

INTEGRATING EVOLUTION EQUATIONS USING FREDHOLM DETERMINANTS

FERIDE TIĞLAY*

Ohio State University, Newark
1179 University Drive
Newark, OH 43055, USA

ABSTRACT. We outline the construction of special functions in terms of Fredholm determinants to solve boundary value problems of the string spectral problem. Our motivation is that the string spectral problem is related to the spectral equations in Lax pairs of at least three nonlinear evolution equations from mathematical physics.

1. Introduction. Our main goal is to outline how to find special functions in terms of Fredholm determinants for the string spectral problem

$$-f_{xx} = \lambda m f, \quad (1)$$

in order to write solutions of integrable PDE with associated spectral problems.

The motivation in considering (1) is that the three integrable equations of interest to us, namely Hunter-Saxton, Camassa-Holm and μ HS equations, all have spectral problems related to the string spectral problem (1).

The μ HS equation

$$u_{txx} - 2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx} = 0, \quad (\mu\text{HS})$$

where $u = u(t, x)$, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $t \in \mathbb{R}^+$ and $\mu(u) = \int_{\mathbb{T}} u \, dx$, is introduced first by Khesin, Lenells and Misiołek in [8]. The equation μ HS is a well-defined PDE rather than a mixed integro-differential equation since $\mu(u)$ is constant in time and is completely determined by the initial condition.

Several different names have been used in the literature for this equation such as μ HS, μ CH, generalized Hunter-Saxton equation etc. All of these names can be justified by different aspects of this equation, some closer to Camassa-Holm equation [2], [5]

$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \quad (\text{CH})$$

and some closer to Hunter-Saxton equation [6]

$$u_{txx} + 2u_xu_{xx} + uu_{xxx} = 0, \quad (\text{HS})$$

from mathematical physics.

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* Corresponding author: Feride Tiğlay.

Like (CH) and (HS), the μ HS equation is “integrable” in the following sense: it has a Lax pair representation, a bi-Hamiltonian structure and peaked soliton (peakon) solutions. The Lax pair given in [8] is

$$\psi_{xx} = \lambda\psi(\mu(u) - u_{xx}), \quad (2)$$

$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi \quad (3)$$

where λ is a spectral parameter. In other words, assuming that (2)-(3) are satisfied for all nonzero $\lambda \in \mathbb{C}$, the relation $\psi_{txx} = \psi_{xxt}$ holds if and only if $u(t, x)$ satisfies the μ HS equation.

Moreover (μ HS) can be written as a Hamiltonian system $m_t = \mathcal{B}_i \frac{\delta H_i}{\delta m}$ in two compatible ways where $m = \mu(u) - u_{xx}$ is called the momentum. The Hamiltonian functionals are

$$H_1 = \frac{1}{2} \int umdx \quad \text{and} \quad H_2 = \int \left(\mu(u)u^2 + \frac{1}{2}uu_x^2 \right) dx$$

and the corresponding Hamiltonian operators are $\mathcal{B}_1 = -(m\partial_x + \partial_x m)$ and $\mathcal{B}_2 = \partial_x^3$ respectively. These two compatible Hamiltonian structures can be used to generate infinitely many conserved quantities [8]. Another manifestation of integrability is the existence of soliton-like solutions. In this respect the μ HS equation resembles Camassa-Holm equation rather than the Hunter-Saxton equation, all of whose periodic solutions have finite lifespan. The μ HS equation admits peaked traveling waves that are analogues of solitons [8], [14].

Like Camassa-Holm and Hunter-Saxton equations, (μ HS) is a geodesic equation with respect to a right invariant Riemannian metric. This metric is induced by the μ inner product: $\langle u, v \rangle_\mu = \int_{S^1} u dx \int_{S^1} v dx + \int_{S^1} u_x v_x dx$. All three equations, namely Camassa-Holm, Hunter-Saxton and (μ HS), fit into the same Lie theoretic framework as well. They can be written as Euler-Arnold equations [1], [10] on the regular dual of the Lie algebra $T_e \text{Diff}(S^1) = \text{vect}(S^1)$ of smooth vector fields on the circle

$$m_t = -\text{ad}_{A^{-1}m}^* m = -um_x - 2u_x m, \quad m = Au. \quad (4)$$

with the following choices of the inertia operator A :

$$A = \begin{cases} 1 - \partial_x^2 & \text{for CH,} \\ \mu - \partial_x^2 & \text{for } \mu\text{HS,} \\ -\partial_x^2 & \text{for HS.} \end{cases} \quad (5)$$

The invariance of coadjoint orbits for these equations [18] leads to the conserved quantity

$$\text{Ad}_\gamma^* m = (m \circ \gamma)(\gamma')^2 = \text{const.} \quad (6)$$

2. The string spectral problem. Note that, a transformation $\lambda \rightarrow -\lambda$ changes the spectral problem (2) to the string spectral problem $-f_{xx} = \lambda m f$ studied in [7]. The name “string spectral problem” derives from the fact that the equation can be seen as the equation for the amplitude of free oscillations with frequency $\sqrt{\lambda}$ of a string with a mass distribution M where $\partial_x M = m$.

In this section we follow closely the discussion in [3] and adapt a similar notation.

The string equation can be reformulated as a first order system

$$V_x = \begin{bmatrix} 0 & 1 \\ \lambda m & 0 \end{bmatrix} V$$

where $V = \begin{bmatrix} f \\ f_x \end{bmatrix}$. Then the isospectral deformation of the string of Zakharov-Shabat type $V_t = AV$ where the entries of the 2×2 matrix A are functions of x, t and λ , satisfy the zero curvature condition

$$(AV)_x = \left(\begin{bmatrix} 0 & 1 \\ \lambda m & 0 \end{bmatrix} V \right)_t.$$

Then we can write all entries of A in terms of one of them as

$$A = \begin{bmatrix} -\frac{1}{2}b_x + \beta & b \\ -\frac{1}{2}b_{xx} - \lambda mb & \frac{1}{2}b_x + \beta \end{bmatrix}$$

and we also have an equation for the evolution of m :

$$\lambda m_t = \frac{1}{2}b_{xxx} + \lambda m_x b + 2\lambda m b_x.$$

Note that different boundary conditions impose different restrictions on b . Dirichlet boundary conditions on the string problem $f(0) = f(1) = 0$ imply

$$b(0) = b(1) = 0, \tag{7}$$

$$b_x(0) = b_x(1) = 0. \tag{8}$$

whereas Neumann boundary conditions $f_x(0) = f_x(1) = 0$ imply

$$\frac{1}{2}b_{xx}(0) + \lambda m(0)b(0) = 0, \tag{9}$$

$$\frac{1}{2}b_{xx}(1) + \lambda m(1)b(1) = 0. \tag{10}$$

If we assume that b is a linear function of λ and set $b = b_0 + b_1\lambda$ this leads to the Harry-Dym equation. For our purposes it is a lot more interesting to assume that b is a linear function of $\frac{1}{\lambda}$. We set $b = b_0 + \frac{b_{-1}}{\lambda}$ and this leads to the set of equations

$$m_t - 2mb_{0,x} - m_x b_0 = 0 \tag{11}$$

$$\frac{1}{2}b_{0,xxx} + 2mb_{-1,x} + m_x b_{-1} = 0 \tag{12}$$

$$b_{-1,xxx} = 0. \tag{13}$$

In this case Dirichlet boundary conditions on the string problem give (CH) after a Liouville transformation [BSS]. On the other hand if we impose Neumann boundary conditions then we obtain the conditions

$$b_0(0) = b_0(1) = 0, \tag{14}$$

$$b_{-1,xx}(0) = b_{-1,xx}(1) = 0, \tag{15}$$

$$\frac{1}{2}b_{0,xx}(0) + mb_{-1}(0) = 0, \tag{16}$$

$$\frac{1}{2}b_{0,xx}(1) + mb_{-1}(1) = 0. \tag{17}$$

which lead to (HS). One open problem is what boundary conditions lead to (μHS) . If the boundary value problem for (1) that gives (μHS) has a simple spectrum than the next section outlines how to integrate (μHS) .

3. Integrating an evolution equation using Fredholm determinants. When the spectral problem

$$-f_{xx} = \lambda m f, \quad 0 < x < 1 \quad (18)$$

associated with an evolution equation has purely discrete and simple spectrum the flow for this evolution equation is a superposition of commuting individual flows.

We construct and express the flows in terms of theta functions following McKean's recipe introduced in [15] for (CH). The breakdown of solutions can be determined using these expressions for the solutions and properties of the theta functions as in the case of (CH) [16].

3.1. Individual flows. We consider any eigenvalue λ_0 of the spectral problem (18) and its associated eigenfunction f_0 normalized by

$$\|f_0\|^2 = \int_{S^1} f_0'^2 = \lambda_0 \int_{S^1} m f_0^2 = 1. \quad (19)$$

The reciprocal of any eigenvalue $H = 1/\lambda_0$ is a conserved quantity and the normalization condition (19) implies that $\delta H/\delta m = f_0^2$. Hence the flow is regulated by $\dot{m} = \mathbb{X}m = (mD + Dm)f_0^2$.

We apply the vector field \mathbb{X} to the spectral equation and solve for $\mathbb{X}f_0$. This gives the formula

$$\mathbb{X}f_0 = -\lambda_0 f_0(x) \int_0^x m(y) f_0^2(y) dy + \frac{f_0(x)}{2}. \quad (20)$$

Our immediate goal is solving for $f_0(\bar{x})$ in terms of $m(x)$ and $f_0(x)$ where the new variable \bar{x} is determined by $\bar{x} = -f_0^2$. We consider from the first summand on the right hand side of (20) evaluated at \bar{x} : $I(\bar{x}) = \lambda_0 \int_0^{\bar{x}} m(y) f_0^2(y) dy$. We differentiate $I(\bar{x})$ in t and observe that it satisfies the transport equation

$$\frac{d}{dt} (I(\bar{x})) = -I^2(\bar{x}) + I(\bar{x}).$$

We solve it and get

$$I(\bar{x}) = \lambda_0 \int_0^{\bar{x}} m(t, y) f_0^2(t, y) dy = \frac{e^t \lambda_0 \int_0^x m(0, y) f_0^2(0, y) dy}{1 + (e^t - 1) \lambda_0 \int_0^x m(0, y) f_0^2(0, y) dy}. \quad (21)$$

Differentiating both sides of the second equality above gives

$$f_0(t, \bar{x}) = \frac{e^{t/2} \sqrt{\bar{x}'} f_0(0, x)}{1 + (e^t - 1) \lambda_0 \int_0^x m(0, y) f_0^2(0, y) dy}. \quad (22)$$

Let us now consider the integral of the spectral equation (18) from 0 to \bar{x} and use the following notation

$$J(\bar{x}) = \lambda_0 \int_0^{\bar{x}} m(t, y) f_0(t, y) dy = -f_0'(t, x) + f_0'(t, 0). \quad (23)$$

Applying d/dt to the first equality leads to the differential equation $\frac{d}{dt} J(\bar{x}) = (\frac{1}{2} - I(\bar{x})) J(\bar{x})$ whose solution is

$$J(\bar{x}) = \lambda_0 \int_0^{\bar{x}} m(t, y) f_0(t, y) dy = \frac{e^{t/2} \lambda_0 \int_0^x m(0, y) f_0(0, y) dy}{1 + (e^t - 1) \lambda_0 \int_0^x m(0, y) f_0^2(0, y) dy}.$$

Differentiating this equality gives, after some algebraic manipulations,

$$\sqrt{\bar{x}'} = \frac{1 + (e^t - 1) \lambda_0 \int_0^x m(0, y) f_0^2(0, y) dy}{1 + (e^t - 1) (\lambda_0 \int_0^x m(0, y) f_0^2(0, y) dy + f_0(0, x) f_0'(0, x))}.$$

We use the following notation for the theta functions that appear repeatedly in the above formulas

$$\vartheta = 1 + (e^t - 1)\lambda_0 \int_0^x m(0, y)f_0^2(0, y)dy \tag{24}$$

and

$$\vartheta_- = 1 + (e^t - 1) \left(\lambda_0 \int_0^x m(0, y)f_0^2(0, y)dy + f_0(0, x)f_0'(0, x) \right). \tag{25}$$

With this notation we have $f_0(t, \bar{x}) = e^{t/2}\sqrt{\bar{x}}f_0(0, x)/\vartheta$ and $\bar{x}' = \vartheta^2/\vartheta_-^2$ hence $f_0(\bar{x}) = e^{t/2}f_0(0, x)/\vartheta_-$. The theta functions ϑ and ϑ_- are related by the identity

$$\vartheta^2 - \vartheta_-^2 = (e^t - 1)^2(f_0(0, x)f_0'(0, x))^2 - 2f_0(0, x)f_0'(0, x)\vartheta.$$

Then we have

$$\bar{x} = x + \frac{(e^t - 1)f_0^2(0, x)}{\vartheta_-}. \tag{26}$$

Observe that $\vartheta_- = 1 + (e^t - 1) \int_0^x (f_0'(0, y))^2 dy$ never vanishes. Therefore both formulas (26) and $f_0(\bar{x}) = e^{t/2}f_0(0, x)/\vartheta_-$ make sense for all time.

On the other hand, by (6), we have $m(t, \bar{x}) = m(0, x)/\bar{x}^2 = m(0, x)\vartheta_-^4/\vartheta^4$.

Summary:

$$\bar{x}' = \frac{\vartheta^2}{\vartheta_-^2}, \quad m(t, \bar{x}) = \frac{m(0, x)\vartheta^4}{\vartheta_-^4}, \quad f_0(t, \bar{x}) = \frac{e^{t/2}f_0(0, x)}{\vartheta_-}.$$

It is easy to check that $-f_0''(\bar{x}) = \lambda_0 m(\bar{x})f_0(\bar{x})$. It is all set up so that $f_0'(0, 0) = 0$ or set the antiderivatives starting at x_0 so that $f_0'(0, x_0) = 0$.

Note that $\vartheta_- = 1 + (e^t - 1) \int_0^x (f_0'(0, y))^2 dy$ never vanishes. Hence the first and third formulas above always make sense. Hence it is natural to wonder whether breakdown of solutions occur when ϑ vanishes.

Furthermore the action of the vector field \mathbb{X} on the eigenfunctions f_n corresponding to the other eigenvalues $\lambda_n, n \neq 0$ is determined as follows. Note that the vector field \mathbb{X} is given by $m = \mathbb{X}f = (mD + Dm)f_0^2$. We apply it to the spectral equation $-f_n'' = \lambda_n m f_n$ and use the identity $f_0'' f_n - f_n'' f_0 = (\lambda_n - \lambda_0)m f_0 f_n$ to obtain

$$\mathbb{X}f_n = -\lambda_n f_0 \int_0^x m f_0 f_n \tag{27}$$

3.2. Composite flows. We now consider the composite flow $e^{t\mathbb{X}} = \prod e^{t_n \mathbb{X}_n}$ with parameters $t_n, n \in \mathbb{Z} - 0$. Here t_n denote the running time of each flow. For convenience we denote the action of $t\mathbb{X} = \sum t_n \mathbb{X}_n$ by a bullet \bullet . Note that the individual flows generated by the Hamiltonians $H_n = \frac{1}{\lambda_n}$ commute $[H_i, H_j] = 0$ for $i \neq j$.

We implement the notation $f \otimes f$ for the matrix $[f_i f_j : i, j \in \mathbb{Z} - 0]$ and denote by t and λ the diagonal matrices $[t_n : n \in \mathbb{Z} - 0]$ and $[\lambda_n : n \in \mathbb{Z} - 0]$ respectively.

Next we construct the special functions to prove our main theorem:

Theorem 3.1. *The solution $u(t, x)$ of μHS with initial data $u(0, x)$ is expressed in the Lagrangian scale $\bar{x}(t, x)$ as*

$$u(t, \bar{x}) = \bar{x}^\bullet = -\frac{A^\bullet \vartheta_- - A\vartheta_-^\bullet}{\vartheta_-^2}. \tag{28}$$

where ϑ, ϑ_- and A satisfy the identity $\vartheta^2 = \vartheta_-^2 - \vartheta_- A' + \vartheta'_- A$.

Proof. Applying $t\mathbb{X}$ to the spectral equation and integrating twice in x leads to

$$f_i^\bullet = \sum t_j \left(-\lambda_i f_j \int_0^x m f_i f_j + \delta_{ij} f_i(x)/2 \right). \tag{29}$$

In analogy with individual flows we consider the term $I(\bar{x}) \equiv \int_0^{\bar{x}} m(t, y) f(t, y) \otimes f(t, y) dy$. The solution of the resulting equation $I^\bullet = -It\lambda I + \frac{1}{2}(tI - It)$ gives

$$I = \int_0^{\bar{x}} m(t, y) f(t, y) \otimes f(t, y) dy = e^{t/2} M(1 + (e^t - 1)\lambda M)^{-1} e^{t/2} \tag{30}$$

where $M = \int_0^x m(0, y) f(0, y) \otimes f(0, y) dy$ is the value of $I = I(\bar{x})$ evaluated at the initial time. Note that for the inverse of the matrix $1 + (e^t - 1)\lambda M$ to exist we assume that $\vartheta := \det(1 + (e^t - 1)\lambda M)$ does not vanish.

We proceed as for the individual flows and differentiate (30) to obtain

$$f(\bar{x}) = \sqrt{\bar{x}} e^{t/2} (1 + MC)^{-1} f(0, x) \tag{31}$$

where $C = (e^t - 1)\lambda$. In order to achieve our goal of expressing $f(\bar{x})$ in terms of $m(0, x)$ and $f(0, x)$ it would be sufficient to derive a formula for \bar{x} in terms of these initial values. For this purpose we integrate the spectral equation from 0 to \bar{x} and introduce the notation $J(\bar{x}) \equiv \int_{x_0}^{\bar{x}} m f$ such that $\lambda J = -f'$. Applying $t\mathbb{X}$ to $J(\bar{x})$ leads to the equation $J^\bullet = -It\lambda J + \frac{1}{2}tJ$. The solution of this equation is $J(\bar{x}) = e^{t/2} (1 + MC)^{-1} \int_{x_0}^x m(0, y) f(0, y) dy$. Then differentiating both the definition and this last formula for $\lambda J(\bar{x})$ in x leads to

$$\frac{1}{\sqrt{\bar{x}}} = \frac{\det(1 + CM + (e^t - 1)f'(0, x) \otimes f(0, x))}{\det(1 + CM)}. \tag{32}$$

We introduce the notation for the theta functions for the composite flow as

$$\begin{aligned} \vartheta_- &= \det(1 + CM + (e^t - 1)f'(0, x) \otimes f(0, x)) \\ &= \det(1 + (e^t - 1) \int_0^x f'(0, x) \otimes f'(0, x)) \end{aligned}$$

and

$$\vartheta = \det(1 + CM). \tag{33}$$

Note that if we set

$$Q \equiv 1 + (e^t - 1) \int_0^x f'(0, y) \otimes f'(0, y) dy$$

we have the identity $\vartheta^2 = \vartheta_-^2 - \vartheta_- A' + \vartheta'_- A$ where

$$A = \det Q \times f(0, x).Q^{-1}(e^t - 1)f(0, x)$$

and the theta functions can be written as $\vartheta_- = \det Q$ and $\vartheta = \det Q(1 - f(0, x).Q^{-1}f'(0, x))$. With this notation we have $\vartheta^2/\vartheta_-^2 = 1 - (A/\vartheta_-)'$ hence

$$\bar{x} = x - \frac{\det Q \times fQ^{-1}(e^t - 1)f}{\vartheta_-} = x - fQ^{-1}(e^t - 1)f.$$

Both theta functions $\vartheta_- = \det Q$ and $\vartheta = \det Q(1 - f(0, x).Q^{-1}f'(0, x))$ converge [4] and ϑ_- is always positive. □

3.3. Breakdown of solutions. Note that the formula (28) makes sense for all times since ϑ_- is strictly positive. On the other hand, for the derivative of the solution u we have

$$u'(t, \bar{x}) = \frac{1}{\bar{x}'} \bar{x}'^\bullet = \frac{2\vartheta^\bullet}{\vartheta} - \frac{2\vartheta_-^\bullet}{\vartheta_-}. \quad (34)$$

Note that ϑ^\bullet can not vanish when ϑ does. Therefore $u'(t, \bar{x})$ blows up when ϑ vanishes.

Since $C \int_0^x mf \otimes fC$ is compact with finite absolute trace with respect to $\langle f_1, f_2 \rangle = \int f_1' f_2' dx$ the condition $\vartheta \neq 0$ means that the spectrum of $C \int_0^x mf \otimes fC$ lies below 1 for all time $t \geq 0$ and $x \in \mathbb{T}$. This requires that the associated quadratic form $\zeta^2 - \int_0^x m |\sum_n \zeta_n C_n f_n|^2$ be positive for $\zeta \neq 0$.

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E-mail address: tiglay.1@osu.edu