SOME RECENT DEVELOPMENTS IN THE UNIQUE DETERMINATIONS IN PHASELESS INVERSE ACOUSTIC SCATTERING THEORY

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Abstract. This article is an overview on some recent advances in the inverse scattering problems with phaseless data. Based upon our previous studies on the uniqueness issues in phaseless inverse acoustic scattering theory, this survey aims to briefly summarize the relevant rudiments comprising prototypical model problems, major results therein, as well as the rationale behind the basic techniques. We hope to sort out the essential ideas and shed further lights on this intriguing field.

1. **Introduction.** Inverse scattering problems are fundamental in many scientific and industrial applications. Several exemplary scenarios involve sonar detection, radar sensing, medical imaging and geophysical exploration. Inverse scattering problems are concerned with the detection and identification of unknown targets from the knowledge of associated wave scattering data. In particular, inverse scattering of time-harmonic waves is of great significance. The typical time-harmonic inverse scattering problems are based on complex-valued data comprising phase and intensity/modulus. On the other hand, the phase information may be unavailable or extremely difficult to be detected accurately in practice. Hence, according to the accessibility to phase, the measured data in time-harmonic inverse scattering problems can be classified into two categories: phased/full data and phaseless or intensity-only/modulus-only data. Over the past several decades, the inverse scattering problems with full measured data received a great deal of attentions in the literature (see, e.g. [11, 12, 22] and the references therein). In fact, the phaseless measurements are usually more feasible in practice. Therefore, the phaseless inverse scattering problems have recently been intensively studied mathematically and numerically [1, 2, 4, 31, 32, 49, 57].

Due to the lack of phase information, the phaseless inverse scattering problems are in general very challenging. From the mathematical perspective, the first challenge is the uniqueness: can one uniquely identify the underlying target from the measured phaseless data? To compensate the missing phase and justify the uniqueness, some additional information should be incorporated into the scattering system. There are several strategies for supplementing more quantities, for instance, superposing incident waves and adding artificial reference/interfering objects. Using these techniques, the phaseless inverse scattering problems can be recast as "*well-posed*" problems in the sense of uniqueness justifications. The uniqueness results for phaseless inverse scattering problems usually rely on the existing uniqueness theorems concerning the phased data. We refer to the monograph [12] as an entry for the vast investigations on the uniqueness issues associated with full data.

Our studies on the uniqueness in phaseless inverse scattering problems are scattered in different papers [44, 50, 52, 55] and thus they are loosely coupled in certain sense. Accordingly, this paper aims to present a unified and consistent framework for the problem formulations, technical treatment and the uniqueness results. We would like to emphasize that the purpose of this short review is mainly to crystallize the major developments that we had participated in, that is, we made no effort to cover all the relevant topics or present a comprehensive investigation on the diverse literature. In fact we only consider the phaseless inverse acoustic scattering problems as models. Nevertheless, it paves the way for many extensions to the cases of electromagnetic and elastic waves, where analogous strategies can be employed but technical details might be more complicated. We refer to the recent survey paper [42] for the phaseless inverse problems for some typical wave equations.

The rest of this paper is arranged as follows. Section 2 is devoted to the reference ball technique for uniquely determining a bounded scatterer from phaseless far-field data. Then in section 3, we present the uniqueness results related to the superposition of exterior point sources and near-field measurements. Finally, the interior problem of imaging the cavity is discussed in section 4.

2. **The reference ball technique.** We present some prototypical inverse acoustic scattering models and the reference ball technique for establishing the uniqueness.

2.1. **Model problems.** In this subsection, we introduce the acoustic direct and inverse scattering models for an incident plane wave. Let the scatterer $D \subset \mathbb{R}^3$ occupy an open and simply-connected domain with C^2 boundary ∂D. Let $u^i(x, d) = e^{\mathrm{i}kx \cdot d}$ be a given incident plane wave, where $d \in \mathbb{S}^2$ and $k > 0$ are the incident direction and wavenumber, respectively.

Denote by ν the unit outward normal to ∂D. Then the forward obstacle scattering problem can be stated as: given D and u^i, find the scattered field $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \backslash \overline{D})$ such that the total field $u = u^i + u^s$ satisfies the following boundary value problem (see [12]):

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{D}, \tag{1}$$

$$\mathscr{B}u = 0 \quad \text{on } \partial D, \tag{2}$$

$$\lim_{r=|x| \to \infty} r \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0, \tag{3}$$

where u^s denotes the scattered field and (3) is the Sommerfeld radiation condition. The boundary operator $\mathscr{B}$ defined in (2) is given by

$$\mathscr{B}u = \begin{cases} u & \text{if } D \text{ is of sound-soft type,} \\ \dfrac{\partial u}{\partial \nu} + \mathrm{i}k\lambda u & \text{if } D \text{ is of impedance type,} \end{cases} \tag{4}$$

where λ is a real parameter signifying the physical impedance. The boundary condition (4) is rather general since it covers the usual Dirichlet/sound-soft boundary condition, the Neumann/sound-hard boundary condition ($\lambda = 0$) and the impedance boundary condition ($\lambda \neq 0$).

In contrast to the obstacle scattering problem dealing with an impenetrable scatterer D, if D is penetrable with respect to wave propagation, then the forward scattering can be formulated as the medium scattering problem: given D and u^i, determine the scattered field $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that the total field $u = u^i + u^s$ fulfills

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3, \tag{5}$$

$$\lim_{r=|x|\to\infty} r\left(\frac{\partial u^s}{\partial r} - \mathrm{i}k u^s\right) = 0, \tag{6}$$

where the refractive index $n(x)$ of the inhomogeneous medium is piecewise continuous such that $\mathrm{Re}(n) > 0, \mathrm{Im}(n) \geq 0$ and $1 - n(x)$ is supported in D.

The forward scattering problems (1)–(3) and (5)–(6) admit a unique solution (see, e.g., [6, 12, 37]), respectively. Moreover, the scattered wave u^s has the following asymptotic behavior

$$u^s(x;d) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|}\left\{u^\infty(\hat{x};d) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty$$

uniformly in all observation directions $\hat{x} = x/|x|$. Denote by $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ the unit sphere in $\mathbb{R}^3$. Then the complex-valued analytic function $u^\infty(\hat{x};d)$ is defined on $\mathbb{S}^2$ and known as the scattering amplitude or the far field pattern (see [12]). In what follows, we shall also employ $u^\infty_D(\hat{x};d)$ to indicate the dependence of the far field pattern u^∞ on the observation direction $\hat{x}$, the incident direction d, the obstacle or medium D and a fixed wavenumber k.

A typical phaseless inverse scattering problem is stated as follows.

Problem 2.1. *Given phaseless far field data $|u^\infty_D(\hat{x};d)|$ for $\hat{x}, d \in \mathbb{S}^2$ and a fixed $k > 0$.*

(i) Suppose that D is an impenetrable obstacle, determine the location and shape ∂D as well as the boundary condition $\mathscr{B}$.

(ii) Suppose that D is a penetrable inhomogeneous medium, determine the refractive index n_D for the medium inclusion.

There is a well-known obstruction of Problem 2.1 due to the translation invariance. Specifically, for the shifted domain $D^h := \{x + h : x \in D\}$ with a fixed vector $h \in \mathbb{R}^3$, the far field pattern $u^\infty_{D^h}(\hat{x};d)$ satisfies the relation [29, 34]

$$u^\infty_{D^h}(\hat{x};d) = \mathrm{e}^{\mathrm{i}kh\cdot(d-\hat{x})}u^\infty_D(\hat{x};d), \quad \hat{x} \in \mathbb{S}^2. \tag{7}$$

Hence the location of the scatterer cannot be uniquely determined from the intensity-only far-field data. More notoriously, this intrinsic ambiguity cannot be remedied by simply using finitely many incident waves with different wavenumbers or distinct directions of incidence. Nevertheless, it is still possible to reconstruct the shape without such phase information. In fact, a great number of inversion schemes have been proposed to recover the shape of scatterer from the intensity-only far-field

data with a single incident plane wave, see [17, 18, 19, 30, 31, 32]. We also refer to [10, 16] for the relevant numerical investigations.

2.2. **Uniqueness with a reference ball.** To compensate the lack of phase and thereby break the translation-invariance obstruction, some supplementing information is required to be incorporated into the scattering system. We made an attempt in [50] to tackle the translation invariance by adding a reference ball in conjunction with the superposition of incident waves. In this subsection, we shall give a brief outline of the reference ball technique for phaseless inverse scattering problems.

Definition 2.1 (Reference ball). For the inverse scattering problem of recovering D, let $B \subset \mathbb{R}^3$ such that $\overline{D} \cap \overline{B} = \emptyset$. The scatterer B is called a reference object if

  (i) The geometrical (location, shape, etc.) and physical properties (boundary conditions, refractive index, etc.) of B are assumed, a priori, to be known.
 (ii) Determination of the target D relies on the scattering data produced by $D \cup B$.

Clearly, a simplest candidate for the reference object is the reference ball $B = B(x_0, R) := \{x \in \mathbb{R}^3 : |x - x_0| < R\}$.

Due to the nonlinearity of typical inverse scattering problems, one should bear in mind that extracting the contribution of D from the data due to $D \cup B$ is not straightforward in most cases.

We also need the following admissible source location.

Definition 2.2 (Admissible source location). Assume that P is a simply-connected convex polyhedron with boundary $\Pi := \partial P$ such that $P \subset \mathbb{R}^3 \backslash (\overline{D} \cup \overline{B})$ and k^2 is not a Dirichlet eigenvalue of $-\Delta$ in P.

Then let us consider the superposition of a plane wave u^i and a point source v^i as the incident wave:

$$u^i(x, d) + v^i(x, z) = \mathrm{e}^{\mathrm{i}kx \cdot d} + \Phi(x, z), \tag{8}$$

where $z \in \Pi$ denotes the location of point source, and

$$\Phi(x; z) = \frac{\mathrm{e}^{\mathrm{i}k|x-z|}}{4\pi|x - z|}$$

is the fundamental solution to the Helmholtz equation.

Let the $u_{D \cup B}^\infty(\hat{x}; d)$ and $v_{D \cup B}^\infty(\hat{x}; z)$ be the far-field pattern generated by $D \cup B$ corresponding to the incident field u^i and v^i, respectively. Then, by the linearity of direct scattering problem, the far-field pattern generated by $D \cup B$ and the incident wave $u^i + v^i$ defined in (8) is given by $u_{D \cup B}^\infty(\hat{x}; d) + v_{D \cup B}^\infty(\hat{x}; z), \hat{x} \in \mathbb{S}^2$, respectively.

Based on Definitions 2.1 and 2.2, we are now in the position to introduce the phaseless datasets.

Definition 2.3 (Far field data with a reference ball). For a fixed wavenumber $k > 0$ and a fixed $d_0 \in \mathbb{S}^2$, define

$$\mathbb{D}_1^{\mathrm{FB}} := \{|u_{D \cup B}^\infty(\hat{x}; d_0)| : \hat{x} \in \mathbb{S}^2\},$$
$$\mathbb{D}_2^{\mathrm{FB}} := \{|v_{D \cup B}^\infty(\hat{x}; z)| : \hat{x} \in \mathbb{S}^2, \ z \in \Pi\},$$
$$\mathbb{D}_3^{\mathrm{FB}} := \{|u_{D \cup B}^\infty(\hat{x}; d_0) + v_{D \cup B}^\infty(\hat{x}; z)| : \hat{x} \in \mathbb{S}^2, \ z \in \Pi\}.$$

The incursion of the reference ball and superposition of incoming waves lead to the following reformulation of phaseless inverse scattering problems:
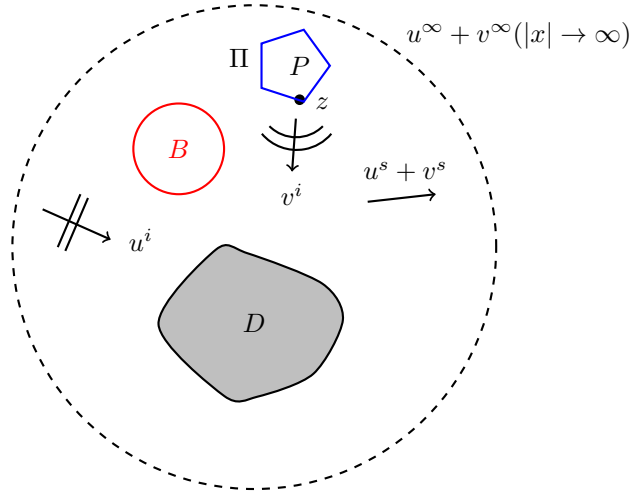
FIGURE 1. An illustration of the reference ball technique.

Problem 2.2 (Inverse obstacle scattering with a reference ball and far-field data)**.** *Let D be the impenetrable obstacle with boundary condition $\mathscr{B}$. Given the phaseless triple $\{\mathbb{D}_1^{\mathrm{FB}}, \mathbb{D}_2^{\mathrm{FB}}, \mathbb{D}_3^{\mathrm{FB}}\}$ due to D and a sound-soft reference ball B, determine the location and shape ∂D as well as the boundary condition $\mathscr{B}$ for the obstacle.*

Problem 2.3 (Inverse medium scattering with a reference ball and far-field data)**.** *Let D be the inhomogeneous medium with refractive index n. Given the phaseless triple $\{\mathbb{D}_1^{\mathrm{FB}}, \mathbb{D}_2^{\mathrm{FB}}, \mathbb{D}_3^{\mathrm{FB}}\}$ due to the inhomogeneity D and a penetrable reference ball B, determine the refractive index n for the medium inclusion.*

We refer to Figure 1 for an illustration of the geometry setup of Problems 2.2 and 2.3. The following theorem shows the uniqueness results for Problem 2.2 and Problem 2.2, i.e., the scatterer can be uniquely determined from the consolidated phaseless data $\{\mathbb{D}_1^{\mathrm{FB}}, \mathbb{D}_2^{\mathrm{FB}}, \mathbb{D}_3^{\mathrm{FB}}\}$.

Theorem 2.1. [50] *Given a fixed $d_0 \in \mathbb{S}^2$ and a reference ball B, the far-field pattern with respect to the incident field $u^i(x; d_0)$ and $v^i(x; z)$ are denoted by $u_{D_j \cup B}^\infty(\hat{x}; d_0)$, and $v_{D_j \cup B}^\infty(\hat{x}; z), j = 1, 2$, respectively. Suppose that*

$$|u_{D_1 \cup B}^\infty(\hat{x}; d_0)| = |u_{D_2 \cup B}^\infty(\hat{x}; d_0)|, \quad \forall \hat{x} \in \mathbb{S}^2,$$

$$|v_{D_1 \cup B}^\infty(\hat{x}; z)| = |v_{D_2 \cup B}^\infty(\hat{x}; z)|, \quad \forall(\hat{x}, z) \in \mathbb{S}^2 \times \Pi,$$

$$|u_{D_1 \cup B}^\infty(\hat{x}; d_0) + v_{D_1 \cup B}^\infty(\hat{x}; z)| = |u_{D_2 \cup B}^\infty(\hat{x}; d_0) + v_{D_2 \cup B}^\infty(\hat{x}; z)|, \quad \forall(\hat{x}, z) \in \mathbb{S}^2 \times \Pi.$$

Then we have

  (i) *If D_1 and D_2 are two obstacles with boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, and B is sound-soft, then $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$.*
  (ii) *If D_1 and D_2 are two inclusions with refractive indices n_1 and n_2 respectively, and $B = B(x_0, R)$ is a homogeneous medium with refractive index $n_0 > 0$ such that $2kR(n_0 + 1) < \pi$, then $n_1 = n_2$.*

To resolve non-uniqueness issues, the idea of utilizing the superposition of distinct incident plane waves was proposed in [56], which led to the multi-frequency Newton

iteration algorithm [56]. Moreover, by the superposition of two incident plane waves, uniqueness results were established in [46] under some a priori assumptions.

We would like to point out that the idea of adding a reference ball to the scattering system in [50] was motivated by [33] and [38, 39]. Later, the reference ball technique was used in [47] to alleviate the requirement of the a priori assumptions in [46]. Recently, similar strategies of adding reference objects or sources to the scattering system have also been intensively applied to the theoretical analysis and numerical approaches for different models of phaseless inverse scattering problems [15, 13, 20, 21, 51, 53, 14].

3. **Superposition of point sources.** Next we will deal with the uniqueness issue concerning the inverse acoustic scattering problems with point excitation sources and associated phaseless near-field measurements. In optics and engineering areas, the phaseless inverse scattering with near-field data is also known as phase retrieval problem [35]. The numerical studies on the inverse scattering problems with phaseless near-field data are intensive (see, e.g., [8, 7, 10, 9, 40, 45]), whereas few theoretical investigations have been made on the uniqueness aspects.

A uniqueness result was established in [25] to the reconstruction of a potential with the phaseless near-field data for point sources on a spherical surface and an interval of frequencies, which was extended to the determination of wave speed in generalized 3-D Helmholtz equation [26]. The uniqueness of a coefficient inverse scattering problem with phaseless near-field data was established in [28]. We also refer to [27, 38, 39] for the inversion algorithms for the inverse medium scattering problems with modulus-only near-field data. The stability analysis for linearized near-field phase retrieval in X-ray phase contrast imaging was given in [36].

In this section, we will be concerned the uniqueness via superposition of incident point sources, which does not rely on any additional reference/interfering scatterer. Toward establishing the uniqueness for exterior inverse scattering problems, superposing certain point sources has the capability of providing more information and overcoming the indispensability of incorporating a reference ball in the previous section. By introducing a general admissible surface/curve, along with the superposition of point sources, we proved that the bounded scatterer (impenetrable obstacle or medium inclusion) and the locally perturbed half-plane (locally rough surface) could be uniquely determined from the phaseless near-field data [52]. The uniqueness can be also established by the superposition of incident point sources and phaseless far-field data, see [44]. For the uniqueness of inverse scattering by locally rough surfaces with phaseless far-field data, we refer to [47]. The crux of our study is the utilization of limited-aperture phaseless near-field data co-produced by the scatterer and point sources. A related study on the electromagnetic case can be found in [48].

The following definition of admissible surfaces is needed.

Definition 3.1 (Admissible surface). An open surface Γ is called an admissible surface with respect to domain Ω if

(i) $\Omega \subset \mathbb{R}^3 \backslash \overline{D}$ is bounded and simply-connected;
(ii) $\partial\Omega$ is analytic homeomorphic to $\mathbb{S}^2$;
(iii) Ω is non-resonant, i.e., k^2 is not a Dirichlet eigenvalue of $-\Delta$ in Ω;
(iv) $\Gamma \subset \partial\Omega$ is a two-dimensional analytic manifold with non-vanishing measure.

For a generic point $z \in \mathbb{R}^3 \backslash \overline{D}$, recall that the incident field due to the point source located at z is given by

$$\Phi(x; z) := \frac{e^{ik|x-z|}}{4\pi|x-z|}, \quad x \in \mathbb{R}^3 \backslash (\overline{D} \cup \{z\}),$$

Denote by $v_D^s(x; z)$ and $v_D^\infty(\hat{x}; z)$ the near-field and far-field pattern generated by D corresponding to the incident field $\Phi(x; z)$. Define

$$v(x; z) := v_D^s(x; z) + \Phi(x; z), \quad x \in \mathbb{R}^3 \backslash (\overline{D} \cup \{z\})$$

and

$$v^\infty(\hat{x}; z) := v_D^\infty(\hat{x}; z) + \Phi^\infty(\hat{x}; z), \quad \hat{x} \in \mathbb{S}^2,$$

where $\Phi^\infty(\hat{x}, z) := e^{-ik\hat{x} \cdot z}/(4\pi)$ is the far-field pattern of $\Phi(x; z)$.

For two generic and distinct source points $z_1, z_2 \in \mathbb{R}^3 \backslash \overline{D}$, we denote by

$$v^i(x; z_1, z_2) := \Phi(x; z_1) + \Phi(x; z_2), \quad x \in \mathbb{R}^3 \backslash (\overline{D} \cup \{z_1\} \cup \{z_2\}),$$

the superposition of these point sources. Then, by the linearity of direct scattering problem, the near- and far-field co-produced by D and the incident wave $v^i(x; z_1, z_2)$ are respectively given by

$$v(x; z_1, z_2) := v(x; z_1) + v(x; z_2), \quad x \in \mathbb{R}^3 \backslash (\overline{D} \cup \{z_1\} \cup \{z_2\}).$$
$$v^\infty(\hat{x}; z_1, z_2) := v^\infty(\hat{x}; z_1) + v^\infty(\hat{x}; z_2), \quad \hat{x} \in \mathbb{S}^2.$$

3.1. **Bounded scatterer with far-field data.** We first outline the results for recovering a bounded scatterer from phaseless far-field data. Following [44], we would like to remark that the phaseless data $|v_D^\infty(\hat{x}, z) + \Phi^\infty(\hat{x}, z)|$ is usually obtainable in practice, while the phaseless data $|v_D^\infty(\hat{x}, z)|$ cannot be directly measured. Hence, in our opinion, it would be more meaningful to use modulus-only data $|v^\infty(\hat{x}, z)| = |v_D^\infty(\hat{x}, z) + \Phi^\infty(\hat{x}, z)|$.

Definition 3.2 (Far-field datasets). For a fixed wavenumber $k > 0$ and a fixed $z_0 \in \mathbb{R}^3 \backslash (\overline{D} \cup \Gamma)$, define

$$\mathbb{D}_1^{\mathrm{F}} := \{|v^\infty(\hat{x}; z_0)| : \hat{x} \in \mathbb{S}^2\},$$
$$\mathbb{D}_2^{\mathrm{F}} := \{|v^\infty(\hat{x}; z)| : \hat{x} \in \mathbb{S}^2, \ z \in \Gamma\},$$
$$\mathbb{D}_3^{\mathrm{F}} := \{|v^\infty(\hat{x}; z_0) + v^\infty(\hat{x}; z)| : \hat{x} \in \mathbb{S}^2, \ z \in \Gamma\},$$

where Γ is an admissible surface.

The phaseless inverse scattering problems are listed in Problems 3.1 and 3.2. These problems are depicted in Figure 2 as an illustration.

Problem 3.1 (Inverse obstacle scattering with far-field data). *Let D be the impenetrable obstacle with boundary condition $\mathscr{B}$. Given the phaseless far-field data triple $\{\mathbb{D}_1^{\mathrm{F}}, \mathbb{D}_2^{\mathrm{F}}, \mathbb{D}_3^{\mathrm{F}}\}$ due to the obstacle D, determine the location and shape ∂D as well as the boundary condition $\mathscr{B}$ for the obstacle.*

Problem 3.2 (Phaseless inverse medium scattering with far-field data). *Let D be the inhomogeneous medium with refractive index n. Given the phaseless far-field data triple $\{\mathbb{D}_1^{\mathrm{F}}, \mathbb{D}_2^{\mathrm{F}}, \mathbb{D}_3^{\mathrm{F}}\}$ due to the inhomogeneity D, determine the refractive index n for the medium inclusion.*

The following theorem tells us that Problem 3.1 (resp. Problem 3.2) admits a unique solution, namely, the geometrical and physical information of the obstacle (resp. the refractive index for the medium) can be uniquely determined from the far-field data triple $\{\mathbb{D}_1^{\mathrm{F}}, \mathbb{D}_2^{\mathrm{F}}, \mathbb{D}_3^{\mathrm{F}}\}$.
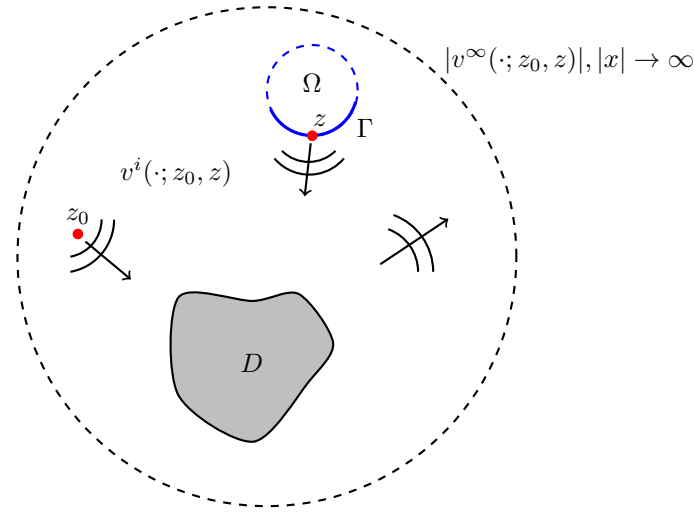
FIGURE 2. An illustration of the inverse scattering with a bounded scatterer and far-field measurements.

Theorem 3.1. [44] *For two scatterers D_1 and D_2, suppose that the corresponding far-field patterns satisfy that*

$$|v_1^\infty(\hat{x}; z_0)| = |v_2^\infty(\hat{x}; z_0)|, \quad \forall \hat{x} \in \mathbb{S}^2,$$
$$|v_1^\infty(\hat{x}; z)| = |v_2^\infty(\hat{x}; z)|, \quad \forall(\hat{x}, z) \in \mathbb{S}^2 \times \Gamma,$$
$$|v_1^\infty(\hat{x}; z_0) + v_1^\infty(\hat{x}; z)| = |v_2^\infty(\hat{x}; z_0) + v_2^\infty(\hat{x}; z)|, \quad \forall(\hat{x}, z) \in \mathbb{S}^2 \times \Gamma$$

for an admissible surface Γ and an arbitrarily fixed $z_0 \in \mathbb{R}^3 \backslash (\overline{D} \cup \Gamma)$. Then we have

  (i) *If D_1 and D_2 are two impenetrable obstacles with boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, then $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$.*
 (ii) *If D_1 and D_2 are two medium inclusions with refractive indices n_1 and n_2 respectively, then $n_1 = n_2$.*

3.2. **Bounded scatterers with near-field data.** We then talk about the inverse scattering problems for a bounded scatterer with near-field data. Let D be the impenetrable obstacle with boundary condition $\mathscr{B}$ or the inhomogeneous medium with refractive index n. The mathematical formulation of forward scattering problems can be found in Section 2.

Definition 3.3 (Near-field datasets for a bounded scatterer)**.** Assume that Γ and Σ are admissible surfaces with respect to Ω and G, respectively, such that $\overline{\Omega} \cap \overline{G} = \emptyset$. Define

$$\mathbb{D}_1^N := \{|v(x; z_0)| : x \in \Sigma\},$$
$$\mathbb{D}_2^N := \{|v(x; z)| : x \in \Sigma, \ z \in \Gamma\},$$
$$\mathbb{D}_3^N := \{|v(x; z_0) + v(x; z)| : x \in \Sigma, \ z \in \Gamma\}$$

for a fixed wavenumber $k > 0$ and a fixed $z_0 \in \mathbb{R}^3 \backslash (\overline{D} \cup \Gamma \cup \Sigma)$.

With these configurations, we formulate the phaseless inverse scattering problems as follows.
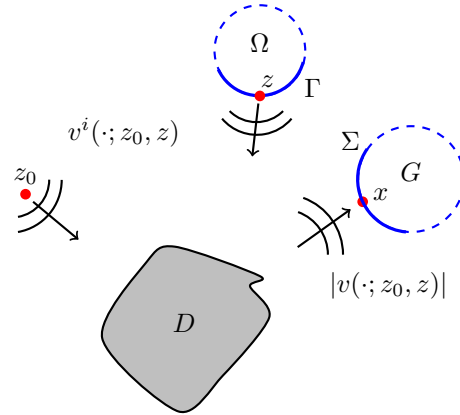
FIGURE 3. An illustration of the inverse scattering with a bounded scatterer and near-field measurements.

Problem 3.3 (Inverse obstacle scattering with near-field data). *Let D be the impenetrable obstacle with boundary condition $\mathscr{B}$. Given the phaseless near-field triple $\{\mathbb{D}_1^{\mathrm{N}}, \mathbb{D}_2^{\mathrm{N}}, \mathbb{D}_3^{\mathrm{N}}\}$ due to the obstacle D, determine the location and shape ∂D as well as the boundary condition $\mathscr{B}$ for the obstacle.*

Problem 3.4 (Inverse medium scattering with near-field data). *Let D be the inhomogeneous medium with refractive index n. Given the phaseless near-field triple $\{\mathbb{D}_1^{\mathrm{N}}, \mathbb{D}_2^{\mathrm{N}}, \mathbb{D}_3^{\mathrm{N}}\}$ due to the inhomogeneity D, determine the refractive index n for the medium inclusion.*

We refer to Figure 3 for an illustration of the geometry setting of Problems 3.3 and 3.4.

Now we present the uniqueness results on phaseless inverse scattering. The following theorem asserts that Problem 3.3 and Problem 3.3 admits a unique solution respectively, namely, the geometrical and physical information of the scatterer boundary or the refractive index for the medium can be simultaneously and uniquely determined from near-field data $\{\mathbb{D}_1^{\mathrm{N}}, \mathbb{D}_2^{\mathrm{N}}, \mathbb{D}_3^{\mathrm{N}}\}$. For similar studies on phaseless inverse problem with superimposed/interfering waves, we refer to [43, 48].

Theorem 3.2. [52] *Assume that Γ and Σ are admissible surfaces with respect to Ω and G respectively, such that $\overline{\Omega} \cap \overline{G} = \emptyset$. For two scatterers D_1 and D_2, suppose that*

$$|v_1(x; z_0)| = |v_2(x; z_0)|, \quad \forall x \in \Sigma,$$
$$|v_1(x; z)| = |v_2(x; z)|, \quad \forall (x, z) \in \Sigma \times \Gamma,$$
$$|v_1(x; z_0) + v_1(x; z)| = |v_2(x; z_0) + v_2(x; z)|, \quad \forall (x, z) \in \Sigma \times \Gamma,$$

for an arbitrarily fixed $z_0 \in \mathbb{R}^3 \backslash (\overline{D} \cup \Gamma \cup \Sigma)$. Then we have

(i) *If D_1 and D_2 are two impenetrable obstacles with boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, then $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$.*

(ii) *If D_1 and D_2 are two medium inclusions with refractive indices n_1 and n_2 respectively, then $n_1 = n_2$.*

Remark 1. In fact Theorem 3.2 (ii) is concerned with the uniqueness for a phaseless coefficient inverse problem of determining the coefficient n in the Helmholtz equation. It is well known that among all inverse problems, with phase and without phase information, the problems about finding coefficients of PDEs are the most challenging ones. So, Theorem 3.2 does not come for "free". In the sense of cardinality counting, the main simplifying assumption in it is of course the over-determination of data: the data depend on more variables than the unknown coefficient. In this regard, the unique method at this point of time allowing to prove uniqueness for coefficient inverse problems with non-overdetermined data is the Bukhgeĭm-Klibanov method [5], also see a survey in [23]. As to the phaseless case, the single paper addressing this issue is the paper [24]. What is done in [24] is that first, the phase is determined. And, second, reference to [23] is given.

3.3. **Locally perturbed half-planes.** This subsection deals with the model scattering problem of a locally perturbed half-plane. Assume that the real-valued function $f \in C^2(\mathbb{R})$ has a compact support. Let $\Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2 |\ x_2 = f(x_1), x_1 \in \mathbb{R}\}$ be the locally perturbed curve and $D = \{x \in \mathbb{R}^2 |\ x_2 > f(x_1), x_1 \in \mathbb{R}\}$ be the locally perturbed half-plane. Denote by $\Gamma_c = \{x \in \mathbb{R}^2 |\ x_2 = 0\}$ and then $\Gamma_p = \Gamma \backslash \Gamma_c$ is the local perturbation. For a generic point $z \in D$, the incident field u^i due to the point source located at z is given by

$$u^i(x; z) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x - z|),\ x \in D \backslash \{z\},$$

which is the 2D fundamental solution to the Helmholtz equation. Here $H_0^{(1)}$ signifies the Hankel function of the first kind and order zero. Then the scattering problem can be formulated as: find the scattered field u^s such that

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } D, \tag{9}$$

$$\mathscr{B}_c u = 0 \quad \text{on } \Gamma_c, \tag{10}$$

$$\mathscr{B}_p u = 0 \quad \text{on } \Gamma_p, \tag{11}$$

$$\lim_{r=|x|\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - \mathrm{i}k u^s\right) = 0, \tag{12}$$

where $u = u^i + u^s$ denotes the total field. The boundary operators $\mathscr{B}_c$ and $\mathscr{B}_p$ appeared in (10)-(11) are defined by

$$\mathscr{B}_c u := \begin{cases} u, & \text{for a perturbation of sound-soft type,} \\ \dfrac{\partial u}{\partial \nu}, & \text{for a perturbation of sound-hard type,} \end{cases}$$

$$\mathscr{B}_p u := \begin{cases} u, & \text{on } \Gamma_{p,D}, \\ \dfrac{\partial u}{\partial \nu} + \lambda u, & \text{on } \Gamma_{p,I}, \end{cases} \tag{13}$$

where ν is the unit normal on Γ directed into D, $\Gamma_{p,D} \cup \Gamma_{p,I} = \Gamma_p$, $\Gamma_{p,D} \cap \Gamma_{p,I} = \emptyset$, $\lambda \in C(\Gamma_{p,I})$ and $\mathrm{Im}\lambda \geq 0$. It can be seen that the mixed boundary condition (13) is rather general since it covers the usual Dirichlet/sound-soft boundary condition ($\Gamma_{p,I} = \emptyset$), the Neumann/sound-hard boundary condition ($\Gamma_{p,D} = \emptyset$ and $\lambda = 0$), and the impedance boundary condition ($\Gamma_{p,D} = \emptyset$ and $\lambda \neq 0$).

The well-posedness of scattering problem (9)–(12) in various specific scenarios can be established by the variational method or the integral equation method [3, 41, 54]. We now consider the inverse scattering problem by the locally perturbed half-plane

for incident point sources with limited-aperture phaseless near-field data. Analogous to Definition 3.1, the admissible curves need to be introduced [52].

Definition 3.4 (Admissible curve)**.** An open curve Λ is called an admissible curve with respect to domain Ω if

(i) $\overline{\Omega} \subset D$ is bounded and simply-connected;
(ii) $\partial\Omega$ is analytic homeomorphic to the unit circle $\mathbb{S}$;
(iii) Ω is non-resonant, i.e., k^2 is not a Dirichlet eigenvalue of $-\Delta$ in Ω;
(iv) $\Lambda \subset \partial\Omega$ is a one-dimensional analytic manifold with non-vanishing measure.

Similar to the arguments in the previous section, for two generic and distinct source points $z_1, z_2 \in D$, we denote by

$$u^i(x; z_1, z_2) := u^i(x; z_1) + u^i(x; z_2), \quad x \in D\backslash(\{z_1\} \cup \{z_2\}),$$

the superposition of them. By the linearity of direct scattering problem, the total near-field is expressed by

$$u(x; z_1, z_2) := u(x; z_1) + u(x; z_2), \quad x \in D\backslash(\{z_1\} \cup \{z_2\}).$$

The phaseless datasets are analogous as well.

Definition 3.5 (Near-field datasets for locally perturbed half-plane)**.** Assume that Γ and Σ are admissible surfaces with respect to Ω and G, respectively, such that $\overline{\Omega} \cap \overline{G} = \emptyset$. Define

$$\mathbb{D}_1^{\mathrm{NL}} := \{|u(x; z_0)| : x \in \Sigma\},$$
$$\mathbb{D}_2^{\mathrm{NL}} := \{|u(x; z)| : x \in \Sigma, \ z \in \Lambda\},$$
$$\mathbb{D}_3^{\mathrm{NL}} := \{|u(x; z_0) + u(x; z)| : x \in \Sigma, \ z \in \Lambda\}$$

for a fixed wavenumber $k > 0$ and a fixed $z_0 \in D\backslash(\Lambda \cup \Sigma)$.

The formulation of the phaseless inverse scattering problems under consideration is given by the following problem, which is geometrically illustrated in Figure 4.

Problem 3.5 (Inverse scattering by locally perturbed half-planes)**.** *Let Γ be the locally perturbed curve with boundary condition $\mathscr{B}_c$ and $\mathscr{B}_p$. Assume that Λ and Σ are admissible curves with respect to Ω and G, respectively. Given the near-field data triple $\{\mathbb{D}_1^{\mathrm{NL}}, \mathbb{D}_2^{\mathrm{NL}}, \mathbb{D}_3^{\mathrm{NL}}\}$, determine the locally perturbed curve Γ as well as the boundary condition $\mathscr{B}_c$ and $\mathscr{B}_p$.*

Let $\Gamma_j = \{x \in \mathbb{R}^2 : x_2 = f_j(x_1), x_1 \in \mathbb{R}\}$ be the locally perturbed curve with compactly supported function $f_j \in C^2(\mathbb{R}), j = 1, 2$. Denote by $D_j = \{x \in \mathbb{R}^2 : x_2 > f_j(x_1), x_1 \in \mathbb{R}\}$ the corresponding domain above $\Gamma_j, j = 1, 2$, and by $D_0 = D_1 \cap D_2$. Denote by u_j^s and u_j the scattered field and the total field produced by Γ_j, respectively, corresponding to the incident field $u^i(x; z), j = 1, 2$. The following theorem shows that Problem 3.5 admits a unique solution, in other words, the geometrical and physical information of the locally perturbed plane can be simultaneously and uniquely determined from the phaseless near-field data $\{\mathbb{D}_1^{\mathrm{NL}}, \mathbb{D}_2^{\mathrm{NL}}, \mathbb{D}_3^{\mathrm{NL}}\}$.

Theorem 3.3. [52] *Let Γ_1 and Γ_2 be two locally perturbed curves with boundary conditions $\mathscr{B}_{c,1}, \mathscr{B}_{p,1}$ and $\mathscr{B}_{c,2}, \mathscr{B}_{p,2}$, respectively. Assume that Λ and Σ are admissible curves with respect to Ω and G, respectively, such that $\overline{\Omega} \subset\subset D_0$, $\overline{G} \subset\subset D_0$*
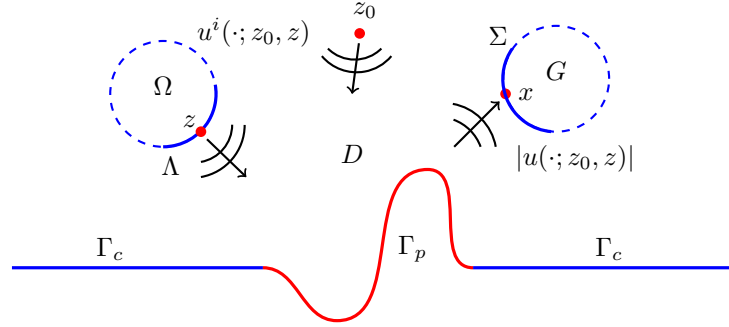
FIGURE 4. An illustration of the phaseless inverse scattering by a locally perturbed half-plane.

and $\overline{\Omega} \cap \overline{G} = \emptyset$. *Suppose that the corresponding total near-fields satisfy that*

$$|u_1(x; z_0)| = |u_2(x; z_0)|, \quad \forall x \in \Sigma,$$
$$|u_1(x; z)| = |u_2(x; z)|, \quad \forall (x, z) \in \Sigma \times \Lambda,$$
$$|u_1(x; z_0) + u_1(x; z)| = |u_2(x; z_0) + u_2(x; z)|, \quad \forall (x, z) \in \Sigma \times \Lambda,$$

for an arbitrarily fixed $z_0 \in D_0 \backslash (\Lambda \cup \Sigma)$. Then we have $\Gamma_1 = \Gamma_2$, $\mathscr{B}_{c,1} = \mathscr{B}_{c,2}$ and $\mathscr{B}_{p,1} = \mathscr{B}_{p,2}$.

4. **An interior problem.** In this last section, a recent result concerning phaseless inverse cavity scattering problems will be presented [55]. To our knowledge, this is the first uniqueness result in inverse cavity scattering problems with phaseless data. As mentioned in the previous sections, the reference ball approach and the superposition of emanating point sources are crucial ingredients for the success of establishing uniqueness for phaseless exterior inverse scattering problems. For the interior problem of an impenetrable cavity, it is reasonable to combine these two strategies in order to guarantee the uniqueness. To this end, we bring together four main ingredients in our analysis: utilization of the reference ball technique, super-position of point sources, the reciprocity relations and the singularity of the total fields. In particular, an impedance reference ball plays an exceptionally important role in our analysis and thus irreplaceable. We refer the interested reader to [55] for more details.

We first formulate the model cavity scattering problem. Let $D \subset \mathbb{R}^3$ be an open and simply connected domain with C^2 boundary ∂D. Then, the interior scattering problem for cavities can be stated as: given the incident field u^i, find the scattered field u^s which satisfies the boundary value problem:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } D, \tag{14}$$
$$\mathscr{B}u = 0 \quad \text{on } \partial D, \tag{15}$$

where $u = u^i + u^s$ denotes the total field and $k > 0$ is the wavenumber. Here $\mathscr{B}$ in (15) is the boundary operator defined by

$$\mathscr{B}u := \begin{cases} u, & \text{for a sound-soft cavity,} \\ \dfrac{\partial u}{\partial \nu} + \lambda u, & \text{for an impedance cavity,} \end{cases} \tag{16}$$

where ν is the unit outward normal to ∂D, and $\lambda \in C(\partial D)$ is the impedance function satisfying $\Im(\lambda) \geq 0$. This boundary condition 16 covers the Dirichlet/sound-soft type, the Neumann/sound-hard type ($\lambda = 0$), and the impedance type ($\lambda \neq 0$). The existence of a solution to the direct scattering problem 14–15 is well known (see, e.g., [12]).

To introduce the interior inverse scattering problem for incident point sources with limited-aperture phaseless near-field data, we again employ a reference ball B as an extra artificial object to the scattering system such that $\overline{B} \subset\subset D$ with the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + i\lambda_0 u = 0 \quad \text{on } \partial B, \tag{17}$$

where λ_0 is a positive constant. Similar to the previous sections, we also need Definition 3.3 as an interior admissible surface.

The superposition of point sources are almost identical to the previous section. For a generic point $z \in D\backslash\overline{B}$, the incident field u^i due to the point source located at z is given by $u^i(x; z) = \Phi(x; z)$, $x \in D\backslash(\overline{B} \cup \{z\})$. Denote by $u^s(x; z)$ the near-field generated by D and B corresponding to the incident field $u^i(x; z)$. Let $u(x; z) = u^s(x; z) + u^i(x; z), x \in D\backslash(\overline{B} \cup \{z\})$ be the total field.

For two generic and distinct source points $z_1, z_2 \in D\backslash\overline{B}$, we denote by

$$u^i(x; z_1, z_2) := u^i(x; z_1) + u^i(x; z_2), \quad x \in D\backslash(\overline{B} \cup \{z_1\} \cup \{z_2\}),$$

the superposition of these point sources. Once again, by the linearity of direct scattering problem, the near-field co-generated by D, B and the incident wave $u^i(x; z_1, z_2)$ is written as

$$u(x; z_1, z_2) := u(x; z_1) + u(x; z_2), \quad x \in D\backslash(\overline{B} \cup \{z_1\} \cup \{z_2\}).$$

Definition 4.1 (Near-field datasets for the cavity). Assume that Γ and Σ are admissible surfaces with respect to Ω and G, respectively, such that $\overline{\Omega} \subset\subset G$ and $\overline{G} \subset\subset D\backslash\overline{B}$. Define

$$\mathbb{D}_1^C := \{|u(x; z_0)| : x \in \Sigma\},$$
$$\mathbb{D}_2^C := \{|u(x; z)| : x \in \Sigma, \ z \in \Gamma\},$$
$$\mathbb{D}_3^C := \{|u(x; z_0) + u(x; z)| : x \in \Sigma, \ z \in \Gamma\}$$

for a fixed wavenumber $k > 0$ and a fixed $z_0 \in D\backslash(\overline{B} \cup \Gamma \cup \Sigma)$.

With these preparations, the inverse problem under consideration is formulated as the following phaseless interior problems. The illustration is shown in Figure 5.

Problem 4.1 (Inverse scattering for cavities). *Let D be the impenetrable cavity with boundary condition $\mathscr{B}$. Given the data triple $\{\mathbb{D}_1^C, \mathbb{D}_2^C, \mathbb{D}_3^C\}$, determine the location and shape ∂D as well as the boundary condition $\mathscr{B}$ for the cavity.*

A rigorous proof shows that Problem 4.1 admits a unique solution, namely, the geometrical and physical information of the cavity can be simultaneously and uniquely determined from the phaseless data $\{\mathbb{D}_1^C, \mathbb{D}_2^C, \mathbb{D}_3^C\}\}$. To sum up, we end this section by the following uniqueness theorem.

Theorem 4.1. [55] *Let D_1 and D_2 be two impenetrable cavities with boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively. Assume that Γ and Σ are admissible surfaces*
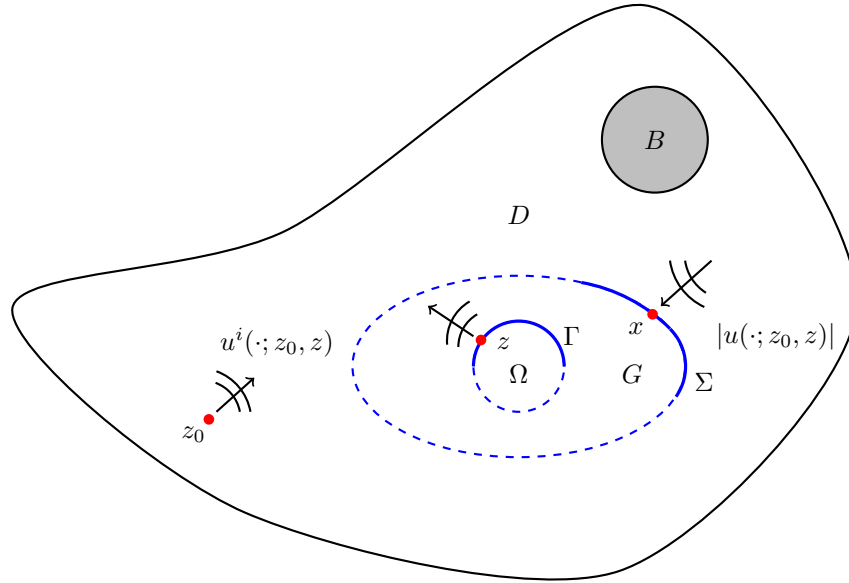
FIGURE 5. An illustration of the interior inverse scattering problem.

with respect to Ω and G, respectively, such that $\overline{\Omega} \subset\subset G$ and $\overline{G} \subset\subset (D_1 \cap D_2)\backslash\overline{B}$. Suppose that the corresponding near-fields satisfy that

$$|u_1(x; z_0)| = |u_2(x; z_0)|, \quad \forall x \in \Sigma,$$
$$|u_1(x; z)| = |u_2(x; z)|, \quad \forall (x, z) \in \Sigma \times \Gamma$$
$$|u_1(x; z_0) + u_1(x; z)| = |u_2(x; z_0) + u_2(x; z)|, \quad \forall (x, z) \in \Sigma \times \Gamma$$

for an arbitrarily fixed $z_0 \in (D_1 \cap D_2)\backslash(\overline{B} \cup \Gamma \cup \Sigma)$. Then we have $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$.

5. **Conclusion.** The phaseless inverse scattering problems are theoretically and practically important. Nevertheless, being lack of phase information in measured data makes it difficult to resolve the uniqueness issue. This paper briefly reviews some recent developments in this field by summarizing the results obtained so far in a series of our publications. Intrinsically speaking, our proofs of the aforementioned uniqueness results hinge on the crux of decoupling the phaseless data via a polarization identity and with an introduction of an additional degree of freedom in the measurement, e.g., with a reference ball, with another point source/plane wave or measurement surface. Consequently, the desired uniqueness can be established by the existing uniqueness results with phase information.

In our view, there are still some open problems in this intriguing and challenging field. Therefore, we finally propose several more topics from our perspective that are interesting for further investigation.

Problem 5.1. *Can one uniquely determine the obstacle using the phaseless far-field data due to incident plane waves with distinct directions with a fixed wavenumber and a reference ball?*

Although there exist numerical results showing that the target obstacle can be accurately reconstructed without any superposition of waves [15], a rigorously mathematical answer to Problem 5.1, to our best knowledge, is still open.

Problem 5.2. *Given the a priori information that the obstacle is of general sound-soft or sound-hard polygon or polyhedron type, can the obstacle be uniquely determined using less phaseless data due to point sources/plane waves in conjunction with a reference object?*

Under the assumption of physical optics approximation, one can indeed determine the exterior unit normal vector of each side/face of the polygonal obstacle from phaseless backscattering data corresponding to a few incident plane waves with suitably chosen incoming directions [31, 32]. By incorporating a reference ball into the scattering system, the phaseless data triple such as the one defined in Definition 2.3 would probably involve redundant data for recovering a polygon/polyhedron. Thus it would be interesting to investigate the proper choice of the dataset so as to establish the corresponding uniqueness.

Problem 5.3. *For the determinations of periodic structures in inverse diffraction grating problems, is it possible to develop similar reference object techniques to obtain the uniqueness from associated amplitude information of scattering data?*

A number of inversion methods have been proposed to numerically reconstruct the grating profiles from phaseless data, e.g., [2, 4, 57] but the theoretical study on the uniqueness seems to be very limited. Whilst various inversion approaches for imaging bounded scatterers can be modified to find unbounded structures, it is hence plausible to consider devising novel methodologies with the techniques presented in this paper.

REFERENCES

[1] H. Ammari, Y. T. Chow and J. Zou, Phased and phaseless domain reconstructions in the inverse scattering problem via scattering coefficients, *SIAM J. Appl. Math.*, **76** (2016), 1000–1030.

[2] G. Bao, P. Li and J. Lv, Numerical solution of an inverse diffraction grating problem from phaseless data, *J. Opt. Soc. Am. A*, **30** (2013), 293–299.

[3] G. Bao and J. Lin, Imaging of local surface displacement on an infinite ground plane: The multiple frequency case, *SIAM J. Appl. Math.*, **71** (2011), 1733–1752.

[4] G. Bao and L. Zhang, Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data, *Inverse Problems*, **32** (2016), 085002, 16 pp.

[5] A. L. Bukhgeĭm and M. V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, *Dokl. Akad. Nauk SSSR*, **260** (1981), 269–272.

[6] F. Cakoni, D. Colton and P. Monk, The direct and inverse scattering problem for partially coated obstacles, *Inverse Problems*, **17** (2001), 1997–2015.

[7] E. J. Candes, X. Li and M. Soltanolkotabi, Phase retrieval via Wirtinger flow: Theory and algorithms, *IEEE Trans. Information Theory*, **61** (2015), 1985–2007.

[8] E. J. Candés, T. Strohmer and V. Voroninski, PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming, *Commun. Pure Appl. Math.*, **66** (2013), 1241–1274.

[9] Z. Chen, S. Fang and G. Huang, A direct imaging method for the half-space inverse scattering problem with phaseless data, *Inverse Probl. Imaging*, **11** (2017), 901–916.

[10] Z. Chen and G. Huang, Phaseless imaging by reverse time migration: Acoustic waves, *Numer. Math. Theor. Meth. Appl.*, **10** (2017), 1–21.

[11] D. Colton and R. Kress, Looking back on inverse scattering theory, *SIAM Rev.*, **60** (2018), 779–807.

[12] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 4th edition, Applied Mathematical Sciences, 93. Springer, Cham, 2019.

[13] H. Dong, J. Lai and P. Li, Inverse obstacle scattering problem for elastic waves with phased or phaseless far-field data, *SIAM J. Imaging Sci.*, **12** (2019), 809–838.

[14] H. Dong, J. Lai and P. Li, An inverse acoustic-elastic interaction problem with phased or phaseless far-field data, *Inverse Problems*, **36** (2020), 035014, 36 pp.

[15] H. Dong, D. Zhang and Y. Guo, A reference ball based iterative algorithm for imaging acoustic obstacle from phaseless far-field data, *Inverse Problems and Imagin*, **13** (2019), 177–195.

[16] P. Gao, H. Dong and F. Ma, Inverse scattering via nonlinear integral equations method for a sound-soft crack from phaseless data, *Applications of Mathematics*, **63** (2018), 149–165.

[17] O. Ivanyshyn, Shape reconstruction of acoustic obstacles from the modulus of the far field pattern, *Inverse Probl. Imaging*, **1** (2007), 609–622.

[18] O. Ivanyshyn and R. Kress, Identification of sound-soft 3D obstacles from phaseless data, *Inverse Probl. Imaging*, **4** (2010), 131–149.

[19] O. Ivanyshyn and R. Kress, Inverse scattering for surface impedance from phaseless far field data, *J. Comput. Phys.*, **230** (2011), 3443–3452.

[20] X. Ji, X. Liu and B. Zhang, Target reconstruction with a reference point scatterer using phaseless far field patterns, *SIAM J. Imaging Sci.*, **12** (2019), 372–391.

[21] X. Ji, X. Liu and B. Zhang, Phaseless inverse source scattering problem: Phase retrieval, uniqueness and direct sampling methods, *J. Comput. Phys. X*, **1** (2019), 100003, 15 pp.

[22] A. Kirsch and N. Grinberg, *The Factorization Methods for Inverse Problems*, Oxford Lecture Series in Mathematics and its Applications, 36. Oxford University Press, Oxford, 2008.

[23] M. V. Klibanov, Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems, *J. Inverse and III-Posed Problems*, **21** (2013), 477–510.

[24] M. V. Klibanov, Uniqueness of two phaseless non-overdetermined inverse acoustics problems in 3-d, *Applicable Analysis*, **93** (2014), 1135–1149.

[25] M. V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.*, **74** (2014), 392–410.

[26] M. V. Klibanov, A phaseless inverse scattering problem for the 3-D Helmholtz equation, *Inverse Probl. Imaging*, **11** (2017), 263–276.

[27] M. V. Klibanov and V. G. Romanov, Reconstruction procedures for two inverse scattering problems without the phase information, *SIAM J. Appl. Math.*, **76** (2016), 178–196.

[28] M. V. Klibanov and V. G. Romanov, Uniqueness of a 3-D coefficient inverse scattering problem without the phase information, *Inverse Problems*, **33** (2017), 095007, 10 pp.

[29] R. Kress and W. Rundell, Inverse obstacle scattering with modulus of the far field pattern as data, *Inverse Problems in Medical Imaging and Nondestructive Testing (Oberwolfach, 1996)*, (1997), 75–92.

[30] K.-M. Lee, Shape reconstructions from phaseless data, *Eng. Anal. Bound. Elem.*, **71** (2016), 174–178.

[31] J. Li and H. Liu, Recovering a polyhedral obstacle by a few backscattering measurements, *J. Differential Equat.*, **259** (2015), 2101–2120.

[32] J. Li, H. Liu and Y. Wang, Recovering an electromagnetic obstacle by a few phaseless backscattering measurements, *Inverse Problems*, **33** (2017), 035001, 20 pp.

[33] J. Li, H. Liu and J. Zou, Strengthened linear sampling method with a reference ball, *SIAM J. Sci. Comput.*, **31** (2009), 4013–4040.

[34] J. Liu and J. Seo, On stability for a translated obstacle with impedance boundary condition, *Nonlinear Anal.*, **59** (2004), 731–744.

[35] M. H. Maleki and A. J. Devaney, Phase-retrieval and intensity-only reconstruction algorithms for optical diffraction tomography, *J. Opt. Soc. Am. A*, **10** (1993), 1086–1092.

[36] S. Maretzke and T. Hohage, Stability estimates for linearized near-field phase retrieval in X-ray phase contrast imaging, *SIAM J. Appl. Math.*, **77** (2017), 384–408.

[37] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[38] R. G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *Bull. Sci. Math.*, **139** (2015), 923–936.

[39] R. G. Novikov, Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions, *J. Geom. Anal.*, **26** (2016), 346–359.

[40] L. Pan, Y. Zhong, X. Chen and S. P. Yeo, Subspace-based optimization method for inverse scattering problems utilizing phaseless data, *IEEE Trans. Geosci. Remote Sensing*, **49** (2011), 981–987.

[41] F. Qu, B. Zhang and H. Zhang, A novel integral equation for scattering by locally rough surfaces and application to the inverse problem: The Neumann case, *SIAM J. Sci. Comput.*, **41** (2019), A3673–A3702.

[42] V. G. Romanov, Phaseless inverse problems for Schrödinger, Helmholtz, and Maxwell Equations, *Comput. Math. Math. Phys.*, **60** (2020), 1045–1062.

[43] V. G. Romanov and M. Yamamoto, Phaseless inverse problems with interference waves, *J. Inverse III-Posed Probl.*, **26** (2018), 681–688.

[44] F. Sun, D. Zhang and Y. Guo, Uniqueness in phaseless inverse scattering problems with known superposition of incident point sources, *Inverse Problems*, **35** (2019), 105007, 10 pp.

[45] T. Takenaka, D. J. N. Wall, H. Harada and M. Tanaka, Reconstruction algorithm of the refractive index of a cylindrical object from the intensity measurements of the total field, *Microwave Opt. Tech. Lett.*, **14** (1997), 139–197.

[46] X. Xu, B. Zhang and H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency, *SIAM J. Appl. Math.*, **78** (2018), 1737–1753.

[47] X. Xu, B. Zhang and H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency. II, *SIAM J. Appl. Math.*, **78** (2018), 3024–3039.

[48] X. Xu, B. Zhang and H. Zhang, Uniqueness in inverse acoustic and electromagnetic scattering with phaseless near-field data at a fixed frequency, *Inverse Probl. Imaging*, **14** (2020), 489–510.

[49] W. Yin, W. Yang and H. Liu, A neural network scheme for recovering scattering obstacles with limited phaseless far-field data, *J. Comput. Phys.*, **417** (2020), 109594, 18 pp.

[50] D. Zhang and Y. Guo, Uniqueness results on phaseless inverse scattering with a reference ball, *Inverse Problems*, **34** (2018), 085002, 12 pp.

[51] D. Zhang, Y. Guo, J. Li and H. Liu, Retrieval of acoustic sources from multi-frequency phaseless data, *Inverse Problems*, **34** (2018), 094001, 21 pp.

[52] D. Zhang, Y. Guo, F. Sun and H. Liu, Unique determinations in inverse scattering problems with phaseless near-field measurements, *Inverse Probl. Imaging*, **14** (2020), 569–582.

[53] D. Zhang, Y. Guo, F. Sun and X. Wang, Reconstruction of acoustic sources from multi-frequency phaseless far-field data, preprint, `arXiv:2002.03279`.

[54] D. Zhang, F. Ma and M. Fang, A finite element method with perfectly matched absorbing layers for the wave scattering from a cavity, *J. Comput. Phys.*, **25** (2008), 301–308.

[55] D. Zhang, Y. Wang, Y. Guo and J. Li, Uniqueness in inverse cavity scattering problem with phaseless near-field data, *Inverse Problems*, **36** (2020), 025004, 10 pp.

[56] B. Zhang and H. Zhang, Recovering scattering obstacles by multi-frequency phaseless far-field data, *J. Comput. Phys.*, **345** (2017), 58–73.

[57] J. Zheng, J. Cheng, P. Li and S. Lu, Periodic surface identification with phase or phaseless near-field data, *Inverse Problems*, **33** (2017), 115004, 35 pp.

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