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THE GLOBAL SUPERSONIC FLOW WITH VACUUM STATE IN A 2D CONVEX DUCT

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Abstract. This paper concerns the motion of the supersonic potential flow in a two-dimensional expanding duct. In the case that two Riemann invariants are both monotonically increasing along the inlet, which means the gases are spread at the inlet, we obtain the global solution by solving the problem in those inner and border regions divided by two characteristics in (x, y)-plane, and the vacuum will appear in some finite place adjacent to the boundary of the duct. In addition, we point out that the vacuum here is not the so-called physical vacuum. On the other hand, for the case that at least one Riemann invariant is strictly monotonic decreasing along some part of the inlet, which means the gases have some local squeezed properties at the inlet, we show that the C^1 solution to the problem will blow up at some finite location in the non-convex duct.

1. **Introduction.** In this paper, we are concerned with the motion of the supersonic potential flow in a two-dimensional expanding duct denoted by Ω (See Fig.1), which is bounded by the lower wall $\Gamma_{\text{low}} = \{(x, y)|y = -f(x), 0 \le x < +\infty\}$, the upper wall $\Gamma_{\text{up}} = \{(x, y)|y = f(x), 0 \le x < +\infty\}$ and the inlet $\Gamma_{\text{in}} = \{(x, y)|x = \varphi(y), y \in [-f(0), f(0)]\}$. Here we assume that $f(x) \in C^2([0, +\infty))$ satisfies

$$f(0) > 0, \quad f'(x) > 0, \quad f''(x) \ge 0, \quad f'_\infty = \lim_{x\to+\infty} f'(x) \quad \text{exists}, \tag{1}$$

and $\varphi(y) \in C^2([-f(0), f(0)])$ is an even function which satisfies

$$\varphi(-y) = \varphi(y), \quad \varphi(\pm f(0)) = 0, \quad \varphi'(\pm f(0)) = \mp f'(0), \quad \varphi'' \le 0. \tag{2}$$

At the inlet, the flow velocity is assumed to be along the normal direction of the inlet and its speed is given by $q_0(y) \in C^1[-f(0), f(0)]$. Moreover, we require that $q_0(y)$ satisfies

$$c_* < c_1 < q_0(y) < c_2 < \hat{q}, \tag{3}$$

where c_1, c_2 are positive constants, c_* is the critical speed of the flow and $\hat{q}$ is the limit speed of the flow. This means the coming flow is supersonic and does not meet vacuum at the inlet. On the two walls, the flow satisfies the solid wall condition, namely

$$\frac{v}{u} = \tan\theta = \pm f'(x), \quad \text{on } y = \pm f(x), \tag{4}$$

where $\theta = \arctan\frac{v}{u}$ is the angle of the velocity inclination to the $x-$axis.

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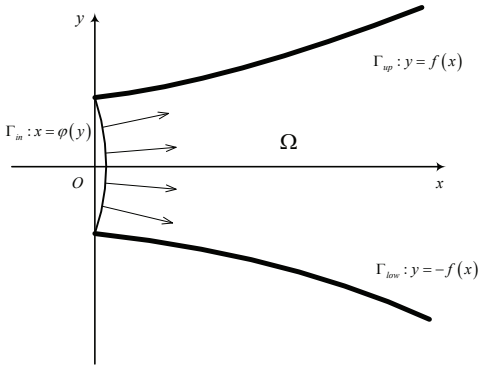

Fig.1 Supersonic flow in 2D convex duct

The supersonic flow in the duct is described by the 2-D steady isentropic compressible Euler equations:

$$\begin{cases} \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) = 0, \\ \dfrac{\partial}{\partial x}(\rho u^2 + p) + \dfrac{\partial}{\partial y}(\rho uv) = 0, \\ \dfrac{\partial}{\partial x}(\rho uv) + \dfrac{\partial}{\partial y}(\rho v^2 + p) = 0, \end{cases} \tag{5}$$

where $(u, v), p$, and ρ stand for the velocity, pressure, and density of the flow. For the polytropic gas, the state equation is given by $p = A\rho^\gamma$, where A is a positive constant and $\gamma > 1$ is the adiabatic exponent. In addition, the gases are assumed as irrotational. Thus the components (u, v) of the velocity satisfy

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \tag{6}$$

then for polytropic gas, the following Bernoulli law holds

$$\frac{1}{2}q^2 + \frac{c^2}{\gamma - 1} = \frac{1}{2}\hat{q}^2, \tag{7}$$

where $q = \sqrt{u^2 + v^2}$ is the speed of the flow, $c = \sqrt{p'(\rho)}$ is the sound speed, and $\hat{q}$ is the limit speed, which is an identical constant over the whole flow. Therefore the density ρ can be expressed by the function of q and the system (5) can be reduced to a 2×2 system with variables (u, v)

$$\begin{cases} (c^2 - u^2)\dfrac{\partial u}{\partial x} - uv\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) + (c^2 - v^2)\dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} = 0, \end{cases} \tag{8}$$

Such problem (8) with initial data (3) and boundary condition (4) has already been studied by Wang and Xin in [25]. In their paper, by introducing the velocity potential φ and stream function ψ, in terms of hodograph transformation, they proved the global solution with vacuum in the phase space (φ, ψ). Inspired by their paper, in order to understand this problem more intuitively, we establish the global existence of such problem in (x, y)-plane, and vacuum will appear in some finite

place adjacent to the boundary of duct. In addition, we prove that the vacuum here is not the so-called physical vacuum.

For better stating our results, we firstly give the description of domains Ω_{non} and Ω_{vac}, which are the domain before the vacuum appearance and the domain adjoining the vacuum respectively (see Fig.2 below). Let M and N stand for the first vacuum point on $\Gamma_{\rm up}$ and $\Gamma_{\rm low}$, where $x_M = x_N$. Choose a curve whose normal direction coincides with the velocity of the incoming flow at $x = x_M$. Denote it as $x = \psi(y)$ with $\psi(\pm f(x_M)) = x_M$. Set Ω_{non} as the open region bounded by $\Gamma_{\rm in}$, $\Gamma_{\rm up}$, $\Gamma_{\rm low}$ and $x = \psi(y)$. Denote $l_{\rm up}$ and $l_{\rm low}$ as the vacuum boundaries adjacent to $\Gamma_{\rm up}$ and $\Gamma_{\rm low}$ respectively. Set Ω_{vac} as the open region bounded by $x = \psi(y)$, $l_{\rm up}$ and $l_{\rm low}$.

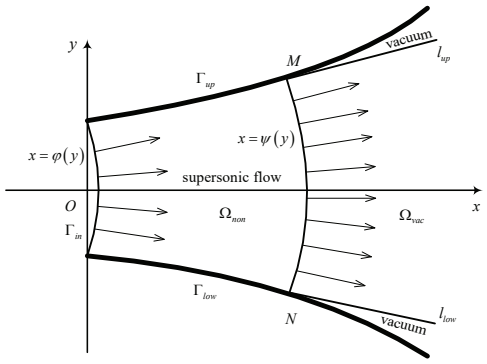

Fig.2 A global smooth solution with vacuum in 2D convex duct

Our main results in the paper are:

Theorem 1.1. *Assume that $f \in C^2([0,+\infty))$ satisfies (1), $\varphi \in C^2([-f(0), f(0)])$ satisfies (2) and $q_0 \in C^1[-f(0), f(0)]$ satisfies (3). If q_0 satisfies*

$$|q_0'(y)| \le -\frac{\varphi''(y)}{1+(\varphi'(y))^2}\frac{q_0 c}{\sqrt{q_0^2 - c^2}} \ \text{on}\ \Gamma_{in}, \tag{9}$$

then two alternative cases will happen in the duct, one contains vacuum, and the other does not. More concretely, the two cases are as follows:

(i) When vacuum actually appears in the duct, there exists a global solution $(u,v) \in C(\overline{\Omega_{non} \cup \Omega_{vac}}) \bigcap C^1(\overline{\Omega_{non} \cup \Omega_{vac}} \setminus \{l_{up}, l_{low}\})$ to the problem (8) with (3) and (4). Moreover,

$$\partial_{\vec{n}} c^2 = 0, \quad on \quad l_{up} \cup l_{low} \setminus \{M, N\}, \tag{10}$$

here $\partial_{\vec{n}}$ stands for the normal derivative of vacuum boundary. This means the vacuum here is not the physical vacuum.

(ii) If vacuum is absent in the duct, the problem (8) with (3) and (4) has a global solution $(u,v) \in C^1(\bar{\Omega})$.

Theorem 1.2. *Assume that $f(x) \in C^2([0,+\infty))$ satisfies:*

$$f(0) > 0, \quad f'(0) > 0, \quad f'_\infty \ge 0, \quad f''(x) \le 0. \tag{11}$$

If

$$|q_0'(y)| > -\frac{\varphi''(y)}{1+(\varphi'(y))^2}\frac{q_0 c}{\sqrt{q_0^2 - c^2}} \ \text{on}\ \Gamma_{in}, \tag{12}$$

then the C^1 solution to the problem (8) with (3) and (4) will blow up at some finite location in the duct.

Remark 1. The condition (9) is equivalent to that the two Riemann invariants are both monotonically increasing along Γ_{in} (see Lemma 2.1). It means the gases are spread at the inlet. In fact, this condition is very important to get the global existence of solution. In order to understand this, we have a glimpse of the Cauchy problem for the Burgers equation

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{13}$$

As well known, if the initial data satisfies

$$\min_{x\in\mathbb{R}} u_0'(x) < 0, \tag{14}$$

then the solution $u(t,x)$ must blow up in finite time, and shock will be formed. On the other hand, by introducing the Riemann invariants $R_\pm = \theta \pm F(q)$ with $F(q) = \int \frac{\sqrt{q^2-c^2}}{qc} dq$, the equation (8) is actually equivalent to (20) (see Section 2). Thus, we consider the global solution of following Cauchy problem for (20) with initial data

$$\begin{cases} (\partial_x + \lambda_-\partial_y)R_+ = 0, \\ (\partial_x + \lambda_+\partial_y)R_- = 0, \\ R_\pm(0,y) = R_\pm^0(y). \end{cases} \tag{15}$$

By the results of [16] and [30], there exists a global solution of the Cauchy problem (15) if and only if the two Riemann invariants $R_\pm^0(y)$ are both monotonically increasing. Therefore, it seems that posing the condition (9) is reasonable to obtain the global solution. As a contrast, in Theorem 1.2, (12) means that at least one Riemann invariant is strictly monotonic decreasing along some part of the inlet. Thus, the gases have some local squeezed properties at the inlet. Similar to the case in Burgers equation, we prove that the C^1 smooth solution to the problem must blow up at some finite location in the straight duct. Actually, for different initial data and structures of boundary, more totally different motions of gases can be found in [7].

Remark 2. By the effect of expanding duct, as far as we know, it is hard to give a necessary and sufficient condition to ensure that vacuum must form at finite location in the duct. In Proposition 4.1, we will give a sufficient condition such that the vacuum will appear in finite place.

Remark 3. For the M-D compressible Euler equations, if the gases are assumed irrotational, by introducing the velocity potential $\nabla\varphi = \vec{u}$, the Euler system can be changed into a quasi-linear hyperbolic equation (i.e. potential flow equation). It is easy to check that this potential flow equation does not fulfill the "null condition" put forward in [1],[2] and [14]. Thus, in terms of the extensive results of [1],[2] and so on, the classical solution will blow up. On the other hand, if the rotation is involved, in the general case, due to the possible compression of gases, the smooth solutions will blow up and the shock is formed (see [6], [20], [21] and [23]). Meanwhile, if the gases are suitably expanded or expanded into the vacuum, the global solutions can exist(see [4], [5], [10], [11], [22], [24] and [27]-[29]).

Remark 4. If the initial data contains a vacuum, especially for physical vacuum, the local well-posedness results of the compressible Euler equations have been studied in [8], [9], [12], [13], [18] and so on. But in general, such a local classical solution will blow up in finite time as shown in [3], [26] and the references therein.

Remark 5. For the case that Riemann invariants are both constants on Γ_{in} which is a straight segment vertical to the velocity of the incoming flow, the problem has been solved by Chen and Qu in [5]. Here we extend their work to a more general case.

Remark 6. The symmetry of $\varphi(y)$ in (2) and the duct with respect to $x-$axis is not essential, by the same analysis as in this paper but more tedious computation, the results can be extended to the non-symmetric case. So for the readers' convenience, we only consider the symmetric case.

The paper is organized as follows. In Section 2, we give some basic structures of the steady plane isentropic flow and discuss the monotonicity of two Riemann invariants $R_{\pm}$ along the inlet. In Section 3, we use the method of the characteristics to divide the duct into several inner and border regions, and the problem is transformed into some Goursat problems in the corresponding inner regions and some boundary value problems in the corresponding border regions. Then the global C^1 solution is obtained before vacuum forms if $R_{\pm}$ are monotonically increasing along the inlet. Meanwhile, for the case that at least one of R_+ or R_- is strictly monotonic decreasing along some part of the inlet, we show that the C^1 solution to the problem will blow up at some finite location in the straight duct. In Section 4, we solve the problem when vacuum appears and obtain that the vacuum here is not the so-called physical vacuum. Combining the results in Section 3 and Section 4, we finally get Theorem 1.1 and Theorem 1.2. At last, We will give a sufficient condition concerning the geometric shape of the duct to ensure the formation of the vacuum.

2. Preliminaries.

In this section, we start with some basic structures of the steady plane isentropic flow, which can be characterized by the Euler system (5).

Firstly, equation (8) can be written as the matrix form

$$\begin{pmatrix} c^2-u^2 & -uv \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -uv & c^2-v^2 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{16}$$

The characteristic equation of (16) is

$$(c^2-u^2)\lambda^2 + 2uv\lambda + (c^2-v^2) = 0.$$

So for the supersonic flow, (8) is hyperbolic and we can get two eigenvalues

$$\lambda_+ = \frac{uv + c\sqrt{u^2+v^2-c^2}}{u^2-c^2}, \quad \lambda_- = \frac{uv - c\sqrt{u^2+v^2-c^2}}{u^2-c^2}. \tag{17}$$

Correspondingly, the two families of characteristics in the (x, y) plane are defined by

$$\frac{dy_{\pm}}{dx} = \lambda_{\pm}. \tag{18}$$

By standard computation, the Riemann invariants $R_{\pm}$ can be defined by

$$R_{\pm} = \theta \pm F(q), \tag{19}$$

where $F(q) = \int \frac{\sqrt{q^2-c^2}}{qc} dq$. Since the Jacobian $\frac{\partial(R_+,R_-)}{\partial(u,v)} = \frac{2\sqrt{q^2-c^2}}{q^2 c} > 0$, then the system (8) without vacuum is equivalent to the following equations

$$\begin{cases} (\partial_x + \lambda_- \partial_y) R_+ = 0, \\ (\partial_x + \lambda_+ \partial_y) R_- = 0, \end{cases} \tag{20}$$

which can also be written as

$$\partial_\pm R_\mp = 0 \tag{21}$$

if we set $\partial_\pm$ as the differential operators $\frac{\partial}{\partial x} + \lambda_\pm \frac{\partial}{\partial y}$.

Let A be the Mach angle defined by $\sin A = c/q$, then we have that

$$\lambda_\pm = \tan(\theta \pm A). \tag{22}$$

Under a proper coordinate rotation, the eigenvalues $\lambda_\pm$ can always be locally bounded, and the system (21) in the new coordinate is also invariant.

From now on, we will discuss the monotonicity of the Riemann invariants $R_\pm$ along the inlet, which will play a key role in the following analysis. Note that $\theta = -\arctan \varphi'(y)$ on Γ_{in}, then $R_\pm = -\arctan \varphi'(y) \pm \int_{c_*}^{q_0} \frac{\sqrt{q^2-c^2}}{qc} dq$ on Γ_{in}. Some properties can be obtained after direct computation.

Lemma 2.1. *For $R_\pm$ defined on Γ_{in} above, we have that $R_\pm$ are both monotonically increasing along Γ_{in} if and only if the initial speed q_0 satisfies*

$$|q_0'(y)| \le -\frac{\varphi''(y)}{1+(\varphi'(y))^2} \frac{q_0 c}{\sqrt{q_0^2 - c^2}} \ \text{on}\ \Gamma_{in}. \tag{23}$$

Proof. Since $R_\pm = -\arctan \varphi'(y) \pm \int_{c_*}^{q_0} \frac{\sqrt{q^2-c^2}}{qc} dq$ on Γ_{in}, then direct computation yields

$$R_\pm'(y) = -\frac{\varphi''(y)}{1+(\varphi'(y))^2} \pm \frac{\sqrt{q_0^2-c^2}}{q_0 c} q_0'(y) \quad \text{on} \quad \Gamma_{\text{in}}. \tag{24}$$

Thus $R_\pm$ are monotonically increasing along Γ_{in} if and only if (24) are nonnegative, which is equivalent to (23). □

Due to Lemma 2.1, the following analysis will be focused on these two cases.

Case I. $R_\pm$ are both monotonically increasing on Γ_{in};

Case II. At least one of R_+, R_- is strictly monotonic decreasing along some part of Γ_{in}.

The next lemma shows the partial derivatives $\frac{\partial \lambda_+}{\partial R_-}$ and $\frac{\partial \lambda_-}{\partial R_+}$ are positive , which will be often used in the following analysis.

Lemma 2.2. *As the q, c, θ, and A defined above, for the supersonic flow, we have*

$$\frac{\partial \lambda_+}{\partial R_-} > 0 \quad \textit{and} \quad \frac{\partial \lambda_-}{\partial R_+} > 0. \tag{25}$$

Proof. By using (22) and the chain rule, it follows from direct computation that

$$\frac{\partial \lambda_+}{\partial R_-} = \frac{(\gamma+1)q^2}{4(q^2-c^2)} \sec^2(\theta + A), \ \frac{\partial \lambda_-}{\partial R_+} = \frac{(\gamma+1)q^2}{4(q^2-c^2)} \sec^2(\theta - A), \tag{26}$$

thus (25) holds true for the supersonic flow. □

3. **The C^1 solution before vacuum formation.** In this section, the existence of the C^1 solution to the problem before vacuum formation will be proved for case I. We shall use characteristics to divide the whole region into several inner and border regions, where solutions are constructed successively by the method of characteristics. As illustrated in Fig.3, we assume that the C_- characteristic issuing from point A_1 intersects the lower wall at point B_2 and the C_+ characteristic issuing from point B_1 intersects the upper wall at point A_2. These two characteristics intersect at point C_1. Denote the inner region surrounded by A_1B_1, A_1C_1, B_1C_1 as region D_1, which is the domain of determination of A_1B_1.

Next we solve the solution in the border region $D_2(D_2'$, resp.)bounded by A_1C_1, $A_1A_2, A_2C_1(B_1C_1, B_1B_2, B_2C_1,$ resp.). Note that R_- (R_+, resp.) has already been defined in region $D_2(D_2'$, resp.). By the fixed wall condition $R_+ + R_- = 2\arctan(\pm f'(x))$, we can know the value of R_+ (R_-, resp.) in region $D_2(D_2'$, resp.).

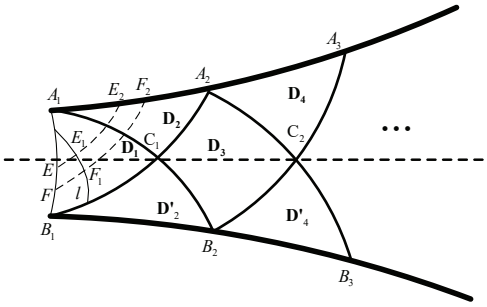

Fig.3 Inner regions and border regions

Let the C_- characteristic issuing from A_2 and the C_+ characteristic issuing from B_2 intersect at C_2. Define the inner region D_3 with boundaries A_2C_1, B_2C_1, A_2C_2 and B_2C_2. If the C_- characteristic issuing from A_2 approaches the lower wall at B_3 and the C_+ characteristic issuing from B_2 approaches the upper wall at A_3, we can get two border regions $D_4(D_4'$, resp.) with boundaries A_2A_3, A_2C_2, A_3C_2 (B_2B_3, B_2C_2, B_3C_2, resp.). Repeat this process until the vacuum forms. The inner regions D_1, D_3 and the border regions $D_2(D_2'$, resp.), $D_4(D_4'$, resp.) are mainly considered in this section.

Lemma 3.1. *If (9) holds true, then for the C^1 solution to the problem we have $\partial_+R_+ \geq 0, \partial_-R_- \leq 0$ in the duct before vacuum forms.*

Proof. We will prove the conclusion between these inner and border regions separately.

Step 1. Inner region D_1.

For any fixed two points E, F on Γ_{in} and any C_- characteristic l in region D_1, suppose the C_+ characteristics issuing from E, F intersect l at E_1, F_1 and intersect A_1A_2 at E_2, F_2 respectively. Since R_- is constant along the C_+ characteristic and R_- is monotonically increasing along Γ_{in}, then the value of R_- at E_1 is no less than the value of R_- at F_1, which implies

$$\partial_-R_- = \frac{d}{dx}(R_-(x, y_-(x))) \leq 0 \tag{27}$$

along l in region D_1. Similarly, $\partial_+ R_+$ is nonnegative in region D_1.

Step 2. Border region D_2.

Next we consider the solution in border region D_2 bounded by A_1C_1, A_1A_2, A_2C_1. From step 1, we know that R_- is monotonically decreasing along A_1C_1. By the same argument in region D_1, we can get that $\partial_- R_- \leq 0$ in region D_2. Since the value of R_- at E_2 is no less than the value of R_- at F_2, which means that R_- is monotonically decreasing along A_1A_2. Due to the solid wall condition

$$R_+ + R_- = 2\arctan f'(x)\ ,\ f''(x) \geq 0 \tag{28}$$

on A_1A_2, R_+ is monotonically increasing along A_1A_2. Therefore we can get $\partial_+ R_+ \geq 0$ in region D_2 by the same argument in region D_1. By analogous method, we also have $\partial_+ R_+ \geq 0, \partial_- R_- \leq 0$ in region D_2'.

Repeating this process in inner region $D_{2i+1}(i \geq 1)$ and border region D_{2i+2} (D_{2i+2}', resp)$(i \geq 1)$ until the vacuum forms, thus we prove Lemma 3.1. □

We can ensure that the hyperbolic direction is always in the x-direction by a proper rotation of coordinates if necessary. Without loss of generality, we may assume that $u^2 > c^2$, which implies that $\lambda_+ > \lambda_-$.

Lemma 3.2. *If (9) holds true, then we have $\partial_y R_\pm \geq 0$ in the duct before vacuum forms.*

Proof. In terms of the definition of $\partial_\pm$, one has

$$\partial_x = \frac{\lambda_+\partial_- - \lambda_-\partial_+}{\lambda_+ - \lambda_-}\ ,\ \partial_y = \frac{\partial_+ - \partial_-}{\lambda_+ - \lambda_-}. \tag{29}$$

Thus we have

$$\partial_y R_+ = \frac{\partial_+ R_+}{\lambda_+ - \lambda_-}\ ,\ \partial_y R_- = \frac{-\partial_- R_-}{\lambda_+ - \lambda_-}. \tag{30}$$

Combining this with Lemma 3.1 yields the result. □

Corollary 1. *Under the assumption that $u^2 > c^2$, by (21) and (29), we have that*

$$|\partial_+ R_+| = \frac{\lambda_+ - \lambda_-}{\sqrt{\lambda_-^2 + 1}}|\nabla R_+| \quad \text{and} \quad |\partial_- R_-| = \frac{\lambda_+ - \lambda_-}{\sqrt{\lambda_+^2 + 1}}|\nabla R_-|. \tag{31}$$

Lemma 3.3. *Let $\partial_l = u\partial_x + v\partial_y$ stand for the derivative along the streamline, under the assumption that $u^2 > c^2$, then the speed of the flow is monotonically increasing along the streamline, that is to say $\partial_l q \geq 0$.*

Proof. It follows from (29) that $\partial_l = u\partial_x + v\partial_y$ can be expressed as

$$\partial_l = \frac{v - u\lambda_-}{\lambda_+ - \lambda_-}\partial_+ + \frac{u\lambda_+ - v}{\lambda_+ - \lambda_-}\partial_-. \tag{32}$$

which implies that

$$\partial_l q = \frac{v - \lambda_- u}{\lambda_+ - \lambda_-}\partial_+ q + \frac{\lambda_+ u - v}{\lambda_+ - \lambda_-}\partial_- q = \frac{u(\frac{v}{u} - \lambda_-)}{\lambda_+ - \lambda_-}\partial_+ q + \frac{u(\lambda_+ - \frac{v}{u})}{\lambda_+ - \lambda_-}\partial_- q. \tag{33}$$

Since $u^2 > c^2$, we know that $\theta \pm A$ is away from $\pm\frac{\pi}{2}$. Thus the coefficients of the right hand part of (33) are positive. So it remains to determine the symbol of $\partial_\pm q$. Due to (20) and Lemma 3.1, one has that $\partial_+ R_+ = \partial_+(R_+ - R_-) = 2F'(q)\partial_+ q \geq 0$, which immediately indicates that $\partial_+ q \geq 0$. Similarly, one can get that $\partial_- q \geq 0$. Then the proof of Lemma 3.3 is completed. □

Corollary 2. *Combining Lemma 3.3 and the Bernoulli law yields that the sound speed is monotonically decreasing along the streamline, that is to say* $\partial_t c \le 0$.

We now focus on the solution in the inner regions D_1 and $D_{2i-1}(i \ge 2)$ bounded by $A_iC_{i-1}, B_iC_{i-1}, A_iC_i$ and B_iC_i. We only need to discuss the region $D_{2i-1}(i \ge 2)$, since same method can be used in region D_1.

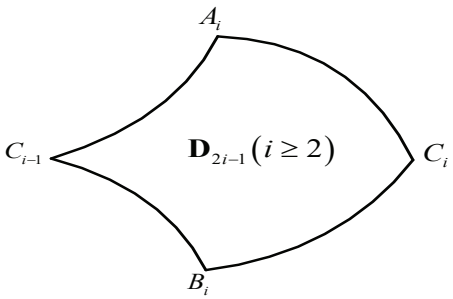

Fig.4 Goursat problem in inner region

Set $R_+(x,y) = W_+(x,y)$ on A_iC_{i-1} and $R_-(x,y) = W_-(x,y)$ on B_iC_{i-1}. Thus the solution in the region D_{2i-1} should be determined by solving a Goursat problem, which is given by

$$\begin{cases} (\partial_x + \lambda_-\partial_y)R_+ = 0, \\ (\partial_x + \lambda_+\partial_y)R_- = 0, \end{cases} \quad \text{in } D_{2i-1}$$
$$\begin{cases} R_+(x,y) = W_+(x,y) & \text{on } A_iC_{i-1}, \\ R_-(x,y) = W_-(x,y) & \text{on } B_iC_{i-1}. \end{cases} \tag{34}$$

By standard iteration method, the local existence of the problem (34) can be achieved, one can also see [4] and [15] for more details. In order to get the global solution to the problem (34) in the region $D_{2i-1}(i \ge 2)$, some prior estimates about $R_\pm$ and $\nabla R_\pm$ are needed.

Lemma 3.4. *If* (9) *holds true, then* $|\partial_+ R_+|$ *is monotonically decreasing along* C_- *characteristics and* $|\partial_- R_-|$ *is monotonically decreasing along* C_+ *characteristics before vacuum formation.*

Proof. It comes from the definition of $\partial_\pm$ that

$$\partial_-\partial_+ R_+ = \partial_-\partial_+ R_+ - \partial_+\partial_- R_+ = (\partial_-\lambda_+ - \partial_+\lambda_-)\partial_y R_+. \tag{35}$$

Due to

$$\partial_-\lambda_+ - \partial_+\lambda_- = \frac{\partial\lambda_+}{\partial R_-}\partial_- R_- - \frac{\partial\lambda_-}{\partial R_+}\partial_+ R_+ \le 0 \tag{36}$$

and $\partial_y R_+ \ge 0$, then $\partial_-\partial_+ R_+ \le 0$, which implies $\partial_+ R_+$ is monotonically decreasing along C_- characteristics. By the same argument, we can get that $\partial_- R_-$ is monotonically increasing along C_+ characteristics.

In terms of Lemma 3.1, one has $\partial_+ R_+ \ge 0$ and $\partial_- R_- \le 0$. This means that $|\partial_+ R_+|$ is monotonically decreasing along C_- characteristics and $|\partial_- R_-|$ is monotonically decreasing along C_+ characteristics before vacuum formation. Thus we complete the proof. □

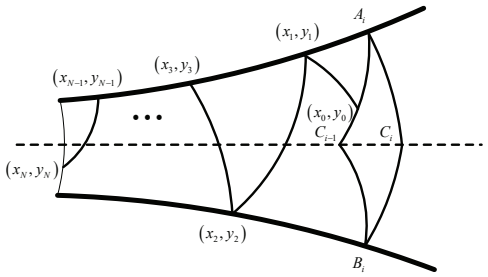

Fig.5 The case that $y(x_{N-1}) = f(x_{N-1})$

Lemma 3.5. *Suppose that* $R_\pm(x,y)$ *are the classical solutions to the problem* (34)*, then* $R_\pm$ *and* $\partial_\pm R_\pm$ *are uniformly bounded in the region* $D_{2i-1}(i \geq 2)$ *if* (9) *holds true.*

Proof. Firstly, we give the estimates of $|R_\pm|$.

For any point (x,y) in $D_{2i-1}(i \geq 2)$, by the method of characteristics, we have that

$$|R_+(x,y)| \leq \|W_+\|_{C(A_iC_{i-1})}\ ,\ |R_-(x,y)| \leq \|W_-\|_{C(B_iC_{i-1})} \tag{37}$$

in the region $D_{2i-1}(i \geq 2)$. To be more specific, we shall point out that $\|W_+(x,y)\|_{C(A_iC_{i-1})}$ and $\|W_-(x,y)\|_{C(B_iC_{i-1})}$ listed above only depend on $\|f\|_{C^1}$ and $\|R_\pm\|_{C(\Gamma_{\rm in})}$.

Fix $(x_0,y_0) \in A_iC_{i-1}$, let $y = y_-^{(1)}(x)$ be the C_- characteristic from (x_0,y_0), which approaches either $\Gamma_{\rm up}$ or $\Gamma_{\rm in}$ at a point (x_1,y_1). If $y_1 = f(x_1)$, then there exists a C_+ characteristic $y = y_+^{(2)}(x)$ from (x_1,y_1), which approaches either $\Gamma_{\rm low}$ or $\Gamma_{\rm in}$ at a point (x_2,y_2). If $y_2 = -f(x_2)$, then there exists a C_- characteristic $y = y_-^{(3)}(x)$ from (x_2,y_2), which approaches either $\Gamma_{\rm up}$ or $\Gamma_{\rm in}$ at a point (x_3,y_3). Since the two eigenvalues $\lambda_\pm$ have the uniform upper or lower bound, then there exists a positive N such that

$$x_0 > x_1 > ... > x_{N-1} > x_N,\ x_N = \varphi(y_N). \tag{38}$$

If $y(x_{N-1}) = f(x_{N-1})$ (see Fig.5), then we get

$$\begin{aligned} W_+(x_0,y_0) &= 2\arctan f'(x_1) - R_-(x_1,y_1) = 2\arctan f'(x_1) - R_-(x_2,y_2) \\ &= 2\arctan f'(x_1) + 2\arctan f'(x_2) + R_+(x_2,y_2) \\ &= \sum_{i=1}^{N} 2\arctan f'(x_i) - R_-(x_N,y_N). \end{aligned} \tag{39}$$

Similarly, if $y(x_{N-1}) = -f(x_{N-1})$ (see Fig.6), then we get

$$W_+(x_0,y_0) = \sum_{i=1}^{N} 2\arctan f'(x_i) + R_+(x_N,y_N). \tag{40}$$

Combining (39) and (40) yields that

$$\|W_+\|_{C(A_iC_{i-1})} \leq 2N\arctan f'_\infty + \max\{\|R_+\|_{C(\Gamma_{\rm in})}, \|R_-\|_{C(\Gamma_{\rm in})}\}. \tag{41}$$

Analogously, the estimate of $\|W_-\|_{C(B_iC_{i-1})}$ can also be obtained as (41).

Next, we give the estimates of $|\partial_\pm R_\pm|$.

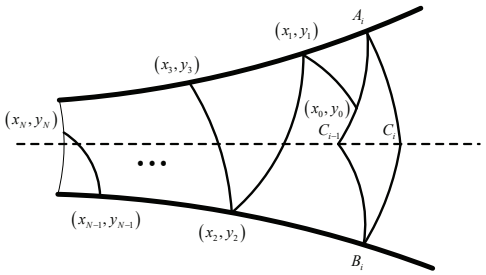

Fig.6 The case that $y(x_{N-1}) = -f(x_{N-1})$

It follows from Lemma 3.4 that

$$|\partial_+ R_+(x,y)| \le \|\partial_+ W_+\|_{C(A_iC_{i-1})}\ ,\ |\partial_- R_-(x,y)| \le \|\partial_- W_-\|_{C(B_iC_{i-1})}. \tag{42}$$

As the process shown in the estimates of $|R_\pm|$, we shall point out that $\|\partial_+ W_+\|_{C(A_iC_{i-1})}$ and $\|\partial_- W_-\|_{C(B_iC_{i-1})}$ listed above only depend on $\|f\|_{C^2}$ and $\|R_\pm\|_{C^1(\Gamma_{\rm in})}$. To this end, we need to derive the boundary condition of $\partial_\pm R_\pm$. It follows from (32) and the solid wall condition that

$$\frac{v-u\lambda_-}{\lambda_+-\lambda_-}\partial_+ R_+ + \frac{u\lambda_+ - v}{\lambda_+-\lambda_-}\partial_- R_- = \pm 2u\frac{f''(x)}{1+(f'(x))^2}, \quad \text{on } y = \pm f(x). \tag{43}$$

By the process of the reflection shown in the estimates of $|R_\pm|$ and Lemma 3.4, if $y(x_{N-1}) = f(x_{N-1})$, we can get

$$\begin{aligned}
\partial_+ W_+(x_0,y_0) &\le \partial_+ R_+(x_1,y_1)\\
&= \left(2u\frac{\lambda_+-\lambda_-}{v-u\lambda_-}\frac{f''}{1+(f')^2} - \frac{u\lambda_+-v}{v-u\lambda_-}\partial_- R_-\right)(x_1,y_1)\\
&\le \left(2u\frac{\lambda_+-\lambda_-}{v-u\lambda_-}\frac{f''}{1+(f')^2}\right)(x_1,y_1) - \left(\frac{u\lambda_+-v}{v-u\lambda_-}\right)(x_1,y_1)\\
&\times \partial_- R_-(x_2,y_2)\\
&= \left(2u\frac{\lambda_+-\lambda_-}{v-u\lambda_-}\frac{f''}{1+(f')^2}\right)(x_1,y_1) + \left(\frac{u\lambda_+-v}{v-u\lambda_-}\right)(x_1,y_1)\\
&\times \left(2u\frac{\lambda_+-\lambda_-}{u\lambda_+-v}\frac{f''}{1+(f')^2}\right)(x_2,y_2) + \left(\frac{u\lambda_+-v}{v-u\lambda_-}\right)(x_1,y_1)\\
&\times \left(\frac{v-u\lambda_-}{u\lambda_+-v}\right)(x_2,y_2)\times\partial_+ R_+(x_2,y_2)
\end{aligned} \tag{44}$$

Since $u^2 > c^2$, then the coefficients

$$2u\frac{f''}{1+(f')^2},\quad \frac{u\lambda_+-v}{v-u\lambda_-},\quad \frac{\lambda_+-\lambda_-}{v-u\lambda_-},\quad \frac{\lambda_+-\lambda_-}{u\lambda_+-v}$$

are uniformly bounded. Denote the maximum one of these bounds as the constant G, thus (44) can be expressed as

$$\begin{aligned}
\partial_+ W_+(x_0,y_0) &\le \partial_+ R_+(x_1,y_1) \le G^2 - G\partial_- R_-(x_1,y_1)\\
&\le G^2 - G\partial_- R_-(x_2,y_2) \le G^2 + G^3 + G^2\partial_+ R_+(x_2,y_2)\\
&\le \sum_{i=1}^{N} G^{i+1} + G^N\partial_+ R_+(x_N,y_N).
\end{aligned} \tag{45}$$

Similarly, if $y(x_{N-1}) = -f(x_{N-1})$, then we also have

$$\partial_+ W_+(x_0, y_0) \leq \sum_{i=1}^{N} G^{i+1} - G^N \partial_- R_-(x_N, y_N). \tag{46}$$

Combining (45), (46) and (31) yields that

$$\|\partial_+ W_+\|_{C(A_i C_{i-1})} \leq \sum_{i=1}^{N} G^{i+1} + G^N \max\{\|\partial_+ R_+\|_{C(\Gamma_{\text{in}})}, \|\partial_- R_-\|_{C(\Gamma_{\text{in}})}\}. \tag{47}$$

Analogously, the same estimate of $\|\partial_- W_-\|_{C(B_i C_{i-1})}$ holds true.

Thus, we complete the proof of Lemma 3.5. □

Corollary 3. *Based on Lemma 3.5, we have that* $|\nabla R_\pm| \lesssim M$, *where* M *depends on* $\|f\|_{C^2}$, $\|R_\pm\|_{C^1(\Gamma_{in})}$ *and the lower bound of the sound speed* c *in* $D_{2i-1}(i \geq 2)$.

Proof. Substituting (31) into (47) gives that

$$|\nabla R_\pm| = \frac{\sqrt{\lambda_\mp^2 + 1}}{\lambda_+ - \lambda_-} |\partial_\pm R_\pm| \lesssim \frac{1}{c} \left(\sum_{i=1}^{N} G^{i+1} + G^N \max\{\|R_+\|_{C^1(\Gamma_{\text{in}})}, \|R_-\|_{C^1(\Gamma_{\text{in}})}\} \right). \tag{48}$$

□

Combining the local existence and the Corollary 3 induces the global C^1 solution to the problem (34) in the inner region $D_{2i-1}(i \geq 2)$ before vacuum forms.

We now turn to consider the border region $D_{2i}(i \geq 1)$, which is bounded by a part of the upper wall $A_i A_{i+1}$, the C_- characteristic $A_i C_i$, and the C_+ characteristic $A_{i+1} C_i$ (see Fig.7). Then the problem in region $D_{2i}(i \geq 1)$ can be generalized as the following problem

$$\begin{cases} (\partial_x + \lambda_- \partial_y) R_+ = 0, & \text{in } D_{2i} \\ (\partial_x + \lambda_+ \partial_y) R_- = 0, & \\ R_-(x, y) = Z_-(x, y) & \text{on } A_i C_i, \\ R_+ + R_- = 2 \arctan f'(x) & \text{on } A_i A_{i+1}, \end{cases} \tag{49}$$

where $Z_-(x, y)$ is a known function. The local existence of the problem (49) can be achieved by [15]. In order to get the global solution to the problem (49) in the region $D_{2i}(i \geq 1)$, some prior estimates about $|R_\pm|$and$|\nabla R_\pm|$ are needed.

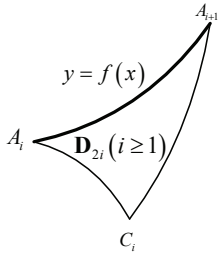

Fig.7 Solution in border region

Lemma 3.6. *Suppose that* $R_\pm(x, y)$ *are the classical solution to the problem (49), then* $R_\pm$ *and* $\partial_\pm R_\pm$ *are uniformly bounded in the region* $D_{2i}(i \geq 1)$.

Proof. For any point (x, y) in $D_{2i}(i \geq 1)$, by the method of characteristics and the solid wall condition, we have that

$$|R_-(x,y)| \leq \|Z_-\|_{C(A_iC_i)},\ |R_+(x,y)| \leq 2\arctan f'_\infty + \|Z_-\|_{C(A_iC_i)}. \tag{50}$$

in the region $D_{2i}(i \geq 1)$. It follows from Lemma 3.4 that

$$|\partial_- R_-(x,y)| \leq \|\partial_- Z_-\|_{C(A_iC_i)},\ |\partial_+ R_+(x,y)| \leq \|\partial_+ R_+\|_{C(A_iA_{i+1})}. \tag{51}$$

Moreover, by (43), we have that

$$\|\partial_+ R_+\|_{C(A_iA_{i+1})} \leq G^2 + G\|\partial_- R_-\|_{C(A_iA_{i+1})} \leq G^2 + G\|\partial_- Z_-\|_{C(A_iC_i)}. \tag{52}$$

By the same process shown in Lemma 3.5, we know that $\|\partial_- Z_-\|_{C(A_iC_i)}$ can be controlled by $\|R_\pm\|_{C^1(\Gamma_{\text{in}})}$. And thus the proof is finished. □

Corollary 4. *Based on Lemma 3.6, we have that* $|\nabla R_\pm| \lesssim M$, *where* M *depends on* $\|f\|_{C^2}$, $\|R_\pm\|_{C^1(\Gamma_{in})}$ *and the lower bound of the sound speed* c *in* $D_{2i}(i \geq 1)$.

Combining the local existence and the Corollary 4 induces the global C^1 solution to the problem (49) in the border region $D_{2i}(i \geq 1)$ before vacuum forms. The problem in the border region $D'_{2i}(i \geq 1)$ can be solved by the same way exactly. After solving the problem in these inner and border regions successively, we have that

Theorem 3.7. *For case I, there exists a global* C^1 *solution to the problem in* $(\overline{\Omega}_{non} \setminus \{M, N\})$ *and*

$$\partial_+ R_+ \geq 0, \quad \partial_- R_- \leq 0 \quad \text{in } (\overline{\Omega}_{non} \setminus \{M, N\}). \tag{53}$$

Moreover, there exists a constant C *which only depends on the initial data and* $\|f\|_{C^2}$, *such that*

$$|\partial_+ R_+| + |\partial_- R_-| \leq C \quad \text{in } (\overline{\Omega}_{non} \setminus \{M, N\}). \tag{54}$$

So far we have established the global C^1 solution before vacuum formation for case I, where the initial speed q_0 satisfies (9). The existence of solution after vacuum formation will be established in Section 4.

Now we begin to consider the problem in case II. We will prove Theorem 1.2.

Proof of Theorem 1.2. Firstly, we consider the special case $f''(x) = 0$, i.e. the duct is straight.

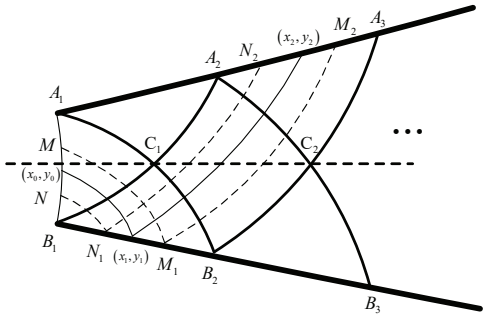

Fig.8 Blowup in 2D straight duct

Suppose that R_+ is strictly monotonic decreasing along arc MN on Γ_{in}(see Fig.8). Assume that the C_- characteristics issuing from M and N intersect Γ_{low}

at M_1 and N_1 respectively. Then it follows from the same argument in Lemma 3.1 that $\partial_+ R_+ < 0$, which implies that $\partial_y R_+ < 0$ in the region MNM_1N_1.

Let $y_-^{(0)}(x)$ be the C_- characteristic from any point (x_0, y_0) on MN, i.e.

$$\begin{cases} \dfrac{dy_-^{(0)}(x)}{dx} = \lambda_-, \\ y_-^{(0)}(x_0) = \varphi^{-1}(x_0). \end{cases} \tag{55}$$

Then R_+ is constant along the curve $y = y_-^{(0)}(x)$, which means that $R_+(x, y_-^{(0)}(x)) = R_+(x_0, \varphi^{-1}(x_0))$. Differentiating the first equation in (20) with respect to y yields that

$$\frac{\partial}{\partial x}\left(\frac{\partial R_+}{\partial y}\right) + \lambda_- \frac{\partial}{\partial y}\left(\frac{\partial R_+}{\partial y}\right) + \frac{\partial \lambda_-}{\partial R_-}\frac{\partial R_-}{\partial y}\frac{\partial R_+}{\partial y} = -\frac{\partial \lambda_-}{\partial R_+}\left(\frac{\partial R_+}{\partial y}\right)^2.$$

Use the second equation in (20), we can get $\frac{\partial R_-}{\partial x} + \lambda_- \frac{\partial R_-}{\partial y} = (\lambda_- - \lambda_+)\frac{\partial R_-}{\partial y}$, i.e.

$$\frac{\partial R_-}{\partial y} = \frac{\partial_x R_- + \lambda_- \partial_y R_-}{\lambda_- - \lambda_+}.$$

Substitute it into the above identity to obtain that

$$\frac{\partial}{\partial x}\left(\frac{\partial R_+}{\partial y}\right) + \lambda_- \frac{\partial}{\partial y}\left(\frac{\partial R_+}{\partial y}\right) + \frac{\partial \lambda_-}{\partial R_-}\frac{\partial_x R_- + \lambda_- \partial_y R_-}{\lambda_- - \lambda_+}\frac{\partial R_+}{\partial y} = -\frac{\partial \lambda_-}{\partial R_+}\left(\frac{\partial R_+}{\partial y}\right)^2. \tag{56}$$

Define $H_0(x) := \exp\{\int_{x_0}^{x} \frac{\partial_x R_- + \lambda_- \partial_y R_-}{\lambda_- - \lambda_+}\frac{\partial \lambda_-}{\partial R_-}(s, y_-^{(0)}(s))ds\}$, then (56) can be transformed into

$$\frac{d}{dx}(H_0(x)\partial_y R_+)^{-1} = \frac{\partial \lambda_-}{\partial R_+} H_0^{-1}, \tag{57}$$

where $\frac{d}{dx}$ stands for the derivative along the curve $y = y_-^{(0)}(x)$.

If $y = y_-^{(0)}(x)$ intersects Γ_{low} at point (x_1, y_1), then by integrating (57) from x_0 to x_1, we get that

$$\frac{\partial R_+}{\partial y}(x_1, y_1) = \frac{\partial_y R_+(x_0, y_0)}{H_0(x)(1 + \partial_y R_+(x_0, y_0)\int_{x_0}^{x_1} \frac{\partial \lambda_-}{\partial R^+} H_0^{-1}(s, y_-^{(0)}(s))ds)}. \tag{58}$$

Since R_+ is constant along $y = y_-^{(0)}(x)$, thus the term $\frac{1}{\lambda_- - \lambda_+}\frac{\partial \lambda_-}{\partial R_-}$ is only the function of R_-. If we set $h_0(R_-) = \int_0^{R_-} \frac{1}{\lambda_- - \lambda_+}\frac{\partial \lambda_-}{\partial R_-} dR_-$, then $H_0(x)$ can be expressed by

$$H_0(x) = \exp\{h_0(R_-(x, y_-^{(0)}(x)) - h_0(R_-(x_0, y_0)\}, \tag{59}$$

and $H_0(x)$ is bounded.

Note that $\partial_y R_+(x_0, y_0) < 0$ and $\frac{\partial \lambda_-}{\partial R^+} > 0$. If we denote the positive lower bound of $\frac{\partial \lambda_-}{\partial R^+} H_0^{-1}$ by K_0, then we have that

$$\frac{\partial R_+}{\partial y}(x_1, y_1) \le \frac{\partial_y R_+(x_0, y_0)}{H_0(x)(1 + \partial_y R_+(x_0, y_0) K_0(x_1 - x_0))}. \tag{60}$$

By the boundary condition (43), one has that

$$\partial_y R_-(x_1, y_1) = \left(\frac{v - u\lambda_-}{u\lambda_+ - v}\partial_y R_+\right)(x_1, y_1) = \left(\frac{\cos(\theta + A)}{\cos(\theta - A)}\partial_y R_+\right)(x_1, y_1). \tag{61}$$

The coefficient $\frac{\cos(\theta+A)}{\cos(\theta-A)}(x_1,y_1)$ is greater than 1, which implies that

$$\partial_y R_-(x_1,y_1) < \partial_y R_+(x_1,y_1). \tag{62}$$

Analogously, let $y_+^{(1)}(x)$ be the C_+ characteristic from (x_1,y_1) on N_1M_1, i.e.

$$\begin{cases} \dfrac{dy_+^{(1)}(x)}{dx} = \lambda_+, \\ y_+^{(1)}(x_1) = y_1. \end{cases} \tag{63}$$

one can obtain the similar equation of $\partial_y R_-$ as (57) by

$$\frac{d}{dx}(H_1(x)\partial_y R_+)^{-1} = \frac{\partial\lambda_+}{\partial R_-}H_1^{-1}, \tag{64}$$

where $H_1(x) = \exp\{\int_{x_1}^{x} \frac{\partial_x R_+ + \lambda_+\partial_y R_+}{\lambda_+-\lambda_-}\frac{\partial\lambda_+}{\partial R_+}(s,y_+^{(1)}(s))ds\}$ and $\frac{d}{dx}$ stands for the derivative along the curve $y = y_+^{(1)}(x)$. In addition, $H_1(x)$ can be written as $H_1(x) = \exp\{h_1(R_+(x,y_+^{(1)}(x)) - h_1(R_+(x_1,y_1)\}$ where $h_1(R_+) = \int_0^{R_+}\frac{1}{\lambda_+-\lambda_-}\frac{\partial\lambda_+}{\partial R_+}dR_+$. If $y = y_+^{(1)}(x)$ intersects Γ_{up} at point (x_2,y_2), integrating (64) from x_1 to x_2 yields that

$$\frac{\partial R_-}{\partial y}(x_2,y_2) = \frac{\partial_y R_-(x_1,y_1)}{H_1(x)(1+\partial_y R_-(x_1,y_1)\int_{x_1}^{x_2}\frac{\partial\lambda_+}{\partial R^-}H_1^{-1}(s,y_+^{(1)}(s))ds)}. \tag{65}$$

Note that $\partial_y R_-(x_1,y_1) < 0$ and $\frac{\partial\lambda_+}{\partial R^-} > 0$. If we denote the positive lower bound of $\frac{\partial\lambda_+}{\partial R^-}H_1^{-1}$ by K_1 and let $K_1^* = \min\{K_0,K_1\}$, then we have that

$$\frac{\partial R_-}{\partial y}(x_2,y_2) \le \frac{\partial_y R_-(x_1,y_1)}{H_1(x)(1+\partial_y R_-(x_1,y_1)K_1^*(x_2-x_1))}. \tag{66}$$

By the same process and after series of reflection if necessary, the denominator of (66) will tend to $-\infty$ at some finite location in the duct. Thus the C^1 solution to the problem will blow up at some finite location in the straight duct.

Next, we consider the case $f''(x) \le 0$.

The proof is very similar to the case $f''(x) = 0$, except for using following boundary condition to replace (61) by

$$\begin{aligned} \big((v-u\lambda_-)\partial_y R_+\big)(x_1,y_1) - \big((u\lambda_+ - v)\partial_y R_-\big)(x_1,y_1) \\ = -2u(x_1,y_1)\frac{f''(x_1)}{1+(f'(x_1))^2}. \end{aligned} \tag{67}$$

Thus we have

$$\begin{aligned} \partial_y R_-(x_1,y_1) &= \left(\frac{v-u\lambda_-}{u\lambda_+ - v}\partial_y R_+\right)(x_1,y_1) \\ &+ \left(\frac{2u}{u\lambda_+ - v}\right)(x_1,y_1)\times\frac{f''(x_1)}{1+(f'(x_1))^2} \\ &< \partial_y R_+(x_1,y_1). \end{aligned} \tag{68}$$

This is same as (62). So we omit the details of proof for the case $f''(x) \le 0$.

Therefore, we complete the proof of Theorem 1.2. □

4. **Formation of vacuum.** Due to the conservation of mass, vacuum can not appear in the inner regions of the duct. So if vacuum exists, it must locate on the two walls. On the other hand, since the two walls are the streamlines, Corollary 2 shows that the sound speed is monotonically decreasing along the two walls. When the sound speed decreases to zero, it corresponds to the formation of vacuum.

First, we list some properties of the vacuum boundary, one can also see [4] and [25] for more details.

Lemma 4.1. *When the vacuum area actually appears, the first vacuum point M and N must form at Γ_{up} and Γ_{low} respectively. Furthermore, the boundary of the vacuum l_{up} or l_{low} is a straight stream line which is tangential to Γ_{up} or Γ_{low} at M or N, where $f''(x_M) = f''(x_N) > 0$.*

Based on Theorem 3.7, we have obtained the C^1 solution to the problem for $(x, y) \in (\overline{\Omega}_{non} \setminus \{M, N\})$ and $\partial_+ R_+ \geq 0, \partial_- R_- \leq 0$ in $(\overline{\Omega}_{non} \setminus \{M, N\})$. We now begin to solve the problem when vacuum forms. Set $R_+(x, y) = \tilde{R}_+(y)$ and $R_-(x, y) = \tilde{R}_-(y)$ on $x = \psi(y)$. Then $\tilde{R}_\pm(y) \in C^1(-f(x_M), f(x_M))$ and $\tilde{R}_\pm(y)$ are monotonically increasing along $x = \psi(y)$. Since the problem is singular at M and N, we choose $Q_+^n = (x_n, y_n)$, $Q_-^n = (x_n, -y_n)$ on $x = \psi(y)$ and $\tilde{R}_\pm^n(y) \in C^1[-f(x_M), f(x_M)]$ such that

$$\begin{aligned} &\tilde{R}_\pm^n(y) = \tilde{R}_\pm(y),\ y \in [-y_n, y_n],\ f(x_M) - y_n = \frac{1}{n},\ n = 1, 2, ... \\ &|q'(y)| \leq -\frac{\psi''(y)}{1 + (\psi'(y))^2} \frac{qc}{\sqrt{q^2 - c^2}}, \quad y \in [-f(x_M), -y_n] \cup [y_n, f(x_M)]. \end{aligned} \tag{69}$$

Consider the problem in the region Ω_{vac} bounded by $x = \psi(y), l_{\text{up}}$ and l_{low} with initial value $\tilde{R}_\pm^n(y)$. The local existence can be obtained by the method in [15]. For any point $(x, y) \in \Omega_{vac}$, the back$-C_+$ characteristic from it must intersect Γ_{low} or Γ_{in} and the back$-C_-$ characteristic from it must intersect Γ_{up} or Γ_{in}. Then by Theorem 3.7, one has that $\partial_\pm R_\pm^n$ are uniformly bounded in Ω_{vac}. Thus we can get the global C^1 solution $R_\pm^n$ to the problem in Ω_{vac} with initial value $\tilde{R}_\pm^n(y)$.

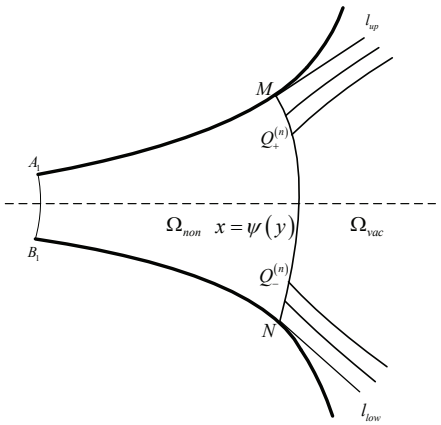

Fig.9 Solution in Ω_{vac}

Let $y = y_+^n$ stand for the C_+ characteristic from (x_n, y_n) and $y = y_-^n$ stand for the C_- characteristic from $(x_n, -y_n)$. Then $y_+^n (y_-^n$, resp.) tends to $l_{\text{up}} (l_{\text{low}}$, resp.)

as $n \to \infty$ and we can set

$$R_{\pm}(x,y) = \lim_{n\to\infty} R_{\pm}^{n}(x,y), \quad (x,y) \in \Omega_{vac}. \tag{70}$$

To sum up, we get that

Theorem 4.2. *Assume that $\tilde{R}_{\pm}(y) \in C^1(-f(x_M), f(x_M))$ and $\tilde{R}_{\pm}(y)$ are monotonically increasing along $x = \psi(y)$. Then $R_{\pm}(x,y)$ defined by (70) is the unique solution to the problem in Ω_{vac}. Furthermore, $R_{\pm}(x,y) \in C^1(\overline{\Omega_{vac}}\backslash\{l_{up} \cup l_{low}\})$ and*

$$\partial_+ R_+ \geq 0, \partial_- R_- \leq 0 \quad in \quad (\overline{\Omega_{vac}}\backslash\{l_{up} \cup l_{low}\}). \tag{71}$$

Next, we consider the regularity near vacuum boundary. In fact, there are many results about physical vacuum singularity. A vacuum boundary (or singularity) is called physical if

$$0 < \left|\frac{\partial c^2}{\partial \vec{n}}\right| < +\infty \tag{72}$$

in a small neighborhood of the boundary, where $\vec{n}$ is the normal direction of the vacuum boundary. This definition of physical vacuum was motivated by the case of Euler equations with damping studied in [17] and [19]. But the following Theorem shows that the vacuum here is not the physical vacuum.

Theorem 4.3. *Let $\partial_{\vec{n}}$ stand for the normal derivative on the vacuum boundary, then for any point (x,y) near the vacuum line, we have $\partial_{\vec{n}} c^2(x,y)$ tends to zero as (x,y) approaches to the vacuum line along the normal direction except the vacuum point M and N.*

Proof. For convenience, we only consider the case near the vacuum line l_{low}, which is tangential to Γ_{low} at N. First, we derive the expression of $\partial_{\vec{n}} c^2$, where $\vec{n} = (n_1, n_2)$. Acting $\partial_{\vec{n}}$ on the Bernoulli law (7) yields that

$$\partial_{\vec{n}} c^2 = -(\gamma - 1) q \partial_{\vec{n}} q. \tag{73}$$

By the relation $R_+ - R_- = 2F(q)$, one has that

$$\partial_x q = \frac{qc}{2\sqrt{q^2 - c^2}}(\partial_x R_+ - \partial_x R_-), \ \partial_y q = \frac{qc}{2\sqrt{q^2 - c^2}}(\partial_y R_+ - \partial_y R_-). \tag{74}$$

Substituting it into (73) gives that

$$\partial_{\vec{n}} c^2 = -\frac{(\gamma - 1) q^2 c}{2\sqrt{q^2 - c^2}}[(n_2 - n_1\lambda_-)\partial_y R_+ + (n_1\lambda_+ - n_2)\partial_y R_-]. \tag{75}$$

Combining this with (30) yields that

$$\partial_{\vec{n}} c^2 = -\frac{(\gamma - 1) q^2 (u^2 - c^2)}{4(q^2 - c^2)}[(n_2 - n_1\lambda_-)\partial_+ R_+ - (n_1\lambda_+ - n_2)\partial_- R_-]. \tag{76}$$

Construct a line NN_h vertical to l_{low} at the first vacuum point N, where h represents the distance between N and N_h. Let C_-^h be the C_- characteristic through N_h and intersect Γ_{up} at P_h. Fix a point S on l_{low}. Make a line from S vertical to l_{low} and it intersects C_-^h at S_h. Then we have that $N_h \to N, S_h \to S$ as $h \to 0$. It follows from (35) that

$$\partial_- \partial_+ R_+ = \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} \partial_+ R_+. \tag{77}$$

Integrating (77) from P_h to S_h yields that

$$\partial_+ R_+(S_h) = \partial_+ R_+(P_h) \exp\left\{ \int_{P_h}^{S_h} \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_+ - \lambda_-} ds \right\}. \tag{78}$$

This together with (26) yields that

$$\partial_+ R_+(S_h) = \partial_+ R_+(P_h) \exp\left\{ \int_{P_h}^{S_h} \frac{\gamma+1}{8c}(-T_1 \partial_+ R_+ + T_2 \partial_- R_-) ds \right\}, \tag{79}$$

where

$$T_1 = \frac{q^2(u^2-c^2)\sec^2(\theta - A)}{(q^2-c^2)^{\frac{3}{2}}}, \quad T_2 = \frac{q^2(u^2-c^2)\sec^2(\theta + A)}{(q^2-c^2)^{\frac{3}{2}}}.$$

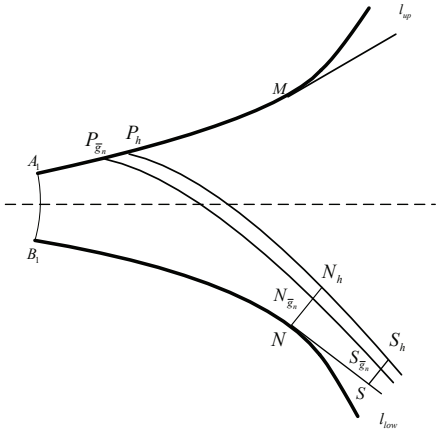

Fig.10 The regularity near vacuum boundary

Since $u^2 > c^2$, then T_1 and T_2 have the uniform positive lower bound denoted as g_0.

We claim that $\overline{\lim_{h\to 0}} \partial_+ R_+(S_h) = 0$ and $\underline{\lim_{h\to 0}} \partial_- R_-(S_h) = 0$, which immediately implies that $\lim_{h\to 0} \partial_{\vec{n}} c^2(S_h) = 0$. For convenience, we only prove $\overline{\lim_{h\to 0}} \partial_+ R_+(S_h) = 0$, for the case $\underline{\lim_{h\to 0}} \partial_- R_-(S_h) = 0$ can be obtained by the same argument.

We will use the proof by contradiction to obtain above assertions. Assume that $\overline{\lim_{h\to 0}} \partial_+ R_+(S_h) = \bar{g} > 0$, where $\bar{g}$ is a positive constant. Then there exists a sequence $S_{\bar{g}_n}$ on $S_h S$ such that $\partial_+ R_+(S_{\bar{g}_n}) \geq \frac{\bar{g}}{2} > 0$ and $|NN_{\bar{g}_n}| \leq \frac{h}{n}$, where $N_{\bar{g}_n}$ is the intersection of the C_- characteristic $C_-^{\bar{g}_n}$ through $S_{\bar{g}_n}$ and the segment NN_h. Let the C_- characteristic $C_-^{\bar{g}_n}$ intersect $\Gamma_{\rm up}$ at $P_{\bar{g}_n}$. Then it follows from Lemma 3.4 that

$$\partial_+ R_+(x,y) \geq \partial_+ R_+(S_{\bar{g}_n}) \geq \frac{\bar{g}}{2} \quad \text{on} \quad (x,y) \in P_{\bar{g}_n} S_{\bar{g}_n}. \tag{80}$$

By the fact that $\partial_- q \geq 0$ and the Bernoulli law, we can get $\partial_- c \leq 0$, which implies that the value of c on $N_{\bar{g}_n} S_{\bar{g}_n}$ is no greater than the value of c at $N_{\bar{g}_n}$ denoted by

$c(N_{\bar{g}_n})$. Thus we have $\lim_{n\to\infty} c(N_{\bar{g}_n}) = 0$ and

$$\begin{aligned}\int_{P_{\bar{g}_n}}^{S_{\bar{g}_n}} \frac{\gamma+1}{8c}(-T_1\partial_+R_+ + T_2\partial_-R_-)ds &\leq \int_{N_{\bar{g}_n}}^{S_{\bar{g}_n}} -\frac{\gamma+1}{8c}T_1\partial_+R_+ds \\ &\leq -\frac{\gamma+1}{16c(N_{\bar{g}_n})}s_0g_0\bar{g}, \end{aligned} \tag{81}$$

where $s_0 = |NS|$. Combining it with (79) yields that

$$\partial_+R_+(S_{\bar{g}_n}) \leq \partial_+R_+(P_{\bar{g}_n}) \exp\left\{-\frac{\gamma+1}{16c(N_{\bar{g}_n})}s_0g_0\bar{g}\right\}. \tag{82}$$

Note that $P_{\bar{g}_n}$ is away from the vacuum as $n \to \infty$. If $\lim_{n\to\infty} P_{\bar{g}_n} = P$, then $\lim_{n\to\infty} \partial_+R_+(P_{\bar{g}_n}) = \partial_+R_+(P)$ and $\partial_+R_+(P)$ is bounded. Then let $n \to \infty$ in (82), we can get that

$$\overline{\lim_{n\to\infty}}\partial_+R_+(S_{\bar{g}_n}) \leqslant \partial_+R_+(P) \lim_{n\to\infty} \exp\left\{-\frac{\gamma+1}{16c(N_{\bar{g}_n})}s_0g_0\bar{g}\right\} = 0, \tag{83}$$

which is a contradiction. Thus, it follows

$$\overline{\lim_{h\to 0}}\partial_+R_+(S_h) = 0, \quad \lim_{h\to 0}\partial_-R_-(S_h) = 0. \tag{84}$$

This together with (71) gives that

$$\lim_{h\to 0}\partial_+R_+(S_h) = 0, \quad \lim_{h\to 0}\partial_-R_-(S_h) = 0. \tag{85}$$

Combining it with (76) yields that

$$\lim_{h\to 0}\frac{\partial c^2}{\partial\vec{n}}(S_h) = 0. \tag{86}$$

Note that $c^2 \in C(\bar{\Omega}_{vac})$, by the derivative limit theorem, we have that

$$\frac{\partial c^2}{\partial\vec{n}} = 0, \quad \text{on} \quad l_{low}\setminus\{N\}. \tag{87}$$

Similarly, one has that

$$\frac{\partial c^2}{\partial\vec{n}} = 0, \quad \text{on} \quad l_{up}\setminus\{M\}. \tag{88}$$

Thus, we prove the Theorem 4.2. □

So far we have established the solution to the problem when vacuum actually appears. When it comes to the case that there is no vacuum in the duct, by the same prior estimates in Lemma 3.5 and Lemma 3.6, one only needs to check whether the duct can be covered by characteristics in finite steps.

Theorem 4.4. *Assume that $f \in C^2([0,+\infty))$ satisfies (1), $\varphi \in C^2([-f(0), f(0)])$ satisfies (2) and $q_0 \in C^1[-f(0), f(0)]$ satisfies (3), (9). If there is no vacuum in the duct, then there exists a global C^1 solution in the duct.*

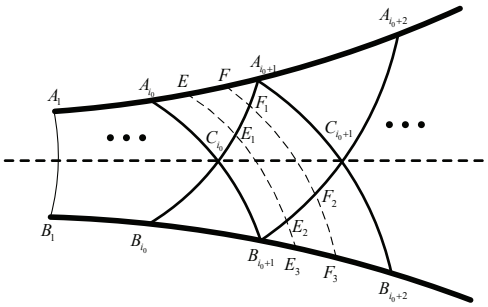

Fig.11 The case without vacuum

Proof. It remains to check whether the duct can be covered by characteristics in finite steps. Since the duct is convex, then there exists i_0 such that we can find an arc EF on $A_{i_0}A_{i_0+1}$ and $f'' > 0$ on EF. Let the C_- characteristics C_-^E, C_-^F intersect $C_{i_0}A_{i_0+1}$ at E_1, F_1, intersect $B_{i_0+1}C_{i_0+1}$ at E_2, F_2 and intersect $B_{i_0+1}B_{i_0+2}$ at E_3, F_3. By Lemma 3.1 and (28), we know that $\partial_+ R_+ > 0$ on E_1F_1 and E_2F_2. Combining it with $\partial_+ R_- = 0$ yields that $\tilde{E}_1\tilde{F}_1, \tilde{E}_2\tilde{F}_2$ which are the images of E_1F_1, E_2F_2 in (u, v) plane have positive length. That is to say

$$|\tilde{C}_{i_0}\tilde{A}_{i_0+1}| > 0,\ |\tilde{B}_{i_0+1}\tilde{C}_{i_0+1}| > 0. \tag{89}$$

Similarly, we can get that

$$|\tilde{C}_{i_0+1}\tilde{B}_{i_0+2}| > 0,\ |\tilde{A}_{i_0+2}\tilde{C}_{i_0+2}| > 0. \tag{90}$$

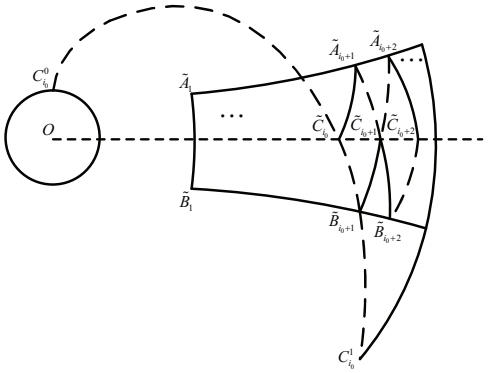

Fig.12 The image in the $(u, v)-$plane

By the induction method, we have that

$$\begin{aligned} &|\tilde{C}_{i_0+2(k-1)}\tilde{A}_{i_0+2k-1}| > 0,\ |\tilde{B}_{i_0+2k-1}\tilde{C}_{i_0+2k-1}| > 0, \\ &|\tilde{C}_{i_0+2(k-1)}\tilde{B}_{i_0+2k}| > 0,\ |\tilde{A}_{i_0+2k}\tilde{C}_{i_0+2k}| > 0, \quad k = 1, 2, ... \end{aligned} \tag{91}$$

Suppose that the image of the C_+ characteristic through C_{i_0} in (u, v) plane intersects the critical circle at $C_{i_0}^0$ and intersects the limiting circle at $C_{i_0}^1$. Then

$$\angle C_{i_0}^0 O C_{i_0}^1 = \frac{\pi}{2\mu}, \quad \theta_{\tilde{B}_{i_0+1}} = \angle \tilde{C}_{i_0} O \tilde{B}_{i_0+1} > 0, \tag{92}$$

where $\mu = \sqrt{\frac{\gamma-1}{\gamma+1}}$. By the same method in [5], we have that

$$i \le \lfloor \frac{\pi}{2\mu\theta_{\tilde{B}_{i_0+1}}} \rfloor, \tag{93}$$

where $\lfloor . \rfloor$ stands for the maximal integer no more than the number inside the bracket. □

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, Theorem 1.1 comes from Theorem 3.1 and Theorem 4.1-4.3. □

Finally, we give a sufficient condition to ensure that vacuum must form at finite location in the duct.

Proposition 1. *If* $f'_\infty > f'(0) + \frac{2}{(\gamma-1)M_{B_1}\sqrt{1+(f'(0))^2}}$, *where* M_{B_1} *represents the Mach number at* B_1, *then the vacuum will form at the finite locations on* Γ_{low}.

Proof. We will use the proof by contradiction. Assume that there is no vacuum on Γ_{low}. Let $\frac{d}{ds}$ stand for the derivative along Γ_{low}. By the relation $R_+ - R_- = 2F(q)$, one has that

$$\frac{dq}{ds} = \frac{d(R_+ - R_-)}{ds}\frac{1}{2F'(q)} = \frac{qc}{2\sqrt{q^2-c^2}}(\partial_x(R_+ - R_-) + \tan\theta\partial_y(R_+ - R_-)). \tag{94}$$

This together with (29) and (43) yields that

$$\frac{dq}{ds} = \frac{qc}{\sqrt{q^2-c^2}}\left(\frac{\tan\theta - \lambda_-}{\lambda_+ - \lambda_-}\partial_+ R_+ + \frac{f''(x)}{1+(f'(x))^2}\right). \tag{95}$$

By (95) and the Bernoulli law (7), we get that

$$\frac{dc}{ds} = -\frac{(\gamma-1)q^2}{2\sqrt{q^2-c^2}}\left(\frac{\tan\theta - \lambda_-}{\lambda_+ - \lambda_-}\partial_+ R_+ + \frac{f''(x)}{1+(f'(x))^2}\right). \tag{96}$$

Since

$$c_{B_1} = -\int_0^{+\infty} \frac{dc}{ds}\sqrt{1+(f'(x))^2}dx,$$

then

$$\begin{aligned} c_{B_1} &\ge \int_0^{+\infty} \frac{(\gamma-1)q^2}{2\sqrt{q^2-c^2}}\frac{f''(x)}{1+(f'(x))^2}\sqrt{1+(f'(0))^2}dx \\ &\ge \frac{q_{B_1}(\gamma-1)}{2}\sqrt{1+(f'(0))^2}\int_0^{+\infty} -\frac{f''(x)}{1+(f'(x))^2}dx \\ &= \frac{q_{B_1}(\gamma-1)}{2}\sqrt{1+(f'(0))^2}(\arctan f'_\infty - \arctan f'(0)) \\ &> c_{B_1}, \end{aligned} \tag{97}$$

which is a contradiction. Thus, we complete the proof. □

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