i(t_n) - J_i(t)|^2 = \|J(t_n) - J(t)\|^2 < \varepsilon. \quad (\text{F.32})$$

Then it follows from (F.28)-(F.32) that for all n sufficiently large,

$$\begin{aligned}
 \|\tilde{f}(t_n, v^n) - \tilde{f}(t, v^0)\|^2 & \leq 5 \sum_{i \in \mathbb{Z}} |f_i(v_i^n(0)) - f_i(v_i^0(0))|^2 + 5 \sum_{i \in \mathbb{Z}} |J_i(t_n) - J_i(t)|^2 \\
 & + 5 \sum_{i \in \mathbb{Z}} \left| \sum_{j=i-N}^{i+N} a_{ij}(t_n)g_{1j}(\theta_{t_n}\omega, v_j^n(0)) - \sum_{j=i-N}^{i+N} a_{ij}(t)g_{1j}(\theta_t\omega, v_j^0(0)) \right|^2 \\
 & + 5 \sum_{i \in \mathbb{Z}} \left| \sum_{j=i-N}^{i+N} b_{ij}(t_n)g_{2j}(\theta_{t_n}\omega, v_j^n(-\hat{h}(t_n))) - \sum_{j=i-N}^{i+N} b_{ij}(t)g_{2j}(\theta_t\omega, v_j^0(-\hat{h}(t))) \right|^2 \\
 & + 5 \sum_{i \in \mathbb{Z}} \left| \sum_{j=i-N}^{i+N} c_{ij}(t_n) \int_{-\infty}^0 g_{3j}(\theta_{t_n}\omega, r, v_j^n(r)) dr \right. \\
 & \left. - \sum_{j=i-N}^{i+N} c_{ij}(t) \int_{-\infty}^0 g_{3j}(\theta_t\omega, r, v_j^0(r)) dr \right|^2 \leq C\varepsilon. \quad (\text{F.33})
 \end{aligned}$$

This implies that $\tilde{f} : \mathbb{R} \times C_{\gamma, l^2} \rightarrow l^2$ is continuous. Thus, Theorem 4 in [5] ensures that for any $\omega \in \Omega$ and $\phi \in C_{\gamma, l^2}$, there exists at least one solution $x(\cdot) \in C^1([\tau, \tau + T(M, \omega)]; l^2)$. \square

Proof of Lemma 4.1. *Proof.* Multiplying (1.1) by x_i we obtain

$$\frac{1}{2} \frac{d}{dt} |x_i(t)|^2 = f_i(x_i(t))x_i(t) + \sum_{j=i-N}^{i+N} a_{ij}(t)g_{1j}(\theta_t\omega, x_j(t))x_i(t)$$

$$\begin{aligned}
& + \sum_{j=i-N}^{i+N} b_{ij}(t)g_{2j}(\theta_t\omega, x_j(t - \hat{h}(t)))x_i(t) \\
& + \sum_{j=i-N}^{i+N} c_{ij}(t)x_i(t) \int_{-\infty}^0 g_{3j}(\theta_t\omega, r, x_j(t+r))dr + J_i(t)x_i(t).
\end{aligned}$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 be positive parameters to be fixed later on. Note that $\hat{h}(t)$ takes the value in $[0, h]$. Then by making use of Young's inequality, Assumptions (C2)-(C4) and $\left(\sum_{j=1}^N u_j\right)^2 \leq N \sum_{j=1}^N u_j^2$, we find that

$$f_i(x_i(t))x_i(t) \leq -h_1x_i^2(t) + h_{2i}^2, \quad (\text{F.34})$$

$$\begin{aligned}
& \sum_{j=i-N}^{i+N} a_{ij}(t)g_{1j}(\theta_t\omega, x_j(t))x_i(t) \\
& \leq \frac{(2N+1)}{4\varepsilon_1} \sum_{j=i-N}^{i+N} a_{ij}^2(t)g_{1j}^2(\theta_t\omega, x_j(t)) + \varepsilon_1|x_i(t)|^2 \\
& \leq \frac{(2N+1)}{4\varepsilon_1} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 p_{1j}^2(\theta_t\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\
& \quad + \frac{(2N+1)}{4\varepsilon_1} \sum_{j=i-N}^{i+N} \tilde{a}_{ij}^2 q_{1j}^2(\theta_t\omega) + \varepsilon_1|x_i(t)|^2,
\end{aligned} \quad (\text{F.35})$$

$$\begin{aligned}
& \sum_{j=i-N}^{i+N} b_{ij}(t)g_{2j}(\theta_t\omega, x_j(t - \hat{h}(t)))x_i(t) \\
& \leq \frac{(2N+1)}{4\varepsilon_2} \sum_{j=i-N}^{i+N} b_{ij}^2(t)g_{2j}^2(\theta_t\omega, x_j(t - \hat{h}(t))) + \varepsilon_2|x_i(t)|^2 \\
& \leq \frac{(2N+1)}{4\varepsilon_2} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 p_{2j}^2(\theta_t\omega) e^{2\gamma h} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\
& \quad + \frac{(2N+1)}{4\varepsilon_2} \sum_{j=i-N}^{i+N} \tilde{b}_{ij}^2 q_{2j}^2(\theta_t\omega) + \varepsilon_2|x_i(t)|^2,
\end{aligned} \quad (\text{F.36})$$

$$\begin{aligned}
& \left| \sum_{j=i-N}^{i+N} c_{ij}(t)x_i(t) \int_{-\infty}^0 g_{3j}(\theta_t\omega, r, x_j(t+r))dr \right| \\
& \leq \sum_{j=i-N}^{i+N} c_{ij}(t) \left(\tilde{p}_{3j}(\theta_t\omega) \sup_{s \in (-\infty, 0]} e^{\gamma s} |x_{jt}(s)| + \tilde{q}_{3j}(\theta_t\omega) \right) |x_i(t)| \\
& \leq \frac{(2N+1)}{4\varepsilon_3} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_t\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jt}(s)|^2 \\
& \quad + \frac{(2N+1)}{4\varepsilon_3} \sum_{j=i-N}^{i+N} \tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_t\omega) + 2\varepsilon_3|x_i(t)|^2,
\end{aligned} \quad (\text{F.37})$$

and

$$J_i(t)x_i(t) \leq \varepsilon_4|x_i(t)|^2 + \frac{1}{4\varepsilon_4}|J_i(t)|^2.$$

Note that $e^{2\gamma h} > 1$. Therefore,

$$\begin{aligned} \frac{d}{dt}|x_i(t)|^2 &\leq (-2h_1 + 2\varepsilon_1 + 2\varepsilon_2 + 4\varepsilon_3 + 2\varepsilon_4)|x_i(t)|^2 \\ &+ \sum_{j=i-N}^{i+N} (2N+1)e^{2\gamma h} \left(\frac{1}{2\varepsilon_1}\tilde{a}_{ij}^2 p_{1j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_2}\tilde{b}_{ij}^2 p_{2j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_3}\tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_t\omega) \right) \\ &\times \sup_{s \in (-\infty, 0]} e^{2\gamma s}|x_{jt}(s)|^2 + \frac{1}{2\varepsilon_4}|J_i(t)|^2 + 2h_{2i}^2 \\ &+ \sum_{j=i-N}^{i+N} (2N+1) \left(\frac{1}{2\varepsilon_1}\tilde{a}_{ij}^2 q_{1j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_2}\tilde{b}_{ij}^2 q_{2j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_3}\tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_t\omega) \right). \end{aligned} \tag{F.38}$$

This implies that

$$\begin{aligned} \frac{d}{dt}(e^{h_1 t}|x_i(t)|^2) &= h_1 e^{h_1 t}|x_i(t)|^2 + e^{h_1 t} \frac{d}{dt}|x_i(t)|^2 \\ &\leq -(h_1 - 2\varepsilon_1 - 2\varepsilon_2 - 4\varepsilon_3 - 2\varepsilon_4) e^{h_1 t}|x_i(t)|^2 + \frac{e^{h_1 t}}{2\varepsilon_4}|J_i(t)|^2 + 2e^{h_1 t}h_{2i}^2 \\ &+ (2N+1)e^{2\gamma h} \sum_{j=i-N}^{i+N} \left(\frac{1}{2\varepsilon_1}\tilde{a}_{ij}^2 p_{1j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_2}\tilde{b}_{ij}^2 p_{2j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_3}\tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_t\omega) \right) \\ &\times e^{h_1 t} \sup_{s \in (-\infty, 0]} e^{2\gamma s}|x_{jt}(s)|^2 \\ &+ (2N+1) \sum_{j=i-N}^{i+N} \left(\frac{1}{2\varepsilon_1}\tilde{a}_{ij}^2 q_{1j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_2}\tilde{b}_{ij}^2 q_{2j}^2(\theta_t\omega) + \frac{1}{2\varepsilon_3}\tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_t\omega) \right) e^{h_1 t}. \end{aligned} \tag{F.39}$$

Integrating (F.39) over $[\tau - t, t^*]$ with $t \geq 0$ and $t^* \geq \tau$, we obtain that for every $\omega \in \Omega$,

$$\begin{aligned} e^{h_1 t^*}|x_i(t^*, \tau - t, \omega, \phi)|^2 &\leq e^{h_1(\tau - t)}|x_i(\tau - t, \tau - t, \omega, \phi)|^2 \\ &- (h_1 - 2\varepsilon_1 - 2\varepsilon_2 - 4\varepsilon_3 - 2\varepsilon_4) \int_{\tau - t}^{t^*} e^{h_1 r}|x_i(r, \tau - t, \omega, \phi)|^2 dr \\ &+ \int_{\tau - t}^{t^*} (2N+1)e^{2\gamma h} \sum_{j=i-N}^{i+N} \left(\frac{1}{2\varepsilon_1}\tilde{a}_{ij}^2 p_{1j}^2(\theta_r\omega) + \frac{1}{2\varepsilon_2}\tilde{b}_{ij}^2 p_{2j}^2(\theta_r\omega) + \frac{1}{2\varepsilon_3}\tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_r\omega) \right) \\ &\times e^{h_1 r} \sup_{s \in (-\infty, 0]} e^{2\gamma s}|x_{jr}(s, \tau - t, \omega, \phi)|^2 dr + \int_{\tau - t}^{t^*} e^{h_1 r} \left(\frac{1}{2\varepsilon_4}|J_i(r)|^2 + 2h_{2i}^2 \right) dr \\ &+ (2N+1) \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \left(\frac{1}{2\varepsilon_1}\tilde{a}_{ij}^2 q_{1j}^2(\theta_r\omega) + \frac{1}{2\varepsilon_2}\tilde{b}_{ij}^2 q_{2j}^2(\theta_r\omega) + \frac{1}{2\varepsilon_3}\tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_r\omega) \right) e^{h_1 r} dr. \end{aligned} \tag{F.40}$$

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{h_1}{16}$. Then we can neglect the second term on the right-hand side of (F.40). Note that $\frac{h_1}{2} < \frac{5}{8}h_1 < \gamma$, so $e^{(2\gamma - h_1)s} \leq 1$ for $s \leq 0$. Setting now $t^* + s$ instead of t^* , multiplying (F.40) by $e^{-h_1(t^* + s)}e^{2\gamma s}$ and replacing ω by $\theta_{-\tau}\omega$,

we find that for all $\tau - t - t^* \leq s \leq 0$,

$$\begin{aligned}
 e^{2\gamma s} |x_{it^*}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 &\leq e^{-h_1(t^* - \tau + t)} |x_i(\tau - t, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &+ \frac{8}{h_1} (2N + 1) e^{2\gamma h} e^{-h_1 t^*} \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) e^{h_1 r} \\
 &\times \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 dr \\
 &+ \frac{8}{h_1} (2N + 1) e^{-h_1 t^*} \int_{\tau - t}^{t^*} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) e^{h_1 r} dr \\
 &+ e^{-h_1 t^*} \int_{\tau - t}^{t^*} e^{h_1 r} \left(\frac{8}{h_1} |J_i(r)|^2 + 2h_{2i}^2 \right) dr,
 \end{aligned} \tag{F.41}$$

where we have used the notations

$$\begin{aligned}
 \alpha_{ij}(\theta_{r-\tau}\omega) &:= \tilde{a}_{ij}^2 p_{1j}^2(\theta_{r-\tau}\omega) + \tilde{b}_{ij}^2 p_{2j}^2(\theta_{r-\tau}\omega) + \tilde{c}_{ij}^2 \tilde{p}_{3j}^2(\theta_{r-\tau}\omega), \\
 \beta_{ij}(\theta_{r-\tau}\omega) &:= \tilde{a}_{ij}^2 q_{1j}^2(\theta_{r-\tau}\omega) + \tilde{b}_{ij}^2 q_{2j}^2(\theta_{r-\tau}\omega) + \tilde{c}_{ij}^2 \tilde{q}_{3j}^2(\theta_{r-\tau}\omega).
 \end{aligned}$$

Note that for all $s \in (-\infty, \tau - t - t^*]$,

$$\begin{aligned}
 &\sum_{i \in \mathbb{Z}} e^{2\gamma s} |x_{it^*}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &= e^{-2\gamma(t^* + t - \tau)} \sum_{i \in \mathbb{Z}} e^{2\gamma(s + t^* - \tau + t)} |x_i(t^* + s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &\leq e^{-h_1(t^* + t - \tau)} \sum_{i \in \mathbb{Z}} e^{2\gamma(s + t^* - \tau + t)} |x_i(t^* + s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &\leq e^{-h_1(t^* + t - \tau)} \|\phi\|_{C_{\gamma, i^2}}^2.
 \end{aligned}$$

Then, it holds

$$\begin{aligned}
 &e^{h_1 t^*} \sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{it^*}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &\leq e^{-h_1(t - \tau)} \|\phi\|_{C_{\gamma, i^2}}^2 + \int_{\tau - t}^{t^*} e^{h_1 r} \left(\frac{8}{h_1} \|J(r)\|^2 + 2\|h_2\|^2 \right) dr \\
 &+ \frac{8}{h_1} (2N + 1) e^{2\gamma h} \int_{\tau - t}^{t^*} e^{h_1 r} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \\
 &\times \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 dr \\
 &+ \frac{8}{h_1} (2N + 1) \int_{\tau - t}^{t^*} e^{h_1 r} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) dr.
 \end{aligned} \tag{F.42}$$

We observe that

$$\begin{aligned}
 &\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \\
 &\leq \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{jr}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \right) \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \right)
 \end{aligned}$$

$$\leq (2N + 1) \left(\sum_{i \in \mathbb{Z}} \sup_{s \in (-\infty, 0]} e^{2\gamma s} |x_{ir}(s, \tau - t, \theta_{-\tau}\omega, \phi)|^2 \right) \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \right).$$

Then (F.42) can be rewritten as

$$\begin{aligned} e^{h_1 t^*} \|x_{t^*}\|_{C_{\gamma, l^2}}^2 &\leq e^{-h_1(t-\tau)} \|\phi\|_{C_{\gamma, l^2}}^2 \\ &+ \frac{8}{h_1} (2N + 1) \int_{\tau-t}^{t^*} e^{h_1 r} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) dr \\ &+ \int_{\tau-t}^{t^*} e^{h_1 r} \left(\frac{8}{h_1} \|J(r)\|^2 + 2\|h_2\|^2 \right) dr \\ &+ \frac{8}{h_1} (2N + 1)^2 e^{2\gamma h} \int_{\tau-t}^{t^*} e^{h_1 r} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) \|x_r\|_{C_{\gamma, l^2}}^2 dr. \end{aligned}$$

Using Gronwall’s lemma, we have

$$\begin{aligned} \|x_{t^*}\|_{C_{\gamma, l^2}}^2 &\leq e^{-h_1(t^*+t-\tau)} e^{\int_{\tau-t}^{t^*} \frac{8}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r-\tau}\omega) dr} \|\phi\|_{C_{\gamma, l^2}}^2 \\ &+ \int_{\tau-t}^{t^*} \left(\frac{8}{h_1} (2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_{r-\tau}\omega) + \frac{8}{h_1} \|J(r)\|^2 + 2\|h_2\|^2 \right) \\ &\times e^{-h_1(t^*-r)+\int_r^{t^*} \frac{8}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{r'-\tau}\omega) dr'} dr \\ &= e^{-h_1(t^*+t-\tau)} e^{\int_{-t}^{t^*-\tau} \frac{8}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s\omega) ds} \|\phi\|_{C_{\gamma, l^2}}^2 \\ &+ \int_{-t}^{t^*-\tau} \left(\frac{8}{h_1} (2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s\omega) + \frac{8}{h_1} \|J(\tau + s)\|^2 + 2\|h_2\|^2 \right) \\ &\times e^{-h_1(t^*-\tau-s)+\int_s^{t^*-\tau} \frac{8}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} ds. \end{aligned} \tag{F.43}$$

Let $t^* = \tau$, then for all $t \geq 0$ we have

$$\begin{aligned} \|x_\tau\|_{C_{\gamma, l^2}}^2 &\leq C e^{-h_1 t + \frac{8}{h_1} (2N+1)^2 e^{2\gamma h} \int_{-t}^0 \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_s\omega) ds} \|\phi\|_{C_{\gamma, l^2}}^2 \\ &+ C \int_{-t}^0 \left(\sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \beta_{ij}(\theta_s\omega) + \|J(\tau + s)\|^2 + \|h_2\|^2 \right) \\ &\times e^{h_1 s + \int_s^0 \frac{8}{h_1} (2N+1)^2 e^{2\gamma h} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \alpha_{ij}(\theta_{s'}\omega) ds'} ds, \end{aligned}$$

and thus the proof of this lemma is finished. □

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Received April 2020; revised August 2020.

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