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ON P_1 NONCONFORMING FINITE ELEMENT APROXIMATION FOR THE SIGNORINI PROBLEM

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Abstract. The main aim of this paper is to study the P_1 nonconforming finite element approximations of the variational inequality arisen from the Signorini problem. We describe the finite dimensional closed convex cone approximation in a meanvalue-oriented sense. In this way, the optimal convergence rate $O(h)$ can be obtained by a refined analysis when the exact solution belongs to $H^2(\Omega)$ without any assumption. Furthermore, we also study the optimal convergence for the case $u \in H^{1+\nu}(\Omega)$ with $\frac{1}{2} < \nu < 1$.

1. **Introduction.** The unilateral contact models are of great practical interest in solid mechanics and many works have been contributed to their numerical analysis. In fact, as a mostly powerful numerical method, the finite element methods for the unilateral contact models have been highlighted in the numerical simulation of variational inequalities for more than fifty years, interested readers please refer to [14, 17, 22] and the references therein.

It is well known that the unilateral contact problems are typically represented by Signorini's model, which may cause some special difficulties in both mathematical theory and numerical approximation. Most often, linear finite elements are used by the practitioners for the approximation of contact problems with unilateral Signorini boundary conditions. However, the numerical analysis of their convergence has been explored a long way. The first error estimate of conforming linear finite element approximations of this problem with frictionless boundary is probably given by Scarpini and Vivaldi (cf. [21]). They proved $\mathcal{O}(h^{3/4})$ convergence rate with the regularity $u \in H^2(\Omega)$ (for simplicity, we call it "assumption A_1"). In the same year, Brezzi, Hager and Raviart (cf. [7]) obtained optimal convergence rate $\mathcal{O}(h)$ with two additional conditions: $u|_{\partial\Omega} \in W^{1,\infty}(\partial\Omega)$ ("assumption A_2") and the number of points in the free boundary where the constraint changes from binding to nonbinding

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is finite ("assumption A_3"). In the subsequent more than twenty years, the finite element methods for the these problems developed very quickly, nevertheless, the convergence rates were still based on the results of Scarpini and Brezzi. Until 2000, Belgacem (cf. [1]) presented the quasi-optimal convergence rate $\mathcal{O}(h|\log h|^{1/2})$ by a detailed analysis and novel approach under "assumption A_1" and "assumption A_3". And later, he and his colleague Renard improved it to $\mathcal{O}(h|\log h|^{1/4})$ (cf. [2]), Hüeber and Wohlmuth [14] obtained $O(h)$ convergence by nonconforming domain decomposition methods based on dual Lagrange multipliers, also with the additional "assumption A_3".

In this work, we consider the nonconforming Crouzeix-Raviart (cf. [9]) finite element approximations to the Signorini problem. Our motivation comes from the numerical investigation, which shows that the convergence rate of P_1 nonconforming finite element method is optimal, please refer to section 5. Therefore, a natural interest arises for a better understanding of the convergence properties of the P_1 nonconforming element method. Our ultimate aim is to propose some locking-free finite element methods for the Signorini problem in incompressible elasticity. It is well-known that the linear nonconforming Crouzeix-Raviart element can avoid locking phenomena in some incompressible flows (see [6, 8, 11, 15]). What's more, the numerical results presented in [22] have showed that the nonconforming Crouzeix-Raviart element behaves better than its conforming counterpart one when used to solve some variational inequalities with small parameter.

In fact, the nonconforming Crouzeix-Raviart method was firstly considered by Wang in [23] and $O(h^{1/2})$ convergence is obtained therein. Later, Hua and Wang improved it by $\mathcal{O}(h|\log h|^{1/4})$ convergence (cf. [13]) with the additional "assumption A_3", exactly the same as the rate of conforming linear finite element method. However, [23] and [13] only consider the case $u \in H^2(\Omega)$. We extend the method to the case $u \in H^{1+\nu}(\Omega)$ with $\frac{1}{2} < \nu \leq 1$, which is more reasonable in practice. Some new techniques in the estimate of the consistency error for nonsmooth solution are developed in this paper.

Though nonconforming Crouzeix-Raviart element contains more degrees of freedom and thus involves more expensive computational cost, the results of linear nonconforming finite element method may have some attractive features. For both linear and nonlinear contact condition models, the optimal convergence rate $\mathcal{O}(h)$ of the meanvalue-oriented approximation can be obtained for $u \in H^2(\Omega)$ without "assumption A_3". This optimal result can also be obtained for the linear contact condition model by the midpoint-oriented approximation. As far as we know, this is the first time to obtain the optimal convergence rate without any supplementary hypotheses. Meanwhile, the numerical investigation presented in section 5 also shows that the P_1 nonconforming finite element method is even slightly better than the conforming one. On the other hand, nonconforming finite element methods have the striking practical advantage that each degrees of freedom belongs to at most two elements, which results in a cheap local communication and the method can be parallelized in a highly efficient manner on MIMD-machines, see e.g. [10] and the references therein.

An outline of this paper is as follows. Section 2 deals with some functional tools and the continuous setting of the Signorini problem. In section 3, we define a meanvalue-oriented type discretized method and obtain the optimal convergence for the case $u \in H^{1+\nu}(\Omega)$ with $\frac{1}{2} < \nu < 1$ with "assumption A_3". When $u \in H^2(\Omega)$, for the meanvalue-oriented approximation, the optimal convergence rate

$\mathcal{O}(h)$ can be obtained without "assumption A_3". Section 4 is concerned with the convergence properties with a midpoint-oriented type discretized method for the Signorini problem. Finally, in section 5, a numerical experiment is presented, where conforming and nonconforming linear finite elements are compared.

2. **Notations and the Signorini problem.**

For the sake of the hereafter analysis, we firstly begin with some necessary notations and functional tools, then we give a brief introduce of the Signorini problem.

Let $\Omega \subset R^2$ be a Lipschitz domain whose generic point of Ω is denoted by $\mathbf{x} = (x_1, x_2)$. The Lebesgue space $L^2(\Omega)$ is endowed with the inner product,

$$(\phi, \psi) = \int_\Omega \phi\psi d\mathbf{x}, \quad \forall \phi, \psi \in L^2(\Omega),$$

and with the norm

$$\|\psi\|_{0,\Omega} = \left(\int_\Omega \psi^2 d\mathbf{x} \right)^{\frac{1}{2}}.$$

Then the standard Sobolev spaces $H^m(\Omega), m \geq 1$, are equipped with the norm

$$\|\psi\|_{m,\Omega} = \left(\sum_{0 \leq |\alpha| \leq m} \|\psi\|_{0,\Omega}^2 \right)^{\frac{1}{2}},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index in N^2 and the symbol $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$ denotes a partial derivative. The convention $H^0(\Omega) = L^2(\Omega)$ is adopted. The fractional order Sobolev space $H^\nu(\Omega), \nu \in R_+ \setminus N$ is defined by the norm:

$$\|\psi\|_{\nu,\Omega} = \left(\|\psi\|_{m,\Omega}^2 + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(D^\alpha\psi(\mathbf{x}) - D^\alpha\psi(\mathbf{y}))^2}{|\mathbf{x} - \mathbf{y}|^{2+2\theta}} d\mathbf{x} d\mathbf{y} \right)^{\frac{1}{2}}, \tag{2.1}$$

where $\nu = m + \theta, m$ is the integer part of ν and $\theta \in (0, 1)$ is the decimal part. The closure in $H^\nu(\Omega)$ of $\mathcal{D}(\Omega)$ is denoted $H_0^\nu(\Omega)$, where $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions whose support is contained in Ω.

For any portion γ of the boundary $\partial\Omega$ and any $\nu > 0$, the Hilbert space $H^\nu(\gamma)$ is defined as the range of $H^{\nu+\frac{1}{2}}(\Omega)$ by the trace operator; it is then endowed with the image norm

$$\|\psi\|_{\nu,\gamma} = \inf_{\chi \in H^{\nu+\frac{1}{2}}(\Omega), \chi|\gamma = \psi} \|\chi\|_{\nu+\frac{1}{2}}.$$

Let the space $H^{-\nu}(\gamma)$ stand for the topological dual space of $H^\nu(\gamma)$ and the duality paring be denoted $< \cdot, \cdot >_{\nu,\gamma}$.

To be complete with the Sobolev functional tools for subsequent use, recall that for $\nu > \frac{3}{2}$, the trace operator

$$T : \psi \mapsto \left(\psi|_{\partial\Omega}, \left(\frac{\partial\psi}{\partial\mathbf{n}} \right) |_{\partial\Omega} \right)$$

is continuous from $H^\nu(\Omega)$ onto $H^{\nu-\frac{1}{2}}(\partial\Omega) \times H^{\nu-\frac{3}{2}}(\partial\Omega)$.

Suppose the boundary $\partial\Omega$ is a union of three nonoverlapping portions Γ_u, Γ_g and Γ_C. The vertices of Γ_C are $\{\mathbf{c}_1, \mathbf{c}_2\}$ and those of Γ_u are $\{\mathbf{c}_1', \mathbf{c}_2'\}$. The part Γ_u of nonzero measure is subjected to Dirichlet conditions while on Γ_g a Neumann condition is prescribed, and Γ_C is the candidate to be in contact with a rigid obstacle.

To avoid technicalities arising from the special Sobolev space $H_{00}^{\frac{1}{2}}(\Gamma_C)$, we assume that Γ_u and Γ_C do not touch.

For the given data $f \in L^2(\Omega), g \in H^{-\frac{1}{2}}(\Gamma_g)$ and $\phi \in H^{\frac{1}{2}}(\Gamma_C)$, the Signorini problem consists of *finding u that verifies in a distributional sense:*

$$-\triangle u = f \quad \text{in} \quad \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_u, \tag{2.3}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on} \quad \Gamma_g, \tag{2.4}$$

$$u - \phi \geq 0, \quad \frac{\partial u}{\partial \mathbf{n}} \geq 0, \quad (u-\phi)\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_C = \Gamma_C^0 \bigcup \Gamma_C^1, \tag{2.5}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ and $\Gamma_C^0 = \{x \in \Gamma_C : u(x) = \varphi\}, \Gamma_C^1 = \{x \in \Gamma_C : u(x) > \phi\}$.

Remark 2.1. If the function $\phi \in P_1(\Gamma_C)$, we can call the above model *linear contact condition model*. Otherwise we call it *nonlinear contact condition model*. Most papers (see [1, 2] only need to consider the case $\phi = 0$, since their analysis can be extended straightforwardly to the case $\phi \neq 0$. However, in this paper, we will show that the same approximation method may have different convergence properties for the different contact condition model. Therefore, we must treat them by different techniques.

The functional framework well suited to solve problem (2.2)-(2.5) consists in working with a subset of the following Sobolev space:

$$H^1_{\Gamma_u}(\Omega) = \left\{ v \in H^1(\Omega), \qquad v|_{\Gamma_u} = 0 \right\},$$

equipedd with the seminorm

$$|v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}.$$

By the Friedrichs inequality, the semi-norm is actually a norm in $H^1_{\Gamma_u}(\Omega)$, which is equivalent to the natural one $\|\cdot\|_{1,\Omega}$. In the weak formulation, the unilateral contact condition on Γ_C is taken into account by incorporating it in the closed cone

$$K(\Omega) = \left\{ v \in H^1_{\Gamma_u}(\Omega), \qquad v|_{\Gamma_C} \geq \phi, \quad \text{a.e.} \right\}.$$

The primal variational principle for the Signorini problem produces the following variational inequality:

$$\begin{cases} \text{find } u \in K(\Omega), \text{ such that} \\ a(u, v-u) \geq L(v-u), \ \forall \ v \in K(\Omega), \end{cases} \tag{2.6}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v d\mathbf{x},$$

$$L(v) = \int_\Omega f v d\mathbf{x} + \int_{\Gamma_g} g v ds.$$

Obviously, the bilinear and linear forms fulfill the Stampacchia theorem's hypothesis, the continuity for both of them and the ellipticity for $a(\cdot,\cdot)$. Thus, the weak formulation (2.6) is well posed and has only one solution in $K(\Omega)$ that depends continuously on the data (f, g, ϕ).

3. **A new Crouzeix-Raviart approximation.** For simplicity and to avoid more technicalities, suppose the domain Ω is polygonal in R^2. Let $\mathcal{J}_h$ be a regular triangulation of Ω with a maximum size h, and $K \in \mathcal{J}_h$ is the triangular element,

$$\overline{\Omega} = \bigcup_{K\in\mathcal{J}_h} \overline{K}.$$

Moreover, the family $\mathcal{J}_h$ is built in such a way that the end points $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_1', \mathbf{c}_2'\}$ are always chosen as the vertices of some triangular elements.

For any $K \in \mathcal{J}_h, P_k(K)$ stands for the set of polynomials of total degree $\leq k$, and $\forall F \subset \partial K, K \in \mathcal{J}_h$, for any $v \in H^{\frac{1}{2}}(K)$, we define

$$M_F(v) = \frac{1}{|F|}\int_F v ds, \quad M_K(v_h) = \frac{1}{|K|}\int_K v d\mathbf{x}.$$

Then we introduce the Crouzeix-Raviart finite element space corresponding to the partition $\mathcal{J}_h$, which is defined as

$$V_h = \{v_h \in L^2(\Omega), v_h|_K \in P_1(K), v_h \text{ is continuous regarding } M_F(\cdot), M_F(v_h) = 0, \forall F \subset \Gamma_u\}.$$

It is easy to see that V_h is not a subspace of $H^1_{\Gamma_u}(\Omega)$ and is so-called nonconforming linear finite element.

Suppose the local interpolation Π_K on an element K is defined as

$$M_F(\Pi_K v) = M_F(v), \quad \forall v \in H^{\frac{1}{2}}(K), \tag{3.1}$$

and the global interpolation Π_h,

$$\Pi_h|_K = \Pi_K, \quad \forall K \in \mathcal{J}_h.$$

Set the broken norm as

$$\|\cdot\|_h = \left(\sum_{K\in\mathcal{J}_h} |\cdot|_{1,K}^2\right)^{\frac{1}{2}}. \tag{3.2}$$

Obviously, $\|\cdot\|_h$ is a norm on V_h.

Then, we work with the following finite dimensional closed convex cone,

$$K_h(\Omega) = \{v_h \in V_h, \quad M_F(v_h) \geq M_F(\phi), \quad \forall F \subset \Gamma_C\}. \tag{3.3}$$

Note that we use a slight different discrete convex cone space from [13].

Now, we are in a position to define and study the nonconforming finite element approximation to problem (2.6), that is to say,

$$\begin{cases} \text{Find } u_h \in K_h(\Omega), \text{ such that} \\ a_h(u_h, v_h - u_h) \geq L(v_h - u_h), \forall\ v_h \in K_h(\Omega), \end{cases} \tag{3.4}$$

where

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{J}_h}\int_K \nabla u_h \nabla v_h d\mathbf{x},$$

$$L(v_h) = \int_\Omega f v_h d\mathbf{x} + \int_{\Gamma_g} g v_h ds.$$

Since the $\|\cdot\|_h$ defined as (3.2) is a norm on V_h, by Stampacchia's Theorem, it can be proved that the discrete problem (3.4) has and only has one solution $u_h \in K_h(\Omega)$. Furthermore, the following abstract error estimate holds.

Theorem 3.1. *Let $u \in K(\Omega)$ be the solution of the variational Signorini inequality (2.6), and $u_h \in K_h(\Omega)$ be the solution of the discrete problem (3.4). Further assume that $f \in L^2(\Omega), u \in H^{1+\nu}(\Omega), \frac{1}{2} < \nu \leq 1$, then we have*

$$\begin{aligned} &\|u - u_h\|_h \\ \leq & C \inf_{v_h \in K_h(\Omega)} \left\{ \|u - v_h\|_h^2 + \sum_{F \subset \Gamma_C} \int_F \frac{\partial u}{\partial \boldsymbol{n}} (v_h - u_h) ds + h^{2\nu} \|u\|_{1+\nu,\Omega}^2 + h^2 \|f\|_{0,\Omega}^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

Proof. Following the same lines of the proof of the second Strang lemma (cf. [5]), for any $v_h \in K_h(\Omega)$, we have

$$\begin{aligned} & \|u - u_h\|_h^2 = a_h(u - u_h, u - u_h) \\ = \;& a_h(u - u_h, u - v_h) + a_h(u, v_h - u_h) - a_h(u_h, v_h - u_h) \\ \leq \;& a_h(u - u_h, u - v_h) + E_h(u, v_h - u_h), \end{aligned} \tag{3.6}$$

where

$$E_h(u, v_h - u_h) = a_h(u, v_h - u_h) - L(v_h - u_h).$$

By Green's formula and the Signorini model (2.2)-(2.5),

$$\begin{aligned} E_h(u, v_h - u_h) &= \sum_{K \in \mathcal{J}_h} \int_K \nabla u \nabla (v_h - u_h) d\mathbf{x} - \int_\Omega f(v_h - u_h) d\mathbf{x} - \int_{\Gamma_g} g(v_h - u_h) ds \\ &= \sum_{K \in \mathcal{J}_h} \sum_{F \subset \partial K} \int_F \nabla u \cdot \mathbf{n} (v_h - u_h) ds - \int_{\Gamma_g} g(v_h - u_h) ds. \end{aligned} \tag{3.7}$$

Let us introduce the interpolation of zero order Raviart-Thomas element RT (cf. [20]), which is defined by

$$\int_{l_i} (\mathbf{v} - RT(\mathbf{v})) \cdot \mathbf{n} ds = 0, \quad i = 1, 2, 3, \quad \mathbf{v} \in H(\mathrm{div}, \Omega)$$

on every element K and $l_i, i = 1, 2, 3$ are three edges of K. $RT(\nabla u)$ does make sense because $\nabla u \in H(\mathrm{div}, \Omega)$. Moreover, from the definition of $RT(\nabla u)$, we know that $RT(\nabla u) \cdot \mathbf{n}$ is constant and continuous on the edges of element, so

$$\begin{aligned} & E_h(u, v_h - u_h) \\ &= \sum_{K \in \mathcal{J}_h} \sum_{F \subset \partial K} \int_F (\nabla u - RT(\nabla u)) \cdot \mathbf{n} (v_h - u_h) ds \\ &+ \sum_{F \subset \Gamma_g} \int_F (RT(\nabla u) - \nabla u) \cdot \mathbf{n} (v_h - u_h) ds + \sum_{F \subset \Gamma_C} \int_F RT(\nabla u) \cdot \mathbf{n} (v_h - u_h) ds \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.8}$$

By $\int_F (\nabla u - RT(\nabla u)) \cdot \mathbf{n} ds = 0$ and Green's formula, we have

$$\begin{aligned} I_1 &= \sum_{K \in \mathcal{J}_h} \sum_{F \subset \partial K} \int_F (\nabla u - RT(\nabla u)) \cdot \mathbf{n} \big((v_h - u_h) - M_K(v_h - u_h) \big) ds \\ &= \sum_{K \in \mathcal{J}_h} \int_K \mathrm{div}(\nabla u - RT(\nabla u)) \big((v_h - u_h) - M_K(v_h - u_h) \big) d\mathbf{x} \end{aligned}$$

$$\begin{aligned} &+ \sum_{K\in\mathcal{J}_h}\int_K (\nabla u - RT(\nabla u))\nabla(v_h-u_h)d\mathbf{x} \\ &= I_{11}+I_{12}. \end{aligned} \tag{3.9}$$

Since $\operatorname{div}(RT(\nabla u)) = M_K(\triangle u)$, so

$$I_{11} = \sum_{K\in\mathcal{J}_h}\int_K f\big((v_h-u_h)-M_K(v_h-u_h)\big)d\mathbf{x} \le Ch\|f\|_{0,\Omega}\|v_h-u_h\|_h. \tag{3.10}$$

A combination of (3.10) and the error of the interpolation RT gives

$$I_1 \le C(h\|f\|_{0,\Omega}+h^\nu|u|_{1+\nu,\Omega})\|v_h-u_h\|_h. \tag{3.11}$$

As for I_2, noticing that $RT(\nabla u)\cdot\mathbf{n}|_F = M_F(\frac{\partial u}{\partial \mathbf{n}})$,

$$\begin{aligned} I_2 &= \sum_{F\subset\Gamma_g}\int_F \left(M_F(\frac{\partial u}{\partial\mathbf{n}})-\frac{\partial u}{\partial\mathbf{n}}\right)\Big((v_h-u_h)-M_K(v_h-u_h)\Big)ds \\ &\le \sum_{F\subset\Gamma_g}\left\|M_F(\frac{\partial u}{\partial\mathbf{n}})-\frac{\partial u}{\partial\mathbf{n}}\right\|_{0,F}\Big\|(v_h-u_h)-M_K(v_h-u_h)\Big\|_{0,F} \\ &\le \sum_{F\subset\Gamma_g} Ch^{\nu-\frac{1}{2}}\left|\frac{\partial u}{\partial\mathbf{n}}\right|_{\nu-\frac{1}{2},F}\Big\{h^{-1}\|(v_h-u_h)-M_K(v_h-u_h)\|_{0,K}^2+h|v_h-u_h|_{1,K}^2\Big\}^{\frac{1}{2}} \\ &\le \sum_{F\subset\Gamma_g} Ch^{\nu}\left|\frac{\partial u}{\partial\mathbf{n}}\right|_{\nu-\frac{1}{2},F}\Big|v_h-u_h\Big|_{1,K} \\ &\le Ch^\nu\|u\|_{1+\nu,\Omega}\|v_h-u_h\|_h. \end{aligned} \tag{3.12}$$

Now we turn to I_3, following the same argument of I_2, we can obtain

$$\begin{aligned} I_3 &= \sum_{F\subset\Gamma_C}\int_F (RT(\nabla u)-\nabla u)\cdot\mathbf{n}(v_h-u_h)ds + \sum_{F\subset\Gamma_C}\int_F \nabla u\cdot\mathbf{n}(v_h-u_h)ds \\ &\le Ch^\nu\|u\|_{1+\nu,\Omega}\|v_h-u_h\|_h + \sum_{F\subset\Gamma_C}\int_F\frac{\partial u}{\partial\mathbf{n}}(v_h-u_h)ds. \end{aligned} \tag{3.13}$$

A combination of (3.6)-(3.13) yields

$$\begin{aligned} \|u-u_h\|_h^2 \le C\inf_{v_h\in K_h(\Omega)}\{&\|u-v_h\|_h^2+(h\|f\|_{0,\Omega}+h^\nu\|u\|_{1+\nu,\Omega}) \\ &(\|u-u_h\|_h+\|u-v_h\|_h)+\sum_{F\subset\Gamma_C}\int_F\frac{\partial u}{\partial\mathbf{n}}(v_h-u_h)ds\}. \end{aligned} \tag{3.14}$$

Then the Young's inequality asserts the desired result.

For the sake of the subsequent analysis, set

$$\Gamma_{Ch} = \{F : F\subset\partial K\cap\Gamma_C, K\in\mathcal{J}_h\}, \tag{3.15}$$

then we can divide Γ_{Ch} into the following three non-overlapping sets:

$$\begin{cases} \Gamma_{Ch}^0 = \left\{F\in\Gamma_{Ch} : F\subset\Gamma_C^0\right\}, \\ \Gamma_{Ch}^1 = \left\{F\in\Gamma_{Ch} : F\subset\Gamma_C^1\right\}, \\ \Gamma_{Ch}^- = \left\{F\in\Gamma_{Ch} : F\cap\Gamma_C^0\ne\emptyset, F\cap\Gamma_C^1\ne\emptyset\right\}. \end{cases} \tag{3.16}$$

Now, we will present the main result of this section.

Theorem 3.2. *Let $u \in K(\Omega), u_h \in K_h(\Omega)$ be the solution of (3.4) and (3.6) respectively.*

i. Assume that $f \in L^2(\Omega), u, \phi \in H^{1+\nu}(\Omega)$ with $\frac{1}{2} < \nu < 1$ and that the number of points in Γ_C, where the constraint changes from binding to nonbinding, is finite. Then, we have the following optimal error estimate,

$$\|u - u_h\|_h \leq Ch^{\nu}(\|u\|_{1+\nu,\Omega} + \|\phi\|_{1+\nu,\Omega} + \|f\|_{0,\Omega}). \tag{3.17}$$

ii. Assume that $f \in L^2(\Omega), u, \phi \in H^2(\Omega)$. Then, we have the following optimal error estimate,

$$\|u - u_h\|_h \leq Ch(\|u\|_{2,\Omega} + \|\phi\|_{2,\Omega} + \|f\|_{0,\Omega}). \tag{3.18}$$

Proof. In view of Theorem 3.1, we only need to bound the approximation error and J, where $J = \sum_{F \subset \Gamma_C} \int_F \frac{\partial u}{\partial \mathbf{n}}(v_h - u_h) ds$. Since $\Pi_h u \in K_h(\Omega)$, we can take $v_h = \Pi_h u$ in (3.5). Then by the classical interpolation result (cf. [9]),

$$\inf_{v_h \in K_h(\Omega)} \|u - v_h\|_h \leq \|u - \Pi_h u\|_h \leq Ch^{\nu}|u|_{1+\nu,\Omega}, \quad \frac{1}{2} < \nu \leq 1. \tag{3.19}$$

Now, let us concentrate on the bound of J, which is also a hard work. Noticing that $(u - \phi)\frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_C} = 0$, we have

$$\begin{aligned} J &= \sum_{F \in \Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\Pi_h u - u_h)\, ds \\ &= \sum_{F \in \Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\Pi_h(u - \phi) - (u - \phi)\Big)\, ds + \sum_{F \in \Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\Pi_h \phi - u_h)\, ds \\ &= J_1 + J_2. \end{aligned} \tag{3.20}$$

By the definition of the interpolation (3.1), J_1 can be written as

$$J_1 = \sum_{F \in \Gamma_{Ch}} \int_F \left(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right)\Big(\Pi_h(u - \phi) - (u - \phi)\Big)\, ds.$$

Observing that

$$\begin{aligned} &\Big\|\Pi_h(u - \phi) - (u - \phi)\Big\|_{0,F} \\ &\leq C\left\{h^{-1}\|\Pi_h(u - \phi) - (u - \phi)\|_{0,K}^2 + h|\Pi_h(u - \phi) - (u - \phi)|_{1,K}^2\right\}^{\frac{1}{2}} \\ &\leq Ch^{\frac{1}{2}+\nu}\Big|u - \phi\Big|_{1+\nu,K}, \end{aligned} \tag{3.21}$$

together with (see Lemma 7.1 of [3])

$$\left\|\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right\|_{0,F} \leq Ch^{\nu - \frac{1}{2}}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu - \frac{1}{2}, F}, \tag{3.22}$$

yield the estimate

$$\begin{aligned} J_1 &\leq \sum_{F \in \Gamma_{Ch}} \Big\|\Pi_h(u - \phi) - (u - \phi)\Big\|_{0,F} \left\|\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right\|_{0,F} \\ &\leq Ch^{2\nu}\Big|u - \phi\Big|_{1+\nu,\Omega}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu - \frac{1}{2}, \Gamma_C} \leq Ch^{2\nu}(\|u\|_{1+\nu,\Omega}^2 + \|\phi\|_{1+\nu,\Omega}^2). \end{aligned} \tag{3.23}$$

We are in a position to bound J_2, it can be written as

$$\begin{aligned} J_2 &= \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\Pi_h\phi - u_h)\,ds \\ &= \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\Pi_h\phi - \phi)\,ds + \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\phi - u_h)\,ds \\ &= J_{21} + J_{22}. \end{aligned} \tag{3.24}$$

By the same argument as J_1, J_{21} can be estimated as

$$\begin{aligned} J_{21} &= \sum_{F\in\Gamma_{Ch}} \int_F \Big(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\Big)\Big(\Pi_h\phi - \phi\Big)\,ds \\ &\le Ch^{2\nu}(\|u\|^2_{1+\nu,\Omega} + \|\phi\|^2_{1+\nu,\Omega}). \end{aligned} \tag{3.25}$$

As for J_{22}, since $\frac{\partial u}{\partial \mathbf{n}}|_F = 0, F \in \Gamma^1_{Ch}$, it can be decomposed as

$$\begin{aligned} J_{22} &= \sum_{F\in\Gamma^0_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\phi - u_h)\,ds + \sum_{F\in\Gamma^-_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\phi - u_h)\,ds \\ &= J_{221} + J_{222}. \end{aligned} \tag{3.26}$$

Considering $(u-\phi)|_F = 0, \forall F \in \Gamma^0_{Ch}$ and $M_F(u_h) \ge M_F(\phi)$, we can derive

$$\begin{aligned} J_{221} = &\sum_{F\in\Gamma^0_{Ch}} \int_F \Big(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\Big)\Big(\phi - u_h\Big)\,ds \\ &+ \sum_{F\in\Gamma^0_{Ch}} M_F(\frac{\partial u}{\partial \mathbf{n}}) \int_F (\phi - u_h)\,ds \\ &\le \sum_{F\in\Gamma^0_{Ch}} \int_F \Big(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\Big)\Big(u - u_h\Big)\,ds \\ &= \sum_{F\in\Gamma^0_{Ch}} \int_F \Big(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\Big)\Big((u - u_h) - M_F(u-u_h)\Big)\,ds \\ &\le \sum_{F\in\Gamma^0_{Ch}} \Big\|\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\Big\|_{0,F} \Big\|(u - u_h) - M_F(u-u_h)\Big\|_{0,F} \\ &\le Ch^{\nu}\|u\|_{1+\nu,\Omega}\|u-u_h\|_h. \end{aligned} \tag{3.27}$$

Now, the last work is to bound the term J_{222}. For a given $F \in \Gamma^-_{Ch}$, if $(\phi - u_h) \le 0$ on F, since $\frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_C} \ge 0$, then

$$\int_F \frac{\partial u}{\partial \mathbf{n}}(\phi - u_h)ds \le 0.$$

Therefore, we only consider $F \in \Gamma^-_{Ch}$ on which there is a segment satisfies $(\phi - u_h) > 0$. On the other hand, for a such F, $M_F(u_h) \ge M_F(\phi)$, namely, $\int_F (\phi - u_h)ds < 0$, so there exists one point $Q_F \in F$, such that $(\phi - u_h)(Q_F) = 0$. Bearing this fact in mind and applying Lemma 8.1 of [3], we have

$$\begin{aligned}\int_F \frac{\partial u}{\partial \mathbf{n}}(\phi-u_h)\,ds &= \int_F \left(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right)\left(\phi-u_h\right)ds + M_F(\frac{\partial u}{\partial \mathbf{n}})\int_F (\phi-u_h)\,ds \\ &\le \int_F \left(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right)\left(\phi-u_h\right)ds \\ &\le \left\|\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right\|_{0,F}\left\|\phi-u_h\right\|_{0,F} \\ &\le Ch^{\frac{1}{2}+\nu}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left\|\frac{d(\phi-u_h)}{ds}\right\|_{0,F} \\ &\le Ch^{\frac{1}{2}+\nu}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left(\left\|\frac{d(\phi-u)}{ds}\right\|_{0,F}+\left\|\frac{d(u-u_h)}{ds}\right\|_{0,F}\right).\end{aligned} \tag{3.28}$$

Let us have a careful analysis of $F\in\Gamma_{Ch}^-$ again. If $\text{meas}(F\bigcap\Gamma_C^0)=0$, then $(u-\phi)>0$ and $\frac{\partial u}{\partial \mathbf{n}}=0$ almost everywhere on F, in this case, the above term $-\int_F \frac{\partial u}{\partial \mathbf{n}}(\phi-u_h)\,ds=0$. Otherwise, $\text{meas}(F\bigcap\Gamma_C^0)>0$, noticing that $u-\phi\in H^{1+\nu}(\Omega)\hookrightarrow C^0(\Omega)$, and by the Lebesgue theory, there must be a line segment $F'\subset F\in\Gamma_{Ch}^-$ such that $(u-\phi)|_{F'}\equiv 0$. Thus we have $\frac{d(u-\phi)(s)}{ds}|_{F'}\equiv 0$. Set $\frac{d(u-\phi)(s)}{ds}=v(s)$, when $\frac{1}{2}<\nu<1$, we can derive

$$\|v\|_{0,F}=\|v-M_{F'}(v)\|_{0,F}\le Ch^{\nu-\frac{1}{2}}\|v\|_{\nu-\frac{1}{2},F}. \tag{3.29}$$

Here the constant C depends on $F'\subset F$, however, since the number of $F\in\Gamma_{Ch}^-$ is finite for the case $\frac{1}{2}<\nu<1$, we can choose the max of them and denote it by a generic constant C.

Let us introduce the interpolation of conforming linear finite element I_h, noticing that $\int_F \frac{d(u-I_hu)}{ds}ds=0$ and $\frac{d(I_hu)}{ds}$ is a constant function, then a combination of (3.28) and (3.29) gives

$$\begin{aligned}J_{222} &= \sum_{F\in\Gamma_{Ch}^-}\int_F \frac{\partial u}{\partial \mathbf{n}}(\phi-u_h)\,ds \\ &\le Ch^{\frac{1}{2}+\nu}\sum_{F\in\Gamma_{Ch}^-}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left(\left\|\frac{d(u-u_h)}{ds}\right\|_{0,F}+\left\|\frac{d(\phi-u)}{ds}\right\|_{0,F}\right) \\ &\le Ch^{\frac{1}{2}+\nu}\sum_{F\in\Gamma_{Ch}^-}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left(\left\|\frac{d(u-I_hu)}{ds}\right\|_{0,F}+\left\|\frac{d(I_hu-u_h)}{ds}\right\|_{0,F}+\left\|\frac{d(\phi-u)}{ds}\right\|_{0,F}\right) \\ &\le Ch^{2\nu}\sum_{F\in\Gamma_{Ch}^-}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left|\frac{du}{ds}\right|_{\nu-\frac{1}{2},F}+Ch^{\nu}\sum_{F\in\Gamma_{Ch}^-}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left|I_hu-u_h\right|_{1,K} \\ &+Ch^{2\nu}\sum_{F\in\Gamma_{Ch}^-}\left|\frac{\partial u}{\partial \mathbf{n}}\right|_{\nu-\frac{1}{2},F}\left\|\frac{d(\phi-u)}{ds}\right\|_{\nu-\frac{1}{2},F} \\ &\le Ch^{2\nu}(\|u\|^2_{1+\nu,\Omega}+\|\phi\|^2_{1+\nu,\Omega})+Ch^{\nu}\|u\|_{1+\nu,\Omega}\|u-u_h\|_h.\end{aligned} \tag{3.30}$$

When $\nu = 1$, considering a point $P_F \in F' \subset F$, $v(P_F) = 0$, then we can derive

$$\|v\|_{0,F} = \left\{ \int_F \left| v^2(s) - v^2(P_F) \right| \, ds \right\}^{\frac{1}{2}}$$
$$= \left\{ \int_F \left| \int_{P_F}^s \frac{dv^2(t)}{dt} dt \right| \, ds \right\}^{\frac{1}{2}} \le C \left\{ \int_F \left\{ \int_{P_F}^s |v(t)| \left| \frac{dv(t)}{dt} \right| dt \right\} \, ds \right\}^{\frac{1}{2}} \quad (3.31)$$
$$\le C|F|^{\frac{1}{2}} \left\{ \|v\|_{\frac{1}{2},F} \|\frac{dv}{dt}\|_{-\frac{1}{2},F} \right\}^{\frac{1}{2}} \le Ch^{\frac{1}{2}} \|v\|_{\frac{1}{2},F}.$$

Combining (3.28) and (3.31), we can estimate J_{222} as

$$J_{222} = \sum_{F \in \Gamma_{Ch}^-} \int_F \frac{\partial u}{\partial \mathbf{n}} (\phi - u_h) \, ds$$
$$\le Ch^{\frac{3}{2}} \sum_{F \in \Gamma_{Ch}^-} \left| \frac{\partial u}{\partial \mathbf{n}} \right|_{\frac{1}{2},F} \left(\left\| \frac{d(u - u_h)}{ds} \right\|_{0,F} + \left\| \frac{d(\phi - u)}{ds} \right\|_{0,F} \right)$$
$$\le Ch^{\frac{3}{2}} \sum_{F \in \Gamma_{Ch}^-} \left| \frac{\partial u}{\partial \mathbf{n}} \right|_{\frac{1}{2},F} \left(\left(h^{-1} |u - u_h|_{1,K}^2 + h |u|_{2,K}^2 \right)^{\frac{1}{2}} + \left\| \frac{d(\phi - u)}{ds} \right\|_{\frac{1}{2},F} \right)$$
$$\le Ch \|u\|_{2,\Omega} \|u - u_h\|_h + Ch^2 (\|u\|_{2,\Omega}^2 + \|\phi\|_{2,\Omega}^2). \quad (3.32)$$

Then a combination of (3.5), (3.19), (3.23), (3.25), (3.27), (3.30) and (3.32) completes the proof. □

Remark 3.1. In fact, the results (ii) of Theorem 3.2 can also be extended to the quadrilateral meanvalue-oriented nonconforming rotated $\mathcal{Q}_1$ finite element (cf. [19]) with a minor modification.

4. **Another Crouzeix-Raviart element discretization.**

In this section, we will discuss an alternative approximation to the numerical model of the contact condition. A distinct idea is to enforce the nonnegativity of the Lagrange degrees of freedom of the discrete solution that are located on the contact region Γ_C, i.e., $(u_h - \phi)(m_F) \ge 0$, where m_F is midpoint of F and $F \in \Gamma_{Ch}$. The approximation of the closed convex cone is defined as

$$\widetilde{K}_h(\Omega) = \left\{ v_h \in V_h, \quad (v_h - \phi)(m_F) \ge 0, \quad F \in \Gamma_{Ch} \right\}. \quad (4.1)$$

The discrete variational inequality is expressed in the same line as the model presented in previous section and can be described to be:

$$\begin{cases} \text{Find } \widetilde{u}_h \in \widetilde{K}_h(\Omega), \text{ such that} \\ a_h(\widetilde{u}_h, \widetilde{v}_h - \widetilde{u}_h) \ge L(\widetilde{v}_h - \widetilde{u}_h), \ \forall \ \widetilde{v}_h \in \widetilde{K}_h(\Omega), \end{cases} \quad (4.2)$$

Using again Stampacchia's Theorem, we know that the approximation problem (4.2) is well posed and the discrete solution is continuous with respect to the data (f, g, ϕ). Moreover, the abstract error estimate in Theorem 3.1 is still valid. The convergent properties can be summarized in the following two theorems.

Theorem 4.1. *As for the linear contact condition model, that is to say, $\phi \in P_1(\Gamma_C)$, let $u \in K(\Omega)$ be the solution of the variational Signorini inequality (2.6), and $\widetilde{u}_h \in \widetilde{K}_h(\Omega)$ be the solution of the discrete problem (4.2).*

i. Assume that $f \in L^2(\Omega), u, \phi \in H^{1+\nu}(\Omega)$ *with* $\frac{1}{2} < \nu < 1$ *and that the number of points in* Γ_C*, where the constraint changes from binding to nonbinding, is finite. Then, we have the following optimal error estimate,*

$$\|u - \widetilde{u}_h\|_h \leq Ch^\nu(\|u\|_{1+\nu,\Omega} + \|\phi\|_{1+\nu,\Omega} + \|f\|_{0,\Omega}). \tag{4.3}$$

ii. Assume that $f \in L^2(\Omega), u, \phi \in H^2(\Omega)$*. Then, we have the following optimal error estimate,*

$$\|u - \widetilde{u}_h\|_h \leq Ch(\|u\|_{2,\Omega} + \|\phi\|_{2,\Omega} + \|f\|_{0,\Omega}). \tag{4.4}$$

Theorem 4.2. *As for the nonlinear contact condition model, let* $u \in K(\Omega)$ *be the solution of the variational Signorini inequality (2.6), and* $\widetilde{u}_h \in \widetilde{K}_h(\Omega)$ *be the solution of the discrete problem (4.2).*

i. Assume that $f \in L^2(\Omega), u, \phi \in H^{1+\nu}(\Omega)$ *with* $\frac{1}{2} < \nu < 1$ *and that the number of points in* Γ_C*, where the constraint changes from binding to nonbinding, is finite. Then, we have the following optimal error estimate,*

$$\|u - \widetilde{u}_h\|_h \leq Ch^\nu(\|u\|_{1+\nu,\Omega} + \|\phi\|_{1+\nu,\Omega} + \|f\|_{0,\Omega}). \tag{4.5}$$

(ii) Assume $f \in L^2(\Omega), u, \phi \in H^2(\Omega)$ *and that the number of points in* Γ_C*, where the constraints changes from binding to nonbinding, is finite, then we have*

$$\|u - \widetilde{u}_h\|_h \leq Ch|\log h|^{\frac{1}{4}}(\|u\|_{2,\Omega} + \|\phi\|_{2,\Omega} + \|f\|_{0,\Omega}). \tag{4.6}$$

(iii) Assume $f \in L^2(\Omega), u, \phi \in H^2(\Omega)$*, and* $u|_{\Gamma_C} \in H^{2-\frac{1}{p}}(\Gamma_C), p > 2$*, then we have*

$$\|u - \widetilde{u}_h\|_h \leq Ch(\|u\|_{2,\Omega} + \|\phi\|_{2,\Omega} + \|f\|_{0,\Omega} + h^{\frac{1}{2}-\frac{1}{p}}\|u\|_{2-\frac{1}{p},\Gamma_C}). \tag{4.7}$$

Proof of Theorem 4.1. Since $\phi \in P_1(\Gamma_C)$, then by Trapezoidal formula, we have

$$M_F(v_h) \geq M_F(\phi), \quad \forall F \in \Gamma_{Ch} \Longleftrightarrow v_h(m_F) \geq \phi(m_F), \quad \forall F \in \Gamma_{Ch}, \tag{4.8}$$

which implies that $\widetilde{K}_h(\Omega) = K_h(\Omega)$. So Theorem 4.1 is followed by the results of Theorem 3.2.

Regarding the nonlinear contact condition case, $\Pi_h u$ does not belong to $K_h(\Omega)$ any more. Thus we need another interpolation $\widetilde{\Pi}_h$, which is defined to be

$$\widetilde{\Pi}_h|_K = \widetilde{\Pi}_K, \quad \forall K \in \mathcal{J}_h, \tag{4.9}$$

and for any $v \in H^2(K)$,

$$\widetilde{\Pi}_K v(m_F) = v(m_F), \quad m_F \text{ is midpoint of } F, \quad \forall F \subset \partial K. \tag{4.10}$$

Since $\widetilde{\Pi}_h u \in \widetilde{K}_h(\Omega)$, we can take $\widetilde{\Pi}_h u$ in (3.5). Then by the known interpolation result, we have

$$\inf_{v_h \in \widetilde{K}_h(\Omega)} \|u - v_h\|_h \leq \|u - \widetilde{\Pi}_h u\|_h \leq Ch^\nu |u|_{1+\nu,\Omega}. \tag{4.11}$$

Therefore, in order to prove Theorem 4.2, we only need to estimate $\widetilde{J} = \sum\limits_{F \in \Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\widetilde{\Pi}_h u - \widetilde{u}_h)\, ds$.

Lemma 4.3. *Let* $u \in K(\Omega)$ *be the solution of the variational Signorini inequality (2.6), and* $\widetilde{u}_h \in \widetilde{K}_h(\Omega)$ *be the solution of the discrete problem (4.2). Assume* $f \in L^2(\Omega), u, \phi \in H^{1+\nu}(\Omega)$ *with* $\frac{1}{2} < \nu < 1$ *and that the number of points in* Γ_C*, where* *the constraint changes from binding to nonbinding, is finite. Then, we have the following optimal error estimate,*

$$\widetilde{J} = \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\widetilde{\Pi}_h u - \widetilde{u}_h)\, ds \leq Ch^{\nu}\|u\|_{1+\nu,\Omega}\|u-\widetilde{u}_h\|_h + Ch^{2\nu}(\|u\|^2_{1+\nu,\Omega} + \|\phi\|^2_{1+\nu,\Omega}). \tag{4.12}$$

Proof. Follows (3.20), we have

$$\widetilde{J} = \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\widetilde{\Pi}_h(u-\phi) - (u-\phi)\Big)\, ds + \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}(\widetilde{\Pi}_h \phi - \widetilde{u}_h)\, ds = \widetilde{J}_1 + \widetilde{J}_2. \tag{4.13}$$

Noticing that $\frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_C^+} = 0$, $\widetilde{J}_1$ can be presented as

$$\begin{aligned}
\widetilde{J}_1 &= \sum_{F\in\Gamma^0_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\widetilde{\Pi}_h(u-\phi) - (u-\phi)\Big)\, ds \\
&\quad + \sum_{F\in\Gamma^-_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\widetilde{\Pi}_h(u-\phi) - (u-\phi)\Big)\, ds \\
&= \widetilde{J}_{11} + \widetilde{J}_{12}.
\end{aligned} \tag{4.14}$$

Since $(u-\phi)|_F = 0, \forall F \in \Gamma^0_{Ch}$, then

$$\begin{aligned}
\widetilde{J}_{11} &= \sum_{F\in\Gamma^0_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\widetilde{\Pi}_h(u-\phi)\Big)\, ds \\
&= \sum_{F\in\Gamma^0_{Ch}} \int_F \left(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right)\Big(\widetilde{\Pi}_h(u-\phi)\Big)\, ds \\
&+ \sum_{F\in\Gamma^0_{Ch}} M_F(\frac{\partial u}{\partial \mathbf{n}}) \int_F \widetilde{\Pi}_h(u-\phi)\, ds \\
&= \sum_{F\in\Gamma^0_{Ch}} \int_F \left(\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right)\Big(\widetilde{\Pi}_h(u-\phi) - (u-\phi)\Big)\, ds \\
&\leq \sum_{F\in\Gamma^0_{Ch}} \left\|\frac{\partial u}{\partial \mathbf{n}} - M_F(\frac{\partial u}{\partial \mathbf{n}})\right\|_{0,F} \Big\|\widetilde{\Pi}_h(u-\phi) - (u-\phi)\Big\|_{0,F} \\
&\leq Ch^{2\nu}\|u\|_{1+\nu,\Omega}\|u-\phi\|_{1+\nu,\Omega}.
\end{aligned} \tag{4.15}$$

As for $\widetilde{J}_{12}$, we use the arguments developed in ([1], Lemma 2.4). Setting $p = \frac{1}{\nu}$ and $p' = \frac{1}{1-\nu}$, noticing that $\frac{\partial u}{\partial \mathbf{n}} \in H^{\nu-\frac{1}{2}}(\Gamma_C)$,$u \in H^{\nu+\frac{1}{2}}(\Gamma_C)$ and the Sobolev embedding theorem

$$H^{\nu-\frac{1}{2}}(\Gamma_C) \subset L^{p'}(\Gamma_C), \qquad H^{\nu+\frac{1}{2}}(\Gamma_C) \subset L^{p}(\Gamma_C),$$

we have $\frac{\partial u}{\partial \mathbf{n}} \in L^{p'}(\Gamma_C)$ and $u \in L^p(\Gamma_C)$. Then Hölder inequality gives

$$\begin{aligned}
\widetilde{J}_{12} &= \sum_{F\in\Gamma^-_{Ch}} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\widetilde{\Pi}_h(u-\phi)(u-\phi)\Big)\, ds \\
&\leq \sum_{F\in\Gamma^-_{Ch}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^{p'}(F)} \Big\|\widetilde{\Pi}_h(u-\phi) - (u-\phi)\Big\|_{L^p(F)}
\end{aligned}$$

$$\leq \sum_{F\in\Gamma_{Ch}^-} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^{p'}(F)} |F|^{\frac{1}{p}} \left\|\widetilde{\Pi}_h(u-\phi)-(u-\phi)\right\|_{L^\infty(F)}. \tag{4.16}$$

Resorting to the Gagliardo-Nirenberg inequality yields

$$\begin{aligned} &\|\widetilde{\Pi}_h(u-\phi)-(u-\phi)\|_{L^\infty(F)} \\ &\leq \|\widetilde{\Pi}_h(u-\phi)-(u-\phi)\|_{L^2(F)}^{\frac{1}{2}} \|\widetilde{\Pi}_h(u-\phi)-(u-\phi)\|_{1,F}^{\frac{1}{2}} \\ &\leq C|F|^\nu |u-\phi|_{\nu+\frac{1}{2},F}. \end{aligned} \tag{4.17}$$

Going back to (4.16), and recalling that the number of $F \in \Gamma_{Ch}^-$ is bounded uniformly in h, we have

$$\widetilde{J}_{12} \leq C \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^{p'}(F)} \sum_{F\in\Gamma_{Ch}^-} |F|^{2\nu} |u-\phi|_{\nu+\frac{1}{2},F} \leq Ch^{2\nu}\|u\|_{1+\nu,\Omega}\|u-\phi\|_{1+\nu,\Omega}. \tag{4.18}$$

Following the same lines of the estimate of J_2 in section 3, one can prove

$$\widetilde{J}_2 \leq Ch^\nu\|u\|_{1+\nu,\Omega}\|u-\widetilde{u}_h\|_h + Ch^{2\nu}(\|u\|_{1+\nu,\Omega}^2 + \|\phi\|_{1+\nu,\Omega}^2). \tag{4.19}$$

Then we complete the proof.

Lemma 4.4. *Let $u \in K(\Omega)$ be the solution of the variational Signorini inequality (2.6), and $\widetilde{u}_h \in \widetilde{K}_h(\Omega)$ be the solution of the discrete problem (4.2). Assume $f \in L^2(\Omega), u, \phi \in H^2(\Omega)$ and that the number of points in Γ_C, where the constraints changes from binding to nonbinding, is finite, then we have*

$$\widetilde{J} = \sum_{F\in\Gamma_{Ch}} \int_F \frac{\partial u}{\partial \boldsymbol{n}}(\widetilde{\Pi}_h u - \widetilde{u}_h)\, ds \leq Ch\|u\|_{2,\Omega}\|u-\widetilde{u}_h\|_h + Ch^2(\|u\|_{2,\Omega}^2 + \|\phi\|_{2,\Omega}^2). \tag{4.20}$$

Proof. Proceeding as the same lines of Lemma 4.3, the bounds of $\widetilde{J}_{11}$ and $\widetilde{J}_2$ are also valid for the case $\nu = 1$, but we need to reestimate $\widetilde{J}_{12}$. We adopt the techniques developed in [1] (see Lemma 5.1),

$$\begin{aligned} \widetilde{J}_{12} &= \sum_{F\in\Gamma_{Ch}^-} \int_F \frac{\partial u}{\partial \mathbf{n}}\Big(\widetilde{\Pi}_h(u-\phi)-(u-\phi)\Big)\, ds \\ &\leq \sum_{F\in\Gamma_{Ch}^-} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{0,p',F} \left\|\widetilde{\Pi}_h(u-\phi)-(u-\phi)\right\|_{0,p,F} \\ &\leq Ch^{1+\frac{1}{p}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{0,p',F} \sum_{F\in\Gamma_{Ch}^-} \Big|u-\phi\Big|_{2,K} \\ &\leq Ch^{1+\frac{1}{p}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{0,p',\Gamma_C} \Big|u-\phi\Big|_{2,\Omega} \\ &\leq C\sqrt{p'}h^{1+\frac{1}{p}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{\frac{1}{2},\Gamma_C} \Big|u-\phi\Big|_{2,\Omega}, \end{aligned} \tag{4.21}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, then we set $p' = |\log h|$, and obtain

$$\widetilde{J}_{12} \leq Ch^2|\log h|^{\frac{1}{2}}\|u\|_{2,\Omega}\|u-\phi\|_{2,\Omega}. \tag{4.22}$$

The proof of the lemma is completed. □

Lemma 4.5. *Let $u \in K(\Omega)$ be the solution of the variational Signorini inequality (2.6), and $\widetilde{u}_h \in \widetilde{K}_h(\Omega)$ be the solution of the discrete problem (4.2). Assume $u, \phi \in H^2(\Omega)$, and $u|_{\Gamma_C} \in H^{2-\frac{1}{p}}(\Gamma_C), p > 2$, then we have*

$$\widetilde{I}_3 \leq Ch^2(\|u\|^2_{2,\Omega} + \|\phi\|^2_{2,\Omega} + h^{1-\frac{2}{p}}\|u\|^2_{2-\frac{1}{p},\Gamma_C}). \tag{4.23}$$

Proof. We also only need to re-estimate $\widetilde{J}_{12}$. Observing that $\frac{\partial u}{\partial \mathbf{n}} \in W^{1-\frac{1}{p},2}(\Gamma_C) \hookrightarrow C^0(\Gamma_C)$, then it is easy to know that $\forall F \in \Gamma_{Ch}$, $\frac{\partial u}{\partial \mathbf{n}}$ vanishes at least once in F. Then by Lemma 8.1 of [3], we have

$$\begin{aligned} \widetilde{J}_{12} &\leq \sum_{F \in \Gamma^-_{Ch}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{0,F} \left\|\widetilde{\Pi}_h(u-\phi) - (u-\phi)\right\|_{0,F} \\ &\leq \sum_{F \in \Gamma^-_{Ch}} Ch^{1-\frac{1}{p}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{1-\frac{1}{p},F} Ch^{\frac{3}{2}} \left|u-\phi\right|_{2,K} \\ &\leq Ch^{\frac{5}{2}-\frac{1}{p}} \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{1-\frac{1}{p},\Gamma_C} \left|u-\phi\right|_{2,\Omega}, \end{aligned} \tag{4.24}$$

which implies the desired result. ∎

Proof of Theorem 4.2. We put together Theorem 3.1 and Lemma 4.3, Lemma 4.4 and Lemma 4.5 to obtain point $(i), (ii)$ and (iii) respectively.

Remark 4.1. Point (iii) in Theorem 4.2 is also valid for the linear conforming finite element approximations considered in [1] and [2]. The optimal convergence rate can be recovered without "Assumption A_3" here, but with a slightly higher regular condition of the exact solution on Γ_C. In fact, from [18], we know that the best u is expect to be of H^σ with $\sigma < \frac{5}{2}$ in the vicinity of Γ_C.

Remark 4.2. If $u|_{\Gamma_C} \in W^{1,\infty}(\Gamma_C)$, the optimal convergence rate can also be recovered with the additional "Assumption A_3". This result is the same as that of linear conforming finite element approximation, which has been proved by Brezzi, Hager and Raviart in [7].

Remark 4.3. The results for the case $u \in H^2(\Omega)$ presented in this section can also be extended to the quadrilateral midpoint-oriented nonconforming rotated $\mathcal{Q}_1$ finite element (cf. [19]) with a slightly modification.

5. **Numerical test.**

In order to investigate the numerical behavior of the P_1 nonconforming finite element approximation to the Signorini problem, we consider the following equation:

$$\begin{cases} -\triangle u = 2\pi \sin(2\pi \mathbf{x}), & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_u, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \Gamma_g, \\ u \geq 0, \quad \frac{\partial u}{\partial \mathbf{n}} \geq 0, \quad u\frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \Gamma_C \end{cases}$$

where $\Omega = [0,1] \times [0,1]$, $\Gamma_u = [0,1] \times \{1\}$ is the Dirichlet boundary, $\Gamma_C = [0,1] \times \{0\}$ is the contact boundary and $\partial\Omega \setminus \{\Gamma_u \cup \Gamma_C\}$ is the Neumman boundary.

Since such a problem does not admit an analytic solution, in order to obtain the convergence order, we must compute a reference solution corresponding to a mesh which is refinement as possible as we can. In this example, we take the discrete solutions of the quadratic triangular finite element with the structured meshes for mesh size $h = \frac{1}{256}$ and $h = \frac{1}{512}$ as the reference solutions (denote by u_{256} and u_{512} respectively). Then we compute u_{Ch} (resp. u_{NCh}) by conforming (resp. nonconforming) linear finite element methods using structured meshes for mesh sizes $h = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\}$, and we compare them with the reference solutions. The detailed numerical results are listed in Table 5.1 and 5.2. Here a.c.e. denotes the average convergence rate.

TABLE 5.1: *L^2-error and norm error based on u_{256}*

h	$\|u_{Ch} - u_{256}\|_{0,\Omega}$	$\|u_{NCh} - u_{256}\|_{0,\Omega}$	$\|u_{Ch} - u_{256}\|_h$	$\|u_{NCh} - u_{256}\|_h$
1/2	0.10075711	0.14182168	0.69049684	0.72541038
1/4	0.02822791	0.02318335	0.34569687	0.31868546
1/8	0.00858496	0.00637438	0.18155419	0.16303402
1/16	0.00199993	0.00163425	0.09269470	0.08283960
1/32	0.00053486	0.00042439	0.04658388	0.04170061
1/64	0.00014356	0.00009835	0.02335961	0.02090943
a.c.r.	1.89100246	2.09877300	0.97710934	1.02331438

TABLE 5.2: *L^2-error and norm error based on u_{512}*

h	$\|u_{Ch} - u_{512}\|_{0,\Omega}$	$\|u_{NCh} - u_{512}\|_{0,\Omega}$	$\|u_{Ch} - u_{512}\|_h$	$\|u_{NCh} - u_{512}\|_h$
1/2	0.10075495	0.14182150	0.69048776	0.72540780
1/4	0.02822581	0.02318398	0.34568311	0.31868764
1/8	0.00858278	0.00637450	0.18152995	0.16304308
1/16	0.00199802	0.00163432	0.09264809	0.08286176
1/32	0.00053274	0.00042440	0.04649183	0.04174806
1/64	0.00014134	0.00009866	0.02317579	0.02100774
a.c.r.	1.89549294	2.09786472	0.97938509	1.02195988

6. **Conclusion.** This paper deals with the convergence properties of the nonconforming Crouzeix-Raviart finite element approximations to the Signorini problem. It is remarkable that the optimal convergence rate $\mathcal{O}(h)$ can be obtained by the meanvalue-oriented discretized method for any $\phi \in H^2(\Omega)$ without the additional assumption that the number of points in Γ_C, where the constraints changes from binding to nonbinding, is finite. Let us mention that the optimal convergence rate is obtained without any additional assumptions. We note that though the conforming linear finite element method exhibits better numerical results than the nonconforming one for many practical problems (second order elliptic problems etc.), but this may not be true for the Signorini problem in incompressible elasticity. Then an important and attractive direction is to develop a locking-free nonconforming Crouzeix-Raviart finite element method for the Signorini problem in incompressible elasticity, which is a future work of us.

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