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ASYMPTOTIC BEHAVIOR OF THE ONE-DIMENSIONAL COMPRESSIBLE MICROPOLAR FLUID MODEL

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Abstract. In this paper, we study the large time behavior of the solution for one-dimensional compressible micropolar fluid model with large initial data. This model describes micro-rotational motions and spin inertia which is commonly used in the suspensions, animal blood, and liquid crystal. We get the uniform positive lower and upper bounds of the density and temperature independent of both space and time. In particular, we also obtain the asymptotic behavior of the micro-rotation velocity.

1. **Introduction and the main theorem.** In this paper, we consider the one-dimensional compressible micropolar fluid model in Lagrange coordinates:

$$\begin{cases} v_t = u_x, \\ u_t + P_x = \mu\big(\frac{u_x}{v}\big)_x, \\ w_t + \nu v w = \lambda\big(\frac{w_x}{v}\big)_x, \\ \big(e + \frac{u^2}{2}\big)_t + (Pu)_x = \big(\mu\frac{uu_x}{v} + k\frac{\theta_x}{v}\big)_x + \nu v w^2 + \lambda\frac{w_x^2}{v}, \\ e = c_v\theta, \qquad P = \frac{R\theta}{v}, \end{cases} \tag{1.1}$$

where $t > 0$ is time, $x \in \Omega \subseteq \mathbb{R}$ denotes the Lagrange mass coordinate, $v = v(x,t) > 0$, $u = u(x,t)$, $w = w(x,t)$, $\theta = \theta(x,t) > 0$, e and P, which represent the specific volume, fluid velocity, micro-rotation velocity, absolute temperature, internal energy and pressure, respectively. Moreover, $\mu > 0$ is the viscosity coefficient, $\kappa > 0$ is the heat conductivity coefficient, $R > 0$ is the specific gas constant, $\nu > 0$ and $\lambda > 0$ are the coefficients of micro-viscosity, c_v is the heat capacity at constant volume. The

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model of micropolar fluid was first introduced by Eringen [7] in 1966. This model can be used to describe the motions of a large variety of complex fluids consisting of dipole elements such as the suspensions, animal blood, liquid crystal, etc. For more physical background on this model, we can refer to [8, 15].

When $w = 0$, the system (1.1) reduces to the standard compressible Navier-Stokes equations. It has attracted great interest among many analysts and there have been many important developments. Kazhikhov and Shelukhin [12] firstly obtained the global existence of solutions in bounded domains for large initial data. Then, Jiang [10, 11] studied the large-time behavior of solutions to the compressible Navier-Stokes system in unbounded domains for large initial data. He proved the specific vomule was pointwise bounded from below and above independent of both space and time. Recently, Li and Liang [13] proved the large-time behavior of solutions to the initial and initial boundary value problems with large initial data. They showed that the temperature was bounded from below and above independent of both space and time. See also [5] for the spherically symmetric and cylindrically symmetric non-barotropic flows.

When $w \neq 0$, we get the micropolar fluid model. Now more and more mathematicians devoted to the research of micropolar fluid model (1.1). For the incompressible fluid, we refer to [3] and references therein. For the complete fluid, Mujakovic [16, 17] obtained a series of results about the local, global in time existence theorem and regularity of solutions for the model with homogeneous boundary condition in one dimension. Next, Chen [2] proved the global existence of strong solutions for the initial boundary problem in one dimension that vacuum can be allowed initially. Yin [19] established the stationary solutions to the inflow problem in a half-line, the time asymptotically stability for the H^1 solutions was also obtained. Recently, Cui and Yin [6] obtained the convergence rates of global solutions toward corresponding stationary solutions if the initial perturbation belongs to the weighted Sobolev space. Qin et al. [18] studied the global existence and asymptotic behavior of H^1 solutions to the Cauchy problem of one-dimensional model with the weighted small initial data. Based on the assumption $\kappa(\theta) = O(1)(1 + \theta^q)$ with $q \geq 0$, Feng and Zhu [9] considered the existence of global classical solution with large initial data and vacuum. For three-dimensional micropolar fluid model, Chen et al. [4] gave the global weak solutions with discontinuous initial data and vacuum. Liu and Zhang [14] obtained the optimal time decay of the 3-dimensional compressible micropolar fluid model.

In this paper, motivated by the result of [13], we get asymptotic behavior of the one-dimensional compressible micropolar fluid model with large initial data. We supplement the system (1.1) with the initial condition

$$(v, u, w, \theta)|_{t=0} = (v_0, u_0, w_0, \theta_0), \tag{1.2}$$

and three types of far-field and boundary conditions:

(1) Cauchy problem

$$\Omega = \mathbb{R}, \quad \lim_{|x|\to\infty} (v(x,t), u(x,t), w(x,t), \theta(x,t)) = (1, 0, 0, 1), \qquad t > 0; \tag{1.3}$$

(2) Boundary and far-field conditions for $\Omega = (0, \infty)$

$$u(0,t) = 0, \quad \theta_x(0,t) = 0, \quad \lim_{x\to\infty} (v(x,t), u(x,t), w(x,t), \theta(x,t)) = (1, 0, 0, 1), \quad t > 0; \tag{1.4}$$

(3) Boundary and far-field conditions for $\Omega = (0,\infty)$

$$u(0,t) = 0, \quad \theta(0,t) = 1, \quad \lim_{x\to\infty}(v(x,t), u(x,t), w(x,t), \theta(x,t)) = (1,0,0,1), \quad t > 0. \tag{1.5}$$

We now state the main result of this paper as follows:

Theorem 1.1. *Consider the initial value problem given by (1.1)-(1.3), and initial-boundary value problem given by (1.1), (1.2), (1.4) and (1.1), (1.2), (1.5). Assume that $v_0(x) > 0$, $\theta_0(x) > 0$, $v_0 - 1$, u_0, w_0, $\theta_0 - 1 \in H^1(\Omega)$ and the initial data are compatible with boundary conditions. Then there exists a unique global strong solution $(v,u,w,\theta) \in C([0,\infty), H^1(\Omega))$, $(v-1,u,w,\theta-1) \in L^\infty([0,\infty), H^1(\Omega))$, and there exists positive constants C_1 and C depending only on μ, κ, R, c_v, ν satisfying*

$$\frac{1}{C_1} \le v(x,t) \le C_1, \tag{1.6}$$

$$\frac{1}{C} \le \theta(x,t) \le C, \tag{1.7}$$

and for any $t \ge 0$, $p > 2$

$$\lim_{t\to\infty}(\| (v-1,u,w,\theta-1)(t) \|_{L^p(\Omega)} + \| (v_x,u_x,w_x,\theta_x)(t) \|_{L^2(\Omega)}) = 0.$$

Remark 1. Compared with the work [13], we have to deal with the strong couple of the micro-rotation velocity w and the fluid motion, which will lead to some difficulties. In particular, we solve it by obtaining some new estimates (2.16), (2.31). We also obtain the asymptotic behavior of $w(x,t)$.

2. Proof of Theorem 1.1.

2.1. Preliminaries.

Lemma 2.1. *(Energy inequality) Under the conditions of Theorem 1.1, the following inequality holds*

$$\begin{aligned}\sup_{0\le t<\infty}\int_\Omega \Big(\frac{1}{2}u^2 + \frac{1}{2}w^2 + R(v - \ln v - 1) + c_v(\theta - \ln\theta - 1)\Big)dx \\ + \int_0^\infty \int_\Omega \Big(\mu\frac{u_x^2}{v\theta} + \kappa\frac{\theta_x^2}{v\theta^2} + \lambda\frac{w_x^2}{v\theta} + \nu\frac{vw^2}{\theta}\Big)dxds \le C_0,\end{aligned} \tag{2.1}$$

$$0 < \alpha_1 \le \int_i^{i+1} vdx \le \alpha_2, \quad 0 < \alpha_1 \le \int_i^{i+1}\theta dx \le \alpha_2, \tag{2.2}$$

and there are $a_i(t), b_i(t) \in [i, i+1]$ satisfying

$$\alpha_1 \le v(a_i(t),t) \le \alpha_2, \qquad \alpha_1 \le \theta(b_i(t),t) \le \alpha_2, \tag{2.3}$$

where α_1, α_2 are two positive roots of the equation $y - \ln y - 1 = \frac{C_0}{\min\{R,c_v\}}$, $i \in \mathbb{N}$ for the initial boundary value problems and $i \in \mathbb{Z}$ for the Cauchy problem.

Proof. The above estimates can been obtained by the similar arguments as that in [10]. □

Lemma 2.2. *Under the conditions of Theorem 1.1, there holds*

$$\sup_{0\le t<\infty}\int_\Omega \frac{1}{2}w^2dx + \int_0^\infty\int_\Omega\Big(\lambda\frac{w_x^2}{v} + \nu vw^2\Big)dxds \le \int_\Omega\frac{1}{2}w_0^2dx, \tag{2.4}$$

and

$$\sup_{0\le t<\infty}\int_\Omega \frac{1}{4}w^4dx+\int_0^\infty\int_\Omega\left(\lambda\frac{w^2w_x^2}{v}+\nu v w^4\right)dxds\le\int_\Omega\frac{1}{4}w_0^4dx. \tag{2.5}$$

Proof. Multiplying $(1.1)_3$ by w, using the boundary condition of w, one directly gets (2.4).

Multiplying $(1.1)_3$ by w^3, using the boundary condition of w, one gets (2.5). □

2.2. **Uniform estimates of v.** In this subsection, the specific volume $v(x,t)$ is proved to be bounded from below and above, independent of both time and space.

Lemma 2.3. *Under the hypotheses of Theorem 1.1, it holds that*

$$0<C_1{}^{-1}\le v(x,t)\le C_1. \tag{2.6}$$

For more detailed proof of Lemma 2.3, refer to [5, 10, 11] and references therein.

2.3. **Uniform estimates of θ.**

Lemma 2.4. *Under the conditions of Theorem 1.1, let (v,u,w,θ) be a solution to the system (1.1) on $\Omega\times[0,T]$, it holds for any $T>0$*

$$\int_\Omega(|\theta-1|^2+u^2\theta+w^2\theta+u^4)dx+\int_0^T\int_\Omega\left(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta+u^2u_x^2\right)dxds\le C. \tag{2.7}$$

Proof. Motivated by [13], the proof will be divided into three steps.

Step 1. At first, for any $t\ge 0$ and $a>1$, denoting

$$\Omega_a(t)\triangleq\{x\in\Omega|\theta(x,t)>a\},$$

and we derive from (2.1) that

$$\sup_{0\le t<\infty}\int_{\Omega_a(t)}\theta\le C(a)\sup_{0\le t<\infty}\int_\Omega(\theta-\ln\theta-1)\le C(a).$$

Using $(1.1)_1$, $(1.1)_3$, $(1.1)_5$, we rewrite $(1.1)_4$ as

$$c_v\theta_t+\frac{R\theta u_x}{v}=(\frac{\kappa\theta_x}{v})_x+\frac{\mu u_x^2}{v}+\nu v w^2+\frac{\lambda w_x^2}{v}. \tag{2.8}$$

Multiplying (2.8) by $(\theta-2)_+=\sup\{\theta-2,0\}$ and integrating the result over $\Omega\times[0,T]$, we have

$$\begin{aligned}
&\frac{c_v}{2}\int_\Omega(\theta-2)_+^2dx+\kappa\int_0^T\int_{\Omega_2(s)}\frac{\theta_x^2}{v}dxds\\
=\ &\frac{c_v}{2}\int_\Omega(\theta_0-2)_+^2dx-R\int_0^T\int_\Omega\frac{\theta u_x}{v}(\theta-2)_+dxds+\mu\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_+dxds\\
&+\lambda\int_0^T\int_\Omega\frac{w_x^2}{v}(\theta-2)_+dxds+\nu\int_0^T\int_\Omega vw^2(\theta-2)_+dxds.
\end{aligned} \tag{2.9}$$

Next, multiplying $(1.1)_2$ by $2u(\theta-2)_+$ and integrating the result over $\Omega\times[0,T]$, we obtain

$$\begin{aligned}
&\int_\Omega u^2(\theta-2)_+dx+2\mu\int_0^T\int_\Omega \frac{u_x^2}{v}(\theta-2)_+dxds\\
=\ &\int_\Omega u_0^2(\theta_0-2)_+dx+2R\int_0^T\int_\Omega\frac{\theta u_x}{v}(\theta-2)_+dxds+2R\int_0^T\int_{\Omega_2(s)}\frac{\theta u\theta_x}{v}dxds\\
&-2\mu\int_0^T\int_{\Omega_2(s)}\frac{uu_x\theta_x}{v}dxds+\int_0^T\int_{\Omega_2(s)}u^2\theta_s dxds.
\end{aligned}\tag{2.10}$$

Then multiplying $(1.1)_3$ by $2w(\theta-2)_+$ and integrating the result over $\Omega\times[0,T]$, yields

$$\begin{aligned}
&\int_\Omega w^2(\theta-2)_+dx+2\lambda\int_0^T\int_\Omega\frac{w_x^2}{v}(\theta-2)_+dxds+2\nu\int_0^T\int_\Omega vw^2(\theta-2)_+dxds\\
=\ &\int_\Omega w_0^2(\theta_0-2)_+dx-2\lambda\int_0^T\int_{\Omega_2(s)}\frac{ww_x\theta_x}{v}dxds+\int_0^T\int_{\Omega_2(s)}w^2\theta_s dxds.
\end{aligned}$$

Adding the above three integral expressions together and using (2.8) to get

$$\begin{aligned}
&\int_\Omega\left(\frac{c_v}{2}(\theta-2)_+^2+u^2(\theta-2)_++w^2(\theta-2)_+\right)dx+\kappa\int_0^T\int_{\Omega_2(s)}\frac{\theta_x^2}{v}dxds\\
&+\mu\int_0^T\int_\Omega\frac{u_x^2}{v}(\theta-2)_+dxds+\lambda\int_0^T\int_\Omega\frac{w_x^2}{v}(\theta-2)_+dxds\\
&+\nu\int_0^T\int_\Omega vw^2(\theta-2)_+dxds\\
=\ &\int_\Omega\left(\frac{c_v}{2}(\theta_0-2)_+^2+u_0^2(\theta_0-2)_++w_0^2(\theta_0-2)_+\right)dx+R\int_0^T\int_\Omega\frac{\theta u_x}{v}(\theta-2)_+dxds\\
&+2R\int_0^T\int_{\Omega_2(s)}\frac{\theta u\theta_x}{v}dxds-2\mu\int_0^T\int_{\Omega_2(s)}\frac{uu_x\theta_x}{v}dxds\\
&+\frac{1}{c_v}\int_0^T\int_{\Omega_2(s)}u^2\Big(\frac{\mu u_x^2}{v}-R\frac{\theta u_x}{v}\Big)dxds+\frac{\kappa}{c_v}\int_0^T\int_{\Omega_2(s)}u^2\Big(\frac{\theta_x}{v}\Big)_xdxds\\
&\frac{\lambda}{c_v}\int_0^T\int_{\Omega_2(s)}\frac{u^2w_x^2}{v}dxds+\frac{\nu}{c_v}\int_0^T\int_{\Omega_2(s)}vu^2w^2dxds+\frac{\lambda}{c_v}\int_0^T\int_{\Omega_2(s)}\frac{w^2w_x^2}{v}dxds\\
&+\frac{1}{c_v}\int_0^T\int_{\Omega_2(s)}w^2\Big(\frac{\mu u_x^2}{v}-R\frac{\theta u_x}{v}\Big)dxds+\frac{\kappa}{c_v}\int_0^T\int_{\Omega_2(s)}w^2\Big(\frac{\theta_x}{v}\Big)_xdxds\\
&+\frac{\nu}{c_v}\int_0^T\int_{\Omega_2(s)}vw^4dxds-2\lambda\int_0^T\int_{\Omega_2(s)}\frac{ww_x\theta_x}{v}dxds\\
=:\ &\int_\Omega\left(\frac{c_v}{2}(\theta_0-2)_+^2+u_0^2(\theta_0-2)_++w_0^2(\theta_0-2)_+\right)dx+\sum_{j=1}^{12}I_j.
\end{aligned}\tag{2.11}$$

Now the main task is to estimate $I_1, I_2, I_3......, I_{12}$ term by term.

For I_1, using Cauchy inequality and (2.6), one obtains

$$\begin{aligned}
|I_1|\ &=R\left|\int_0^T\int_\Omega\frac{\theta u_x}{v}(\theta-2)_+dxds\right|\\
&\le\epsilon\int_0^T\int_{\Omega_2(s)}\frac{u_x^2}{v}(\theta-2)_+dxds+C(\epsilon)\int_0^T\int_\Omega\theta^2(\theta-2)_+dxds
\end{aligned}$$

$$\leq \epsilon \int_0^T \int_{\Omega_2(s)} \frac{u_x^2}{v}(\theta-2)_+ dxds + C(\epsilon)\int_0^T \sup_{\Omega}(\theta-\frac{3}{2})_+^2 ds.$$

It follows from Cauchy inequality and (2.6) that for any $\epsilon > 0$

$$\begin{aligned} |I_2| &= R\Big|\int_0^T \int_{\Omega_2(s)} \frac{\theta u \theta_x}{v} dxds\Big| \\ &\leq \epsilon \int_0^T \int_{\Omega_2(s)} \theta_x^2 dxds + C(\epsilon)\int_0^T \int_{\Omega_2(s)} u^2\theta^2 dxds \\ &\leq \epsilon \int_0^T \int_{\Omega} \theta_x^2 dxds + C(\epsilon)\int_0^T \sup_{\Omega}(\theta-\frac{3}{2})_+^2 ds, \end{aligned}$$

where we have used the fact that

$$\int_0^T \int_{\Omega_2(s)} u^2\theta^2 dxds \leq 16\int_0^T \int_{\Omega} u^2(\theta-\frac{3}{2})_+^2 dxds \leq C\int_0^T \sup_{\Omega}(\theta-\frac{3}{2})_+^2 ds. \tag{2.12}$$

Then for I_3, it follows from Cauchy inequality and (2.12)

$$\begin{aligned} |I_3| &= 2\mu\Big|\int_0^T \int_{\Omega_2(s)} \frac{u u_x \theta_x}{v} dxds\Big| \\ &\leq \epsilon \int_0^T \int_{\Omega} \theta_x^2 dxds + C(\epsilon)\int_0^T \int_{\Omega} u^2 u_x^2 dxds. \end{aligned}$$

For I_4,

$$|I_4| \leq C\int_0^T \int_{\Omega} u^2 u_x^2 dxds + C\int_0^T \sup_{\Omega}(\theta-\frac{3}{2})_+^2 ds.$$

For any $\eta > 0$, set

$$\chi_\eta(\theta) = \begin{cases} 0, & \theta \leq 2, \\ \dfrac{\theta-2}{\eta}, & 2 \leq \theta \leq \eta+2, \\ 2, & \theta \geq \eta+2. \end{cases}$$

For I_5, we immediately get for any $\epsilon > 0$

$$\begin{aligned} I_5 &= \frac{\kappa}{c_v}\int_0^T \int_{\Omega} \lim_{\eta\to 0} \chi_\eta(\theta) u^2 (\frac{\theta_x}{v})_x dxds \\ &= \lim_{\eta\to 0} \frac{\kappa}{c_v}\int_0^T \int_{\Omega} \Big(-2\chi_\eta(\theta) u u_x \frac{\theta_x}{v} - \chi_\eta'(\theta) u^2 \frac{\theta_x^2}{v}\Big) dxds \\ &\leq \epsilon \int_0^T \int_{\Omega} \theta_x^2 dxds + C(\epsilon)\int_0^T \int_{\Omega} u^2 u_x^2 dx. \end{aligned}$$

It follows from Cauchy inequality that

$$|I_6| + |I_7| \leq C\int_0^T \int_{\Omega} u^2 w_x^2 dxds + C\int_0^T \int_{\Omega} u^2 w^2 dxds.$$

For I_8,

$$\begin{aligned} |I_8| &\leq C\int_0^T \int_{\Omega} u_x^2 w^2 dxds + C\int_0^T \int_{\Omega_2(s)} w^2\theta^2 dxds \\ &\leq C\int_0^T \int_{\Omega} u_x^2 w^2 dxds + C\int_0^T \sup_{\Omega}(\theta-\frac{3}{2})_+^2 ds. \end{aligned}$$

For I_9, using (2.5), we have

$$\begin{aligned}
I_9 &= \frac{\kappa}{c_v}\int_0^T\int_\Omega \lim_{\eta\to 0}\chi_\eta(\theta)w^2(\frac{\theta_x}{v})_x dxds\\
&= \lim_{\eta\to 0}\frac{\kappa}{c_v}\int_0^T\int_\Omega\Big(-2\chi_\eta(\theta)ww_x\frac{\theta_x}{v}-\chi_\eta'(\theta)w^2\frac{\theta_x^2}{v}\Big)dxds\\
&\le \epsilon\int_0^T\int_{\Omega_2(s)}\theta_x^2dxds+C\int_0^T\int_\Omega w^2w_x^2dxds\\
&\le \epsilon\int_0^T\int_{\Omega_2(s)}\theta_x^2dxds+C.
\end{aligned}$$

For the terms I_{10}-I_{12}, it follows from Cauchy inequality that

$$|I_{10}|+|I_{11}|\le C,$$

and

$$|I_{12}|\le \epsilon\int_0^T\int_\Omega\theta_x^2dxds+C.$$

Noticing that

$$\begin{aligned}
&\int_0^T\int_\Omega\Big(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta\Big)dxds\\
=\ &\int_0^T\int_{\Omega_3(s)}\Big(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta\Big)dxds\\
&+\int_0^T\int_{\Omega-\Omega_3(s)}\Big(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta\Big)dxds\\
\le\ & 3\int_0^T\int_{\Omega_3(s)}\Big(\theta_x^2+u_x^2(\theta-2)_++w_x^2(\theta-2)_++w^2(\theta-2)_+\Big)dxds\\
&+C\int_0^T\int_\Omega\Big(\mu\frac{u_x^2}{v\theta}+\kappa\frac{\theta_x^2}{v\theta^2}+\lambda\frac{w_x^2}{v\theta}+\nu\frac{vw^2}{\theta}\Big)dxds\\
\le\ & C\int_0^T\int_{\Omega_2(s)}\Big(\frac{\kappa\theta_x^2}{v}+\frac{\mu u_x^2}{v}(\theta-2)_++\frac{\lambda w_x^2}{v}(\theta-2)_++\nu vw^2(\theta-2)_+\Big)dxds+C.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_\Omega\Big(\frac{c_v}{2}(\theta-2)_+^2+u^2(\theta-2)_++w^2(\theta-2)_+\Big)dx\\
&+\int_0^T\int_\Omega\Big(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta\Big)dxds\\
\le\ & C+C\int_0^T\int_\Omega u^2u_x^2dx+C\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds\\
&+C\int_0^T\int_\Omega u^2w_x^2dxds+C\int_0^T\int_\Omega u^2w^2dxds+C\int_0^T\int_\Omega u_x^2w^2dx.
\end{aligned}\tag{2.13}$$

Step 2. Multiplying $(1.1)_3$ by $2u^2w$ and integrating the result over $\Omega\times[0,T]$, we have

$$\begin{aligned}
&\int_\Omega u^2w^2dx+2\lambda\int_0^T\int_\Omega\frac{u^2w_x^2}{v}dxds+2\nu\int_0^T\int_\Omega vu^2w^2dxds\\
=\ &\int_\Omega u_0^2w_0^2dx-4\lambda\int_0^T\int_\Omega\frac{uu_xww_x}{v}dxds+2\int_0^T\int_\Omega uu_sw^2dxds.
\end{aligned}$$

Using $(1.1)_2$, we obtain

$$\begin{aligned}&\int_\Omega u^2w^2dx+2\lambda\int_0^T\int_\Omega \frac{u^2w_x^2}{v}dxds+2\nu\int_0^T\int_\Omega vu^2w^2dxds\\&+2\mu\int_0^T\int_\Omega\frac{u_x^2w^2}{v}dxds\\=&\int_\Omega u_0^2w_0^2dx-4\lambda\int_0^T\int_\Omega\frac{uu_xww_x}{v}dxds-4\mu\int_0^T\int_\Omega\frac{uu_xww_x}{v}dxds\\&+2R\int_0^T\int_\Omega\frac{\theta u_xw^2}{v}dxds+4R\int_0^T\int_\Omega\frac{u\theta ww_x}{v}dxds=:\int_\Omega u_0^2w_0^2dx+\sum_{j=1}^4J_j.\end{aligned}\tag{2.14}$$

What's more, it follows from (2.1) and (2.6) that for any $\alpha\in[2,3]$,

$$\begin{aligned}&\sup_{0\le t\le T}\int_\Omega(v-1)^2dx+\sup_{0\le t\le T}\int_{\Omega\backslash\Omega_\alpha}(\theta-1)^2dx\\\le&\ C\sup_{0\le t\le T}\int_\Omega(v-\ln v-1)dx+C\sup_{0\le t\le T}\int_\Omega(\theta-\ln\theta-1)dx\le C.\end{aligned}\tag{2.15}$$

Then we can estimate $J_j(j=1,2,3,4)$,

$$\begin{aligned}|J_1|+|J_2|&\le C\int_0^T\int_\Omega u^2u_x^2dxds+C\int_0^T\int_\Omega w^2w_x^2dxds\\&\le C\int_0^T\int_\Omega u^2u_x^2dxds+C,\end{aligned}$$

$$\begin{aligned}|J_3|&\le\mu\int_0^T\int_\Omega\frac{u_x^2w^2}{v}dx+C\int_0^T\int_{\Omega_2(s)}w^2\theta^2dxds\\&\le\mu\int_0^T\int_\Omega\frac{u_x^2w^2}{v}dx+C\int_0^T\int_{\Omega_2(s)}w^2\theta^2dxds+C\int_0^T\int_{\Omega-\Omega_2(s)}w^2\theta^2dxds\\&\le\mu\int_0^T\int_\Omega\frac{u_x^2w^2}{v}dx+C\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds+C,\end{aligned}$$

and

$$\begin{aligned}|J_4|&\le\lambda\int_0^T\int_\Omega\frac{u^2w_x^2}{v}dx+C\int_0^T\int_{\Omega_2(s)}w^2\theta^2dxds+C\int_0^T\int_{\Omega-\Omega_2(s)}w^2\theta^2dxds\\&\le\lambda\int_0^T\int_\Omega\frac{u^2w_x^2}{v}dx+C\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds+C.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_\Omega u^2w^2dx+\int_0^T\int_\Omega\left(u^2w_x^2+u^2w^2+u_x^2w^2\right)dxds\\\le&\ C+C\int_0^T\int_\Omega u^2u_x^2dxds+C\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds.\end{aligned}\tag{2.16}$$

And similarly,

$$\begin{aligned}&\sup_{0\le t<\infty}\int_\Omega u^4dx+\int_0^\infty\int_\Omega u^2u_x^2dxds\\\le&\ C+C\delta\int_0^T\int_\Omega\theta u_x^2dxds+C\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds.\end{aligned}$$

Adding all the above inequalities, one obtains

$$\begin{aligned}
&\int_\Omega\Big(\frac{c_v}{2}(\theta-2)_+^2+u^2(\theta-2)_++w^2(\theta-2)_++u^4\Big)dx\\
&+\int_0^T\int_\Omega\Big(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta+u^2u_x^2\Big)dxds\\
\le\ & C+C\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds.
\end{aligned}\tag{2.17}$$

Step 3. We estimate the last term on the right hand side of (2.17).

$$\int_0^T\sup_\Omega(\theta-\frac{3}{2})_+^2ds\le C(\epsilon)+\epsilon\int_0^T\int_\Omega\theta_x^2dxds.\tag{2.18}$$

Then

$$\begin{aligned}
&\int_\Omega\Big(\frac{c_v}{2}(\theta-2)_+^2+u^2(\theta-2)_++w^2(\theta-2)_++u^4\Big)dx\\
&+\int_0^T\int_\Omega\Big(\theta_x^2+u_x^2\theta+w_x^2\theta+w^2\theta+u^2u_x^2\Big)dxds\\
\le\ & C.
\end{aligned}\tag{2.19}$$

The proof of Lemma 3.4 is complete. □

2.4. **Uniform estimates on the derivatives of v, u, w, θ.**

Lemma 2.5. *There exists a positive constant C such that for any $T>0$, it holds that*

$$\sup_{0\le t\le T}\int_\Omega(v_x^2+u_x^2+\theta_x^2+w_x^2)dx+\int_0^T\int_\Omega(\theta v_x^2+u_{xx}^2+\theta_{xx}^2+w_{xx}^2)dxds\le C.\tag{2.20}$$

Moreover,

$$\sup_{\Omega\times[0,T]}\theta(x,t)\le C.\tag{2.21}$$

Proof. The process will be divided into five steps:

Step 1. Integrating $(1.1)_2\times\frac{v_x}{v}$ over Ω, we get by using $(1.1)_1$

$$\frac{\mu}{2}\frac{d}{dt}\int_\Omega(\frac{v_x}{v})^2dx+R\int_\Omega\frac{\theta v_x^2}{v^3}dx=\frac{d}{dt}\int_\Omega u\frac{v_x}{v}dx+R\int_\Omega\frac{v_x\theta_x}{v^2}dx+\int_\Omega\frac{u_x^2}{v}dx.$$

Integrating the above equality on $[0,T]$, we have

$$\begin{aligned}
&\int_\Omega(\frac{v_x}{v})^2dx+R\int_0^T\int_\Omega\frac{\theta v_x^2}{v^3}dxds\\
\le\ & C+\int_\Omega u\frac{v_x}{v}dx+C\int_0^T\int_\Omega(\frac{v_x\theta_x}{v^2}+\frac{u_x^2}{v})dxds\\
\le\ & C+\delta\int_\Omega(\frac{v_x}{v})^2dx+C(\delta)\int_\Omega u^2dx\\
&+\delta\int_0^T\int_\Omega\theta v_x^2dxds+C(\delta)\int_0^T\int_\Omega\frac{\theta_x^2}{\theta}dxds.
\end{aligned}$$

It follows from Cauchy inequality, (1.6), (2.1), (2.6), (2.7) and (2.18) that

$$\sup_{0\le t\le T}\int_\Omega v_x^2dx+\int_0^T\int_\Omega\theta v_x^2dxds\le C.\tag{2.22}$$

Step 2. Integrating $(1.1)_2 \times (-u_{xx})$ over Ω leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 dx + \mu\int_\Omega \frac{u_{xx}^2}{v}dx = \mu\int_\Omega \frac{u_x v_x u_{xx}}{v^2}dx - R\int_\Omega \frac{u_{xx}\theta_x}{v}dx + R\int_\Omega \frac{\theta v_x u_{xx}}{v^2}dx. \tag{2.23}$$

Using (2.7), (2.22) and Sobolev inequality, we get

$$\begin{aligned}
&\int_0^T \Big| \mu\int_\Omega \frac{u_x v_x u_{xx}}{v^2}dx - R\int_\Omega \frac{u_{xx}\theta_x}{v}dx + R\int_\Omega \frac{\theta v_x u_{xx}}{v^2}dx \Big| ds \\
\leq\ & \frac{\mu}{4}\int_0^T\int_\Omega \frac{u_{xx}^2}{v}dxds + C\int_0^T\int_\Omega u_x^2 v_x^2 dxds + C\int_0^T\int_\Omega \theta_x^2 dxds + C\int_0^T\int_\Omega \theta^2 v_x^2 dxds \\
\leq\ & C + \frac{\mu}{4}\int_0^T\int_\Omega \frac{u_{xx}^2}{v}dxds + C\sup_{\Omega\times[0,T]}\theta\int_0^T\int_\Omega \theta v_x^2 dxds + C\int_0^T \| u_x \|_{L^\infty(\Omega)}^2 ds \\
\leq\ & C + \frac{\mu}{2}\int_0^T\int_\Omega \frac{u_{xx}^2}{v}dxds + C\sup_{\Omega\times[0,T]}\theta,
\end{aligned} \tag{2.24}$$

then by using (2.23), one has

$$\sup_{0\leq t\leq T}\int_\Omega u_x^2 dx + \int_0^T\int_\Omega u_{xx}^2 dxds \leq C + C\sup_{\Omega\times[0,T]}\theta. \tag{2.25}$$

Step 3. Integrating $(2.8)\times(-\theta_{xx})$ over Ω leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \theta_x^2 dx + \kappa\int_\Omega \frac{\theta_{xx}^2}{v}dx \quad &= \kappa\int_\Omega \frac{\theta_x v_x \theta_{xx}}{v^2}dx - \mu\int_\Omega \frac{u_x^2\theta_{xx}}{v}dx \\
&+R\int_\Omega \frac{\theta u_x \theta_{xx}}{v}dx - \int_\Omega \nu v w^2\theta_{xx}dx - \lambda\int_\Omega \frac{w_x^2\theta_{xx}}{v}dx,
\end{aligned} \tag{2.26}$$

then using Cauchy inequality, (2.7), (2.22), (2.25), Sobolev inequality, we get

$$\begin{aligned}
&\int_0^T \Big| \kappa\int_\Omega \frac{\theta_x v_x \theta_{xx}}{v^2}dx - \mu\int_\Omega \frac{u_x^2\theta_{xx}}{v}dx + R\int_\Omega \frac{\theta u_x\theta_{xx}}{v} - \int_\Omega \nu v w^2 \theta_{xx}dx \\
&-\lambda\int_\Omega \frac{w_x^2\theta_{xx}}{v}dx \Big| ds \\
\leq\ & C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} \| \theta_x \|_{L^\infty(\Omega)} \| v_x \|_{L^2(\Omega)} ds \\
&+C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} \| u_x \|_{L^\infty(\Omega)} \| u_x \|_{L^2(\Omega)} ds \\
&+C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} \| \theta \|_{L^\infty(\Omega)} \| u_x \|_{L^2(\Omega)} ds + C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} ds \\
\leq\ & C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} \| \theta_x \|_{L^2(\Omega)}^{\frac{1}{2}} \| \theta_{xx} \|_{L^2(\Omega)}^{\frac{1}{2}} \| v_x \|_{L^2(\Omega)} ds \\
&+C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} \| u_x \|_{H^1(\Omega)} \| u_x \|_{L^2(\Omega)} ds \\
&+C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} \| u_x \|_{H^1(\Omega)} \| \theta \|_{L^\infty(\Omega)} ds + C\int_0^T \| \theta_{xx} \|_{L^2(\Omega)} ds \\
\leq\ & \frac{\kappa}{4}\int_0^T\int_\Omega \frac{\theta_{xx}^2}{v}dxds + C + C\sup_{\Omega\times[0,T]}\theta^3.
\end{aligned} \tag{2.27}$$

Hence we obtain by using (2.26) that

$$\sup_{0\le t\le T}\int_\Omega \theta_x^2dx+\int_0^T\int_\Omega \theta_{xx}^2dxds\le C+C\sup_{\Omega\times[0,T]}\theta^3. \tag{2.28}$$

Step 4. Integrating $(1.1)_3\times(-w_{xx})$ over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w_x^2dx+\lambda\int_\Omega\frac{w_{xx}^2}{v}dx=\lambda\int_\Omega\frac{w_xv_xw_{xx}}{v^2}dx-\nu\int_\Omega vww_{xx}dx, \tag{2.29}$$

using (2.1), (2.22) and Lemma 2.2, one has

$$\int_0^T\mid\lambda\int_\Omega\frac{w_xv_xw_{xx}}{v^2}dx-\nu\int_\Omega vww_{xx}dx\mid ds\le\frac{\lambda}{4}\int_0^T\int_\Omega\frac{w_{xx}^2}{v}dxds+C. \tag{2.30}$$

Then, by using (2.29), we get

$$\sup_{0\le t\le T}\int_\Omega w_x^2dx+\int_0^T\int_\Omega w_{xx}^2dxds\le C. \tag{2.31}$$

Step 5. Using Sobolev inequality and (2.7), for any $0\le t\le T$

$$\begin{aligned}\|\ \theta-1\ \|_{C(\Omega)}^2&\le C\ \|\ \theta-1\ \|_{L^2(\Omega)}\|\ \theta_x\ \|_{L^2(\Omega)}\\&\le C\ \|\ \theta_x\ \|_{L^2(\Omega)}\ .\end{aligned} \tag{2.32}$$

Combining (2.28), yields

$$\sup_{\Omega\times[0,T]}(\theta-1)^2\le C+C\max_{\bar\Omega\times[0,T]}\theta^{\frac{3}{2}}.$$

It implies that there exists a positive constant C for any $(x,t)\in\bar\Omega\times[0,T]$

$$\theta(x,t)\le C. \tag{2.33}$$

Hence, putting together (2.22), (2.25), (2.28), (2.31) and (2.33), we finish the proof of Lemma 2.5. □

2.5. **Large-time behavior of global solutions.**

Lemma 2.6. *Under the conditions of Theorem 1.1, it holds that*

$$\lim_{t\to\infty}(\|\ (v-1,u,w,\theta-1)(t)\ \|_{L^p(\Omega)}+\|\ (v_x,u_x,w_x,\theta_x)(t)\ \|_{L^2(\Omega)})=0, \tag{2.34}$$

for any $p>2$, there also exists a positive constant C_2

$$C_2^{-1}\le\theta(x,t)\le C_2,\quad \textit{for}\quad (x,t)\in\bar\Omega\times[0,\infty). \tag{2.35}$$

Proof. First, it follows from (2.7), (2.20), (2.23), (2.24), (2.26), (2.27), (2.29) and (2.30) that

$$\begin{aligned}&\int_0^\infty(\|\ u_x\ \|_{L^2(\Omega)}^2+|\frac{d}{dt}\|\ u_x\ \|_{L^2(\Omega)}^2|)dt+\int_0^\infty(\|\ \theta_x\ \|_{L^2(\Omega)}^2\\&+|\frac{d}{dt}\|\ \theta_x\ \|_{L^2(\Omega)}^2|)dt+\int_0^\infty(\|\ w_x\ \|_{L^2(\Omega)}^2+|\frac{d}{dt}\|\ w_x\ \|_{L^2(\Omega)}^2|)dt\\\le\ &C,\end{aligned}$$

which directly gives

$$\lim_{t\to\infty}(\|\ u_x\ \|_{L^2(\Omega)}+\|\ \theta_x\ \|_{L^2(\Omega)}+\|\ w_x\ \|_{L^2(\Omega)})=0. \tag{2.36}$$

By applying (2.36) to (2.32), we get

$$\lim_{t\to\infty} \| \theta - 1 \|_{C(\bar{\Omega})} = 0.$$

Hence, there exists some $T^* > 0$

$$\frac{1}{2} \le \theta \le \frac{3}{2}, \quad \text{for any} \quad (x,t) \in \bar{\Omega} \times [T^*, \infty). \tag{2.37}$$

Combining (2.20), leads to

$$\int_{T^*}^{\infty} \| v_x \|_{L^2(\Omega)}^2 ds \le C. \tag{2.38}$$

Then combining $(1.1)_1$ and (2.20), one has

$$\begin{aligned} & \int_{T^*}^{\infty} \left| \frac{d}{dt} \| v_x \|_{L^2(\Omega)}^2 \right| ds \\ = \quad & 2 \int_{T^*}^{\infty} \left| \int_{\Omega} u_{xx} v_x dx \right| ds \\ \le \quad & \int_{T^*}^{\infty} \int_{\Omega} u_{xx}^2 dx ds + \int_{T^*}^{\infty} \int_{\Omega} v_x^2 dx ds \le C, \end{aligned}$$

which together with (2.38) implies

$$\lim_{t\to\infty} \| v_x \|_{L^2(\Omega)} = 0. \tag{2.39}$$

Therefore, combining (2.1), (2.7), (2.36), (2.39) and (2.15), we can get (2.34).

Finally, we will establish the lower bound of θ. According to [1, 12], there exists a constant $C_3 > 2$

$$C_3^{-1} e^{-C_3 t} \le \theta, , \quad \text{for any} \quad (x,t) \in \bar{\Omega} \times [0, \infty),$$

which together with (2.37), yield

$$C_3^{-1} e^{-C_3 T^*} \le \theta.$$

Combining (2.33) and (2.35), we choose $C \triangleq \max\{C_2, C_3 e^{C_3 T^*}\}$. The proof of Lemma 3.6 is finished. $\square$

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