

## FINITE/FIXED-TIME SYNCHRONIZATION FOR COMPLEX NETWORKS VIA QUANTIZED ADAPTIVE CONTROL

YU-JING SHI\* AND YAN MA

Department of Applied Mathematics, Harbin University of Science and Technology  
Harbin 150080, China

**ABSTRACT.** In this paper, a unified theoretical method is presented to implement the finite/fixed-time synchronization control for complex networks with uncertain inner coupling. The quantized controller and the quantized adaptive controller are designed to reduce the control cost and save the channel resources, respectively. By means of the linear matrix inequalities technique, two sufficient conditions are proposed to guarantee that the synchronization error system of the complex networks is finite/fixed-time stable in virtue of the Lyapunov stability theory. Moreover, two types of setting time, which are dependent and independent on the initial values, are given respectively. Finally, the effectiveness of the control strategy is verified by a simulation example.

**1. Introduction.** In practice, many real systems can be described by complex networks, which are composed of a large number of interconnected nodes, such as social networks, food Webs, electric power grids, biological networks, and so on[1, 9]. Synchronization, which means that all dynamic nodes tend common dynamic behavior, is a typical dynamic phenomenon in complex networks. In recent years, various synchronization control strategies have been provided, which mainly include impulsive control [7], adaptive control [5, 12], sliding mode control [10], periodic intermittent control [18], and pinning control [20] etc.

The stability of synchronization error dynamic systems is one of the main research issues of synchronization control for complex networks. Recently, finite-time synchronization has received more and more attention due to its faster convergence rate and better disturbance rejection ability [32, 25, 24, 13]. However, one critical problem is that the settling time of finite-time control is heavily dependent on the initial conditions. Especially for some large complex networks in practical applications, it is often difficult even impossible to get information on the initial conditions, which could directly result in the inaccessibility of the control time. In this case, the finite-time synchronization control technology cannot be appropriately applied. In order to overcome the above-mentioned drawbacks of finite-time control, the concept of fixed-time convergence is proposed in [23]. Recently, according to this idea, the corresponding research results of fixed-time synchronization control have been presented in [30, 31, 8]. In [30], the continuous pinning controller has been designed

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\* Corresponding author: Yu-Jing Shi.

for the complex networks, and the sufficient condition has been obtained to ensure the fixed time synchronization of complex networks. In some practical applications, the stability of many systems is always affected by various random perturbations. Therefore, more researches have been focused on the fixed-time synchronization control for a class of complex networks with random noise disturbance, and the Itô integral principle has been proposed to solve the problem such as [31]. For a type of cluster networks with linear coupling and discontinuous nodes, the differential inclusion principle has been used to obtain the fixed-time cluster synchronization criterion in [8].

On the other hand, signal transmission is usually affected by bandwidth and communication channels. Due to the limited transmission capacity, signal distortion is caused, which affects the control performance of the system. Therefore, in order to solve this problem, signal quantization is an effective method to improve communication efficiency. For example, by designing aperiodically intermittent pinning controllers with logarithmic quantization, the sufficient condition for finite-time synchronization of the nonlinear systems is obtained in [26]. Based on the convex combination technique, the finite-time synchronization criteria for dynamic switching systems via quantized control is proposed in [27]. To the best of our knowledge, a quantized adaptive controller, which can make complex networks achieve synchronization in fixed-time, is not proposed. Furthermore, it is assumed that the internal coupling is known in most of the existing literature for finite-time synchronous control of complex networks. However, complex networks with a large number of nodes always encounter uncertain or unknown internal coupling [16].

Motivated by the discussions mentioned above, the finite/fixed-time synchronization control problem is investigated for complex networks with uncertain inner coupling via quantized adaptive control technology. The main contributions of this paper can be summarized as follows: (1) This paper studies a more extensive complex networks whose inner coupling is uncertain, and the interval matrix method is used to deal with the problem of the uncertain internal coupling; (2) By designing a quantized adaptive controller and adaptive law of parameters, it can effectively solve the problem of bandwidth limitations of networks; (3) A unified method is given for quantized adaptive finite/fixed time synchronization control. By adjusting the controller parameters, two types of setting time, which are dependent and independent on the initial values, can be realized, respectively; (4) Two finite/fixed-time synchronization criteria are established by the linear matrix inequalities technique.

**Notations.**  $\mathbb{R}$  is the space of real number,  $\mathbb{R}^n$  denotes the  $n$  dimensional Euclidean space and  $\|\cdot\|$  is the Euclidean norm.  $I$  denotes the identity matrix of compatible dimensions.  $\min\{\dots\}$  and  $\max\{\dots\}$  are the minimum and maximum values of this set. For real symmetric matrices,  $X$  and  $Y$  the notation  $X > Y$  means that the matrix  $X - Y$  is positive definite.  $B^T$  represents the transpose of the matrix  $B$ .  $\otimes$  is the Kronecker product.  $\text{diag}\{\dots\}$  stands for a block-diagonal matrix. The asterisk  $*$  in a matrix is used to denote the term that is induced by symmetry.

**2. Problem formulation and preliminaries.** In this paper, consider the following complex networks model:

$$\dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) + \sum_{j=1}^N l_{ij}\Gamma x_j(t) + u_i(t), \quad i = 1, \dots, N. \quad (1)$$

where  $x_i(t) \in \mathbb{R}^n$  and  $u_i(t) \in \mathbb{R}^n$  represents the states vector and control input of the  $i$ th node, respectively,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous nonlinear vector-valued function,  $A, B \in \mathbb{R}^{n \times n}$  are constant matrices.  $L = (l_{ij})_{N \times N}$  denotes outer-coupling matrix, if there exists a link from node  $j$  to node  $i$ , then  $l_{ij} > 0$ , otherwise  $l_{ij} = 0$ , for  $i \neq j$ ,  $l_{ii} = -\sum_{j=1, j \neq i}^N l_{ij}$ .  $\Gamma = \text{diag}\{r_1, r_2, \dots, r_n\}$  is an inner-coupling matrix.

Here, the coupling strength  $r_i (i = 1, 2, \dots, n)$  is unknown, but it belongs to a certain interval  $[\underline{r}_i, \bar{r}_i]$ , where  $\underline{r}_i$  and  $\bar{r}_i$  are known constants satisfying  $\underline{r}_i \leq \bar{r}_i$ . By denoting

$$\begin{aligned}\Gamma_0 &= \text{diag} \left\{ \frac{\underline{r}_1 + \bar{r}_1}{2}, \frac{\underline{r}_2 + \bar{r}_2}{2}, \dots, \frac{\underline{r}_n + \bar{r}_n}{2} \right\} \\ \Gamma_1 &= \text{diag} \left\{ \frac{\bar{r}_1 - \underline{r}_1}{2}, \frac{\bar{r}_2 - \underline{r}_2}{2}, \dots, \frac{\bar{r}_n - \underline{r}_n}{2} \right\}\end{aligned}$$

the inner-coupling matrix  $\Gamma$  can be written as  $\Gamma = \Gamma_0 + \tilde{\Gamma}$ , where  $\tilde{\Gamma} \in [-\Gamma_1, \Gamma_1]$ . Defining  $F = \tilde{\Gamma}\Gamma_1^{-1}$ , the matrix  $\Gamma$  can be further expressed as follows:

$$\Gamma = \Gamma_0 + F\Gamma_1. \quad (2)$$

and  $F$  satisfies  $FF^T = F^T F \leq I$ .

Furthermore, let  $s(t) \in \mathbb{R}^n$  be a solution of the isolate node, which is regarded as the target state and is assumed to be unique.

$$\dot{s}(t) = As(t) + Bf(s(t)). \quad (3)$$

where  $s(t)$  can be an equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit. Let the synchronization error be  $e_i(t) = x_i(t) - s(t)$  and  $f(e_i(t)) = f(x_i(t)) - f(s(t))$ . Then, subtracting (3) from (1) gives the dynamic of synchronization error in the following as:

$$\dot{e}_i(t) = Ae_i(t) + Bf(e_i(t)) + \sum_{j=1}^N l_{ij}\Gamma e_j(t) + u_i(t), \quad i = 1, \dots, N. \quad (4)$$

**Definition 2.1.** [17] The complex networks (1) is said to be finite-timely (or fixed-timely) synchronized onto (3), if there exists a settling time  $T$  depending (or independent) on the initial values, such that

$$\lim_{t \rightarrow T} \|x_i(t) - s(t)\| = 0 \text{ and } \|x_i(t) - s(t)\| \equiv 0 \text{ for } t \geq T, \quad i = 1, \dots, N.$$

**Assumption 1.** There exists a constant  $d > 0$  such that

$$\|f(z_1(t)) - f(z_2(t))\| \leq d\|z_1(t) - z_2(t)\|, \quad \forall z_1(t), z_2(t) \in \mathbb{R}^n$$

**Lemma 2.2.** [26] Let  $\eta_1, \eta_2, \dots, \eta_n \geq 0$ ,  $0 < \zeta \leq 1$ ,  $\omega > 1$ , then one can derive that

$$\sum_{i=1}^n \eta_i^\zeta \geq \left( \sum_{i=1}^n \eta_i \right)^\zeta, \quad \sum_{i=1}^n \eta_i^\omega \geq n^{1-\omega} \left( \sum_{i=1}^n \eta_i \right)^\omega$$

**Lemma 2.3.** [16] For any dimension-compatible matrices  $M, H$  and  $E$  with  $HH^T \leq I$  and a scalar  $\varepsilon > 0$ , then the following inequality holds:

$$MHE + E^T H^T M^T \leq \varepsilon MM^T + \varepsilon^{-1} E^T E.$$

**Lemma 2.4.** [19] Suppose that  $V(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C$ -regular and that  $x(t)$  is absolutely continuous on any compact interval of  $[0, +\infty)$ . If there exists a continuous function  $K(V(t)) : [0, +\infty) \rightarrow (0, +\infty)$ , with  $K(\sigma) > 0$  for  $\sigma \in (0, +\infty)$ , such that

the derivative  $\dot{V}(t) \leq -K(V(t)) = -k_1V^\alpha(t) - k_2V^\beta(t)$  and  $T = \int_0^{V(0)} \frac{1}{K(\sigma)} d\sigma < +\infty$ . Then, we have  $V(t) = 0$  for  $t \geq T$ .

1) If  $0 < \alpha < 1, 0 < \beta < 1$ ,  $V(t)$  will reach zero in a finite time, and the settling time is estimated as

$$T = \min \left\{ \frac{V^{1-\alpha}(0)}{k_1(1-\alpha)}, \frac{V^{1-\beta}(0)}{k_2(1-\beta)} \right\},$$

2) If  $0 < \alpha < 1, \beta > 1$ ,  $V(t)$  will reach zero in a fixed time, and the settling time is estimated as

$$T = \frac{1}{k_1(1-\alpha)} + \frac{1}{k_2(\beta-1)}.$$

**Remark 1.** There are two cases in Lemma 2.4, for a given parameter  $\alpha \in (0, 1)$ , the value of  $\beta$  will determine the settling time is finite or fixed, that is to say, the finite-time convergence can be achieved if  $0 < \beta < 1$  as claimed in case 1 and the fixed-time convergence can be realized if  $\beta > 1$  as described in case 2.

**3. Quantized synchronization controller design.**  $\varphi(\cdot) : \mathbb{R} \rightarrow \Omega$  is a logarithmic quantizer, where  $\Omega = \{\pm\omega_i : \omega_i = \rho^i\omega_0, i = 0, \pm 1, \pm 2, \dots\} \cup \{0\}$ , with  $\omega_0 > 0$ . According to the analysis in [26, 6], for  $\tau \in \mathbb{R}$ , the quantizer  $\varphi(\tau)$  is constructed as follows:

$$\varphi(\tau) = \begin{cases} \omega_i, & \text{if } \frac{1}{1+\delta}\omega_i \leq \tau \leq \frac{1}{1-\delta}\omega_i \\ 0, & \text{if } \tau = 0 \\ -\varphi(-\tau), & \text{if } \tau < 0 \end{cases} \quad (5)$$

where  $\delta = \frac{1-\rho}{1+\rho}$  and  $0 < \rho < 1$  is the quantized density. From equation (5) there exists  $\Delta \in [-\delta, \delta]$  such that

$$\varphi(\tau) = (1 + \Delta)\tau. \quad (6)$$

The design of the quantized synchronization controller is as follows:

$$u_i(t) = -\xi_i\varphi(e_i(t)) - \lambda_i[\varphi(e_i(t))]^{\frac{r}{\theta}} - \chi_i[\varphi(e_i(t))]^{\frac{s}{v}}. \quad (7)$$

where  $\xi_i > 0$  to be determined,  $\lambda_i > 0, \chi_i > 0$  are tunable parameters and  $r, \theta, s$  and  $v$  are positive odd integers,  $e_i(t) = [e_{i1}(t), e_{i2}(t), \dots, e_{in}(t)]^T$ ,  $\varphi(e_i(t)) = [\varphi(e_{i1}(t)), \varphi(e_{i2}(t)), \dots, \varphi(e_{in}(t))]^T$  and  $[\varphi(e_i(t))]^\sigma = [(\varphi(e_{i1}(t)))^\sigma, (\varphi(e_{i2}(t)))^\sigma, \dots, (\varphi(e_{in}(t)))^\sigma]^T$ , here  $\sigma = \frac{r}{\theta}$  or  $\sigma = \frac{s}{v}$ .

Let the matrix  $\Lambda(t) = \text{diag}\{\Lambda_1(t), \Lambda_2(t), \dots, \Lambda_N(t)\}$ ,  $\Lambda_i(t) = \text{diag}\{\Lambda_{i1}(t), \Lambda_{i2}(t), \dots, \Lambda_{in}(t)\}$ , and satisfy  $\Lambda_{ij}(t) \in [-\delta, \delta]$ ,  $i = 1, 2, \dots, N, j = 1, 2, \dots, n$ . Therefore, according to (6) we have

$$\varphi(e_{ij}(t)) = (1 + \Lambda_{ij}(t))e_{ij}(t). \quad (8)$$

**Theorem 3.1.** Let Assumption 1 hold, the complex networks (1) can be finite/fixed-time synchronization with (3) via the quantized controller (7), if there exist positive definite diagonal matrices  $P, M$  and scalar  $\varepsilon_1 > 0$ , such that the following LMI holds:

$$\begin{bmatrix} \bar{\Omega}_1 & P \otimes B & \bar{L} \\ * & -\mu I & 0 \\ * & * & -\varepsilon_1 I \end{bmatrix} < 0, \quad (9)$$

where  $\bar{\Omega}_1 = \mu d^2 I + [P \otimes A + (PL) \otimes \Gamma_0 - (1 - \delta)M \otimes I_n] + [P \otimes A + (PL) \otimes \Gamma_0 - (1 - \delta)M \otimes I_n]^T + \varepsilon_1 \bar{\Gamma}_1^T \bar{\Gamma}_1$ ,  $\bar{L} = (PL) \otimes I_n$ ,  $\bar{\Gamma}_1 = \text{diag}_N\{\Gamma_1\}$ ,  $M = P\xi$ ,  $P = \text{diag}\{p_1, p_2, \dots, p_N\}$ ,  $\xi = \text{diag}\{\xi_1, \xi_2, \dots, \xi_N\}$ . Then,

1) when  $r < \theta$ ,  $s < v$ , finite-time synchronization can be realized and the settling time is estimated as

$$T = \min \left\{ \frac{2\theta V^{\frac{\theta-r}{2\theta}}(0)}{\hat{\lambda}(\theta - r)}, \frac{2v V^{\frac{v-s}{2v}}(0)}{\hat{\chi}_1(v - s)} \right\}, \tag{10}$$

where  $\hat{\lambda} = 2\lambda\eta_1(1 - \delta)^{\frac{r}{\theta}}$ ,  $\hat{\chi}_1 = 2\chi\eta_2(1 - \delta)^{\frac{s}{v}}$ ,  $\lambda = \min_i \lambda_i$ ,  $\chi = \min_i \chi_i$ ,  $\eta_1 = \min_i \{p_i^{\frac{\theta-r}{2\theta}}\}$ ,  $\eta_2 = \min_i \{p_i^{\frac{v-s}{2v}}\}$ ,  $i = 1, \dots, N$ ;

2) when  $r < \theta$ ,  $s > v$ , fixed-time synchronization can be realized and the settling time is estimated as

$$T = \frac{2\theta}{\hat{\lambda}(\theta - r)} + \frac{2v}{\hat{\chi}_2(s - v)}. \tag{11}$$

where  $\hat{\chi}_2 = 2\chi\eta_2(1 - \delta)^{\frac{s}{v}}(Nn)^{\frac{v-s}{2v}}$ .

*Proof.* Choose Lyapunov function as follows

$$V(t) = \sum_{i=1}^N p_i e_i^T(t) e_i(t). \tag{12}$$

where  $p_i > 0$  is the scalar. According to the dynamic equation (4) of synchronization error and the controller (7), we can obtain

$$\begin{aligned} \dot{V}(t) &= 2 \sum_{i=1}^N p_i e_i^T(t) \{ A e_i(t) + B f(e_i(t)) + \sum_{j=1}^N l_{ij} \Gamma e_j(t) - \xi_i \varphi(e_i(t)) \\ &\quad - \lambda_i [\varphi(e_i(t))]^{\frac{r}{\theta}} - \chi_i [\varphi(e_i(t))]^{\frac{s}{v}} \} \\ &= 2 \sum_{i=1}^N p_i e_i^T(t) A e_i(t) + 2 \sum_{i=1}^N p_i e_i^T(t) B f(e_i(t)) + 2 \sum_{i=1}^N p_i e_i^T(t) \sum_{j=1}^N l_{ij} \Gamma_0 e_j(t) \\ &\quad + 2 \sum_{i=1}^N p_i e_i^T(t) \sum_{j=1}^N l_{ij} F \Gamma_1 e_j(t) - 2 \sum_{i=1}^N p_i \xi_i e_i^T(t) \varphi(e_i(t)) \\ &\quad - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{r}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{s}{v}}. \end{aligned} \tag{13}$$

According to (8) we have

$$\begin{aligned} 2 \sum_{i=1}^N p_i \xi_i e_i^T(t) \varphi(e_i(t)) &= 2 \sum_{i=1}^N p_i \xi_i \sum_{j=1}^n e_{ij}(t) \varphi(e_{ij}(t)) \\ &= 2 \sum_{i=1}^N p_i \xi_i \sum_{j=1}^n e_{ij}(t) (1 + \Lambda_{ij}(t)) e_{ij}(t) \geq 2 \sum_{i=1}^N p_i \xi_i (1 - \delta) \sum_{j=1}^n e_{ij}^2(t) \\ &= 2(1 - \delta) \sum_{i=1}^N p_i \xi_i e_i(t)^T e_i(t) = 2(1 - \delta) e^T(t) [(P\xi) \otimes I_n] e(t). \end{aligned} \tag{14}$$

From (13) and (14), we get

$$\begin{aligned} \dot{V}(t) \leq & e^T(t)(P \otimes A)e(t) + e^T(t)(P \otimes A^T)e(t) + e^T(t)(P \otimes B)f(e(t)) \\ & + f^T(e(t))(P \otimes B)^T e(t) + e^T(t)[(PL) \otimes \Gamma_0]e(t) \\ & + e^T(t)[(PL) \otimes \Gamma_0]^T e(t) + e^T(t)[(PL) \otimes (F\Gamma_1)]e(t) \\ & + e^T(t)[(PL) \otimes (F\Gamma_1)]^T e(t) - 2(1 - \delta)e^T(t)[(P\xi) \otimes I_n]e(t) \\ & - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}}. \end{aligned} \tag{15}$$

where  $e(t) = [e_1^T(t), e_2^T(t), \dots, e_N^T(t)]^T$ ,  $f(e(t)) = [f(e_1(t)), f(e_2(t)), \dots, f(e_N(t))]^T$

By Assumption 1, one can derive that

$$\|f(e_i(t))\| = \|f(x_i(t)) - f(s(t))\| \leq d\|x_i(t) - s(t)\| = d\|e_i(t)\|, \quad i = 1, \dots, N$$

such that

$$d^2 e^T(t)e(t) - f^T(e(t))f(e(t)) \geq 0. \tag{16}$$

According to (15) and (16), the following inequality can be obtained for  $\mu > 0$ :

$$\begin{aligned} \dot{V}(t) \leq & e^T(t)(P \otimes A)e(t) + e^T(t)(P \otimes A)^T e(t) + e^T(t)(P \otimes B)f(e(t)) \\ & + f^T(e(t))(P \otimes B)^T e(t) + e^T(t)[(PL) \otimes \Gamma_0]e(t) + e^T(t)[(PL) \otimes \Gamma_0]^T e(t) \\ & + e^T(t)[(PL) \otimes (F\Gamma_1)]e(t) + e^T(t)[(PL) \otimes (F\Gamma_1)]^T e(t) \\ & - 2(1 - \delta)e^T(t)[(P\xi) \otimes I_n]e(t) + \mu d^2 e^T(t)e(t) - \mu f^T(e(t))f(e(t)) \\ & - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}} \\ = & z^T \Pi z(t) - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}}. \end{aligned} \tag{17}$$

where  $z(t) = [e^T(t), f^T(e(t))]^T$ ,  $\Pi = \begin{bmatrix} \Omega_1 & P \otimes B \\ (P \otimes B)^T & -\mu I \end{bmatrix}$ ,  $\Omega_1 = \mu d^2 I + [P \otimes A + (PL) \otimes \Gamma_0 + (PL) \otimes (F\Gamma_1) - (1 - \delta)(P\xi) \otimes I_n] + [P \otimes A + (PL) \otimes \Gamma_0 + (PL) \otimes (F\Gamma_1) - (1 - \delta)(P\xi) \otimes I_n]^T$ .

Noting that the matrix  $(PL) \otimes (F\Gamma_1)$  contains an unknown parameter  $F$ , we rewrite the matrix  $(PL) \otimes (F\Gamma_1)$  as

$$(PL) \otimes (F\Gamma_1) = \bar{L} \bar{F} \bar{\Gamma}_1$$

where  $\bar{L} = (PL) \otimes I_n$ ,  $\bar{F} = \text{diag}_N\{F\}$ ,  $\bar{\Gamma}_1 = \text{diag}_N\{\Gamma_1\}$ . Furthermore, by Lemma 2.3, we can get

$$\Pi = \Pi_0 + T_1 \bar{F}^T N_1 + N_1^T \bar{F} T_1^T \leq \Pi_0 + \varepsilon_1 T_1 T_1^T + \varepsilon_1^{-1} N_1^T N_1. \tag{18}$$

where  $\Pi_0 = \begin{bmatrix} \Omega_2 & P \otimes B \\ (P \otimes B)^T & -\mu I \end{bmatrix}$ ,  $\Omega_2 = \mu d^2 I + [P \otimes A + (PL) \otimes \Gamma_0 - (1 - \delta)(P\xi) \otimes I_n] + [P \otimes A + (PL) \otimes \Gamma_0 - (1 - \delta)(P\xi) \otimes I_n]^T$ ,  $T_1 = [\bar{\Gamma}_1 \quad 0]^T$ ,  $N_1 = [\bar{L}^T \quad 0]$ . From (18) and by using the Schur complement, it is easily known that  $\Pi < 0$  is implied by the matrix inequality (9) in Theorem 3.1. Therefore, we can obtain

$$\dot{V}(t) \leq -2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}}. \tag{19}$$

If  $r < \theta$ ,  $s < v$ , by Lemma 2.2 and (8), it can be derive that

$$\begin{aligned}
& 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{r}{\theta}} = 2 \sum_{i=1}^N p_i \lambda_i \sum_{j=1}^n e_{ij}(t) [1 + \Lambda_{ij}(t)]^{\frac{r}{\theta}} e_{ij}(t)^{\frac{r}{\theta}} \\
& \geq 2 \sum_{i=1}^N p_i \lambda_i \sum_{j=1}^n (1 - \delta)^{\frac{r}{\theta}} (e_{ij}^2(t))^{\frac{r+\theta}{2\theta}} \geq 2\lambda(1 - \delta)^{\frac{r}{\theta}} \sum_{i=1}^N p_i \sum_{j=1}^n (e_{ij}^2(t))^{\frac{r+\theta}{2\theta}} \\
& \geq 2\lambda(1 - \delta)^{\frac{r}{\theta}} \sum_{i=1}^N p_i \left( \sum_{j=1}^n e_{ij}^2(t) \right)^{\frac{r+\theta}{2\theta}} = 2\lambda(1 - \delta)^{\frac{r}{\theta}} \sum_{i=1}^N p_i n^{\frac{\theta-r}{2\theta}} (p_i e_i^T(t) e_i(t))^{\frac{r+\theta}{2\theta}} \quad (20) \\
& \geq 2\lambda\eta_1(1 - \delta)^{\frac{r}{\theta}} \sum_{i=1}^N (p_i e_i^T(t) e_i(t))^{\frac{r+\theta}{2\theta}} \geq 2\lambda\eta_1(1 - \delta)^{\frac{r}{\theta}} \left( \sum_{i=1}^N p_i e_i^T(t) e_i(t) \right)^{\frac{r+\theta}{2\theta}} \\
& = 2\lambda\eta_1(1 - \delta)^{\frac{r}{\theta}} V(t)^{\frac{r+\theta}{2\theta}}.
\end{aligned}$$

Similarly, the following inequality can be obtained

$$2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{s}{v}} \geq 2\chi\eta_2(1 - \delta)^{\frac{s}{v}} V(t)^{\frac{s+v}{2v}}. \quad (21)$$

It follows from (19)-(21) that

$$\dot{V}(t) \leq -2\lambda\eta_1(1 - \delta)^{\frac{r}{\theta}} V(t)^{\frac{r+\theta}{2\theta}} - 2\chi\eta_2(1 - \delta)^{\frac{s}{v}} V(t)^{\frac{s+v}{2v}} \leq -\hat{\lambda}V^{\frac{r+\theta}{2\theta}} - \hat{\chi}_1V^{\frac{s+v}{2v}}. \quad (22)$$

If  $r < \theta$ ,  $s > v$ , by Lemma 2.2 and (8), it can be derive that

$$\begin{aligned}
& 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{s}{v}} = 2 \sum_{i=1}^N p_i \chi_i \sum_{j=1}^n e_{ij}^T(t) [1 + \Lambda_{ij}(t)]^{\frac{s}{v}} e_{ij}(t)^{\frac{s}{v}} \\
& \geq 2 \sum_{i=1}^N p_i \chi_i \sum_{j=1}^n (1 - \delta)^{\frac{s}{v}} (e_{ij}^2(t))^{\frac{s+v}{2v}} \geq 2\chi(1 - \delta)^{\frac{s}{v}} \sum_{i=1}^N p_i n^{\frac{v-s}{2v}} \left( \sum_{j=1}^n e_{ij}^2(t) \right)^{\frac{s+v}{2v}} \\
& \geq 2\chi\eta_2(1 - \delta)^{\frac{s}{v}} n^{\frac{v-s}{2v}} \sum_{i=1}^N (p_i e_i^T(t) e_i(t))^{\frac{s+v}{2v}} \quad (23) \\
& \geq 2\chi\eta_2(1 - \delta)^{\frac{s}{v}} n^{\frac{v-s}{2v}} N^{\frac{v-s}{2v}} \left( \sum_{i=1}^N p_i e_i^T(t) e_i(t) \right)^{\frac{s+v}{2v}} \\
& = 2\chi\eta_2(1 - \delta)^{\frac{s}{v}} (Nn)^{\frac{v-s}{2v}} V(t)^{\frac{s+v}{2v}}.
\end{aligned}$$

From (19),(20) and (23), it can be obtained

$$\begin{aligned}
\dot{V}(t) & \leq -2\lambda\eta_1(1 - \delta)^{\frac{r}{\theta}} V(t)^{\frac{r+\theta}{2\theta}} - 2\chi\eta_2(1 - \delta)^{\frac{s}{v}} (Nn)^{\frac{v-s}{2v}} V(t)^{\frac{s+v}{2v}} \\
& \leq -\hat{\lambda}V^{\frac{r+\theta}{2\theta}} - \hat{\chi}_2V^{\frac{s+v}{2v}}. \quad (24)
\end{aligned}$$

By Lemma 2.4, we can get  $V(t) \equiv 0$  for  $t \geq T$ , the synchronization error system (4) can be finite/fixed-time stabilized under quantized controller (7), where the settling time  $T$  satisfies (10) and (11), respectively.  $\square$

**4. Adaptive quantized synchronization controller design.** This section discusses the quantized adaptive finite/fixed-time synchronization of complex networks (1). We design a quantized adaptive controller and adaptive parameter updating law as follows

$$u_i(t) = -c\xi_i(t)\varphi(e_i(t)) - \lambda_i[\varphi(e_i(t))]^{\frac{r}{\theta}} - \chi_i[\varphi(e_i(t))]^{\frac{s}{v}}. \quad (25)$$

where  $\lambda_i > 0$ ,  $\chi_i > 0$  are tunable parameters and  $r$ ,  $\theta$ ,  $s$  and  $v$  are positive odd integers.

$$\begin{aligned} \dot{\xi}_i(t) = & (1 - \delta)q_i p_i \|e_i(t)\|^2 - k_1 \left(\frac{c}{q_i}\right)^{\frac{r-\theta}{2\theta}} \text{sign}(\xi_i(t) - \xi^*) |\xi_i(t) - \xi^*|^{\frac{r}{\theta}} \\ & - k_2 \left(\frac{c}{q_i}\right)^{\frac{s-v}{2v}} \text{sign}(\xi_i(t) - \xi^*) |\xi_i(t) - \xi^*|^{\frac{s}{v}}. \end{aligned} \quad (26)$$

where  $\xi^* > 0$  is a constant to be determined,  $k_1 > 0$ ,  $k_2 > 0$ ,  $c > 0$ ,  $q_i > 0$  are the constant.

**Theorem 4.1.** *Let Assumption 1 hold, the complex networks (1) can be finite/fixed-time synchronization with (3) via the quantized adaptive controller (25) and adaptive parameter updating law (26), if there exist positive definite diagonal matrices  $P$ ,  $M^*$  and scalars  $\xi^* > 0$ ,  $\varepsilon_1 > 0$ , such that the following LMI holds:*

$$\begin{bmatrix} \bar{\Omega}_2 & P \otimes B & \bar{L} \\ * & -\mu I & 0 \\ * & * & -\varepsilon_1 I \end{bmatrix} < 0, \quad (27)$$

where  $\bar{\Omega}_2 = \mu d^2 I + [P \otimes A + (PL) \otimes \Gamma_0 - c(1 - \delta)(M^* \otimes I_n)] + [P \otimes A + (PL) \otimes \Gamma_0 - c(1 - \delta)(M^* \otimes I_n)]^T + \varepsilon_1 \bar{\Gamma}^T \bar{\Gamma}$ ,  $M^* = \xi^* P$ .

1) when  $r < \theta$ ,  $s < v$ , finite-time synchronization can be realized and the settling time is estimated as

$$T = \min \left\{ \frac{2\theta V^{\frac{\theta-r}{2\theta}}(0)}{\hat{k}_1(\theta-r)}, \frac{2v V^{\frac{v-s}{2v}}(0)}{\hat{k}_2(v-s)} \right\}, \quad (28)$$

where  $\hat{k}_1 = \min\{\hat{\lambda}, 2k_1\}$ ,  $\hat{k}_2 = \min\{\hat{\chi}_1, 2k_2\}$ ;

2) when  $r < \theta$ ,  $s > v$ , fixed-time synchronization can be realized and the settling time is estimated as

$$T = \frac{2\theta}{\hat{k}_1(\theta-r)} + \frac{2v}{\hat{k}_3(s-v)}. \quad (29)$$

where  $\hat{k}_3 = \min\{\hat{\chi}_2, 2k_2\}$ .

*Proof.* Consider Lyapunov function as follows

$$\tilde{V}(t) = \sum_{i=1}^N p_i e_i^T(t) e_i(t) + \sum_{i=1}^N \frac{c}{q_i} (\xi_i(t) - \xi^*)^2. \quad (30)$$



It follows from (4) and (30) that

$$\begin{aligned} \dot{V}(t) \leq & 2 \sum_{i=1}^N p_i e_i^T(t) A e_i(t) + 2 \sum_{i=1}^N p_i e_i^T(t) B f(e_i(t)) + 2 \sum_{i=1}^N p_i e_i^T(t) \sum_{j=1}^N l_{ij} \Gamma e_j(t) \\ & - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}} \\ & - 2c(1-\delta) \sum_{i=1}^N p_i \xi^* \|e_i(t)\|^2 - 2k_1 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{r+\theta}{2\theta}} |\xi_i(t) - \xi^*|^{\frac{r+\theta}{\theta}} \\ & - 2k_2 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{s+v}{2v}} |\xi_i(t) - \xi^*|^{\frac{s+v}{v}}. \end{aligned} \quad (31)$$

According to (16) and (31), for  $\mu > 0$  it is derived that

$$\begin{aligned} \dot{V}(t) \leq & e^T(t)(P \otimes A)e(t) + e^T(t)(P \otimes A)^T e(t) + e^T(t)(P \otimes B)f(e(t)) \\ & + f^T(e(t))(P \otimes B)^T e(t) + e^T(t)[(PL) \otimes \Gamma_0]e(t) + e^T(t)[(PL) \otimes \Gamma_0]^T e(t) \\ & + e^T(t)[(PL) \otimes (F\Gamma_1)]e(t) + e^T(t)[(PL) \otimes (F\Gamma_1)]^T e(t) \\ & - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}} \\ & - 2c(1-\delta) \xi^* e^T(t)(P \otimes I_n)e(t) - 2k_1 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{r+\theta}{2\theta}} |\xi_i(t) - \xi^*|^{\frac{r+\theta}{\theta}} \\ & - 2k_2 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{s+v}{2v}} |\xi_i(t) - \xi^*|^{\frac{s+v}{v}} + \mu d^2 e^T(t)e(t) - \mu f^T(e(t))f(e(t)) \\ \leq & z^T(t) \tilde{\Pi} z(t) - 2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{\alpha}{\nu}} \\ & - 2k_1 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{r+\theta}{2\theta}} |\xi_i(t) - \xi^*|^{\frac{r+\theta}{\theta}} - 2k_2 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{s+v}{2v}} |\xi_i(t) - \xi^*|^{\frac{s+v}{v}}. \end{aligned} \quad (32)$$

where  $\tilde{\Pi} = \begin{bmatrix} \Omega_3 & P \otimes B \\ (P \otimes B)^T & -\mu I \end{bmatrix}$ ,  $\Omega_3 = \mu d^2 I + [P \otimes A + (PL) \otimes \Gamma_0 + (PL) \otimes (F\Gamma_1) - c(1-\delta)\xi^*(P \otimes I_n)] + [P \otimes A + (PL) \otimes \Gamma_0 + (PL) \otimes (F\Gamma_1) - c(1-\delta)\xi^*(P \otimes I_n)]^T$ .

Using the similar proof method of Theorem 3.1, the following inequality can be obtained

$$\tilde{\Pi} = \tilde{\Pi}_0 + T_1 \bar{F}^T N_1 + N_1^T \bar{F} T_1^T \leq \tilde{\Pi}_0 + \varepsilon_1 T_1 T_1^T + \varepsilon_1^{-1} N_1^T N_1. \quad (33)$$

where  $\tilde{\Pi}_0 = \begin{bmatrix} \Omega_4 & P \otimes B \\ (P \otimes B)^T & -\mu I \end{bmatrix}$ ,  $\Omega_4 = \mu d^2 I + [P \otimes A + (PL) \otimes \Gamma_0 - c(1-\delta)(M^* \otimes I_n)] + [P \otimes A + (PL) \otimes \Gamma_0 - c(1-\delta)(M^* \otimes I_n)]^T$ .

From (33) and by using the Schur complement, it is easily known that  $\tilde{\Pi} < 0$  is implied by the matrix inequality (27) in Theorem 4.1. Therefore, it can be obtained

that

$$\begin{aligned} \dot{V}(t) &\leq -2 \sum_{i=1}^N p_i \lambda_i e_i^T(t) [\varphi(e_i(t))]^{\frac{r}{\theta}} - 2 \sum_{i=1}^N p_i \chi_i e_i^T(t) [\varphi(e_i(t))]^{\frac{s}{v}} \\ &\quad - 2k_1 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{r+\theta}{2\theta}} |\xi_i(t) - \xi^*|^{\frac{r+\theta}{\theta}} - 2k_2 \sum_{i=1}^N \left(\frac{c}{q_i}\right)^{\frac{s+v}{2v}} |\xi_i(t) - \xi^*|^{\frac{s+v}{v}}. \end{aligned} \quad (34)$$

If  $r < \theta$ ,  $s < v$ , from (20), (21), (34) and Lemma 2.2, we have

$$\begin{aligned} \dot{V}(t) &\leq -\hat{\lambda} \left( \sum_{i=1}^N p_i e_i^T(t) e_i(t) \right)^{\frac{r+\theta}{2\theta}} - \hat{\chi}_1 \left( \sum_{i=1}^N p_i e_i^T(t) e_i(t) \right)^{\frac{s+v}{2v}} \\ &\quad - 2k_1 \left( \sum_{i=1}^N \frac{c}{q_i} |\xi_i(t) - \xi^*|^2 \right)^{\frac{r+\theta}{2\theta}} - 2k_2 \left( \sum_{i=1}^N \frac{c}{q_i} |\xi_i(t) - \xi^*|^2 \right)^{\frac{s+v}{2v}} \\ &\leq -\hat{k}_1 [\tilde{V}(t)]^{\frac{r+\theta}{2\theta}} - \hat{k}_2 [\tilde{V}(t)]^{\frac{s+v}{2v}}. \end{aligned} \quad (35)$$

If  $r < \theta$ ,  $s > v$ , From (20),(23), (34) and Lemma 2.2, we can get

$$\begin{aligned} \dot{V}(t) &\leq -\hat{\lambda} \left( \sum_{i=1}^N p_i e_i^T(t) e_i(t) \right)^{\frac{r+\theta}{2\theta}} - \hat{\chi}_2 \left( \sum_{i=1}^N p_i e_i^T(t) e_i(t) \right)^{\frac{s+v}{2v}} \\ &\quad - 2k_1 \left( \sum_{i=1}^N \frac{c}{q_i} |\xi_i(t) - \xi^*|^2 \right)^{\frac{r+\theta}{2\theta}} - 2k_2 \left( \sum_{i=1}^N \frac{c}{q_i} |\xi_i(t) - \xi^*|^2 \right)^{\frac{s+v}{2v}} \\ &= -\hat{k}_1 [\tilde{V}(t)]^{\frac{r+\theta}{2\theta}} - \hat{k}_3 [\tilde{V}(t)]^{\frac{s+v}{2v}}. \end{aligned} \quad (36)$$

By Lemma 2.4, we can get  $V(t) \equiv 0$  for  $t \geq T$ , the synchronization error system (4) can be finite/fixed-time stabilized under adaptive quantized controller (25) and adaptive parameter updating law (26). The settling time  $T$  satisfies (28) and (29), respectively.  $\square$

**Remark 2.** In Theorem 3.1 and Theorem 4.1, by adjusting different parameters of the unified quantized controller, both finite time and fixed time synchronization goals are achieved. By choosing of parameters  $r, \theta$ , such that  $\frac{r}{\theta} \in (0, 1)$  and  $s$  and  $v$  are in different parameter ranges, which results in different settling time. When  $s < v$ , it can be seen from (10) and (28) that the settling time is dependent of the initial value. While  $s > v$ , it can be seen from (11) and (29) that the settling time is independent of the initial value, which is only dependent of the parameters of the controller.

**Remark 3.** From Theorem 4.1, we can see that the parameters  $k_1, k_2, \lambda_i$  and  $\chi_i (i = 1, \dots, N)$  can affect the settling time for synchronization. From (28) and (29), we can see that  $k_1, k_2, \lambda_i, \chi_i (i = 1, \dots, N)$  are larger, the settling time  $T$  will decrease. Furthermore, from (25) and (26) it is known that the increase of  $k_1, k_2, \lambda_i, \chi_i (i = 1, \dots, N)$  will lead to the controller gain and adaptive updating law gain raising. Therefore, the choice of  $k_1, k_2, \lambda_i, \chi_i (i = 1, \dots, N)$  need a compromise with control performance and settling time.

**5. Simulation examples.** Consider the complex networks composed of five Chua’s circuits based on [2]. The signal circuit model is depicted in Figure 1. According to [11], the feedback controller and the inductor are connected in series to form the voltage  $u_i(t)$ . So the complex networks can be expressed as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} &= \begin{bmatrix} p & s & 0 \\ 1 & -1 & 1 \\ 0 & -v & 0 \end{bmatrix} \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \end{bmatrix} + \begin{bmatrix} \frac{27}{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -wf(x_{i1}) \\ 0 \\ 0 \end{bmatrix} \\ &+ \sum_{j=1}^N l_{ij}\Gamma x_j(t) + u_i(t). \end{aligned} \tag{37}$$

where  $x_{i1} = v_{i1}$ ,  $x_{i2} = v_{i2}$ ,  $x_{i3} = i_{i3}$ , and  $f(v_1) = G_{b1}v_1 + 0.5(G_{a1} - G_{b1})(|v_1 + 1| - |v_1 - 1|)$ .  $v_1$  and  $v_2$  represent the voltage at both sides of capacitor  $C_1$  and  $C_2$  in order.  $i_3$  represents the current through the inductor  $L$ .  $R_0$  and  $R$  are linear resistance.

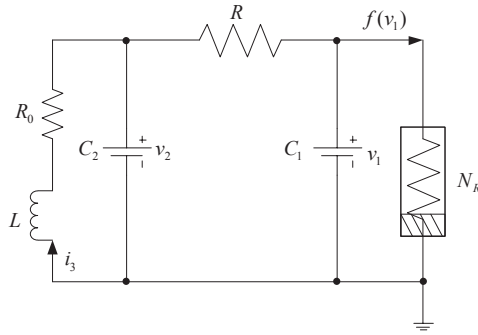


FIGURE 1. Chua’s circuit

The coupling configuration matrix is described by:

$$L = \begin{pmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 0 & 1 \\ 2 & 2 & -7 & 0 & 3 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 2 & 0 & 2 & -5 \end{pmatrix}$$

From [30], select  $p = -\frac{19}{7}$ ,  $s = 9$ ,  $v = 14.28$ ,  $G_{a1} = -0.8$ ,  $G_{b1} = -0.5$ , and  $w = -1$ . Let the quantization density as  $\rho = 0.7$  and  $\Gamma = \text{diag}\{1 + 0.7\sin(t), 1 + 0.8\cos(t), 1 + 0.9\sin(t)\cos(t)\}$  is an inner-coupling matrix. We can get  $r_1 = 0.3$ ,  $r_2 = 0.2$ ,  $r_3 = 0.1$ ,  $\bar{r}_1 = 1.7$ ,  $\bar{r}_2 = 1.8$ ,  $\bar{r}_3 = 1.9$  and calculate  $\Gamma_0 = \text{diag}\{1, 1, 1\}$ ,  $\Gamma_1 = \text{diag}\{0.7, 0.8, 0.9\}$ .

The initial states of each node in the networks are  $x_1(0) = [-2, 2.6, 0]^T$ ,  $x_2(0) = [-1.5, 1.3, -6]^T$ ,  $x_3(0) = [-2.3, 6, -3.4]^T$ ,  $x_4(0) = [3, -4.1, 5.3]^T$ ,  $x_5(0) = [3, -4.4, -3]^T$ , and the initial value of the isolated node is  $s(0) = [0, 65, 0.2, 0.8]^T$ .

First of all, considering the complex network (1) and the target system (3) under uncontrolled, the open-loop responses is shown in Figure 2. It can be seen from Figure 2 that the synchronization error trajectories diverge in the case of open-loop. And then, the quantized adaptive controller (25) is applied to the Chua’s

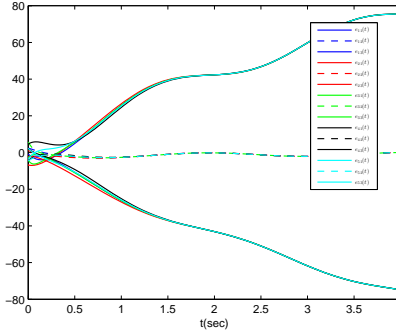


FIGURE 2. The trajectories of synchronization error without control

circuit network. The parameters of the controller are selected as:  $\lambda_i = 6$ ,  $\chi_i = 6$ ,  $i = 1, 2, \dots, 5$ ,  $r = 3$ ,  $\theta = 5$ ,  $s = 5$ ,  $v = 7$ ,  $c = 1.6$ ,  $q_i = 2$ ,  $k_1 = 4$ , and  $k_2 = 6$ . It is obvious that the Assumption 1 is satisfied with  $d = 2$ . For given  $\xi_0 = \{30.8, 29, 18, 15, 24\}$  and  $\mu = 0.075$ , the following parameters can be obtained by using MATLAB to solve the LMIs (27):

$$\varepsilon_1 = 0.1908, \xi^* = 74.3160, P = \text{diag}\{0.0030, 0.0031, 0.0022, 0.0031, 0.0025\}.$$

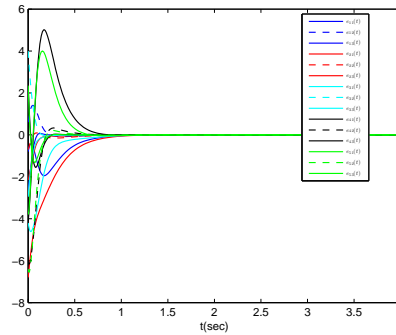


FIGURE 3. The trajectories of finite-time synchronization errors  $e_i(t)$  ( $i = 1, \dots, 5$ ) with adaptive control

Figure 3 shows the synchronization error trajectories of the complex network (1) and the target system (3) under the action of the quantized adaptive controller (25). It can be seen from the Figure 3 that the synchronization error system is stable in finite time and the settling time is estimated as follows

$$T = \min \left\{ \frac{2\theta V^{\frac{\theta-r}{2\theta}}(0)}{\hat{k}_1(\theta-r)}, \frac{2v V^{\frac{v-s}{2v}}(0)}{\hat{k}_2(v-s)} \right\} = \min\{2.7415, 2.2911\} = 2.2911. \quad (38)$$

Adjust the parameters of the controller such that  $r = 3, \theta = 5, s = 5$  and  $v = 3$ . Figure 4 shows the trajectories of fixed-time synchronization error between the complex network (1) and the target system (3) under the action of a quantized

adaptive controller (25). The corresponding settling time is estimated as follows

$$T = \frac{1}{\hat{k}_1} \frac{2\theta}{\theta - r} + \frac{1}{\hat{k}_3} \frac{2v}{s - v} = 1.3934 + 0.25 = 1.6434. \quad (39)$$

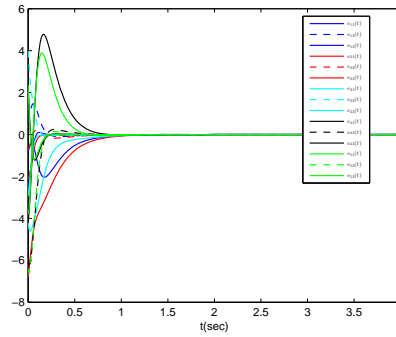


FIGURE 4. The trajectories of fixed-time synchronization errors  $e_i(t)$  ( $i = 1, \dots, 5$ ) with adaptive control

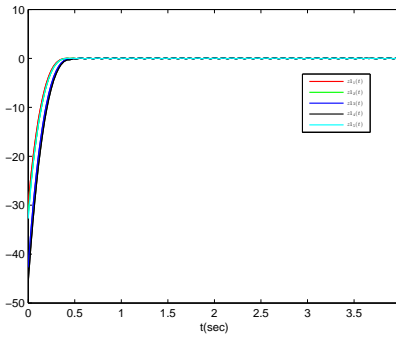


FIGURE 5. The trajectory of adaptive parameter  $\xi_i(t)$  ( $i = 1, \dots, 5$ ) for finite-time synchronization

It can be seen from Figure 4 that fixed-time stability of the synchronization error dynamic system can be achieved. Figure 5 and Figure 6 show the variation curves of adaptive parameter  $\xi_i(t)$ ,  $i = 1, 2, \dots, 5$  for the finite/fixed-time stabilization, respectively. It can be seen from the simulation results that the time-varying controller gain parameters  $\xi_i(t)$  also converges to some constants. It is obvious that the effectiveness of presented method is proved by the simulation.

**6. Conclusions.** In this paper, a unified parameterized quantized controller and quantized adaptive controller have been designed to simultaneously realize the finite/fixed-time synchronization of complex networks with uncertain internal coupling. It can be decided just by regulating the power parameters in one common controller that the setting time is either dependent or independent of the initial condition. The interval matrix method is used to describe the uncertainty of coupling

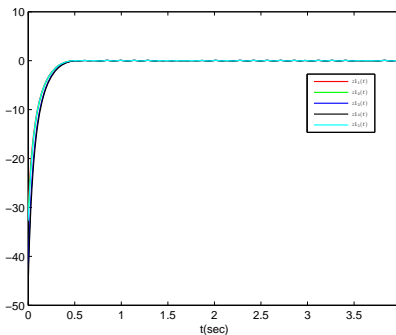


FIGURE 6. The trajectory of adaptive parameter  $\xi_i(t)$  ( $i = 1, \dots, 5$ ) for fixed-time synchronization

in networks. Based on Lyapunov stability theory and linear matrix inequalities technology, the criterion of finite/fixed-time stability for synchronization error systems of complex networks is obtained. Finally, the effectiveness of the proposed control scheme is verified by simulation. Moreover, many factors can influence the dynamic behavior of complex networks, such as random disturbances and actuator faults. Therefore, the corresponding results with the aforementioned factors will realize in the near future. It is brought to our attention that the synchronization control problem in our study is closely related to inverse problems for differential equations, see e.g. [3, 4, 21, 22, 28, 29] in the deterministic setting and [14, 15] in the random setting. It would be interesting for us to consider inverse problem techniques to the synchronization control problem in our future study.

#### REFERENCES

- [1] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, [Complex networks: Structure and dynamics](#), *Phys. Rep.*, **424** (2006), 175–308.
- [2] L. O. Chua, M. Itoh, L. Kocarev and K. Eckert, [Chaos synchronization in Chua's circuit](#), *J. Circuits Systems Comput.*, **3** (1993), 93–108.
- [3] Y. Deng, J. Li and H. Liu, [On identifying magnetized anomalies using geomagnetic monitoring within a magnetohydrodynamic model](#), *Arch. Ration. Mech. Anal.*, **235** (2020), 691–721.
- [4] Y. Deng, H. Liu and G. Uhlmann, [On an inverse boundary problem arising in brain imaging](#), *J. Differential Equations*, **267** (2019), 2471–2502.
- [5] Z. Ding and Z. Li, [Distributed adaptive consensus control of nonlinear output-feedback systems on directed graphs](#), *Automatica J. IFAC*, **72** (2016), 46–52.
- [6] N. Elia and S. K. Mitter, [Stabilization of linear systems with limited information](#), *IEEE Trans. Automat. Contr.*, **46** (2001), 1384–1400.
- [7] J. Feng, F. Yu and Y. Zhao, [Exponential synchronization of nonlinearly coupled complex networks with hybrid time-varying delays via impulsive control](#), *Nonlinear Dynam.*, **85** (2016), 621–632.
- [8] Q. Gan, F. Xiao, Y. Qin and J. Yang, [Fixed-time cluster synchronization of discontinuous directed community networks via periodically or aperiodically switching control](#), *IEEE Access*, **7** (2019), 83306–83318.
- [9] X. Ge, F. Yang and Q.-L. Han, [Distributed networked control systems: A brief overview](#), *Inf. Sci.*, **380** (2017), 117–131.
- [10] H. Hou, Q. Zhang and M. Zheng, [Cluster synchronization in nonlinear complex networks under sliding mode control](#), *Nonlinear Dynam.*, **83** (2016), 739–749.
- [11] C.-C. Hwang, J.-Y. Hsieh and R.-S. Lin, [A linear continuous feedback control of Chua's circuit](#), *Chaos Solitons Fract.*, **8** (1997), 1507–1516.

- [12] A. Khan and M. Shahzad, [Synchronization of circular restricted three body problem with Lorenz hyper chaotic system using a robust adaptive sliding mode controller](#), *Complexity*, **18** (2013), 58–64.
- [13] Q. Li, J. Guo, C. Sun, Y. Wu and Z. Ding, [Finite-time synchronization for a class of dynamical complex networks with nonidentical nodes and uncertain disturbance](#), *J. Syst. Sci. Complex.*, **32** (2019), 818–834.
- [14] J. Li, H. Liu and S. Ma, [Determining a random Schrödinger operator: Both potential and source are random](#), preprint, [arXiv:1906.01240](#).
- [15] J. Li, H. Liu and S. Ma, [Determining a random Schrödinger equation with unknown source and potential](#), *SIAM J. Math. Anal.*, **51** (2019), 3465–3491.
- [16] Q. Li, B. Shen, J. Liang and H. Shu, [Event-triggered synchronization control for complex networks with uncertain inner coupling](#), *Int. J. Gen. Syst.*, **44** (2015), 212–225.
- [17] X. Liu and T. Chen, [Finite-time and fixed-time cluster synchronization with or without pinning control](#), *IEEE Trans. Cybern.*, **48** (2018), 240–252.
- [18] X. Liu and T. Chen, [Cluster synchronization in directed networks via intermittent pinning control](#), *IEEE Trans. Neural Netw.*, **22** (2011), 1009–1020.
- [19] X. Liu, D. W. C. Ho, Q. Song and J. Cao, [Finite/fixed-time robust stabilization of switched discontinuous systems with disturbances](#), *Nonlinear Dynam.*, **90** (2017), 2057–2068.
- [20] X. Liu, D. W. C. Ho, Q. Song and W. Xu, [Finite/fixed-time pinning synchronization of complex networks with stochastic disturbances](#), *IEEE Trans. Cybern.*, **49** (2019), 2398–2403.
- [21] H. Liu and G. Uhlmann, [Determining both sound speed and internal source in thermo- and photo-acoustic tomography](#), *Inverse Problems*, **31** (2015), 10pp.
- [22] H. Liu and J. Zou, [Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers](#), *Inverse Problems*, **22** (2006), 515–524.
- [23] A. Polyakov, [Nonlinear feedback design for fixed-time stabilization of linear control systems](#), *IEEE Trans. Automat. Control*, **57** (2012), 2106–2110.
- [24] S. Qiu, Y. Huang and S. Ren, [Finite-time synchronization of multi-weighted complex dynamical networks with and without coupling delay](#), *Neurocomputing*, **275** (2018), 1250–1260.
- [25] J. Wang, T. Ru, J. Xia, Y. Wei and Z. Wang, [Finite-time synchronization for complex dynamic networks with semi-Markov switching topologies: An  \$H\_\infty\$  event-triggered control scheme](#), *Appl. Math. Comput.*, **356** (2019), 235–251.
- [26] C. Xu, X. Yang, J. Lu, J. Feng, F.E. Alsaadi and T. Hayat, [Finite-time synchronization of networks via quantized intermittent pinning control](#), *IEEE Trans. Cybern.*, **48** (2018), 3021–3027.
- [27] X. Yang, J. Cao, C. Xu and J. Feng, [Finite-time stabilization of switched dynamical networks with quantized couplings via quantized controller](#), *Sci. China Technol. Sci.*, **61** (2018), 299–308.
- [28] W. Yin, W. Yang and H. Liu, [A neural network scheme for recovering scattering obstacles with limited phaseless far-field data](#), *J. Comput. Phys.*, **417** (2020), 18pp.
- [29] D. Zhang, Y. Guo, J. Li and H. Liu, [Retrieval of acoustic sources from multi-frequency phaseless data](#), *Inverse Problems*, **34** (2018), 21pp.
- [30] W. Zhang, H. Li, C. Li, Z. Li and X. Yang, [Fixed-time synchronization criteria for complex networks via quantized pinning control](#), *ISA Trans.*, **91** (2019), 151–156.
- [31] W. Zhang, X. Yang and C. Li, [Fixed-time stochastic synchronization of complex networks via continuous control](#), *IEEE Trans. Cybern.*, **49** (2019), 3099–3104.
- [32] C. Zhou, W. Zhang, X. Yang, C. Xu and J. Feng, [Finite-time synchronization of complex-valued neural networks with mixed delays and uncertain perturbations](#), *Neural Process. Lett.*, **46** (2017), 271–291.

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*E-mail address:* [yjshi168@126.com](mailto:yjshi168@126.com)

*E-mail address:* [1023104318@qq.com](mailto:1023104318@qq.com)