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A GEOMETRIC-ANALYTIC STUDY OF LINEAR DIFFERENTIAL EQUATIONS OF ORDER TWO

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Abstract. We study second order linear differential equations with analytic coefficients. One important case is when the equation admits a so called regular singular point. In this case we address some untouched and some new aspects of Frobenius methods. For instance, we address the problem of finding formal solutions and studying their convergence. A characterization of regular singularities is given in terms of the space of solutions. An analytic-geometric classification of such linear polynomial homogeneous ODEs is obtained by the use of techniques from geometric theory of foliations means. This is done by associating to such an ODE a rational Riccati differential equation and therefore a *global holonomy group*. This group is a computable group of Moebius maps. These techniques apply to classical equations as Bessel and Legendre equations. We also address the problem of deciding which such polynomial equations admit a Liouvillian solution. A normal form for such a solution is then obtained. Our results are concrete and (computationally) constructive and are aimed to shed a new light in this important subject.

1. **Introduction and main results.** Since the first appearance of Newton's laws of motion (see Axioms or Laws of Motion in [17] page 13), the study of ordinary differential equations has been associated with fundamental problems in physics and science in general. Many are the applications as universal gravitation and planetary dynamics, dynamics of particles under the action of a force field as the electromagnetic field, thermodynamics, meteorology and weather forecast, study of climate phenomena as typhoons and hurricanes, aerodynamics and hydrodynamics, atomic models, etc. Thanks to the nature of Newton's laws and other laws as Maxwell's equations or Faraday's and Kepler's laws [12], most of the pioneering work is first or second order ordinary differential equations (ODEs). Of special interest are the laws of the oscillatory movement (pendulum equation and Hill lunar movement

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equation [11]) and Hooke's law (spring extension or compression). A number of classical equations are, or have nice approximations by, linear equations with analytic coefficients. Among the linear equations the homogeneous case is a first step and quite meaningful. To be able to solve classical ordinary linear homogeneous differential equations is an important and active subject in mathematics. The arrival of features like scientific computing brings back the problem of finding solutions via power series. In this direction, a classical and powerful method is due to Frobenius. The main point is that Frobenius method works pretty well in a suitable class of second order linear ODEs, so called *regular singular* ODEs.

In this paper we study second order linear differential equations with analytic coefficients under the viewpoint of finding solutions and studying their convergence. In very few words, we study forgotten as well as new aspects of Frobenius method. We start with the convergence of formal solutions. We also discuss the characterization of the so called *regular singularities* in terms of the space of solutions. An analytic-geometric classification of these polynomial ODEs is obtained via associating to such an ODE a Riccati differential equation and therefore a *global holonomy group*. This group is a computable group of Moebius maps. Next we apply these techniques and results to classical equations as Bessel and Legendre equations. Finally, we study the existence and form of Liouvillian solutions for polynomial ODEs.

Next we give a more detailed description of our results.

1.1. **Convergence of formal solutions for second order linear homogenous ODEs.** In Section 3 we discuss the problem of convergence of formal solutions for linear homogeneous ODEs of order two of the form

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{1}$$

where a, b, c are analytic functions at $x_0 \in \mathbb{R}$. We recall that there are examples of ODEs admitting a formal solution that is nowhere convergent (cf. Example 3.13). Our next result may be seen as a version of a theorem due to Malgrange (see [14,15]) and also to Mattei-Moussu (see [16]) for holomorphic integrable systems of differential forms. By a *formal solution centered at* $x_0 \in \mathbb{R}$ of an ODE we shall mean a formal power series $\hat{y}(x) = \sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ with complex coefficients $a_n \in \mathbb{C}$. We prove:

Theorem A. *Consider a second order ordinary differential equation given by (1). Suppose also that there exist two linearly independent formal solutions $\hat{y}_1(x)$ and $\hat{y}_2(x)$ centered at x_0 of equation (1). Then x_0 is an ordinary point or a regular singular point of (1). Moreover, $\hat{y}_1(x)$ and $\hat{y}_2(x)$ are convergent.*

A formal solution associated to a regular singularity is always convergent:

Theorem B. *Consider a second order ordinary differential equation given by (1). Suppose (1) has at x_0 an ordinary point or a regular singular point. Given a formal solution $\hat{y}(x)$ of (1) then this solution is convergent. Indeed, this solution converges in the same disc type neighborhood where the coefficients $a(x), b(x), c(x)$ are analytic.*

1.2. **Characterization of regular singular points in order two.** We shall say that a function $u(x)$ for x in a disc $|x| < R$ centered at the origin $0 \in \mathbb{R}, \mathbb{C}$ is *an analytic combination of log and power* (*anclop* for short) if it can be written as $u(x) = \alpha(x) + \log(x)\beta(x) + \gamma(x)x^r$ for some analytic functions $\alpha(x), \beta(x), \gamma(x)$ defined in the disc $|x| < R$ and $r \in \mathbb{R}$ or $r \in \mathbb{C}$. In the real case we assume that $x > 0$ in case we have $\beta \not\equiv 0$ or $\gamma \not\equiv 0$ and a power x^r with $r \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 1.1. A one-variable complex function $u(z)$ considered in a domain $U \subset \mathbb{C}$ will be called *analytic up to log type singularities* (*autlos* for short) if:

1. $u(z)$ is holomorphic in $U \setminus \sigma$ where $\sigma \subset U$ is a discrete set of points, called *singularities*.
2. Given a singularity $p \in \sigma$ either p is a removable singularity of $u(z)$ or there is a germ of real analytic curve $\gamma\colon [0, \epsilon) \to U$ such that $\gamma(0) = p$ and $u(z)$ is holomorphic in $D \setminus \gamma[0, \epsilon)$ for some disc $D \subset U$ centered at p.

A one variable real function $u(x)$ defined in an interval $J \subset \mathbb{R}$ will be called *analytic up to log type singularities* (*autlos* for short) if, after complexification, the corresponding function $u_{\mathbb{C}}(z)$, which is defined in some neighborhood $J \times \{0\} \subset U \subset \mathbb{C}$ is analytic up to log type singularities, as defined above.

With such notions we obtain the following characterization of regular singularities:

Theorem C (characterization of regular points)**.** *Consider a second order ordinary differential equation given by (1). Then the following conditions are equivalent:*

(i) The equation admits two linearly independent solutions $y_1(x), y_2(x)$ which are anclop (analytic combinations of log and power). Then x_0 is an ordinary point or a regular singular point for the ODE.

(ii) The equation admits two solutions $y_1(x), y_2(x)$ which are autlos (analytic up to logarithmic singularities).

(iii) The equation has an ordinary point or a regular singular point at x_0.

1.3. **Riccati model and holonomy of a second order equation.** We start with a polynomial second order linear equation of the form

$$a(z)u'' + b(z)u' + c(z)u = 0 \tag{2}$$

with a, b, c are complex polynomials of a variable z. By introducing the change of coordinates $t = u'/u$ we obtain a first order Riccati equation which writes as

$$\frac{dt}{dz} = -\frac{a(z)t^2 + b(z)t + c(z)}{a(z)}.$$

Definition 1.2. The Riccati differential equation above is called *Riccati model* of the ODE (2).

By its turn, since the work of Paul Painlevé (see [18]), a polynomial Riccati equation is studied from the point of view of its transversality with respect to the vertical fibers $z = constant$, even at the points at the infinity $u = \infty$. With the advent of the theory of foliations, due to Ehresmann, the notion of holonomy was introduced as well as the notion of global holonomy of a foliation transverse to the fibers of a fibration. This is the case of a polynomial Riccati foliation once placed in the ruled surface $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, where $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. This allows us to introduce the notion of *global holonomy* of a second order linear equation as above. Then we proceed the study of the equation from this group theoretical point of view, since the global holonomy will be a group of Moebius maps of the form $t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}$. We do calculate this group in some special cases and reach some interesting consequences for the original ODE.

Theorem D. *Consider a second order ODE given by $u'' + b(z)u' + c(z)u = 0$ where b, c are complex polynomials of a variable z. Then the equation above admits a* *general solution of the form*

$$u_{\ell,k}(z) = k \exp\left(\int_0^z \frac{\ell D(\xi) - B(\xi)}{A(\xi) - \ell C(\xi)} d\xi\right),\ k, \ell \in \mathbb{C}$$

where A, B, C, D *are entire functions satisfying* $AD - BC \neq 0$.

We apply these techniques for studying the classical Bessel and Legendre equations (cf. Examples 5.10 and 5.13).

1.4. **Liouvillian solutions.** One important class of solutions for ODEs is the class of *Liouvillian solutions*, following the work of Liouville, Rosenlicht and Ross among other authors. The question whether a polynomial first order ODE admits a Liouvillian solution or first integral has been addressed by M. Singer in [24] and others. We refer to [24] for the notion of Liouvillian function in n complex variables. Such a function has (holomorphic) analytic branches in some Zariski dense open subset of $\mathbb{C}^n$. In particular we can ask whether an ODE admits such a solution.

Question 1.3. *What are the polynomial ODEs of the form (2) admitting a Liouvillian solution* $u(z)$*? What are the possible Liouvillian solutions?*

Our contribution to the above problem is:

Theorem E. *Consider a complex ODE of the form* $L(u) := a(z)u'' + b(z)u' + c(z)u = 0$ *where the coefficients* $a(z), b(z), c(z)$ *are complex polynomials. Then we have the following:*

(i) If $L[u] = 0$ *admits a solution satisfying a Liouvillian relation then it has a Liouvillian first integral (cf. Corollary pages 674,675 [24]).*
(ii) If $L[u] = 0$ *admits a Liouvillian solution then it has a Liouvillian first integral (cf. Corollary page 674,675 [24]).*
(iii) If $L[u] = 0$ *admits a Liouvillian first integral then its solutions are Liouvillian and given by one of the forms below:*
   (a) $u(z) = \exp\left(-\int^z \gamma(\eta)d\eta\right)\left[k\int^z \exp\left(\int^\eta \frac{2\gamma(\xi)-b(\xi)}{a(\xi)}d\xi\right)d\eta + \ell\right]$ *for constants* $k,\ \ell \in \mathbb{C}$ *and* $\gamma(z)$ *a rational solution for the Riccati equation.*
   (b) $u(z) = k_1 + k_2\int^z \exp\left(-\int^\eta \frac{b(\xi)}{a(\xi)}d\xi\right)d\eta$*, for constants* $k_1, k_2 \in \mathbb{C}$.

2. **Frobenius method: Characterization of Euler equations.** In this section we recall briefly the classical Frobenius method. We consider equations that write in the form $a(x)y'' + b(x)y' + c(x)y = 0$ for some real analytic functions $a(x), b(x), c(x)$ at some point $x_0 \in \mathbb{R}$. We say that x_0 is an *ordinary point* if $a(x_0) \neq 0$. Nevertheless, most of the relevant equations are connected to the *singular* (non-ordinary) case. We can mention the *Bessel equation* $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, whose range of applications goes from heat conduction, to the model of the hydrogen atom (see [1]). This equation has the origin $x = 0$ as a singular point. Another remarkable equation is the *Laguerre equation* $xy'' + (\nu + 1 - x)y' + \lambda y = 0$ where $\lambda, \nu \in \mathbb{R}$ are parameters. This equation is quite relevant in quantum mechanics, since it appears in the modern quantum mechanical description of the hydrogen atom. According to Frobenius a singular point $x = x_0$ of the ODE $a(x)y'' + b(x)y' + c(x)y = 0$ is *regular* if the following limits $\lim\limits_{x\to x_0}(x - x_0)\frac{b(x)}{a(x)}$ and $\lim\limits_{x\to x_0}(x - x_0)^2\frac{c(x)}{a(x)}$ exist and are finite.

The Frobenius method consists in associating to the original ODE an *Euler equation*, i.e., an equation of the form $A(x - x_0)^2y'' + B(x - x_0)y' + Cy = 0$ and looking for solutions (to this equation) of the form $y_0(x) = (x - x_0)^r$. This gives an algebraic equation of degree two $Ar(r-1) + Br + C = 0$, so called *indicial equation*, whose zeroes r give solutions $y_0(x) = (x - x_0)^r$ of the Euler equation. The Euler equation associated to the original ODE with a regular singular point at $x = x_0$ is given by $(x - x_0)^2 y'' + p_0(x - x_0)y' + q_0 y = 0$ where $p_0 = \lim_{x \to x_0} (x - x_0)\frac{b(x)}{a(x)}$ and $q_0 = \lim_{x \to x_0} (x - x_0)^2 \frac{c(x)}{a(x)}$.

In this case we have the following classical theorem of Frobenius:

Theorem 2.1 (Frobenius theorem, Theorem 5.6.1 in [2] pages 293,294, [8], Theorem 3 and 4 in [6] pages 158,175)**.** *Assume that the ODE*

$$(x - x_0)^2 y'' + (x - x_0)b(x)y' + c(x)y = 0$$

has a regular singularity at $x = x_0$, where the functions $b(x), c(x)$ are analytic with convergent power series in $|x - x_0| < R$. Then there is at least one solution of the form

$$y(x) = |x - x_0|^r \sum_{n=0}^{\infty} d_n (x - x_0)^n$$

where r is a root of the indicial equation, $d_0 = 1$ where the series converges for $|x - x_0| < R$.

Frobenius method actually consists in looking for solutions of the form

$$y_1(x) = |x - x_0|^r \sum_{n=0}^{\infty} d_n (x - x_0)^n \tag{3}$$

where r is the zero of the indicial equation having greater real part. Whether there is a second linearly independent solution is related to the roots of the indicial equation. Indeed, there is some zoology and in general the second solution is of the form

$$y_2(x) = |x - x_0|^{\tilde{r}} \sum_{n=0}^{\infty} \tilde{d}_n (x - x_0)^n \tag{4}$$

in case there is a second root $\tilde{r}$ of the indicial equation and this root is such that $r - \tilde{r} \notin \mathbb{Z}$. If $\tilde{r} = r$ then there is a solution of the form

$$y_2(x) = y_1(x) \log |x - x_0| + |x - x_0|^{r+1} \sum_{n=0}^{\infty} \hat{d}_n (x - x_0)^n. \tag{5}$$

Finally, if $0 \neq r - \tilde{r} \in \mathbb{N}$ then we have a second solution of the form

$$y_2(x) = k y_1(x) \log |x - x_0| + |x - x_0|^{\tilde{r}} \sum_{n=0}^{\infty} \check{d}_n (x - x_0)^n. \tag{6}$$

Each of the series in equations (3), (4), (5) and (6) converges for $|x - x_0| < R$ and defines a function that is analytic in some neighborhood of $x = x_0$.

As referred above, Frobenius theorem statement is found in the book of Boyce-DiPrima and in the book of E. Coddington. Nevertheless, we are afraid that a more detailed and complete proof of the convergence of the formal part of the solutions in the disc $|x - x_0| < R$ (ie., in the common disc where the coefficients of the ODE are analytic) can only be found in Coddington's book.

3. Convergence of formal solutions: Theorems A and B. Consider a second order ordinary differential equation given by $a(z)u''+b(z)u'+c(z)u=0$ where a, b, c are holomorphic in a neighborhood of the origin $0 \in \mathbb{C}$. We shall mainly address two questions:

Question 3.1. *(i) Under what conditions can we assure that the origin is an ordinary point or a regular singular point of the equation?*
(ii) Is it that a formal solution of the ODE is always convergent?

Let us fix some notations that will be used from now on:

- $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ is the ring of holomorphic functions at $0 \in \mathbb{C}^n$.
- $\hat{\mathcal{O}}_n = \mathbb{C}\{\{z_1, \dots, z_n\}\}$ is the ring of formal series in n indeterminates over $\mathbb{C}$.
- $\mathcal{M}_n$ is the field of meromorphic functions.
- $\hat{\mathcal{M}}_n$ is the *field of fractions* of $\hat{\mathcal{O}}_n$.

Recall that the field of fractions of an integral domain D is the smallest field in which the field embeds. It consists of fractions having elements of D as numerator and nonzero elements of D as denominator in the standard way. Thus $\hat{\mathcal{M}}_n$ is the field consisting of fractions of the form $\frac{\hat{f}}{\hat{g}}$ where $\hat{f}$, $\hat{g} \in \hat{\mathcal{O}}_n$ and $\hat{g} \neq 0$.

Let us give a first proof of the convergence in Theorem A:

Proof of the convergence in Theorem A. In order to simplify our notation we shall assume $x_0 = 0$. We consider equation

$$a(z)u'' + b(z)u' + c(z)u = 0 \tag{7}$$

where a, b, c are complex analytic (holomorphic) functions in neighborhood $|z| < R$. According to [19] there is an integrable complex analytic one-form Ω in $\mathbb{C}^3$ defined as follows

$$\Omega = -a(z)ydx + a(z)xdy + [a(z)y^2 + b(z)xy + c(z)x^2]dz.$$

Indeed,

$$d\Omega = [2xc(z)+yb(z)+ya'(z)]dx\wedge dz+[2ya(z)+xb(z)-xa'(z)]dy\wedge dz+2a(z)dx\wedge dy$$

$$\begin{aligned}
\Omega \wedge d\Omega &= [-2y^2a(z)^2 - xya(z)b(z) + xya(z)a'(z)]dx \wedge dy \wedge dz \\
&\quad +[-2x^2a(z)c(z) - xya(z)b(z) - xya(z)a'(z)]dx \wedge dy \wedge dz \\
&\quad +[2y^2a(z)^2 + 2xya(z)b(z) + 2x^2a(z)c(x)]dx \wedge dy \wedge dz \\
&= 0.
\end{aligned}$$

This one-form is tangent to the vector field in $\mathbb{C}^3$ associated to the reduction of order of the ODE. Indeed, as a first step we perform the classical order reduction process where equation (7) is rewritten after the following ‘change of coordinates’: $x = u,\ y = u',\ z = z$. We then obtain

$$x' = u' = y,\ y' = u'' = -\frac{b(z)}{a(z)}u' - \frac{c(z)}{a(z)}u = -\frac{b(z)}{a(z)}y - \frac{c(z)}{a(z)}x,\ z' = 1.$$

Therefore, a natural vector field X associated to equation (7) is given by

$$X(x, y, z) = y\frac{\partial}{\partial x} - \left(\frac{b(z)}{a(z)}y + \frac{c(z)}{a(z)}x\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Note that

$$\Omega(X) = -y^2a(z) - xa(z)\left(\frac{b(z)}{a(z)}y + \frac{c(z)}{a(z)}x\right) + [y^2a(z) + xyb(z) + x^2c(z)] = 0.$$

Moreover, given two linearly independent solutions $u_1(z)$ and $u_2(z)$ of the ODE the function

$$H(x,y,z) = \frac{xu_1'(z) - yu_1(z)}{xu_2'(z) - yu_2(z)}$$

is a first integral for the form Ω, ie., $dH \wedge \Omega = 0$. Indeed,

$$\begin{aligned} dH &= \frac{1}{(xu_2' - yu_2)^2}\Big[y(u_1u_2' - u_1'u_2)dx - x(u_1u_2' - u_1'u_2)dy \\ &\quad -[y^2(u_1u_2' - u_1'u_2) + xy(u_1''u_2 - u_1u_2'') + x^2(u_1'u_2'' - u_1''u_2')]dz\Big] \end{aligned}$$

note that

$$u_1''u_2 - u_1u_2'' = \frac{b}{a}(u_1u_2' - u_1'u_2) \quad \text{and} \quad u_1'u_2'' - u_1''u_2' = \frac{c}{a}(u_1u_2' - u_1'u_2)$$

then

$$\begin{aligned} dH &= \frac{1}{(xu_2' - yu_2)^2}\Bigg[y(u_1u_2' - u_1'u_2)dx - x(u_1u_2' - u_1'u_2)dy \\ &\quad -\Big[y^2(u_1u_2' - u_1'u_2) + xy\frac{b}{a}(u_1u_2' - u_1'u_2) + x^2\frac{c}{a}(u_1u_2' - u_1'u_2)\Big]dz\Bigg] \\ &= -\frac{u_1u_2' - u_1'u_2}{a(xu_2' - yu_2)^2}\Big(- aydx + axdy + [ay^2 + bxy + cx^2]dz\Big) \\ &= -\frac{u_1u_2' - u_1'u_2}{a(xu_2' - yu_2)^2}\Omega. \end{aligned}$$

By hypothesis there exist two linearly independent formal solutions $\hat{u}_1$ and $\hat{u}_2$ of equation (7). Each solution writes as a formal complex power series

$$\hat{u}_j(z) = \sum_{n=0}^{\infty} a_n^j z^n \in \mathbb{C}\{\{z\}\}.$$

According to the above, there exists a formal first integral

$$H(x,y,z) = \frac{x\hat{u}_1'(z) - y\hat{u}_1(z)}{x\hat{u}_2'(z) - y\hat{u}_2(z)}$$

of the integrable one-form Ω above. Now note that $H, 1/H \notin \hat{\mathcal{O}}_3$. Indeed, if $H \in \hat{\mathcal{O}}_3$ then there exist $a_{i,j,k} \in \mathbb{C}$ such that

$$\frac{x\hat{u}_1'(z) - y\hat{u}_1(z)}{x\hat{u}_2'(z) - y\hat{u}_2(z)} = \sum_{i+j+k=0}^{\infty} a_{i,j,k}x^iy^jz^k$$

then

$$\underbrace{x\hat{u}_1'(z) - y\hat{u}_1(z)}_{\substack{\text{formal series containing}\\ \text{only one term in } x \text{ and } y}} = \underbrace{(x\hat{u}_2'(z) - y\hat{u}_2(z))}_{\substack{\text{formal series containing}\\ \text{only one term in } x \text{ and } y}} \underbrace{\left(\sum_{i+j+k=0}^{\infty} a_{i,j,k} x^i y^j z^k\right)}_{\Rightarrow \text{ only depends on } z}$$

so there is $f \in \hat{\mathcal{O}}_1$ such that

$$x\hat{u}_1'(z) - y\hat{u}_1(z) = (x\hat{u}_2'(z) - y\hat{u}_2(z))f(z)$$

equivalently

$$x[\hat{u}_1'(z) - f(z)\hat{u}_2'(z)] = y[\hat{u}_1(z) - f(z)\hat{u}_2(z)]$$

so we have

$$\hat{u}_1(z) = f(z)\hat{u}_2(z) \quad \text{and} \quad \hat{u}_1'(z) = f(z)\hat{u}_2'(z)$$

therefore $\hat{u}_1$ and $\hat{u}_2$ are linearly dependent which contradicts the hypothesis. Similarly $1/H \notin \hat{\mathcal{O}}_3$.

Now we recall the following convergence theorem:

Theorem 3.2 (Cerveau-Mattei, [5], Theorem 1.1 page 106)**.** *Let Ω be a germ at $0 \in \mathbb{C}^n$ of an integrable holomorphic 1-form and $H = \frac{f}{g} \in \hat{\mathcal{M}}_n$ a purely formal meromorphic first integral of Ω, i.e. $\Omega \wedge dH = 0$ and $H, 1/H \notin \hat{\mathcal{O}}_n$. Then H converges, i.e., $H \in \mathcal{M}_n$.*

From the above theorem $H = \frac{f}{g}$ where $f, g \in \mathcal{O}_3$ are relatively prime. Hence we have

$$[x\hat{u}_1'(z) - y\hat{u}_1(z)]g(x,y,z) = [x\hat{u}_2'(z) - y\hat{u}_2(z)]f(x,y,z)$$

as f and g are relatively prime then f divides $x\hat{u}_1'(z) - y\hat{u}_1(z)$ e g divides $x\hat{u}_2'(z) - y\hat{u}_2(z)$ from there exists $\alpha, \beta, \xi, \eta \in \mathcal{O}_1$ and $k \in \hat{\mathcal{O}}_1$ such that

$$x\hat{u}_1'(z) - y\hat{u}_1(z) = k(z)[x\alpha(z) - y\beta(z)] \text{ and } x\hat{u}_2'(z) - y\hat{u}_2(z) = k(z)[x\xi(z) - y\eta(z)]$$

equivalently we have

$$x[\hat{u}_1'(z) - k(z)\alpha(z)] = y[\hat{u}_1(z) - k(z)\beta(z)] \text{ and } x[\hat{u}_2'(z) - k(z)\xi(z)] = y[\hat{u}_2(z) - k(z)\eta(z)]$$

so we have

$$\hat{u}_1'(z) = k(z)\alpha(z),\ \hat{u}_1(z) = k(z)\beta(z),\ \hat{u}_2'(z) = k(z)\xi(z) \text{ and } \hat{u}_2(z) = k(z)\eta(z)$$

therefore

$$\frac{\hat{u}_1'(z)}{\hat{u}_1(z)} = \frac{\alpha(z)}{\beta(z)} \text{ and } \frac{\hat{u}_2'(z)}{\hat{u}_2(z)} = \frac{\xi(z)}{\eta(z)}$$

thus

$$\hat{u}_1(z) = A\exp\left(\int^z \frac{\alpha(w)}{\beta(w)}dw\right) \text{ and } \hat{u}_2(z) = B\exp\left(\int^z \frac{\xi(w)}{\eta(w)}dw\right),$$

for some A, B constants, are convergent. □

We stress the fact that we are not assuming the ODE to be regular at x_0.

3.1. **The wronskian I.** Consider the linear homogeneous second order ODE

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{8}$$

where $a(x), b(x), c(x)$ are differentiable real or complex functions defined in some open subset $U \subset \mathbb{R}, \mathbb{C}$. We may assume that U is an open disc centered at the origin $0 \in \mathbb{R}, \mathbb{C}$. We make no hypothesis on the nature of the point $x = 0$ as a singular or ordinary point of (8). Given two solutions y_1 and y_2 of (8) their *wronskian* is defined by $W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

Claim 3.3. *The wronskian $W(y_1, y_2)$ satisfies the following first order ODE*

$$a(x)w' + b(x)w = 0. \tag{9}$$

This is a well-known fact and we shall not present a proof, which can be done by straightforward computation. Most important, the above fact allows us to introduce the notion of *wronskian* of a general second order linear homogeneous ODE as (8) as follows:

Definition 3.4. *The wronskian of (8) is defined as the general solution of (9).*

Hence, in general the wronskian is of the form

$$W(x) = K \exp\left(-\int^x \frac{b(\eta)}{a(\eta)} d\eta\right) \tag{10}$$

where K is a constant.

A well-known consequence of the above formula is the following:

Lemma 3.5. *Given solutions $y_1(x), y_2(x)$ the following conditions are equivalent:*

(i) $W(y_1, y_2)(x)$ *is identically zero.*
(ii) $W(y_1, y_2)(x)$ *vanishes at some point $x = x_0$.*
(iii) $y_1(x), y_2(x)$ *are linearly dependent.*

Let us analyze the consequences of this form. We shall consider the origin as the center of our disc domain. In what follows the coefficients are analytic in a neighborhood of the origin.

Case (1). If $\frac{b}{a}$ has poles of order $r > 1$ at the origin: In this case we can write

$$\frac{b(x)}{a(x)} = \frac{A_r}{x^r} + \ldots + \frac{A_2}{x^2} + \frac{A_1}{x} + d(x)$$

where $A_1, A_2, \ldots, A_r$ are constant, $A_r \neq 0$ and d is analytic at the origin. Hence

$$W(x) = K|x|^{-A_1} \exp\left(\frac{A_r}{(r-1)x^{r-1}} + \ldots + \frac{A_2}{x}\right) \exp\big(\tilde{d}(x)\big)$$

where $\tilde{d}$ is analytic. Now observe that $\exp\big(\frac{A_r}{(r-1)x^{r-1}} + \ldots + \frac{A_2}{x}\big)$ is neither analytic nor formal. Therefore, in this case, W is neither analytic nor formal.

Case (2): $\frac{b}{a}$ has poles of order ≤ 1 at the origin. In this case $\frac{b(x)}{a(x)} = \frac{A_1}{x} + d(x)$ and $W(x) = K|x|^{-A_1} \exp\big(\tilde{d}(x)\big)$. If W is analytic or formal then we must have $A_1 \in \{0, -1, -2, -3, \ldots\}$.

Summarizing we have:

Lemma 3.6. *Assume that the wronskian W of the ODE $a(x)y'' + b(x)y' + c(x)y = 0$, with analytic coefficients, is analytic or formal. Then $\frac{b}{a}$ has a pole of order $r \leq 1$ at the origin. Moreover, we must have $W(x) = K|x|^{-A} \exp\big(f(x)\big)$, where $A \in \{0, -1, -2, -3, \ldots\}$ and f is analytic.*

Now we are able to prove the remaining part of Theorem A:

End of the proof of Theorem A. We have already proved the first part. Let us now prove that the origin is an ordinary point or a regular singularity of the ODE. This is done by means of the two following claims:

Claim 3.7. *The quotient $\frac{b}{a}$ has poles of order ≤ 1 at the origin.*

Proof. Indeed, since by hypothesis there are two formal linearly independent functions, the wronskian is formal. Thus, from the above discussion we conclude. □

The last part is done below. For simplicity we shall assume that $x = z \in \mathbb{C}$ and that the coefficients are complex analytic (holomorphic) functions.

Claim 3.8. *We have $\lim_{z\to 0} z^2 \frac{c(z)}{a(z)} \in \mathbb{C}$.*

Proof. Write $a(z)u'' + b(z)u' + c(z)u = 0$ and $a(z) = z^k$ according to the local form of holomorphic functions. Since $\lim_{z\to 0} z\dfrac{b(z)}{a(z)} \in \mathbb{C}$ we must have $\frac{b(z)}{a(z)} = \frac{\tilde{b}(z)}{z}$ for some holomorphic function $\tilde{b}(z)$ at 0. Assume that the Claim is not true, then $\frac{c(z)}{a(z)}$ must have a pole of order ≥ 3 at 0. Thus we may write $\frac{c(z)}{a(z)} = \frac{\tilde{c}(z)}{z^{3+\nu}}$ for some holomorphic function $\tilde{c}(z)$ at 0 and some $\nu \in \mathbb{N}$. We may choose ν such that $\tilde{c}(0) \neq 0$. We have for the ODE above $z^{3+\nu}u'' + z^{2+\nu}\tilde{b}(z)u' + \tilde{c}(z)u = 0$. For sake of simplicity we will assume that $\tilde{c}(0) = 1$ and $\nu = 0$. This does not affect the argumentation below. We write $\tilde{b}(z) = b_0 + b_1 z + b_2 z^2 + \ldots$ and $\tilde{c}(z) = 1 + c_1 z + c_2 z^2 + \ldots$ in power series. Substituting this in the ODE we obtain $z^3 u'' + z^2(b_0 + b_1 z + b_2 z^2 + \ldots)u' + (1 + c_1 z + c_2 z^2 + \ldots)u = 0$. Now we write $u(z) = \sum_{n=0}^{\infty} a_n z^n$ in power series. Thus we have

$$\begin{aligned}\sum_{n=3}^{\infty}\left[(n-1)(n-2)a_{n-1} + \sum_{k=2}^{n}(k-1)a_{k-1}b_{n-k} + \sum_{k=0}^{n} a_k c_{n-k}\right] z^n & \\ + [a_2 c_0 + a_1(b_0 + c_1) + a_1 b_0]z^2 + (a_0 c_1 + a_1 c_0)z + a_0 c_0 & = 0\end{aligned}$$

then $a_0 = a_1 = a_2 = 0$ and for $n \geq 3$ we have

$$(n-1)(n-2)a_{n-1} + \sum_{k=2}^{n}(k-1)a_{k-1}b_{n-k} + \sum_{k=0}^{n} a_k c_{n-k} = 0.$$

Hence $a_n = 0$ for all $n \geq 0$, ie., $u = 0$ is the only possible formal solution. This proves the claim by contradiction. □

The two claims above end the proof of Theorem A. □

Next we present a proof that also implies Theorem A.

Proof of Theorem B. First of all we are assuming that the origin is an ordinary point or a regular singularity of the ODE. If it is an ordinary point, then by the classical existence theorem for ODEs there are two linearly independent analytic solutions and any solution, formal or convergent, will be a linear combination of these two solutions. Such a solution is therefore convergent.

Thus we may write the ODE as $x^2 y'' + xb(x)y' + c(x)y = 0$ where the new coefficients $b(x)$ and $c(x)$, obtained after renaming $xb(x)/a(x)$ and $x^2 c(x)/a(x)$ conveniently, are analytic.

Let us consider a formal solution $\hat{y}(x)=\sum\limits_{n=0}^{\infty} d_n x^n$. We can write $\hat{y}(x)=x^{r_1}(1+\varphi(x))$ for some $r_1\geq 0$ and $\varphi(x)$ a formal function with $\varphi(0)=0$. In other words, $r_1\in\{0,1,2,\dots\}$ is the order of $\hat{y}(x)$ at the origin. Then we have $\hat{y}'(x)=r_1x^{r_1-1}(1+\varphi(x))+x^{r_1}\varphi'(x)$ and $\hat{y}''(x)=r_1(r_1-1)x^{r_1-2}(1+\varphi(x))+2r_1x^{r_1-1}\varphi'(x)+x^{r_1}\varphi''(x)$. Substituting this in the ODE $x^2\hat{y}''(x)+xb(x)\hat{y}'(x)+c(x)\hat{y}(x)=0$ and dividing by x^{r_1} we obtain

$$x^2\varphi''(x)+x(2r_1+b(x))\varphi'(x)+(r_1(r_1-1)+r_1b(x)+c(x))(1+\varphi(x))=0.$$

For $x=0$, since $\varphi(0)=0$, we then obtain the equation $r_1(r_1-1)+r_1b(0)+c(0)=0$. The above is exactly the indicial equation associated to the original ODE. We then conclude that the original ODE has an indicial equation with a root r_1 that belongs to the set of non-negative integers. Let now $r\in\mathbb{Z}$ be the other root of the indicial equation. There are two possibilities:
(i) $r\geq r_1$. In this case, then according to Frobenius classical theorem we conclude that there is at least one solution $y_r(x)=x^r\sum\limits_{n=0}^{\infty}e_nx^n$ which is convergent. There are two possibilities:
(i.1) $y_r(x)$ and $\hat{y}(x)$ are linearly dependent: in this case, $y_r(x)=\ell\cdot\hat{y}(x)$ for some constant $\ell\in\mathbb{R},\mathbb{C}$. Then $r=r_1$ and therefore $y_r(x)$ is analytic and the same holds for $\hat{y}(x)$. More precisely, $\hat{y}(x)$ is analytic in the same neighborhood $|x|<R$ where $b(x),c(x)$ are convergent.
(i.2) $y_r(x)$ and $\hat{y}(x)$ are linearly independent: Since $y_r(x)$ is analytic and seeing $y_r(x)$ as a formal solution, we have two linearly independent formal solutions. From what we have seen above in Theorem A both solutions are convergent in the common disc domain of analyticity of the functions $b(x),c(x)$.
(ii) $r_1\geq r$. In this case, then according to Frobenius classical theorem we conclude that there is at least one solution $\tilde{y}_{r_1}(x)=x^{r_1}\sum\limits_{n=0}^{\infty}f_nx^n$, where the power series is convergent. There are two possibilities:
(ii.1) $\tilde{y}_{r_1}(x)$ and $\hat{y}(x)$ are linearly dependent: in this case, $\tilde{y}_{r_1}(x)=\tilde{\ell}\cdot\hat{y}(x)$ for some constant $\tilde{\ell}\in\mathbb{R},\mathbb{C}$. Then $r=r_1$ and therefore $\tilde{y}_{r_1}(x)$ is analytic and the same holds for $\hat{y}(x)$. More precisely, $\hat{y}(x)$ is analytic in the same neighborhood $|x|<R$ where $b(x),c(x)$ are convergent.
(ii.2) $\tilde{y}_{r_1}(x)$ and $\hat{y}(x)$ are linearly independent: in this case, $\tilde{y}_{r_1}(x)$ is analytic and seeing $\tilde{y}_{r_1}(x)$ as a formal solution, we have two linearly independent formal solutions. From what we have seen above in Theorem A both solutions are convergent in the common disc domain of analyticity of the functions $b(x),c(x)$. □

The above proof still makes use of the convergence part in Theorem A, thus it cannot be used to give an alternative proof of Theorem A. Let us work on a totally independent proof of Theorem A based only on classical methods of Frobenius and ODEs. For this sake we shall need a few lemmas.

3.2. **The wronskian II.** We consider the ODE $x^2y''+xb(x)y'+c(x)y=0$ with a regular singular point at the origin.

Lemma 3.9. *Let $\hat{y}(x)$ be a formal solution of the ODE. Then we must have $\hat{y}(x)=x^r(1+\sum\limits_{n=1}^{\infty}a_nx^n)$ where r is a root of the indicial equation of the ODE.*

Remark 3.10. Let $r \in \{0, 1, 2, \ldots\}$ be a root of the indicial equation and assume that we have two solutions $\hat{y}_1(x) = x^r(1 + \varphi_1(x))$ and $\hat{y}_2(x) = x^r(1 + \varphi_2(x))$ which are formal. Then we have two cases: **(i)** $r \geq 1$. In this case $W(\hat{y}_1, \hat{y}_2)(0) = 0$. In this situation we must have $W(\hat{y}_1, \hat{y}_2)(x) = 0$ and therefore $\hat{y}_1, \hat{y}_2$ are linearly dependent. **(ii)** $r = 0$.

Let us proceed. We are assuming now that we have two formal solutions $\hat{y}_1, \hat{y}_2$ for the ODE above. We write $\hat{y}_j(x) = x^{r_j}(1 + \varphi_j(x))$ for some formal series $\varphi_j(x)$ that satisfies $\varphi_j(0) = 0$. The exponents r_j are non-negative integers and from what we have seen above, these are roots of the indicial equation $r(r-1)+rb(0)+c(0) = 0$ of the ODE. We may assume that $r_1 \geq r_2$.

So we have the following possibilities:

(i) $r_1 = r_2$. If this is the case we cannot a priori assure that the indicial equation has only the root $r = r_1 = r_2$. Anyway, if $r \neq 0$ then from what we have seen above the formal solutions $\hat{y}_1, \hat{y}_2$ are linearly dependent. This is a contradiction. Thus we must have $r = 0$. If $r = 0$ is the only root of the indicial equation then we have a basis of the solution space given by $y_1(x) = 1 + \sum\limits_{n=1}^{\infty} e_n x^n$ and $y_2(x) = y_1(x) \log|x| + \sum\limits_{n=1}^{\infty} f_n x^n$. If a linear combination $\hat{y}(x) = c_1 y_1(x) + c_2 y_2(x)$ is a formal function then necessarily $c_2 = 0$. Thus any two formal solutions are linearly dependent. Assume now that $r = 0$ is not the only root of the indicial equation. Denote by $\tilde{r} \in \mathbb{Z}^*$ the other root of the indicial equation. There are two possibilities:

(a) $\tilde{r} > 0$ then there is a basis of solutions given by $y_1(x) = x^{\tilde{r}}(1 + \sum\limits_{n=1}^{\infty} g_n x^n)$ and $y_2(x) = a y_1(x) \log|x| + |x|^0(1 + \sum\limits_{n=1}^{\infty} h_n x^n)$. Let $y(x) = c_1 y_1(x) + c_2 y_2(x)$ be a formal power series. Then $y(x) = (c_1 + a c_2 \log|x|) x^{\tilde{r}}(1 + \sum\limits_{n=1}^{\infty} g_n x^n) + c_2(1 + \sum\limits_{n=1}^{\infty} h_n x^n)$. If $y(x)$ is a formal power series then we must have $a c_2 = 0$ and therefore $y(x) = c_1 x^{\tilde{r}}(1 + \sum\limits_{n=1}^{\infty} g_n x^n) + c_2(1 + \sum\limits_{n=1}^{\infty} h_n x^n)$. In particular, since $\tilde{r} \in \mathbb{N}$, $y(x)$ is convergent. This shows that the formal solutions $\hat{y}_1(x), \hat{y}_2(x)$ are convergent and this is the only possible case where they can be linearly independent.

(b) $\tilde{r} < 0$ then there is a basis of solutions given by $y_1(x) = 1 + \sum\limits_{n=1}^{\infty} p_n x^n$ and $y_2(x) = a y_1(x) \log|x| + x^{\tilde{r}}(1 + \sum\limits_{n=1}^{\infty} q_n x^n)$. Write $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for a linear combination of $y_1(x)$ and $y_2(x)$. Then $y(x) = (c_1 + a c_2 \log|x|)(1 + \sum\limits_{n=1}^{\infty} p_n x^n) + c_2(x^{\tilde{r}}(1 + \sum\limits_{n=1}^{\infty} q_n x^n))$. If $y(x)$ is a formal series then necessarily $a c_2 = 0$ (because of the term $\log|x|$) and also $c_2 = 0$ in this case because $\tilde{r} < 0$. Thus we get $y(x) = c_1 y_1(x)$ which is convergent. This shows that again we must have that $\hat{y}_1$ and $\hat{y}_2$ are multiple of y_1 and therefore they are linearly dependent, contradiction again.

(ii) $0 < r_1 - r_2 = N \in \mathbb{N}$. This case follows from facts already used above. Since $r_1 > r_2$ and since each r_j is a root of the indicial equation, we conclude that these are the roots of the indicial equation. By Frobenius theorem there is a basis of the solutions given by $y_1(x) = x^{r_1}(1 + \sum\limits_{n=1}^{\infty} s_n x^n)$ and $y_2(x) = ay_1(x)\log|x| + |x|^{r_2}(1 + \sum\limits_{n=1}^{\infty} t_n x^n)$. If $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is a formal power series then we must have $ac_2 = 0$ and $y(x) = c_1 x^{r_1}(1 + \sum\limits_{n=1}^{\infty} s_n x^n) + c_2 x^{r_2}(1 + \sum\limits_{n=1}^{\infty} t_n x^n)$ which is convergent. This shows that $\hat{y}_1, \hat{y}_2$ must be convergent.

We are now in conditions of giving a second proof to Theorem A.

Alternative proof of Theorem A. Indeed, from the second part of the proof (which is based only on classical methods of Frobenius and ODEs) we know that the origin is an ordinary point or a regular singular point of the ODE. Given the two linearly independent formal solutions $\hat{y}_j(x),\ j = 1, 2$, from the above discussion, the solutions $\hat{y}_1(x), \hat{y}_2(x)$ are analytic. □

3.3. **The wronskian III: Some examples.** The next couple of examples show that the information on the wronskian (whether it is convergent, formal,etc) is not enough to infer about the nature of the solutions.

Example 3.11 (convergent wronskian but no formal solution)**.** This is an example of an ODE with a convergent wronskian but admitting no formal solution. The ODE $x^3y'' - x^2y' - y = 0$ has a non-regular singular point at the origin. From what we have observed above the wronskian W of two linearly independent solutions of the ODE satisfies the following first order ODE $x^3w' - x^2w = 0$ whose solution is of the form $W(x) = K\exp\Big(\int^x \frac{\eta^2}{\eta^3}d\eta\Big) = Kx$ for some constant K. It is now easy to check that there are no formal solutions besides the trivial.

3.4. **A couple of nonconvergent examples.** We now give an example of an ODE with non-convergent wronskian and admitting no formal solution but the trivial one.

Example 3.12 (non-convergent wronskian no formal solution)**.** The ODE $x^3y'' - xy' - y = 0$ has a non-regular singular point at the origin. Indeed, the wronskian is solution of the first order ODE $x^3w' - xw = 0$ which has solutions of the form $W(x) = K\exp\Big(\int^x \frac{\eta}{\eta^3}d\eta\Big) = K\exp\Big(-\frac{1}{x}\Big)$ where K is a constant. The ODE only admits the trivial formal solution.

Next we give an example of an equation admitting a formal but not a convergent solution.

Example 3.13. Consider the equation $x^2y'' - y' - \frac{1}{2}y = 0$. The origin $x_0 = 0$ is a singular point, but not is regular singular point, since the coefficient -1 of y' does not have the form $xb(x)$, where b is analytic for 0. Nevertheless, we can formally solve this equation by power series $\sum_{k=0}^{\infty} a_k x^k$, where the coefficients a_k satisfy the following recurrence formula $(k+1)a_{k+1} = \big[k^2 - k - \frac{1}{2}\big]a_k, \quad$ for every $k = 0, 1, 2, \ldots$. If $a_0 \neq 0$, applying the quotient test to this expression we have that

$$\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \left|\frac{k^2 - k - \frac{1}{2}}{k+1}\right| \cdot |x| \to \infty,$$

when $k \to \infty$, provided that $|x| \neq 0$. Hence, the series converges only for $x = 0$, and therefore does not represent a function in a neighborhood of $x = 0$.

4. **Characterization of regular singular points: Proof of Theorem C.** We shall now prove Theorem C.

Proof of Theorem C. We shall first consider the complex analytic case. We start then with a complex analytic ODE of the form $a(z)u'' + b(z)u' + c(z)u = 0$. Let us assume that this equation admits two linearly independent solutions $u_1(z), u_2(z)$, which are of autlos type in some neighborhood of the origin $z = 0 \in \mathbb{C}$.

The wronskian $W(u_1, u_2)(z)$ satisfies the first order ODE $a(z)w' + b(z)w = 0$ and since it is given by $W(u_1, u_2)(z) = u_1'(z)u_2(z) - u_1(z)u_2'(z)$, it is also of autlos type in some neighborhood of the origin $z = 0 \in \mathbb{C}$. Using then the above first order ODE and arguments similar to those in the proof of Lemma 3.6 we conclude that $b(z)/a(z)$ must have a pole of order ≤ 1 at the origin, otherwise $W(u_1, u_2)(z)$ would have an essential singularity at the origin. Following now a similar reasoning in the proof of Claim 3.8 in the second part of the proof of Theorem A we conclude that $c(z)/a(z)$ must have a pole of order ≤ 2 at the origin. This shows that the singularity at the origin is regular, or the origin is an ordinary point. If we start with a real analytic ODE then we consider its complexification. The fact that there are two linearly independent solutions of autlos type for the original ODE implies that there are two linearly independent solutions for the corresponding complex ODE, by definition these solutions will be of autlos type. Once we have concluded that the complex ODE has a regular singularity or an ordinary point at the origin, the same holds for the original real analytic ODE. Thus $(ii) \implies (iii)$. The classical Frobenius theorem shows that $(iii) \implies (i)$. Finally, it is clear from the definitions that $(i) \implies (ii)$. □

The next examples show how sharp is the statement of Theorem C.

Example 4.1. Consider the equation

$$z^3u'' - zu' + u = 0. \tag{11}$$

The origin $z_0 = 0$ is a singular point, but not is regular singular point. It is easy to see that $u_1 = z$ is a solution of equation (11). Making use of the method of reduction of order we can construct a second solution u_2 linearly independent with u_1. Hence we have that

$$u_2(z) = z\int^z \left(\frac{\exp\big(-\int^w \frac{-v}{v^3}dv\big)}{w^2}\right) dw = z\exp\Big(-\tfrac{1}{z}\Big).$$

Note that u_2 is not holomorphic.

Remark 4.2. Consider a second order differential equation of the form

$$z^3a(z)u'' + z^2b(z)u' + c(z)u = 0 \tag{12}$$

where a, b, c are holomorphic at the origin with $a(0) \neq 0$ and $c(0) \neq 0$. We shall see that (12) admits no formal solution. Indeed we assume that $u(z) = \sum_{n=0}^{\infty} d_n z^n$ is a formal solution of (12) then we have $d_n = 0$ for all $n \geq 0$. Observe that there exists the limit $\lim_{x\to 0} \frac{x^3b(x)}{x^3a(x)} = \frac{b_0}{a_0}$ and the limit below does not exist $\lim_{x\to 0} \frac{x^2c(x)}{x^3a(x)}$.

Remark 4.3. Consider a second order differential equation of the form

$$z^2a(z)u'' + b(z)u' + c(z)u = 0 \tag{13}$$

where a, b, c are holomorphic at the origin with $a(0) \neq 0$, $b(0) \neq 0$ and $c(0) \neq 0$. We shall see that (13) always admits non trivial formal solution. Indeed we assume that $u(z) = \sum_{n=0}^{\infty} d_n z^n$ is a formal solution of (13) then $d_1 = -\frac{c_0 d_0}{b_0}$, $d_2 = \frac{(b_1 c_0 + c_0^2 + c_1 b_0) d_0}{b_0^2}$ and for $n \geq 2$ we have

$$d_{n+1} = \frac{1}{b_0(n+1)}\Big(-c_n d_0 - \sum_{k=1}^{n} [k(k-1)a_{n-k} + k b_{n-k+1} + c_{n-k}] d_k \Big).$$

Observe that the coefficients of the series depend on d_0, since we look for non trivial formal solutions it suffices to choose $d_0 \neq 0$. Hence, there exist non trivial formal solution. Also note that there exists the limit $\lim_{x\to 0} \frac{x^2 c(x)}{x^2 a(x)} = \frac{c_0}{a_0}$ and the following limit is not finite $\lim_{x\to 0} \frac{x b(x)}{x^2 a(x)}$.

Example 4.4. Consider a second order differential equation given by

$$z^2 u'' + bu' + cu = 0 \tag{14}$$

where b and c are nonzero constants. Observe that the origin is a non regular singular point of (14). Next we shall see that there exist non trivial formal solutions for (14). Let us assume that

$$u(z) = \sum_{n=0}^{\infty} a_n z^n \tag{15}$$

is a non trivial formal solution of (14). We have $a_1 = -\frac{c a_0}{b}$, $a_2 = \frac{c^2 a_0}{2b^2}$ and

$$a_{n+1} = -\frac{(n^2 - n + c)a_n}{b(n+1)}, \quad \text{for all } n = 2, 3, \ldots. \tag{16}$$

Observe that the coefficients of the series depend on a_0, since we look for non trivial formal solutions it suffices to choose $a_0 \neq 0$. Hence, there exist non trivial formal solution. Observe now that this formal solution is not convergent. Applying the ratio test to the expressions (15) and (16), we have that

$$\left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = \left| \frac{n^2 - n + c}{b(n+1)} \right| \cdot |z| \to \infty,$$

when $n \to \infty$, whenever $|z| \neq 0$. Hence, the series converges only for $z = 0$.

5. Riccati model for a second order linear ODE.

We shall now exhibit method of associating to a homogeneous linear second order ODE a Riccati differential equation. Consider a second order ODE given by

$$a(z)u'' + b(z)u' + c(z)u = 0$$

where a, b, c are analytic functions, real or complex, of a variable z real or complex, defined in a domain $U \subset \mathbb{R}, \mathbb{C}$. According to [19] there is an integrable one-form

$$\Omega = -a(z)y dx + a(z)x dy + [a(z)y^2 + b(z)xy + c(z)x^2]dz$$

that vanishes at the vector field corresponding to the reduction of order of the ODE, i.e., $\omega(X) = 0$ where

$$X(x, y, z) = y\frac{\partial}{\partial x} - \left(\frac{b(z)}{a(z)} y + \frac{c(z)}{a(z)} x \right) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

As a consequence the orbits of X are tangent to the foliation $\mathcal{F}_\Omega$ given by the Pfaff equation $\Omega = 0$.

First of all we remark that we can write Ω as follows
$$\frac{\Omega}{x^2} = a(z)d\left(\frac{y}{x}\right) + \left[a(z)\left(\frac{y}{x}\right)^2 + b(z)\left(\frac{y}{x}\right) + c(z)\right]dz.$$

Thus, by introducing the variable $t = \frac{y}{x}$ we see that the same foliation $\mathcal{F}_\Omega$ can be defined by the one-form ω below:
$$\omega = a(z)dt + [a(z)t^2 + b(z)t + c(z)]dz.$$

By its turn $\omega = 0$ defines a Riccati foliation which writes as
$$\frac{dt}{dz} = -\frac{a(z)t^2 + b(z)t + c(z)}{a(z)}.$$

Definition 5.1. The Riccati differential equation above is called *Riccati model* of the ODE $\quad a(z)u'' + b(z)u' + c(z)u = 0$.

Remark 5.2. The Riccati model can be obtained in a less geometrically clear way by setting $t = u'/u$ as a new variable. Sometimes it is also useful to consider the change of variable $w = u/u'$ which leads to the Riccati equation $\frac{dw}{dz} = \frac{c(z)w^2+b(z)w+a(z)}{a(z)}$.

5.1. **Holonomy of a second order equation.** Let $\eta = (E, \pi, B, F)$ be a (locally trivial) fibration with total space E, fiber F, base B and projection $\pi : E \to B$. A foliation $\mathcal{F}$ on E is *transverse to the fibre bundle* η if: (a) for each $p \in E$, the leaf L_p of $\mathcal{F}$ with $p \in L_p$ is transverse to the fiber $\pi^{-1}(q)$, $q = \pi(p)$; (b) $\dim(\mathcal{F}) + \dim(F) = \dim(E)$; (c) for each leaf L of $\mathcal{F}$, the restriction $\pi|L : L \to B$ is a covering map. According to a theorem of Ehresmann (see [4, Ch. V], [9, Ch. II]) if the fiber F is compact, then conditions (a) and (b) together already imply (c).

Let us state the exact notion of Riccati foliation we consider. We shall consider a complex manifold M admitting a locally trivial holomorphic fibration $\pi\colon M \to B$, onto a complex manifold B, with fiber F. A singular holomorphic foliation $\mathcal{F}$ on M will be called a *Riccati foliation* on M if (i) $\operatorname{codim}\mathcal{F} = \dim \mathrm{F}$ and (ii) there is a *ramification set* $\sigma \subset B$, analytic of codimension ≥ 1, such that $\pi^{-1}(\sigma)$ is invariant (a union of $\mathcal{F}$-invariant fibers), and the restriction of $\mathcal{F}$ to $M \setminus \pi^{-1}(\sigma)$ is a foliation transverse to the fibre bundle $\pi\big|_{M\setminus\pi^1(\sigma)}\colon M \setminus \pi^{-1}(\sigma) \to B \setminus \sigma$ in the classical sense of Ehresmann (see for instance [4, Ch. V]). In particular we have $\operatorname{sing}(\mathcal{F}) \subset \pi^{-1}(\sigma)$. Most relevant is the fact, well-known for foliations transverse to fibre bundles, that (iii) in case the fundamental group of $B \setminus \sigma$ is finitely generated, $\mathcal{F}|_{M\setminus\pi^{-1}(\sigma)}$ *is conjugate to the suspension of a subgroup of holomorphic diffeomorphisms* $G \subset \operatorname{Diff}(F)$, given by a (so called *holonomy or monodromy*) homomorphism $\varphi\colon \pi_1(B \setminus \sigma) \to \operatorname{Diff}(F)$.

In view of (iii) our notion of Riccati foliation is quite general. Indeed, in our framework, a Riccati foliation on a fibered space $\pi\colon M \to B$, with fiber F, may be seen as an extension, holomorphic with singularities, of a foliation given by a suspension of a group of complex diffeomorphisms $G \subset \operatorname{Diff}(F)$, obtained as a representation $\pi_1(B \setminus \sigma) \to \operatorname{Diff}(F)$, where σ is a codimension ≥ 1 analytic subset of B. This idea is reinforced by the following. When $M = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, a holomorphic foliation with singularities, is of Riccati type $\frac{dy}{dx} = \frac{a(x)y^2+b(x)y+c(x)}{p(x)}$, if and only if, it is transverse to a generic vertical line $\mathbb{P}^1(\mathbb{C})_{x_0} = \{x_0\} \times \mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This fact, widely used by Paul Painlevé in his memoire (see [18]), has been extended in a natural way in [21]. Indeed, in [21] the notion of Riccati foliation adopted is pretty much the same we use here, by considering suitable (natural) fibrations.

It is well-known that a complex rational Riccati differential equation $\frac{dy}{dx} = \frac{a(x)y^2+b(x)y+c(x)}{p(x)}$ induces in the complex surface $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ a foliation $\mathcal{F}$ with singularities, having the following characteristics:

(i) The foliation has a finite number of invariant vertical lines $\{x_0\}\times\mathbb{P}^1(\mathbb{C})$. These lines are given by the zeroes of $p(x)$ and possibly by the line $\{\infty\} \times \mathbb{P}^1(\mathbb{C})$.
(ii) For each non-invariant vertical line $\{x_0\} \times \mathbb{P}^1(\mathbb{C})$ the foliation has its leaves transverse to this line.
(iii) From Ehresmann we conclude that the restriction of $\mathcal{F}$ to $(\mathbb{P}^1(\mathbb{C})\setminus\sigma)\times\mathbb{P}^1(\mathbb{C})$, where $\sigma \times \mathbb{P}^1(\mathbb{C})$ is the set of invariant vertical lines, is a foliation transverse to the fibers of the fiber space $(\mathbb{P}^1(\mathbb{C}) \setminus \sigma) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \setminus \sigma$ with fiber $\mathbb{P}^1(\mathbb{C})$ and projection given by $\pi(x, y) = x$.
(iv) The restriction $\pi\big|_L$ of the projection to each leaf L of the Riccati foliation defines a covering map $L \to \mathbb{P}^1(\mathbb{C}) \setminus \sigma$.

In particular, there is a global holonomy map which is defined as follows: choose any point $x_0 \notin \sigma$ as base point and consider the lifting of the closed paths $\gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \sigma)$ to each leaf $L \in \mathcal{F}$ by the restriction $\pi\big|_L$ above. Denote the lift of γ starting at the point $(x_0, z) \in \{x_0\} \times \mathbb{P}^1(\mathbb{C})$ by $\tilde{\gamma}_z$. If the end point of $\tilde{\gamma}_z$ is denoted by $(x_0, h_\gamma(z))$ then the map $z \mapsto h_\gamma(z)$ depends only on the homotopy class of $\gamma \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \sigma)$. Moreover, this defines a complex analytic diffeomorphism $h_{[\gamma]} \in \text{Diff}(\mathbb{P}^1(\mathbb{C}))$ and the map $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \sigma) \to \text{Diff}(\mathbb{P}^1(\mathbb{C}))$, $[\gamma] \mapsto h_{[\gamma]}$ is a group homomorphism (see Proposition 1.16 in [13] page 24). The image $Hol(\mathcal{F}) \subset \text{Diff}(\mathbb{P}^1(\mathbb{C}))$ is called *global holonomy* of the Riccati equation. It is well-known from the theory of foliations transverse to fiber spaces that the global holonomy classifies the foliation up to fibered conjugacy (see Theorem 3 in [4] page 99). This will be useful to us in what follows. Recall that $\text{Diff}(\mathbb{P}^1(\mathbb{C}))$ as meant above is the projectivization of the special linear group ie., $\text{Diff}(\mathbb{P}^1(\mathbb{C})) \cong \mathbb{P}SL(2, \mathbb{C})$ (see [10] page 64) meaning that every global holonomy map can be represented by a Moebius map $T(z) = \frac{a_1z+a_2}{a_3z+a_4}$ where $a_1, a_2, a_3, a_4 \in \mathbb{C}$ and $a_1a_4 - a_2a_3 = 1$. Thus the global holonomy group of a Riccati foliation identifies with a group of Moebius maps. Another important fact which comes from the general theory of suspension foliations is:

(v) Given a non-vertical leaf L of a Riccati foliation $\mathcal{F}$ in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, the leaf L is conformally equivalent, via the projection $\pi\big|L\colon L \to B = \mathbb{P}^1(\mathbb{C}) \setminus \sigma$, to the holomorphic covering of B associate to the subgroup $Fix(L) \subset Hol(\mathcal{F})$ stabilizer of L in the global holonomy $Hol(\mathcal{F})$.

(vi) Fix a point $p \in \{x_0\} \times \mathbb{P}^1(\mathbb{C})$, $x_0 \notin \sigma$, and consider as transverse section the germ of disc induced by the vertical fiber $\{x_0\}\times\mathbb{P}^1(\mathbb{C})$. The holonomy group of the non-vertical leaf L through this point p is conjugate to the subgroup of germs at p of elements of the global holonomy group $Hol(\mathcal{F})$ that fix the point p.

In particular, *if the group $Fix(L)$ is trivial then L is conformally equivalent to the basis $B = \mathbb{P}^1(\mathbb{C}) \setminus \sigma$.*

Definition 5.3 (holonomy of a second order ODE)**.** Given a linear homogeneous second order ODE with complex polynomial coefficients

$$a(z)u'' + b(z)u' + c(z)u = 0$$

we call the *holonomy* of the ODE the global holonomy group of the corresponding Riccati model.

Remark 5.4. As we have seen above we can also obtain a Riccati model by any of the changes of variables $t = u'/u$ or $w = u/u'$. From the viewpoint of ODEs these models may seem distinct. Nevertheless, they differ only up to the change of coordinates $t = 1/w$. Moreover, both have the same global holonomy group, since the point at infinity is always considered in the definition of global holonomy group. Indeed, the ideal space for considering a Riccati equation from the geometrical viewpoint, is the space $\mathbb{C} \times \mathbb{C}$.

5.2. **Trivial Holonomy: Theorem D.** Let us investigate some interesting cases. First consider a Riccati foliation $\mathcal{F}$ assuming that σ is a single point. Thus we may assume that in affine coordinates (x, y) the ramification point is the point $x = \infty$. Then we may write $\mathcal{F}$ as given by a polynomial differential equation $\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$. The global holonomy of $\mathcal{F}$ is given by an homomorphism $\phi\colon \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \sigma) \to \mathrm{Diff}(\mathbb{P}^1(\mathbb{C}))$. Since σ is a single point we have $\mathbb{P}^1(\mathbb{C}) \setminus \sigma = \mathbb{C}$ is simply-connected and therefore the global holonomy is trivial. By the classification of foliations transverse to fibrations (see [4, Ch. V]) there is a fibered biholomorphic map $\Phi\colon \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{C} \times \mathbb{P}^1(\mathbb{C})$ that takes the foliation $\mathcal{F}$ into the foliation $\mathcal{H}$ given by the horizontal fibers $\mathbb{C} \times \{y\}, y \in \mathbb{P}^1(\mathbb{C})$.

Lemma 5.5. A holomorphic diffeomorphism $\Phi\colon \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{C} \times \mathbb{P}^1(\mathbb{C})$ preserving the vertical fibration writes in affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{C} \times \mathbb{P}^1(\mathbb{C})$ as $\Phi(x, y) = \Big(Ax + B, \frac{a(x)y+b(x)}{c(x)y+d(x)}\Big)$ where a, b, c, d are entire functions satisfying $ad - bc \neq 0$, $0 \neq A, B \in \mathbb{C}$.

Proof of Lemma 5.5. The fact that the vertical fibration is preserved means that in coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}P^1(\mathbb{C})$ the map Φ is of the form $\Phi(x, y) = (f(x), g(x, y))$. Similarly, the inverse map Φ^{-1} is also of the form $\Phi^{-1}(x, y) = (\tilde{f}(x), \tilde{g}(x, y))$. The holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ is then an entire automorphism (diffeomorphism of $\mathbb{C}$). It is well-known that the automorphisms of $\mathbb{C}$ is the group of affine maps, so that $f(x) = Ax+B$ for some $A, B \in \mathbb{C}$ with $A \neq 0$. Again, since the vertical fibration $x =$ *constant* is preserved, for each fixed $x \in \mathbb{C}$ the map $\Phi^x\colon \{x\} \times \mathbb{P}^1(\mathbb{C}) \to \{Ax+B\} \times \mathbb{P}^1(\mathbb{C}), (x, y) \mapsto \Phi(x, y) = (Ax + B, g(x, y))$ is a diffeomorphism. In particular the map $g^x : y \mapsto g(x, y)$ is a holomophic diffeomorphism of $\mathbb{P}^1(\mathbb{C})$. It is well-known that the group of automorphisms (holomorphic diffeomorphisms) of the Riemann sphere is the group of Moebius maps of the form $y \mapsto \frac{ay+b}{cy+d}$ with $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. Hence, locally we must have $g(x, y) = \frac{a_j(x)y+b_j(x)}{c_j(x)y+d_j(x)}$ for some holomorphic functions $a_j(x), b_j(x), c_j(x), d_j(x)$ satisfying satisfying $a_j(x)d_j(x) - b_j(x)c_j(x) \neq 0$, $\forall x \in D_j$, defined in some open subset $U_j \subset \mathbb{C}$ such that the union $\bigcup\limits_{j \in J} U_j$ is a cover of the complex plane $\mathbb{C}$. These functions $a_j(x), b_j(x), c_j(x), d_j(x)$ are not unique, but they are unique up to multiplication by a nonzero complex number, i.e., they correspond to a unique element of the projectivization $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$. Because $\mathbb{C}$ is a simply-connected domain, by standard analytic continuation we conclude that the map g induces a holomorphic map $\mathbb{C} \to \mathrm{GL}(2, \mathbb{C}), \mathbb{C} \ni x \mapsto g^x(y) = g(x, y)$, ie, there are entire functions $a(x), b(x), c(x), d(x)$ satisfying $ad - bc \neq 0$ such that $g(x, y) = \frac{a(x)y+b(x)}{c(x)y+d(x)}$. □

The map $\Phi(x,y) = (Ax+B, \frac{a(x)y+b(x)}{c(x)y+d(x)})$ takes leaves of $\mathcal{F}$ into leaves of the horizontal fibration $\mathcal{H} : y = constant$. Since the second projection map $\sigma_2 : \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, $(x,y) \mapsto y$ is a first integral for $\mathcal{H}$, it follows that the composition $g = \sigma_2 \circ \Phi : \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic first integral for $\mathcal{F}$. Another point is that, as already mentioned in (iv) above, given any leaf L of $\mathcal{F}$, the restriction $\pi\big|_L$ of the projection to each leaf L of the Riccati foliation defines a covering map $L \to \mathbb{P}^1(\mathbb{C}) \setminus \sigma$. In our current situation we have $\sigma = \{(\infty, 0)\}$, so that we have a covering map $L \to \mathbb{C}$. In particular we conclude that the leaves of $\mathcal{F}$ are diffeomorphic with $\mathbb{C}$ (including the one contained in the invariant fiber $\{(0,\infty)\} \times \mathbb{P}^1(\mathbb{C})$).

Let us now apply this to our framework of second order linear ODEs.

Proof of Theorem D. Beginning with the ODE $a(z)u'' + b(z)u' + c(z)u = 0$ the Riccati model is

$$\frac{dt}{dz} = -\frac{a(z)t^2 + b(z)t + c(z)}{a(z)}.$$

Thus if we assume that $a(z) = 1$ then we have for this Riccati equation that $\sigma = \{\infty\}$ as considered in § 5.2 above. This implies that $\mathcal{F}$ admits a meromorphic first integral $g : \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of the above form $g(z,t) = \frac{A(z)t+B(z)}{C(z)t+D(z)}$ for some entire functions $A(t), B(t), C(t), D(t)$. Given a leaf L of the Riccati foliation there is a constant $\ell \in \mathbb{P}^1(\mathbb{C})$ such that $g(z,t) = \ell$ for all $(t,z) \in L$. Hence $t = \frac{\ell D(z) - B(z)}{A(z) - \ell C(z)}$ for all $(t,z) \in L$. This defines a *meromorphic* parametrization $z \mapsto t(z)$ of the leaf. Since we have $t = \frac{y}{x} = \frac{u'}{u}$ therefore $u(z) = k \exp\big(\int_0^z t(\xi) d\xi\big)$ is a solution of the ODE with $k \in \mathbb{C}$ a constant. This gives

$$u_{\ell,k}(z) = k \exp\left(\int_0^z \frac{\ell D(\xi) - B(\xi)}{A(\xi) - \ell C(\xi)} d\xi\right), \; k, \ell \in \mathbb{C};$$

as general solution of the original ODE. Notice that $\frac{u'_{\ell,k}(z)}{u_{\ell,k}(z)} = \frac{\ell D(z) - B(z)}{A(z) - \ell C(z)}$ so that if $\ell_1 \neq \ell_2$ then the corresponding solutions u_{ℓ_1,k_1} and u_{ℓ_2,k_2} generate a nonzero wronskian, and therefore they are linearly independent solutions for all $k_1 \neq 0 \neq k_2$. □

Illustrating Theorem D we have:

Example 5.6. Consider the equation given by

$$u'' - zu' - u = 0. \tag{17}$$

From what we have observed above we known that for $a(z) = 1$, $b(z) = -z$ and $c(z) = -1$ there exists a Riccati equation given by

$$\frac{dt}{dz} = 1 + zt - t^2.$$

It is not difficult to see that $t = z$ is a solution of the differential Riccati model. Hence

$$t = \frac{\exp\big(-\frac{z^2}{2}\big)}{\ell + \int^z \exp\big(-\frac{\eta^2}{2}\big) d\eta} + z,$$

where ℓ is constant, is a solution of the differential Riccati model. In the construction of the Riccati equation associated to (17) it is considered that $t = \frac{u'}{u}$ where u is a solution of (17). Hence

$$u(z) = k \exp\Big(\frac{z^2}{2}\Big)\Big(\ell + \int^z \exp\Big(-\frac{\eta^2}{2}\Big) d\eta\Big)$$

where k is constant. It is a straightforward computation to show that u is a solution of (17). Rewriting u we have

$$\begin{aligned} u(z) &= k\exp\left(\int^z \left[\eta + \frac{\exp\big(-\frac{\eta^2}{2}\big)}{\ell + \int^\eta \exp\big(-\frac{\xi^2}{2}\big)d\xi}\right]d\eta\right) \\ &= k\exp\left(\int^z \frac{\ell\eta - \big[-\exp\big(-\frac{\eta^2}{2}\big) - \eta\int^\eta \exp\big(-\frac{\xi^2}{2}\big)d\xi\big]}{\int^\eta \exp\big(-\frac{\xi^2}{2}\big)d\xi - \ell(-1)}d\eta\right). \end{aligned}$$

Note that $A(\eta) = \int^\eta \exp\big(-\frac{\xi^2}{2}\big)d\xi$, $B(\eta) = -\exp\big(-\frac{\eta^2}{2}\big) - \eta\int^\eta \exp\big(-\frac{\xi^2}{2}\big)d\xi$, $C(\eta) = -1$ and $D(\eta) = \eta$ are entire functions satisfying

$$A(\eta)D(\eta) - B(\eta)C(\eta) = -\exp\big(-\tfrac{\eta^2}{2}\big) \neq 0.$$

5.3. **Cyclic holonomy.** Next we investigate the case where the ODE generates a Ricatti foliation $\mathcal{R}$ having a ramification set σ that consists of two points. In this case the holonomy group of the ODE is cyclic generated by a single Moebius map. Let us make a general study of this case. We start by assuming that $\sigma = \{0, \infty\}$ i.e., that the invariant vertical lines are $\{0\} \times \mathbb{P}^1(\mathbb{C})$ and $\{\infty\} \times \mathbb{P}^1(\mathbb{C})$ in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. The basis $B \setminus \sigma$ is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ minus two points, the origin and the point at infinity. This corresponds to $\mathbb{C}$ minus the origin, which is kwnon as the cylinder $\mathbb{C}^*$. Since the fundamental group of the cylinder $\mathbb{C}^*$ is isomorphic to $\mathbb{Z}$, the global holonomy group of $\mathcal{R}$ is cyclic generated by a single Moebius map. About the conformal and topological type of the leaves of $\mathcal{R}$ we have:

Lemma 5.7. *Each non-vertical leaf of $\mathcal{R}$ is either conformally equivalent to $\mathbb{C}$ or to the cylinder $\mathbb{C}^*$.*

Proof. By a vertical leaf of $\mathcal{R}$ we mean a leaf contained in a vertical line $z = constant$. Since $\mathcal{R}$ is a foliation transverse to a fibration in $\mathbb{C}^* \times \mathbb{P}^1(\mathbb{C})$, a non-vertical leaf L of $\mathcal{R}$ is, by the second coordinate projection, a holomorphic covering of the basis $\mathbb{C}^*$. This leaf is then either conformally equivalent to $\mathbb{C}$ or to the cylinder $\mathbb{C}^*$. This is a simple consequence of the classical Riemann-Koebe uniformization theorem (see [7]). □

Indeed, we can state:

Proposition 5.8. *The non-vertical leaves of $\mathcal{R}$ are diffeomorphic to the cylinder $\mathbb{C}^*$.*

Proof. As we already observed a non-vertical leaf L of $\mathcal{R}$ is conformally equivalent to $\mathbb{C}$ or to $\mathbb{C}^*$. If L has trivial holonomy group then (as we have seen in the final remarks preceding Definition 5.3) L is diffeomorphic to the basis $B = \mathbb{C}^*$. Assume now that the holonomy of L is not trivial. We claim that still in this case L is diffeomorphic to $\mathbb{C}^*$. Indeed, otherwise L mus be diffeomorphic to $\mathbb{C}$, in which case L is simply-connected and then its holonomy group must be trivial, a contradiction. □

Next we give a concrete example of a second order ODE having a cyclic global holonomy group. This is an example that fits into Frobenius approach, since the origin is a regular singularity.

Example 5.9. Consider the equation given by

$$z^2u'' + u = 0.$$

From what we observed above we have that for $a(z) = z^2$, $b(z) = 0$ and $c(z) = 1$ there exists a Riccati equation given by

$$\frac{dt}{dz} = \frac{z^2 + t^2}{z^2}.$$

This is a Riccati equation with a single vertical invariant line in $\mathbb{C} \times \mathbb{P}^1(\mathbb{C})$, the projective line $\{0\} \times \mathbb{P}^1(\mathbb{C})$. The line at infinity $\{\infty\} \times \mathbb{P}^1(\mathbb{C})$ is also invariant, as it follows from a simple change of coordinates $\tilde{z} = \frac{1}{z}$.

Before finishing this example, we give a word about the computation of the global holonomy. Since it is a homogeneous equation, the solutions of the Riccati differential equation are of the form

$$t = \frac{K(1+i\sqrt{3})z^{1-i\sqrt{3}} - (1-i\sqrt{3})z}{2Kz^{-i\sqrt{3}} - 2}$$

where K is constant. It is enough to compute a simple loop holonomy map with base point at $t = 0$ for instance, $z(\theta) = z_0e^{i\theta}$ with $0 \le \theta \le 2\pi$. After the corresponding computations we get a holonomy map as

$$h(t_0) = t(z(2\pi)) = z_0\frac{(t_0(1+i\sqrt{3}) - 2z_0)e^{2\pi\sqrt{3}} - t_0(1-i\sqrt{3}) + 2z_0}{(2t_0 - (1-i\sqrt{3})z_0)e^{2\pi\sqrt{3}} - 2t_0 + (1+i\sqrt{3})z_0}.$$

A fairly classical regular singularity type example is given below.

Example 5.10 (Bessel equation)**.** Consider the complex Bessel equation given by

$$z^2u'' + zu' + (z^2 - \nu^2)u = 0$$

where $z, \nu \in \mathbb{C}$. Since $a(z) = z^2$, $b(z) = z$ and $c(z) = z^2 - \nu^2$ the corresponding Riccati model is

$$\frac{dt}{dz} = -\frac{z^2t^2 + zt + z^2 - \nu^2}{z^2}$$

A change of coordinates to $w = \frac{1}{z}$ shows that the ramification set is $\sigma = \{0, \infty\}$. We are then in the cyclic global holonomy case. At this moment we shall not compute the Moebius map that generates the global holonomy. We are more interested in the geometry of the leaves which is given by Proposition 5.8.

A non-regular singularity example is given below. This example is specially interesting in view of our Proposition 5.8. Indeed, it cannot be studied by the classical Frobenius approach since the origin is not a regular singularity.

Example 5.11. Let us consider the following polynomial ODE

$$z^nu'' + b(z)u' + c(z)u = 0$$

where b, c are complex polynomials of a variable z. If $n \ge 2$ and $b(0) \neq 0$ or if $n \ge 3$ and $c(0) \neq 0$ or $b(0).b'(0) \neq 0$ then $z = 0$ is a non-regular singular point. Let us assume that this is the case. The corresponding Riccati equation is

$$\frac{dt}{dz} = -\frac{z^nt^2 + b(z)t + c(z)}{z^n}.$$

Changing coordinates $w = 1/z$ we obtain

$$\frac{dt}{dw} = \frac{t^2 + w^nb(1/w)t + w^nc(1/w)}{w^2} = \frac{w^kt^2 + \tilde{b}(w)t + \tilde{c}(w)}{w^{2+k}}$$

for some polynomials $\tilde{b}(w), \tilde{c}(w)$ and some $k \in \mathbb{N}$. This shows that the ramification set $\sigma \subset \mathbb{P}^1(\mathbb{C})$ consists of the points $z = 0$ and $z = \infty$. The basis is then the Riemman sphere minus two points. This corresponds to the complex plane $\mathbb{C}$ minus one point, ie., to the cylinder $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The fundamental group of the basis is then isomorphic to $\mathbb{Z}$, being cyclic generated by a simple non-trivial loop homotopy class. The holonomy of the ODE is then generated by a single Moebius map.

5.4. **Two generators global holonomy case.** Now we consider the case where σ consists of three points. We may assume that these are the points $z = \pm 1$ and $z = \infty$, i.e., $\sigma = \{1, -1, \infty\}$. The considered Riccati foliation $\mathcal{R}$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has three invariant vertical lines: the lines $\{1\} \times \mathbb{P}^1(\mathbb{C}), \{-1\} \times \mathbb{P}^1(\mathbb{C})$ and the line $\{\infty\} \times \mathbb{P}^1(\mathbb{C})$ where ∞ stands for the point at infinity in the horizontal coordinate. The basis $B = \mathbb{P}^1(\mathbb{C}) \setminus \sigma$ corresponds to the Riemann sphere minus three points. The fundamental group $\pi_1(B)$ is then free with two generators. The global holonomy group is therefore generated by two Moebius maps which we shall not compute. Instead we shall work with the conformal type of the leaves as follows:

Proposition 5.12. *The non-vertical leaves of $\mathcal{R}$ are parabolic (in the sense of potential theory) Riemman surfaces of the form $\mathbb{D}/G$ where $\mathbb{D}$ is the unit disc and $G \subset SL(2, \mathbb{R})$ is a properly discontinous group.*

Proof. Again this is a consequence of the uniformization theorem of Riemman-Koebe: a non-vertical leaf admits a holomorphic covering onto the basis $\mathbb{P}^1(\mathbb{C}) \setminus \{1, -1, \infty\}$. Thanks to Picard little theorem there is no nonconstant holomorphic map $\mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \setminus \{1, -1, \infty\}$ and therefore the universal covering of L is conformally equivalent to the unit disc $\mathbb{D}$. The fact that these leaves are parabolic is a consequence of the fact that the basis $\mathbb{P}^1(\mathbb{C}) \setminus \{1, -1, \infty\}$ is clearly parabolic (it is a compact Riemann surface minus a finite number of points, therefore it does not admit a non-constant bounded from above subharmonic function). □

The following is an example with a holonomy group generated by two Moebius maps.

Example 5.13 (Legendre equation)**.** Consider the equation of Legendre given by

$$(1 - z^2)u'' - 2zu' + \alpha(\alpha + 1)u = 0$$

where $\alpha \in \mathbb{C}$. From what we observed above we have that for $a(z) = 1 - z^2$, $b(z) = -2z$ and $c(z) = \alpha(\alpha + 1)$ there exists a Riccati equation given by

$$\frac{dt}{dz} = -\frac{(1 - z^2)t^2 - 2zt + \alpha(\alpha + 1)}{1 - z^2}.$$

Putting $w = \frac{1}{z}$ we have

$$\frac{dt}{dw} = \frac{(w^2 - 1)t^2 - 2wt + \alpha(\alpha + 1)w^2}{w^2(w^2 - 1)}.$$

As above, we are not aiming to compute the global holonomy at this time. We want to hightlight the qualitative information we obtain about the nature of the leaves by applying Proposition 5.12.

5.5. **Application.** As a final application of our methods we have:

Example 5.14 (an equation without solutions)**.** We consider the ODE $zu'' + u' + zu = 0$. In the Euler form we have $z^2u'' + zu' + z^2u = 0$ which was obtained from Besssel equation (Example 5.10) for $\nu = 0$. This last has indicial equation

$r(r-1)+r=0$ which gives as only solution $r=0$. Then Frobenius theorem assures the existence of an analytic solution of the form $u(z) = z^0 \sum_{n=0}^{\infty} a_n z^n$. Let us take $t = u'/u$ and consider the corresponding Riccati equation

$$\frac{dt}{dz} = -\frac{zt^2 + t + z}{z}.$$

Rewriting this equation we have

$$zt'(z) = -zt^2(z) - t(z) - z \tag{18}$$

Claim 5.15. *Equation (18) admits no non-trivial formal solution.*

Proof. Indeed, let us assume that $t(z) = \sum_{n=0}^{\infty} a_n z^n$ is a formal solution of (18). Thus we have

$$\sum_{n=2}^{\infty}[(n+1)a_n + c_{n-1}]z^n + (2a_1 + c_0 + 1)z + a_0 = 0$$

then $a_0 = 0$, $2a_1 + c_0 + 1 = 0$ and $(n+1)a_n + c_{n-1} = 0$ for all $n = 2, 3, \ldots$. Hence $a_n = 0$ for all $n = 0, 1, 2, \ldots$. □

The conclusion is that the solution given by Frobenius method vanishes at the origin (easy to see already from Frobenius type computations of the solution) and (more interesting) the solutions of the original ODE are Riemann surfaces of the logarithmic, since each solution u is of the form $u = Ke^{\int t}$ for some constant K.

6. Liouvillian solutions: Theorem E.

In this section we shall refer to the notion of Liouvillian function as introduced in [24]. We stress the fact that the generating basis field is the one of rational functions. Thus a Liouvillian function of n complex variables $x_1, \ldots, x_n$ will be a function belonging to a Liouvillian tower of differential extensions $k_0 \subset k_1 \subset \cdots \subset k_r$ starting the field k_0 of rational functions $k_0 = \mathbb{C}(x_1, \ldots, x_n)$ equipped with the partial derivatives $\frac{\partial}{\partial x_j}$.

Recall that a Liouvillian function is always holomorphic in some Zariski open subset of the space $\mathbb{C}^n$. Nevertheless, it may have several branches. Let us denote by $\mathrm{Dom}(F) \subset \mathbb{C}^n$ the *domain of* F as the biggest open subset where F has local holomorphic branches. This allows the following definition:

Definition 6.1 (Liouvillian solution, Liouvillian first integral, Liouvillian relation)**.** Given an equation $a(z)u'' + b(z)u' + c(z)u = 0$, a Liouvillian function $u(z)$ of the variable z will be called *a solution* of the ODE if we have $a(z)u'' + b(z)u' + c(z)u = 0$ in some nonempty open subset where $u(z)$ is holomorphic. A three variables Liouvillian function $F(x_1, x_2, x_3)$ will be called *a first integral* of the ODE if given any local solution $u_0(z)$ of the ODE, defined for z in a disc $D(z_0, r) \subset \mathbb{C}$, we have that $F(z, u_0(z), u_0'(z))$ is constant for $|z - z_0| < r$ provided that $(z, u_0(z), u_0'(z)) \subset \mathrm{Dom}(F)$, for all $z \in D(z_0, r)$. Similarly we shall say that a solution $u_0(z)$ of the ODE, defined for $z \in \mathrm{Dom}(u_0) \subset \mathbb{C}$, *satisfies a Liouvillian relation* if there is a Liouvillian function $F(x_1, x_2, x_3)$ such that $\{(z, u(z), u'(z)) \in \mathbb{C}^3, z \in \mathrm{Dom}(u)\} \cap \mathrm{Dom}(F) \neq \emptyset$ and $F(z, u_0(z), u_0'(z)) = 0$ in some dense open subset of $\mathrm{Dom}(u_0)$.

Let us recall a couple of classical results:

Theorem 6.2 (Singer, [24])**.** *Assume that the polynomial first order ODE* $\frac{dx}{dz} = P(x, y)$, $\frac{dy}{dz} = Q(x, y)$ *admits a Liouvillian first integral. Then there are rational* *functions* $U(x,y)$, $V(x,y)$ *such that* $\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}$ *and the differential form* $Q(x,y)dx - P(x,y)dy$ *admits the integrating factor* $R(x,y) = \exp\big[\int_{(x_0,y_0)}^{(x,y)} U(x,y)dx + V(x,y)dy\big]$.

Theorem 6.3 (Rosenlicht [20], Singer [24])**.** *Let* $p(z), q(z)$ *be Liouvillian functions and* $L(y) = y'' + p(z)y' + q(z)y$. *If* $L(y) = 0$ *has a Liouvillian first integral then all solutions are Liouvillian. If* $L(y) = 0$ *has a nontrivial Liouvillian solution, then this equation has a Liouvillian first integral.*

Example 6.4 (Bernoulli ODEs)**.** Recall that a *Bernoulli differential equation of power* 1 is one of the form $\frac{dy}{dx} = \frac{a_1(x)y + a_2(x)y^2}{p(x)}$. If we perform a change of variables as $(x,y) \mapsto (x, y^{1/k})$ then we obtain an equation of the form $\frac{dy}{dx} = \frac{y^{k+1}a_2(x) + ya_1(x)}{kp(x)}$ which will be called a Bernoulli equation of power k.

We prove the existence of a first integral for $\Omega = 0$ of Liouvillian type. First we observe that $\Omega = 0$ can be given by

$$\frac{\Omega}{p(x)y^{k+1}} = k\frac{dy}{y^{k+1}} - \left(\frac{a_2(x)}{p(x)} + \frac{a_1(x)}{p(x)y^k}\right)dx = 0.$$

Let now $f(x)$ be such that $\frac{f'(x)}{f(x)} = -\frac{a_1(x)}{p(x)}$ and let $g(x)$ be such that $g'(x) = -\frac{a_2(x)}{p(x)f(x)}$. Then $\Omega = 0$ can be given by

$$k\frac{dy}{y^{k+1}} + f(x)g'(x)dx + \frac{f'(x)}{y^k f(x)}dx = 0.$$

Therefore $F(x,y) = g(x) - \frac{1}{f(x)y^k}$ defines a first integral for $\Omega = 0$ which is clearly of Liouvillian type.

Before proving Theorem E we shall need a lemma:

Lemma 6.5. *Let* $\frac{dy}{dx} = \frac{c(x)y^2 + b(x)y + a(x)}{a(x)}$ *be a rational Riccati ODE, where* $a(x), b(x)$, $c(x)$ *are complex polynomials. Assume that there is a Liouvillian first integral. Then we have the following possibilities:*

1. *The equation is linear of the form* $a(x)y' - b(x)y = a(x)$.
2. *Up to a rational change of coordinates of the form* $Y = y - A(x)/B(x)$, *the equation is a Bernoulli equation* $\frac{dY}{dx} = \frac{\tilde{c}(x)Y^2 + \tilde{b}(x)Y}{\tilde{a}(x)}$.

Proof. Let $\Omega = [c(x)y^2 + b(x)y + a(x)]dx - a(x)dy$. The ODE is equivalent to $\Omega = 0$. According to Singer [24] (Theorem 6.2 above) there is a rational 1-form $\eta = U(x,y)dx + V(x,y)dy$ such that $d\eta = (\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x})dy \wedge dx = 0$ and $\exp(\int \eta)$ is an integrating factor for Ω. This means that $d(\Omega/\exp(\int\eta)) = 0$ and therefore $d\Omega = \eta \wedge \Omega$.

We shall split our argumentation in two cases according to the existence of an invariant *algebraic* curve other than one of the vertical lines:

Case 1. $\Omega = 0$ admits some invariant *algebraic* curve $C \subset \mathbb{C}^2$ which is not a vertical line $x = c \in \mathbb{C}$. First we let us prove that this curve is a graph of a rational function $y = R(x)$. We may choose an irreducible polynomial $f(x,y)$ such that $f(x,y) = 0$ describes this non-vertical algebraic solution. Now we observe that the leaves of the Riccati foliation defined by $\Omega = 0$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ are, except for those contained in the invariant vertical fibers, all transverse to the vertical fibers $\{x\} \times \mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Thus we conclude that the partial derivative $f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$ never vanishes for each x such that $a(x) \neq 0$. Since $f(x,y)$ is a polynomial the partial derivative $f_y(x,y)$ is also a polynomial. Take a value of x such that $y \mapsto f_y(x,y)$ has no zero. Since this map is a polynomial, it must be constant say $f_y(x) = B(x)$. In particular the second partial derivative $f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y)$ is identically zero for all (x,y) except maybe for a finite number of values of x. Then the polynomial f_{yy} is identically zero and because f is a polynomial we must have $f(x,y) = A(x) + B(x)y$ for some polynomial $A(x)$. This shows that the non-vertical solution is a graph of the form $y(x) = \frac{A(x)}{B(x)}$.

Now, as a classical procedure we may perform a change of variables as follows: write $Y = y - y(x)$ to obtain:

$$\frac{dY}{dx} = \frac{c(x)Y^2 + (2c(x)y(x) + b(x))Y}{a(x)} = \frac{B(x)c(x)Y^2 + (b(x)B(x) + 2c(x)A(x))Y}{a(x)B(x)}.$$

This is a Bernoulli type equation (see [23] Example 4.2 page 771).

Case 2. There is no invariant *algebraic* curve other than the vertical lines. Denote by $(\eta)_\infty$ the polar divisor of η in $\mathbb{C}^2$. Let us prove that this polar divisor is a finite union of invariant vertical lines: Firstly notice that the polar set of η is invariant by $\Omega = 0$ (see [3] or [23] Lemma 5.5 page 772). By the hypothesis we then conclude that $(\eta)_\infty \subseteq \{x \in \mathbb{C}, a(x) = 0\} \times \mathbb{C}$ is a union of vertical invariant lines.

Now, from the *Integration lemma* in [22, Page 174; 5, Page 5] we have

$$\eta = \sum_{j=1}^{r} \lambda_j \frac{dx}{x - x_j} + d\left(\frac{g(x,y)}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}}\right)$$

where $g(x,y)$ is a polynomial function, $x_j \in \mathbb{C}$, $\lambda_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ is the order of the poles of Ω in the component $(x = x_j)$ of the polar set for every $j = 1, ..., r$. Now we use the equation $d\Omega = \eta \wedge \Omega$ to obtain

$$-[a'(x) + 2yc(x) + b(x)]dx \wedge dy = -a(x)\sum_{j=1}^{r} \frac{\lambda_j}{x - x_j} dx \wedge dy + d\left(\frac{g(x,y)}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}}\right) \wedge \Omega$$

where

$$d\left(\frac{g(x,y)}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}}\right) \wedge \Omega = \frac{dg}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}} \wedge \Omega - ga(x)d\left(\frac{1}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}}\right) \wedge dy.$$

Notice that

$$\frac{dg}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}} \wedge \Omega = \frac{g_y[c(x)y^2 + b(x)y + a(x)]}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}} dx \wedge dy - \frac{g_x a(x)}{\prod_{j=1}^{r}(x - x_j)^{n_j - 1}} dx \wedge dy$$

Notice that the left side is $-[a'(x) + 2yc(x) + b(x)]$ has no term in y^2 so that we must have $g_y c(x) = 0$ in the right side. If $g_y = 0$ then $g = g(x)$ and from the left side we must have $c(x) = 0$. This shows that we must always have $c(x) = 0$. This implies that the original equation is a linear homogeneous equation of the form $\frac{dy}{dx} = \frac{b(x)y + a(x)}{a(x)}$ which can be written as $a(x)y' - b(x)y = a(x)$. □

Proof of Theorem E. Let us prove the second part, ie., the equivalence. We assume that $L[u](z) = a(z)u'' + b(z)u' + c(z)u = 0$ admits a Liouvillian first integral. We consider the change of coordinates $t = \frac{u'}{u}$ which gives the following Riccati model $\mathcal{R} : \frac{dt}{dz} = -\frac{a(z)t^2+b(z)t+c(z)}{a(z)}$. We claim:

Claim 6.6. *The Riccati model $\mathcal{R}$ also admits a Liouvillian first integral.*

Proof. By hypothesis the ODE $L[u] = a(z)u'' + b(z)u' + c(z)u = 0$ has a Liouvillian first integral. By the Corollary in [24] page 674 all solutions of $L[u] = 0$ are Liouvillian. This implies, by Theorem 1 in [24] page 674, that $\mathcal{R}$ admits a Liouvillian first integral. □

From the Lemma 6.5 have then two possibilities:

Case 1. There is a solution $\gamma(z) = A(z)/B(z)$ for the Riccati equation, where A, B are polynomials. In this case there is a rational change of coordinates of the form $T = t - A(z)/B(z)$ that takes the Riccati model $\mathcal{R}$ into a Bernoulli foliation $\mathcal{B}$: $\frac{dT}{dz} = -T^2 - \tilde{b}(z)T$ where $\tilde{b}(z) = \frac{b(z)}{a(z)} + 2\gamma(z)$. In this case the original ODE $L[u](z) = 0$ becomes $\tilde{L}[U](z) = U'' + \tilde{b}(z)U' = 0$ after a rational change of coordinates

$$U = \exp(\int^z T d\eta) = u \exp(-\int^z \gamma(\eta)d\eta)$$

where $\gamma(z) = \frac{A(z)}{B(z)}$. This shows that we have Liouvillian solutions to the ODE which are given by

$$u(z) = \exp\big(\int^z \gamma(\eta)d\eta\big)\big[\ell + k\int^z \exp\big(-\int^\eta \frac{b(\xi)}{a(\xi)}d\xi\big)\cdot \exp\big(\int^\eta -2\gamma(\xi)d\xi\big)d\eta\big]$$

for constants $k, \ell \in \mathbb{C}$.

Case 2. We have $c(z) = 0$ and therefore the original ODE is of the form $L[u] = a(z)u'' + b(z)u' = 0$. Thus the solutions are Liouvillian given by $u(z) = k_1 + k_2\int^z \exp(-\int^\eta \frac{b(\xi)}{a(\xi)}d\xi)d\eta$ for constants $k_1, k_2 \in \mathbb{C}$. □

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