REGULAR RANDOM ATTRACTORS FOR NON-AUTONOMOUS STOCHASTIC REACTION-DIFFUSION EQUATIONS ON THIN DOMAINS

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Abstract. This paper deals with the limiting dynamical behavior of non-autonomous stochastic reaction-diffusion equations on thin domains. Firstly, we prove the existence and uniqueness of the regular random attractor. Then we prove the upper semicontinuity of the regular random attractors for the equations on a family of $(n + 1)$-dimensional thin domains collapses onto an n-dimensional domain.

1. **Introduction.** Let $\mathcal{Q} \subset \mathbb{R}^n$ be a bounded C^2-domain and $\mathcal{O}_\varepsilon \subset \mathbb{R}^{n+1}$ be the domain

$$\mathcal{O}_\varepsilon = \{x = (x^*, x_{n+1}) \, | x^* = (x_1, \ldots, x_n) \in \mathcal{Q} \text{ and } 0 < x_{n+1} < \varepsilon g(x^*)\},$$

where $g \in C^2(\overline{\mathcal{Q}}, (0, +\infty))$ and $0 < \varepsilon \leq 1$. Since $g \in C^2(\overline{\mathcal{Q}}, (0, +\infty))$, there exist two positive constants γ_1 and γ_2 such that

$$\gamma_1 \leq g(x^*) \leq \gamma_2, \quad \forall x^* \in \overline{\mathcal{Q}}. \tag{1}$$

Denote $\mathcal{O} = \mathcal{Q} \times (0, 1)$ and $\widetilde{\mathcal{O}} = \mathcal{Q} \times (0, \gamma_2)$ which contains $\mathcal{O}_\varepsilon$ for $0 < \varepsilon \leq 1$. Given $\tau \in \mathbb{R}$, we will study the limit of asymptotical behavior of the following stochastic reaction-diffusions equation with multiplicative noise defined on the thin domain $\mathcal{O}_\varepsilon$ as ε tends to 0:

$$
\begin{cases}
d\hat{u}^\varepsilon - \Delta\hat{u}^\varepsilon dt = (H(t, x, \hat{u}^\varepsilon(t)) + G(t, x)) \, dt + \sum_{j=1}^{m} c_j \hat{u}^\varepsilon \circ dw_j, & x \in \mathcal{O}_\varepsilon, \ t > \tau, \\
\frac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, & x \in \partial\mathcal{O}_\varepsilon,
\end{cases}
\tag{2}
$$

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with the initial condition

$$\hat{u}^{\varepsilon}(\tau, x) = \hat{\phi}^{\varepsilon}(x), \quad x \in \mathcal{O}_{\varepsilon}, \tag{3}$$

where ν_{ε} is the unit outward normal vector to $\partial \mathcal{O}_{\varepsilon}$, H is a superlinear source term, G is a function defined on $\mathbb{R} \times \widetilde{\mathcal{O}}$, $c_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$, w_j, $j = 1, 2, \ldots, m$, are independent two-sided real-valued Wiener processes on a probability space, and the symbol $\circ$ indicates that the equation is understood in the sense of Stratonovich integration.

As $\varepsilon \to 0$, we will show in certain sense that the limiting behavior of (2) is governed by the following equation:

$$\begin{cases} du^0 - \frac{1}{g} \sum\limits_{i=1}^{n} \left(g u_{y_i}^0 \right)_{y_i} dt = \left(H(t, (y^*, 0), u^0(t)) + G\left(t, (y^*, 0)\right) \right) dt \\ \qquad\qquad\qquad\qquad\qquad + \sum\limits_{j=1}^{m} c_j u^0 \circ dw_j, \quad y^* = (y_1, \ldots, y_n) \in \mathcal{Q}, \, t > \tau, \\ \frac{\partial u^0}{\partial \nu_0} = 0, \quad y^* \in \partial \mathcal{Q}, \end{cases} \tag{4}$$

with the initial condition

$$u^0(\tau, y^*) = \phi^0(y^*), \quad y^* \in \mathcal{Q}, \tag{5}$$

where ν_0 is the unit outward normal to $\partial \mathcal{Q}$.

Random attractors have been investigated in [2, 5, 10, 19, 9] in the autonomous stochastic case, and in [3, 21, 22, 23] in the non-autonomous stochastic case. Recently, the limiting dynamical behavior of stochastic partial differential equations on thin domain was studied in [16, 20, 13, 14, 11, 12, 17, 4]. However, in [17, 13], we only investigated the limiting behavior of random attractors in $L^2(\mathcal{O})$ of stochastic evolution equations on thin domain. In this paper, we will prove the existence and uniqueness of bi-spatial pullback attractor for the systems defined on fixed domain $\mathcal{O}$ converted from (2)-(3) when the initial space is $L^2(\mathcal{O})$ and the terminate space is $H^1(\mathcal{O})$ and establish upper semicontinuity result for the corresponding family of random attractors in $H^1(\mathcal{O})$ as ε approaches 0.

Let X be a Banach space. The norm of X is written as $\| \cdot \|_X$. Let $\mathcal{M} = L^2(\mathcal{Q})$ and $\mathcal{N} = L^2(\mathcal{O})$. We denote by $(\cdot, \cdot)_Y$ the inner product in a Hilbert space Y. The letter c and c_i, $i \in \mathbb{N}$, are generic positive constants which may change its values from line to line.

We organize the paper as follows. In the next section, we establish the existence of a continuous cocycle in $\mathcal{N}$ for the stochastic equation defined on the fixed domain $\mathcal{O}$ converted from (2)-(3). We also describe the existence of a continuous cocycle in $\mathcal{M}$ for the stochastic equation (4)-(5). Section 3 contains all necessary uniform estimates of the solutions. We then prove the existence and uniqueness of regular random attractors for the stochastic equations in section 4, and analyze convergence properties of the solutions as well as the random attractors in $H^1(\mathcal{O})$ in section 5.

2. **Cocycles associated with non-autonomous stochastic equations.** Here we show that there is a continuous cocycle generated by the reaction-diffusion equation defined on $\mathcal{O}_{\varepsilon}$ with multiplicative noise and deterministic non-autonomous forcing:

$$\begin{cases} d\hat{u}^\varepsilon - \Delta\hat{u}^\varepsilon dt = \left(H(t,x,\hat{u}^\varepsilon(t)) + G\left(t,x\right)\right)dt + \sum_{j=1}^{m} c_j\hat{u}^\varepsilon \circ dw_j, \\ \qquad\qquad x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon, \ t > \tau, \\ \frac{\partial\hat{u}^\varepsilon}{\partial\nu_\varepsilon} = 0, \quad x \in \partial\mathcal{O}_\varepsilon, \end{cases} \tag{6}$$

with the initial condition

$$\hat{u}_\tau^\varepsilon(x) = \hat{\phi}^\varepsilon(x), \quad x \in \mathcal{O}_\varepsilon, \tag{7}$$

where ν_ε is the unit outward normal to $\partial\mathcal{O}_\varepsilon$, $G : \mathbb{R} \times \widetilde{\mathcal{O}} \to \mathbb{R}$ belongs to $L_{loc}^2(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$, $c_j \in \mathbb{R}$, w_j $(j = 1, 2, \ldots, m)$ are independent two-sided real-valued Wiener processes on a probability space which will be specified later, and H is a nonlinear function satisfying the following conditions: for all $x \in \widetilde{\mathcal{O}}$ and $t, s \in \mathbb{R}$,

$$H\left(t,x,s\right)s \leq -\lambda_1\left|s\right|^p + \varphi_1(t,x), \tag{8}$$

$$\left|H\left(t,x,s\right)\right| \leq \lambda_2\left|s\right|^{p-1} + \varphi_2(t,x), \tag{9}$$

$$\frac{\partial H\left(t,x,s\right)}{\partial s} \leq \lambda_3, \tag{10}$$

$$\left|\frac{\partial H\left(t,x,s\right)}{\partial x}\right| \leq \psi_3\left(t,x\right), \tag{11}$$

where $p > 2$, λ_1 λ_2 and λ_3 are positive constants, $\varphi_1 \in L_{loc}^\infty(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$ and $\varphi_2, \psi_3 \in L_{loc}^2(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$.

Throughout this paper, we fix a positive number $\lambda \in (0, \lambda_1)$ and write

$$h(t, x, s) = H(t, x, s) + \lambda s \tag{12}$$

for all $x \in \widetilde{\mathcal{O}}$ and $t, s \in \mathbb{R}$. Then it follows from (8)-(11) that there exist positive numbers α_1, α_2, β, b_1 and b_2 such that

$$h\left(t,x,s\right)s \leq -\alpha_1\left|s\right|^p + \psi_1(t,x), \tag{13}$$

$$\left|h\left(t,x,s\right)\right| \leq \alpha_2\left|s\right|^{p-1} + \psi_2(t,x), \tag{14}$$

$$\frac{\partial h\left(t,x,s\right)}{\partial s} \leq \beta, \tag{15}$$

$$\left|\frac{\partial h\left(t,x,s\right)}{\partial x}\right| \leq \psi_3\left(t,x\right), \tag{16}$$

where $\psi_1(t,x) = \varphi_1(t,x) + b_1$ and $\psi_2(t,x) = \varphi_2(t,x) + b_2$ for $x \in \widetilde{\mathcal{O}}$ and $t, s \in \mathbb{R}$.

Substituting (12) into (6) we get for $t > \tau$,

$$\begin{cases} d\hat{u}^\varepsilon - \left(\Delta\hat{u}^\varepsilon - \lambda\hat{u}^\varepsilon\right)dt = \left(h(t,x,\hat{u}^\varepsilon(t)) + G\left(t,x\right)\right)dt + \sum_{j=1}^{m} c_j\hat{u}^\varepsilon \circ dw_j, \\ \qquad\qquad x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon, \\ \frac{\partial\hat{u}^\varepsilon}{\partial\nu_\varepsilon} = 0, \quad x \in \partial\mathcal{O}_\varepsilon, \end{cases} \tag{17}$$

with the initial condition

$$\hat{u}_\tau^\varepsilon(x) = \hat{\phi}^\varepsilon(x), \quad x \in \mathcal{O}_\varepsilon. \tag{18}$$

We now transfer problem (17)-(18) into an initial boundary value problem on the fixed domain $\mathcal{O}$. To that end, we introduce a transformation $T_\varepsilon : \mathcal{O}_\varepsilon \to \mathcal{O}$ by $T_\varepsilon(x^*, x_{n+1}) = \left(x^*, \frac{x_{n+1}}{\varepsilon g(x^*)}\right)$ for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$. Let $y = (y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1})$. Then we have

$$x^* = y^*, \qquad x_{n+1} = \varepsilon g\left(y^*\right)y_{n+1}.$$

It follows from [18] that the Laplace operator in the original variable $x \in \mathcal{O}_\varepsilon$ and in the new variable $y \in \mathcal{O}$ are related by

$$\Delta_x \hat{u}(x) = |J| \mathrm{div}_y(|J|^{-1} JJ^* \nabla_y u(y)) = \frac{1}{g} \mathrm{div}_y(P_\varepsilon u(y)),$$

where we denote by $u(y) = \hat{u}(x)$ and P_ε is the operator given by

$$P_\varepsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_{n+1} u_{y_{n+1}} \\ \vdots \\ g u_{y_n} - g_{y_n} y_{n+1} u_{y_{n+1}} \\ -\sum_{i=1}^{n} y_{n+1} g_{y_i} u_{y_i} + \frac{1}{\varepsilon^2 g}(1 + \sum_{i=1}^{n} (\varepsilon y_{n+1} g_{y_i})^2) u_{y_{n+1}} \end{pmatrix}.$$

In the sequel, we abuse the notation a little bit by writing $h(t, x, s)$ and $G(t, x)$ as $h(t, x^*, x_{n+1}, s)$ and $G(t, x^*, x_{n+1})$ for $x = (x^*, x_{n+1})$, respectively. With this agreement, for any function $F(t, y, s)$, we introduce

$$F_\varepsilon (t, y^*, y_{n+1}, s) = F (t, y^*, \varepsilon g (y^*) y_{n+1}, s), \qquad F_0 (t, y^*, s) = F (t, y^*, 0, s),$$

where $y = (y^*, y_{n+1}) \in \mathcal{O}$ and $t, s \in \mathbb{R}$, Then problem (17)-(18) is equivalent to the following system for $t > \tau$,

$$\begin{cases} du^\varepsilon - (\frac{1}{g} \mathrm{div}_y(P_\varepsilon u^\varepsilon) - \lambda u^\varepsilon) dt = (h_\varepsilon (t, y, u^\varepsilon(t)) + G_\varepsilon (t, y)) \, dt \\ \qquad\qquad\qquad + \sum_{j=1}^{m} c_j u^\varepsilon \circ dw_j, \quad y = (y^*, y_{n+1}) \in \mathcal{O}, \qquad (19) \\ P_\varepsilon u^\varepsilon \cdot \nu = 0, \quad y \in \partial\mathcal{O}, \end{cases}$$

with the initial condition

$$u_\tau^\varepsilon(y) = \phi^\varepsilon(y) = \hat{\phi}^\varepsilon \circ T_\varepsilon^{-1}(y), \quad y \in \mathcal{O}, \qquad (20)$$

where ν is the unit outward normal to $\partial\mathcal{O}$.

Given $t \in \mathbb{R}$, define a translation $\theta_{1,t}$ on $\mathbb{R}$ by

$$\theta_{1,t} (\tau) = \tau + t, \quad \text{for all } \tau \in \mathbb{R}. \qquad (21)$$

Then $\{\theta_{1,t}\}_{t \in \mathbb{R}}$ is a group acting on $\mathbb{R}$. We now specify the probability space. Denote by

$$\Omega = \{\omega \in C (\mathbb{R}, \mathbb{R}) : \omega (0) = 0\}.$$

Let $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $(\Omega, \mathcal{F}, P)$, which is defined by

$$\theta_t \omega (\cdot) = \omega (\cdot + t) - \omega (t), \ \omega \in \Omega, \ t \in \mathbb{R}. \qquad (22)$$

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system (see [1]). On the other hand, let us consider the one-dimensional stochastic differential equation

$$dz + \alpha z dt = dw (t), \qquad (23)$$

for $\alpha > 0$. This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process (see [6] for more details). In fact, we have

Lemma 2.1. *There exists a $\{\theta_t\}_{t \in \mathbb{R}}$-invariant subset $\Omega' \in \mathcal{F}$ of full measure such that*

$$\lim_{t \to \pm\infty} \frac{|\omega (t)|}{t} = 0 \quad \text{for all} \quad \omega \in \Omega',$$

and, for such ω, the random variable given by

$$z^*(\omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s} \omega(s)\, ds$$

is well defined. Moreover, for $\omega \in \Omega^{'}$, the mapping

$$(t,\omega) \to z^*(\theta_t \omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s} \theta_t \omega(s)\, ds = -\alpha \int_{-\infty}^{0} e^{\alpha s} \omega(t+s)\, ds + \omega(t)$$

is a stationary solution of (23) with continuous trajectories. In addition, for $\omega \in \Omega^{'}$

$$\lim_{t \to \pm\infty} \frac{|z^*(\theta_t \omega)|}{t} = 0, \quad \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z^*(\theta_s \omega)\, ds = 0, \tag{24}$$

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_s \omega)|\, ds = \mathbb{E}\,|z^*| < \infty. \tag{25}$$

Denote by z_j^* the associated Ornstein-Uhlenbeck process corresponding to (23) with $\alpha = 1$ and w replaced by w_j for $j = 1, \ldots, m$. Then for any $j = 1, \ldots, m$, we have a stationary Ornstein-Uhlenbeck process generated by a random variable $z_j^*(\omega)$ on $\Omega_j^{'}$ with properties formulated in Lemma 2.1 defined on a metric dynamical system $(\Omega_j^{'}, \mathcal{F}_j, P_j, \{\theta_t\}_{t\in\mathbb{R}})$. We set

$$\tilde{\Omega} = \Omega_1^{'} \times \cdots \times \Omega_m^{'} \quad \text{and} \quad \mathcal{F} = \mathop{\otimes}_{j=1}^{m} \mathcal{F}_j,$$

Then $(\tilde{\Omega}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

Denote by

$$S_{C_j}(t)\, u = e^{c_j t} u, \quad \text{for } u \in L^2(\mathcal{O}),$$

and

$$\mathcal{T}(\omega) := S_{C_1}(z_1^*(\omega)) \circ \cdots \circ S_{C_m}(z_m^*(\omega)) = e^{\sum_{j=1}^{m} c_j z_j^*(\omega)} Id_{L^2(\mathcal{O})}, \quad \omega \in \Omega^{'}.$$

Then for every $\omega \in \Omega^{'}$, $\mathcal{T}(\omega)$ is a homeomorphism on $L^2(\mathcal{O})$, and its inverse operator is given by

$$\mathcal{T}^{-1}(\omega) := S_{C_m}(-z_m^*(\omega)) \circ \cdots \circ S_{C_1}(-z_1^*(\omega)) = e^{-\sum_{j=1}^{m} c_j z_j^*(\omega)} Id_{L^2(\mathcal{O})}.$$

It follows that $\|\mathcal{T}^{-1}(\theta_t \omega)\|$ has sub-exponential growth as $t \to \pm\infty$ for any $\omega \in \tilde{\Omega}$. Hence $\|\mathcal{T}^{-1}\|$ is tempered. Analogously, $\|\mathcal{T}\|$ is also tempered. Obviously, $\sup_{s\in[s_0-a,s_0+a]} \|\mathcal{T}(\theta_s \omega)\|$ is still tempered for every $s_0 \in \mathbb{R}$ and $a \in \mathbb{R}^+$.

On the other hand, since $z_j^*, j = 1, \ldots, m$, are independent Gaussian random variables, by the ergodic theorem we still have a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\hat{\Omega} \in \mathcal{F}$ of full measure such that

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \|\mathcal{T}(\theta_\tau \omega)\|^2\, d\tau = \mathbb{E}\,\|\mathcal{T}\|^2 = \prod_{j=1}^{m} \mathbb{E}(e^{2c_j z_j^*}) < \infty,$$

and

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \|\mathcal{T}^{-1}(\theta_\tau \omega)\|^2\, d\tau = \mathbb{E}\,\|\mathcal{T}^{-1}\|^2 = \prod_{j=1}^{m} \mathbb{E}(e^{-2c_j z_j^*}) < \infty.$$

Remark 1. We now consider θ defined in (22) on $\tilde{\Omega} \cap \hat{\Omega}$ instead of Ω. This mapping possesses the same properties as the original one if we choose $\mathcal{F}$ as the trace σ-algebra with respect to $\tilde{\Omega} \cap \hat{\Omega}$. The corresponding metric dynamical system is still denoted by $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ throughout this paper.

Next, we define a continuous cocycle for system (19)-(20) in $\mathcal{N}$. This can be achieved by transferring the stochastic system into a deterministic one with random parameters in a standard manner. Let u^ε be a solution to (19)-(20) and denote by $v^\varepsilon(t) = \mathcal{T}^{-1}(\theta_t \omega) u^\varepsilon(t)$ and $\delta(\omega) = \sum\limits_{j=1}^{m} c_j z_j^*(\omega)$. Then v^ε satisfies

$$\begin{cases} \frac{dv^\varepsilon}{dt} - \frac{1}{g}\mathrm{div}_y(P_\varepsilon v^\varepsilon) = (-\lambda + \delta(\theta_t \omega))v^\varepsilon + \mathcal{T}^{-1}(\theta_t \omega) h_\varepsilon(t, y, \mathcal{T}(\theta_t \omega) v^\varepsilon(t)) \\ \qquad\qquad\qquad + \mathcal{T}^{-1}(\theta_t \omega) G_\varepsilon(t, y), \ y \in \mathcal{O}, \ t > \tau, \\ P_\varepsilon v^\varepsilon \cdot \nu = 0, \quad y \in \partial\mathcal{O}, \end{cases}$$

$$(26)$$

with the initial conditions

$$v_\tau^\varepsilon(y) = \psi^\varepsilon(y), \quad y \in \mathcal{O}, \tag{27}$$

where $\psi^\varepsilon = (\mathcal{T}^{-1}(\theta_\tau \omega))\phi^\varepsilon$.

Since (26) is a deterministic equation, by the Galerkin method, one can show that if H satisfies (8)-(11), then for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $\psi^\varepsilon \in \mathcal{N}$, (26)-(27) has a unique solution $v^\varepsilon(t, \tau, \omega, \psi^\varepsilon) \in C\left([\tau, \tau + T], L^2(\mathcal{O})\right) \cap L^2\left((\tau, \tau + T), H^1(\mathcal{O})\right) \cap C\left([\tau + \epsilon, \tau + T), H^1(\mathcal{O})\right)$ with $v_\tau(\cdot, \tau, \omega, \psi^\varepsilon) = \psi^\varepsilon$ for every $T > 0$ and $0 < \epsilon < T$. Furthermore, one may show that $v^\varepsilon(t, \tau, \omega, \psi^\varepsilon)$ is $(\mathcal{F}, \mathcal{B}(\mathcal{N}))$-measurable in $\omega \in \Omega$ and continuous with respect to ψ^ε in $\mathcal{N}$ for all $t \geq \tau$. Since $u^\varepsilon(t, \tau, \omega, \phi^\varepsilon) = \mathcal{T}(\theta_t \omega) v^\varepsilon(t, \tau, \omega, \psi^\varepsilon)$ with $\phi^\varepsilon = (\mathcal{T}(\theta_\tau \omega))\psi^\varepsilon$, we find that $u^\varepsilon(t)$ is continuous in both $t \geq \tau$ and $\phi^\varepsilon \in \mathcal{N}$ and is $(\mathcal{F}, \mathcal{B}(\mathcal{N}))$-measurable in $\omega \in \Omega$. In addition, it follows from (26) that u^ε is a solution of problem (19)-(20). We now define $\Phi_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{N} \to \mathcal{N}$ by

$$\begin{aligned} \Phi_\varepsilon(t, \tau, \omega, \phi^\varepsilon) &= u^\varepsilon(t + \tau, \tau, \theta_{-\tau}\omega, \phi^\varepsilon) = \mathcal{T}(\theta_{t+\tau}\omega) v^\varepsilon(t + \tau, \tau, \theta_{-\tau}\omega, \psi^\varepsilon), \\ &\qquad \text{for all } (t, \tau, \omega, \phi^\varepsilon) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{N}. \end{aligned} \tag{28}$$

By the properties of u^ε, we find that Φ_ε is a continuous cocycle on $\mathcal{N}$ over $(\mathbb{R}, \{\theta_{1,t}\}_{t \in \mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$, where $\{\theta_{1,t}\}_{t \in \mathbb{R}}$ and $\{\theta_t\}_{t \in \mathbb{R}}$ are given by (21) and (22), respectively. In this paper, we will first prove the asymptotic compactness of solutions in $H^1(\mathcal{O})$ and then establish the existence and upper semicontinuity in $H^1(\mathcal{O})$ of $(\mathcal{N}, H^1(\mathcal{O}))$-random attractors.

Let $R_\varepsilon : L^2(\mathcal{O}_\varepsilon) \to L^2(\mathcal{O})$ be an affine mapping of the form

$$(R_\varepsilon \hat{\phi}_\varepsilon)(y) = \hat{\phi}_\varepsilon(T_\varepsilon^{-1} y), \quad \forall \hat{\phi}_\varepsilon \in L^2(\mathcal{O}_\varepsilon).$$

Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\hat{\phi}_\varepsilon \in L^2(\mathcal{O}_\varepsilon)$, we can define a continuous cocycle $\hat{\Phi}_\varepsilon$ for problem (6)-(7) by the formula

$$\hat{\Phi}_\varepsilon(t, \tau, \omega, \hat{\phi}_\varepsilon) = R_\varepsilon^{-1} \Phi_\varepsilon(t, \tau, \omega, R_\varepsilon \hat{\phi}_\varepsilon).$$

The same change of unknown variable $v^0(t) = \mathcal{T}^{-1}(\theta_t \omega) u^0(t)$ transforms equation (4) into the following random partial differential equation on $\mathcal{Q}$:

$$\begin{cases} \frac{dv^0}{dt} - \sum_{i=1}^{n} \frac{1}{g} \left(g v_{y_i}^0 \right)_{y_i} = (-\lambda + \delta(\theta_t \omega)) v^0 + \mathcal{T}^{-1}(\theta_t \omega) h_0 \left(t, y^*, \mathcal{T}(\theta_t \omega) v^0(t) \right) \\ \qquad\qquad\qquad\qquad + \mathcal{T}^{-1}(\theta_t \omega) G_0(t, y^*), \quad y^* \in \mathcal{Q}, \ t > \tau, \\ \frac{\partial v^0}{\partial \nu_0} = 0, \quad y^* \in \partial \mathcal{Q}, \end{cases}$$

(29)

with the initial conditions

$$v_\tau^0(y^*) = \psi^0(y^*), \quad y^* \in \mathcal{Q},$$

(30)

where $\psi^0 = (\tilde{\mathcal{T}}^{-1}(\theta_\tau \omega)) \phi^0$.

The same argument as above allows us to prove that problem (4) and (5) generates a continuous cocycle $\Phi_0(t, \tau, \omega, \phi^0)$ in the space $\mathcal{M}$.

Now we want to write equation (26)-(27) as an abstract evolutionary equation. We introduce the inner product $(\cdot, \cdot)_{H_g(\mathcal{O})}$ on $\mathcal{N}$ defined by

$$(u, v)_{H_g(\mathcal{O})} = \int_{\mathcal{O}} g u v \, dy, \quad \text{for all } u, v \in \mathcal{N}$$

and denote by $H_g(\mathcal{O})$ the space equipped with this inner product. Since g is a continuous function on $\overline{\mathcal{Q}}$ and satisfies (1), one easily shows that $H_g(\mathcal{O})$ is a Hilbert space with norm equivalent to the natural norm of $\mathcal{N}$.

For $0 < \varepsilon \le 1$, we introduce a bilinear form $a_\varepsilon(\cdot, \cdot) : H^1(\mathcal{O}) \times H^1(\mathcal{O}) \to \mathbb{R}$, given by

$$a_\varepsilon(u, v) = (J^* \nabla_y u, J^* \nabla_y v)_{H_g(\mathcal{O})},$$

(31)

where

$$J^* \nabla_y u = (u_{y_1} - \frac{g_{y_1}}{g} y_{n+1} u_{y_{n+1}}, \ldots, u_{y_n} - \frac{g_{y_n}}{g} y_{n+1} u_{y_{n+1}}, \frac{1}{\varepsilon g} u_{y_{n+1}}).$$

By introducing on $H^1(\mathcal{O})$ the equivalent norm, for every $0 < \varepsilon \le 1$,

$$\|u\|_{H_\varepsilon^1(\mathcal{O})} = \left(\int_{\mathcal{O}} (|\nabla_{y^*} u|^2 + |u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy \right)^{\frac{1}{2}},$$

(32)

we see that there exist positive constants ε_0, η_1 and η_2 such that for all $0 < \varepsilon \le \varepsilon_0$ and $u \in H^1(\mathcal{O})$,

$$\eta_1 \int_{\mathcal{O}} (|\nabla_{y^*} u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy \le a_\varepsilon(u, u) \le \eta_2 \int_{\mathcal{O}} (\nabla_{y^*} u|^2 + \frac{1}{\varepsilon^2} u_{y_{n+1}}^2) dy$$

(33)

and

$$\eta_1 \|u\|_{H_\varepsilon^1(\mathcal{O})}^2 \le a_\varepsilon(u, u) + \|u\|_{L^2(\mathcal{O})}^2 \le \eta_2 \|u\|_{H_\varepsilon^1(\mathcal{O})}^2.$$

(34)

Denote by A_ε an unbounded operator on $H_g(\mathcal{O})$ with domain

$$D(A_\varepsilon) = \left\{ v \in H^2(\mathcal{O}), P_\varepsilon v \cdot \nu = 0 \text{ on } \partial \mathcal{O} \right\}$$

as defined by

$$A_\varepsilon v = -\frac{1}{g} \text{div} P_\varepsilon v, \quad v \in D(A_\varepsilon).$$

Then we have

$$a_\varepsilon(u, v) = (A_\varepsilon u, v)_{H_g(\mathcal{O})}, \forall u \in D(A_\varepsilon), \forall v \in H^1(\mathcal{O}).$$

(35)

Using A_ε, (26)-(27) can be written as

$$\begin{cases} \frac{dv^\varepsilon}{dt} + A_\varepsilon v^\varepsilon = (-\lambda + \delta(\theta_t\omega))v^\varepsilon + \mathcal{T}^{-1}(\theta_t\omega) h_\varepsilon(t, y, \mathcal{T}(\theta_t\omega) v^\varepsilon(t)) \\ \qquad\qquad\qquad + \mathcal{T}^{-1}(\theta_t\omega) G_\varepsilon(t, y), \ y \in \mathcal{O}, \ t > \tau, \\ v_\tau^\varepsilon = \psi^\varepsilon. \end{cases} \quad (36)$$

To reformulate system (29)-(30), we introduce the inner product $(\cdot, \cdot)_{H_g(\mathcal{Q})}$ on $\mathcal{M}$ defined by

$$(u, v)_{H_g(\mathcal{Q})} = \int_{\mathcal{Q}} guv dy^*, \quad \text{for all } u, v \in \mathcal{M},$$

and denote by $H_g(\mathcal{Q})$ the space equipped with this inner product. Let $a_0(\cdot, \cdot)$: $H^1(\mathcal{Q}) \times H^1(\mathcal{Q}) \to \mathbb{R}$ be a bilinear form given by

$$a_0(u, v) = \int_{\mathcal{Q}} g \nabla_{y^*} u \cdot \nabla_{y^*} v dy^*.$$

Denote by A_0 an unbounded operator on $H_g(\mathcal{Q})$ with domain

$$D(A_0) = \left\{ v \in H^2(\mathcal{Q}), \frac{\partial v}{\partial \nu_0} = 0 \text{ on } \partial\mathcal{Q} \right\}$$

as defined by

$$A_0 v = -\frac{1}{g} \sum_{i=1}^{n} (gv_{y_i})_{y_i} \quad v \in D(A_0).$$

Then we have

$$a_0(u, v) = (A_0 u, v)_{H_g(\mathcal{Q})}, \quad \forall u \in D(A_0), \forall v \in H^1(\mathcal{Q}).$$

Using A_0, (29)-(30) can be written as

$$\begin{cases} \frac{dv^0}{dt} + A_0 v^0 = (-\lambda + \delta(\theta_t\omega))v^0 + \mathcal{T}^{-1}(\theta_t\omega) h_0(t, y^*, \mathcal{T}(\theta_t\omega) v^0(t)) \\ \qquad\qquad\qquad + \mathcal{T}^{-1}(\theta_t\omega) G_0(t, y^*), \ y^* \in \mathcal{Q}, \ t > \tau, \\ v_\tau^0(s) = \psi^0(s), \qquad s \in [-\rho, 0]. \end{cases} \quad (37)$$

Hereafter, we set $X_0 = \mathcal{M}$, $X_\varepsilon = L^2(\mathcal{O}_\varepsilon)$ and $X_1 = \mathcal{N}$. For every $i = \varepsilon, 0$ or 1, a family $B_i = \{B_i(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of nonempty subsets of X_i is called tempered if for every $c > 0$, we have:

$$\lim_{t \to -\infty} e^{ct} \|B_i(\tau + t, \theta_t\omega)\|_{X_i} = 0,$$

where $\|B_i\|_{X_i} = \sup_{x \in B_i} \|x\|_{X_i}$. The collection of all families of tempered nonempty subsets of X_i is denoted by $\mathcal{D}_i$, i.e.,

$$\mathcal{D}_i = \{B_i = \{B_i(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : B_i \text{ is tempered in } X_i\}.$$

Our main purpose of the paper is to prove that the cocycle $\hat{\Phi}_\varepsilon$ and Φ_0 possess a unique $(L^2(\mathcal{O}_\varepsilon), H^1(\mathcal{O}_\varepsilon))$-random attractor $\hat{\mathcal{A}}_\varepsilon$ and $(\mathcal{M}, \mathcal{J})$-random attractor $\mathcal{A}_0$, respectively. Furthermore $\hat{\mathcal{A}}_\varepsilon$ is upper-semicontinuous at $\varepsilon = 0$, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \sup_{u_\varepsilon \in \hat{\mathcal{A}}_\varepsilon} \inf_{u_0 \in \mathcal{A}_0} \varepsilon^{-1} \|u_\varepsilon - u_0\|_{H^1(\mathcal{O}_\varepsilon)}^2 = 0. \quad (38)$$

To prove (38), we only need to show that the cocycle Φ_ε has a unique $(\mathcal{N}, \mathcal{H})$-random attractor $\mathcal{A}_\varepsilon$ and it is upper-semicontinuous at $\varepsilon = 0$ in the sense that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \text{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0,$$

which will be established in the last section of the paper.

Furthermore, we suppose that there exists $\lambda_0 > 0$ such that

$$\overline{\gamma} \triangleq \lambda_0 - 2\mathbb{E}(|\delta(\omega)|) > 0. \tag{39}$$

Let us consider the mapping

$$\gamma(\omega) = \lambda_0 - 2|\delta(\omega)|. \tag{40}$$

By the ergodic theory and (39) we have

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \gamma(\theta_l \omega) dl = \mathbb{E}\gamma = \overline{\gamma} > 0. \tag{41}$$

The following condition will be needed when deriving uniform estimates of solutions:

$$\int_{-\infty}^{\tau} e^{\frac{1}{2}\overline{\gamma}s}(\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} + \|\varphi_1(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_3(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{42}$$

When constructing tempered pullback attractors, we will assume

$$\lim_{r \to -\infty} e^{\sigma r} \int_{-\infty}^0 e^{\frac{1}{2}\overline{\gamma}s} \left(\|G(s+r,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} \right.$$

$$+ \|\varphi_1(s+r,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_3(s+r,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} \right) ds = 0, \ \forall \sigma > 0. \tag{43}$$

Since $\psi_1 = \varphi_1 + b_1$ for some positive constant b_1, it is evident that (42) and (43) imply

$$\int_{-\infty}^{\tau} e^{\frac{1}{2}\overline{\gamma}s}(\|G(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_3(s,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})ds < \infty, \quad \forall \tau \in \mathbb{R} \tag{44}$$

and

$$\lim_{r \to -\infty} e^{\sigma r} \int_{-\infty}^0 e^{\frac{1}{2}\overline{\gamma}s} \left(\|G(s+r,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_1(s+r,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} \right.$$

$$\left. + \|\psi_3(s+r,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} \right) ds = 0, \tag{45}$$

for any $\sigma > 0$.

3. **Uniform estimates of solutions.** In this section, we recall and generalize some results in [17] and derive some new uniform estimates of solutions of problem (36) or (19)-(20) which are needed for proving the existence of $\mathcal{D}_1$-pullback absorbing sets and the $\mathcal{D}_1$-pullback asymptotic compactness in $H^1(\mathcal{O})$ of the cocycle Φ_ε.

Lemma 3.1. *Assume that (8)-(11), (39) and (42) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, there exists $T = T(\tau,\omega,D_1) \geq 2$, independent of ε, such that for all $t \geq T$, $\lambda_1 > \lambda_0$ and $\psi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, the solution v^ε of (36) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$\sup_{-1 \leq s \leq 0} \|v^\varepsilon(\tau+s, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon)\|^2_{H^1_\varepsilon(\mathcal{O})} \leq R_2(\tau,\omega), \tag{46}$$

where $R_2(\tau,\omega)$ is determined by

$$R_2(\tau,\omega) = r_1(\omega)R_1(\tau,\omega)$$

$$+ c \int_{-\infty}^0 e^{\overline{\gamma}r} \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 (\|G(r+\tau,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} + \|\psi_3(r+\tau,\cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})})dr, \tag{47}$$

where $R_1(\tau,\omega)$ is determined by

$$
R_1(\tau,\omega) \;=\; c\int_{-\infty}^{0} e^{\int_0^r \gamma(\theta_l\omega)dl}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|G(r+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr
$$

$$
+\, c\int_{-\infty}^{0} e^{\int_0^r \gamma(\theta_l\omega)dl}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2 \|\psi_1(r+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr, \quad (48)
$$

and $r_1(\omega)$ is a tempered function, and c is independent of ε.

Proof. The proof is similar as that of Lemma 3.4 in [17], so we only sketch the proof here. Taking the inner product of (36) with v^ε in $H_g(\mathcal{O})$, we find that

$$
\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|_{H_g(\mathcal{O})}^2 \;\leq\; -a_\varepsilon(v^\varepsilon,v^\varepsilon) + (-\lambda_0 + \delta(\theta_t\omega))\|v^\varepsilon\|_{H_g(\mathcal{O})}^2
$$

$$
+\left(\mathcal{T}^{-1}(\theta_t\omega)\,h_\varepsilon(t,y,\mathcal{T}(\theta_t\omega)v^\varepsilon(t)),v^\varepsilon\right)_{H_g(\mathcal{O})}
$$

$$
+\left(\mathcal{T}^{-1}(\theta_t\omega)\,G_\varepsilon(t,y),v^\varepsilon\right)_{H_g(\mathcal{O})}. \quad (49)
$$

By (13), we have

$$
\frac{d}{dt}\|v^\varepsilon\|_{H_g(\mathcal{O})}^2 + 2a_\varepsilon(v^\varepsilon,v^\varepsilon) + \frac{\lambda_0}{2}\|v^\varepsilon\|_{H_g(\mathcal{O})}^2 + 2\alpha_1\gamma_1\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|u^\varepsilon\|_{L^p(\mathcal{O})}^p
$$

$$
\leq\; (-\lambda_0 + 2\delta(\theta_t\omega))\|v^\varepsilon\|_{H_g(\mathcal{O})}^2 + \frac{2}{\lambda_0}\gamma_2|\widetilde{\mathcal{O}}|\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|G(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2
$$

$$
+\, 2\gamma_2|\widetilde{\mathcal{O}}|\|\mathcal{T}^{-1}(\theta_t\omega)\|^2\|\psi_1(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}. \quad (50)
$$

Then, we have for any $\sigma\geq\tau$,

$$
e^{\int_\tau^\sigma \gamma(\theta_l\omega)dl}\|v^\varepsilon(\sigma)\|_{H_g(\mathcal{O})}^2 + 2\int_\tau^\sigma e^{\int_\tau^r \gamma(\theta_l\omega)dl}a_\varepsilon(v^\varepsilon(r),v^\varepsilon(r))dr
$$

$$
+\frac{\lambda_0}{2}\int_\tau^\sigma e^{\int_\tau^r \gamma(\theta_l\omega)dl}\|v^\varepsilon(r)\|_{H_g(\mathcal{O})}^2 dr
$$

$$
+ 2\alpha_1\gamma_1\int_\tau^\sigma \|\mathcal{T}^{-1}(\theta_r\omega)\|^2 e^{\int_\tau^r \gamma(\theta_l\omega)dl}\|u^\varepsilon(r)\|_{L^p(\mathcal{O})}^p dr
$$

$$
\leq\; \|v^\varepsilon(\tau)\|_{H_g(\mathcal{O})}^2 + \frac{2}{\lambda_0}\gamma_2|\widetilde{\mathcal{O}}|\int_\tau^\sigma e^{\int_\tau^r \gamma(\theta_l\omega)dl}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|G(r,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr
$$

$$
+ 2\gamma_2|\widetilde{\mathcal{O}}|\int_\tau^\sigma e^{\int_\tau^r \gamma(\theta_l\omega)dl}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|\psi_1(r,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr, \quad (51)
$$

where $\gamma(\theta_t\omega) = -\lambda_0 + \delta(\theta_t\omega)$.

Thus by the similar arguments as Lemma 3.1 in [17] we get for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, and $D_1\in\mathcal{D}_1$, there exists $T = T(\tau,\omega,D_1) > 0$ such that for all $t\geq T$,

$$
\|v_\tau^\varepsilon(\cdot,\tau-t,\theta_{-\tau}\omega,\psi)\|_{L^2(\mathcal{O})}^2 \leq c\int_{-\infty}^0 e^{\int_0^r \gamma(\theta_l\omega)dl}\|\psi_1(r+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr
$$

$$
+ c\int_{-\infty}^0 e^{\int_0^r \gamma(\theta_l\omega)dl}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|G(r+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr
$$

$$
+ c\int_{-\infty}^0 e^{\int_0^r \gamma(\theta_l\omega)dl}\|\mathcal{T}^{-1}(\theta_r\omega)\|^2\|\psi_1(r+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr. \quad (52)
$$

Moreover, taking the inner product of (36) with $A_\varepsilon v^\varepsilon$ in $H_g(\mathcal{O})$, we find that

$$
\frac{1}{2}\frac{d}{dt}a_\varepsilon\left(v^\varepsilon, v^\varepsilon\right) + \|A_\varepsilon v^\varepsilon\|^2_{H_g(\mathcal{O})}
$$
$$
\leq \quad (-\lambda_0 + \delta(\theta_t\omega))a_\varepsilon\left(v^\varepsilon, v^\varepsilon\right) + \left(\mathcal{T}^{-1}\left(\theta_t\omega\right)h_\varepsilon\left(t, y, \mathcal{T}\left(\theta_t\omega\right)v^\varepsilon(t)\right), A_\varepsilon v^\varepsilon\right)_{H_g(\mathcal{O})}
$$
$$
+ \left(\mathcal{T}^{-1}\left(\theta_t\omega\right)G_\varepsilon\left(t, y\right), A_\varepsilon v^\varepsilon\right)_{H_g(\mathcal{O})}. \tag{53}
$$

By (15)-(16) we have

$$
\frac{d}{dt}a_\varepsilon\left(v^\varepsilon, v^\varepsilon\right) + \|A_\varepsilon v^\varepsilon\|^2_{H_g(\mathcal{O})} \tag{54}
$$
$$
\leq \quad (c + 2\delta\left(\theta_t\omega\right))a_\varepsilon\left(v^\varepsilon, v^\varepsilon\right) + c\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\|^2(\|G\left(t, \cdot\right)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3\left(t, \cdot\right)\|^2_{L^\infty(\widetilde{\mathcal{O}})}),
$$

The left proof is similar of that Lemma 3.4 in [17], so we omit it here. $\qquad\square$

We are now in a position to establish the uniform estimates for the solution u^ε of the stochastic equation (19)-(20) by using those estimates for the solution v^ε of (36) and the relation between v^ε and u^ε.

Lemma 3.2. *Assume that (8)-(11), (39) and (42) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1\left(\tau, \omega\right) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, there exists $T = T(\tau, \omega, D_1) \geq 2$, independent of ε, such that for all $t \geq T$, $\lambda_1 > \lambda_0$ and $\phi^\varepsilon \in D_1\left(\tau - t, \theta_{-t}\omega\right)$, the solution u^ε of (19)-(20) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$
\sup_{-1 \leq s \leq 0} \|u^\varepsilon\left(\tau + s, \tau - t, \theta_{-\tau}\omega, \phi^\varepsilon\right)\|^2_{H^1_\varepsilon(\mathcal{O})} \leq r_2(\omega)R_2\left(\tau, \omega\right), \tag{55}
$$

where $r_2(\omega)$ is a tempered function and $R_2(\tau, \omega)$ is given by (47).

Lemma 3.3. *Assume that (8)-(11), (39) and (42) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1\left(\tau, \omega\right) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, there exists $T = T(\tau, \omega, D_1) \geq 2$, independent of ε, such that for all $t \geq T$, $\lambda_1 > \lambda_0$ and $\psi^\varepsilon \in D_1\left(\tau - t, \theta_{-t}\omega\right)$, the solution v^ε of (36) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$
\sup_{-1 \leq s \leq 0} \|v^\varepsilon\left(\tau + s, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon\right)\|^p_{L^p(\mathcal{O})}
$$
$$
+ \int_{\tau-\rho}^{\tau} \|v^\varepsilon\left(r, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon\right)\|^{2p-2}_{L^{2p-2}(\mathcal{O})} dr \leq R_3\left(\tau, \omega\right), \tag{56}
$$

where $R_3(\tau, \omega) < \infty$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Proof. The proof is similar as that of Lemma 3.6 in [14], so we omit it here. $\qquad\square$

Lemma 3.4. *Assume that (8)-(11), (39) and (42) hold. Then for every $\eta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D_1 = \{D_1\left(\tau, \omega\right) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, there exist $T = T(\tau, \omega, D_1) \geq 2$, $\gamma = \gamma(\omega) > 0$, a large $M = M(\tau, \omega, \eta) > 0$ and $0 < \varepsilon_1 < \varepsilon_0$ such that for all $t \geq T$, $\lambda_1 > \lambda_0$, $0 < \varepsilon < \varepsilon_1$ and $\psi^\varepsilon \in D_1\left(\tau - t, \theta_{-t}\omega\right)$, the solution v^ε of (36) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$
\int_{-1}^{0} e^{\gamma M^{p-2}s}\int_{\{y\in\mathcal{O}:\, v^\varepsilon(s+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\geq 2M\}} |v^\varepsilon(s+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)|^{2p-2}dyds \leq \eta, \tag{57}
$$

$$
\int_{-1}^{0} e^{\gamma M^{p-2}s}\int_{\{y\in\mathcal{O}:\, v^\varepsilon(s+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\leq -2M\}} |v^\varepsilon(s+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)|^{2p-2}dyds \leq \eta. \tag{58}
$$

Proof. Let M be a positive number to be specified later. Taking the scalar product of (36) with $(v^\varepsilon - M)_+^{p-1}$, where $(v^\varepsilon - M)_+ = \max\{v^\varepsilon - M, 0\}$, we have

$$\frac{1}{p}\frac{d}{dt}\left\|(v^\varepsilon - M)_+\right\|_{L^p(\mathcal{O})}^p + (p-1)\int_{v^\varepsilon \geq M}(v^\varepsilon - M)^{p-2}a_\varepsilon(v^\varepsilon, v^\varepsilon)dx$$

$$\leq \left(\delta\left(\theta_t\omega\right)v^\varepsilon, (v^\varepsilon - M)_+^{p-1}\right) + \left(\mathcal{T}^{-1}\left(\theta_t\omega\right)h_\varepsilon\left(t, y, \mathcal{T}\left(\theta_t\omega\right)v^\varepsilon\right), (v^\varepsilon - M)_+^{p-1}\right)$$

$$+ \left(\mathcal{T}^{-1}\left(\theta_t\omega\right)G_\varepsilon\left(t, y\right), (v^\varepsilon - M)_+^{p-1}\right). \tag{59}$$

For the first term on the right side of (59) we have

$$\left|\left(\delta\left(\theta_t\omega\right)v^\varepsilon, (v^\varepsilon - M)_+^{p-1}\right)\right|$$

$$\leq \frac{1}{p}|\delta\left(\theta_r\omega\right)|^p\int_{\mathcal{O}}|v^\varepsilon|^p dx + \frac{p-1}{p}\int_{\mathcal{O}}(v^\varepsilon - M)_+^p dx. \tag{60}$$

For the second term on the right-hand side of (59), by (8), we obtain, for $v^\varepsilon > M$,

$$h_\varepsilon(t, y, \mathcal{T}\left(\theta_t\omega\right)v^\varepsilon)\,(v^\varepsilon - M)_+^{p-1} \leq -\alpha_1\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-1}(v^\varepsilon)^{p-1}(v - M)_+^{p-1}$$

$$+\|\mathcal{T}\left(\theta_t\omega\right)\|^{-1}\psi_1(t, y^*, \varepsilon g(y^*)y_{n+1})(v^\varepsilon)^{-1}(v^\varepsilon - M)_+^{p-1}$$

$$\leq -\frac{1}{2}\alpha_1 M^{p-2}\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-1}(v^\varepsilon - M)_+^p - \frac{1}{2}\alpha_1\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-1}(v^\varepsilon - M)_+^{2p-2}$$

$$+\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\|^{-1}|\psi_1(t, y^*, \varepsilon g(y^*)y_{n+1})|(v^\varepsilon - M)_+^{p-2}$$

which implies

$$(\mathcal{T}^{-1}\left(\theta_t\omega\right)h_\varepsilon(t, y, \mathcal{T}\left(\theta_t\omega\right)v^\varepsilon),\ (v^\varepsilon - M)_+^{p-1})$$

$$\leq -\frac{1}{2}\alpha_1 M^{p-2}\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-2}\int_{\mathcal{O}}(v^\varepsilon - M)_+^p dx - \frac{1}{2}\alpha_1\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\|^{p-2}\int_{\mathcal{O}}(v^\varepsilon - M)_+^{2p-2}dx$$

$$+\|\mathcal{T}\left(\theta_t\omega\right)\|^{-2}\int_{\mathcal{O}}|\psi_1(t, y^*, \varepsilon g(y^*)y_{n+1})|(v^\varepsilon - M)_+^{p-2}dx$$

$$\leq -\frac{1}{2}\alpha_1 M^{p-2}\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-2}\int_{\mathcal{O}}(v^\varepsilon - M)_+^p dx - \frac{1}{2}\alpha_1\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-2}\int_{\mathcal{O}}(v^\varepsilon - M)_+^{2p-2}dx$$

$$+\frac{p-2}{p}\int_{\mathcal{O}}(v^\varepsilon - M)_+^p dx + \frac{2}{p}\|\mathcal{T}\left(\theta_t\omega\right)\|^{-p}\int_{\mathcal{O}}|\psi_1(t, y^*, \varepsilon g(y^*)y_{n+1})|^{\frac{p}{2}}dy. \tag{61}$$

The last term in (59) is bounded by

$$\left(\mathcal{T}^{-1}\left(\theta_t\omega\right)G_\varepsilon\left(t, y\right), (v^\varepsilon - M)_+^{p-1}\right) \leq \frac{1}{8}\alpha_1\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-2}\int_{\mathcal{O}}(v^\varepsilon - M)_+^{2p-2}dx$$

$$+\frac{2}{\alpha_1}\|\mathcal{T}\left(\theta_t\omega\right)\|^{-p}\int_{v^\varepsilon \geq M}|G_\varepsilon\left(t, y\right)|^2 dy. \tag{62}$$

All above estimates yield

$$\frac{d}{dt}\left\|(v^\varepsilon - M)_+\right\|_{L^p(\mathcal{O})}^p - \left(2p - 3 - \frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-2}\right)\int_{\mathcal{O}}(v^\varepsilon - M)_+^p dx$$

$$+\frac{1}{4}p\alpha_1\|\mathcal{T}\left(\theta_t\omega\right)\|^{p-2}\int_{\mathcal{O}}(v^\varepsilon - M)_+^{2p-2}dx$$

$$\leq |\delta\left(\theta_r\omega\right)|^p\int_{\mathcal{O}}|v^\varepsilon|^p dx + 2\|\mathcal{T}\left(\theta_t\omega\right)\|^{-p}\int_{\mathcal{O}}|\psi_1(t, y^*, \varepsilon g(y^*)y_{n+1})|^{\frac{p}{2}}dy$$

$$+\frac{2p}{\alpha_1}\|\mathcal{T}\left(\theta_t\omega\right)\|^{-p}\int_{\mathcal{O}}|G_\varepsilon\left(t, y\right)|^2 dy. \tag{63}$$

Multiplying (63) by $e^{-\int_0^t (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}$, then integrating on $(\tau-1,\tau)$ we obtain

$$
\| \left(v^\varepsilon(\tau, \tau-t, \omega, \psi^\varepsilon) - M\right)_+ \|^p_{L^p(\mathcal{O})}
$$

$$
+\frac{1}{4}p\alpha_1 \int_{\tau-1}^{\tau} \|\mathcal{T}(\theta_\zeta\omega)\|^{p-2} e^{-\int_\tau^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}
$$

$$
\times \int_{\mathcal{O}} (v^\varepsilon(\zeta, \tau-t, \omega, \psi^\varepsilon) - M)_+^{2p-2} dx d\zeta
$$

$$
\leq e^{-\int_\tau^{\tau-1}(2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \| \left(v^\varepsilon(\tau-1, \tau-t, \omega, \psi^\varepsilon) - M\right)_+ \|^p_{L^p(\mathcal{O})}
$$

$$
+ \int_{\tau-1}^{\tau} |\delta(\theta_\zeta\omega)|^p e^{-\int_\tau^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \|v^\varepsilon(\zeta, \tau-t, \omega, \psi^\varepsilon)\|^p_{L^p(\mathcal{O})} d\zeta
$$

$$
+2|\mathcal{O}| \int_{\tau-1}^{\tau} \|\mathcal{T}(\theta_\zeta\omega)\|^{-p} e^{-\int_{\tau+\xi}^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \|\psi_1(\zeta, \cdot)\|^{\frac{p}{2}}_{L^\infty(\tilde{\mathcal{O}})} d\zeta.
$$

$$
+ \frac{2p|\mathcal{O}|}{\alpha_1} \int_{\tau-1}^{\tau} \|\mathcal{T}(\theta_\zeta\omega)\|^{-p} e^{-\int_{\tau+\xi}^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \|G(\zeta, \cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} d\zeta,
$$

$$(64)$$

where $|\mathcal{O}|$ stands for the Lebesgue measure of $\mathcal{O}$. Replacing ω by $\theta_{-\tau}\omega$ in (64) we get

$$
\frac{1}{4}p\alpha_1 \int_{-1}^{0} \|\mathcal{T}(\theta_\zeta\omega)\|^{p-2} e^{-\int_0^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}
$$

$$
\times \int_{\mathcal{O}} (v^\varepsilon(\zeta+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon) - M)_+^{2p-2} dx d\zeta
$$

$$
\leq e^{-\int_0^{-1}(2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \| \left(v^\varepsilon(\tau-1, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon) - M\right)_+ \|^p_{L^p(\mathcal{O})}
$$

$$
+ \int_{-1}^{0} |\delta(\theta_{\zeta+\xi}\omega)|^p e^{-\int_0^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \|v^\varepsilon(\zeta+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon)\|^p_{L^p(\mathcal{O})} d\zeta
$$

$$
+2|\mathcal{O}| \int_{-1}^{0} \|\mathcal{T}(\theta_\zeta\omega)\|^{-p} e^{-\int_0^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \|\psi_1(\zeta+\tau, \cdot)\|^{\frac{p}{2}}_{L^\infty(\tilde{\mathcal{O}})} d\zeta.
$$

$$
+ \frac{2p|\mathcal{O}|}{\alpha_1} \int_{-1}^{0} \|\mathcal{T}(\theta_\zeta\omega)\|^{-p} e^{-\int_0^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr} \|G(\zeta+\tau, \cdot)\|^2_{L^\infty(\tilde{\mathcal{O}})} d\zeta.
$$

$$(65)$$

Since ω is continuous on $[-1, 0]$, there exist $c_1 = c_1(\omega, p, \alpha_1) > 0$ and $c_2 = c_2(\omega, p, \alpha_1) > 0$ such that

$$
c_1 \leq \frac{1}{2}p\alpha_1 \|\mathcal{T}(\theta_r\omega)\|^{p-2} \leq c_2 \quad \text{for all} \ \ r \in [-\rho-1, 0]. \tag{66}
$$

By (66) we obtain

$$
e^{c_2 M^{p-2}\zeta} \leq e^{\int_\xi^{\zeta+\xi} \frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2}dr} \leq e^{c_1 M^{p-2}\zeta} \quad \text{for all} \ \zeta \in [-1, 0] \text{ and } \xi \in [-\rho, 0]. \tag{67}
$$

For the left-hand side of (65), by (67) we find that there exists $c_3 = c_3(\omega) > 0$ such that

$$
\frac{1}{4}p\alpha_1 \int_{-1}^{0} \|\mathcal{T}(\theta_\zeta\omega)\|^{p-2} e^{-\int_0^\zeta (2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}
$$

$$
\int_{\mathcal{O}} (v^\varepsilon(\zeta+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon) - M)_+^{2p-2} dx d\zeta
$$

$$
\geq c_3 \int_{-1}^{0} e^{c_2 M^{p-2}\zeta} \int_{\mathcal{O}} (v^\varepsilon(\zeta+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon) - M)_+^{2p-2} dx d\zeta. \tag{68}
$$

For the first term on the right-hand side of (65), by (67) we obtain

$$e^{-\int_0^{-1}(2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}\|\left(v^\varepsilon(\tau-1,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)-M\right)_+\|^p_{L^p(\mathcal{O})}$$

$$\leq e^{2p-3}e^{-c_1 M^{p-2}}\|\left(v^\varepsilon(\tau-1,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)-M\right)_+\|^p_{L^p(\mathcal{O})}$$

$$\leq e^{2p-3}e^{-c_1 M^{p-2}}\|v^\varepsilon(\tau-1,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\|^p_{L^p(\mathcal{O})}. \qquad (69)$$

Similarly, for the second terms on the right-hand side of (65), we have from (67) there exists $c_4 = c_4(\omega) > 0$ such that

$$\int_{-1}^0|\delta(\theta_\zeta\omega)|^p e^{-\int_0^\zeta(2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}\|v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\|^p_{L^p(\mathcal{O})}d\zeta$$

$$\leq c_4\int_{-1}^0 e^{c_1 M^{p-2}\zeta}\|v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\|^p_{L^p(\mathcal{O})}d\zeta \qquad (70)$$

Since $\varphi_1 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$ and $G \in L^2_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$, for the two three terms on the right-hand side of (65), by (67) we obtain there exists $c_5 = c_5(\tau,\omega) > 0$ such that

$$2|\mathcal{O}|\int_{-1}^0\|\mathcal{T}(\theta_\zeta\omega)\|^{-p}e^{-\int_0^\zeta(2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}\|\psi_1(\zeta+\tau,\cdot)\|^{\frac{p}{2}}_{L^\infty(\widetilde{\mathcal{O}})}d\zeta.$$

$$+\frac{2p|\mathcal{O}|}{\alpha_1}\int_{-1}^0\|\mathcal{T}(\theta_\zeta\omega)\|^{-p}e^{-\int_0^\zeta(2p-3-\frac{1}{2}p\alpha_1 M^{p-2}\|\mathcal{T}(\theta_r\omega)\|^{p-2})dr}\|G(\zeta+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}d\zeta$$

$$\leq c_5\int_{-1}^0 e^{c_1 M^{p-2}\zeta}d\zeta \leq c_1^{-1}c_5 M^{2-p}. \qquad (71)$$

By (68)-(71) we get from (65) that

$$c_3\int_{-1}^0 e^{c_2 M^{p-2}\zeta}\int_{\mathcal{O}}(v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)-M)^{2p-2}_+dyd\zeta$$

$$\leq e^{2p-3}e^{-c_1 M^{p-2}}\|v^\varepsilon(\tau-1,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\|^p_{L^p(\mathcal{O})}$$

$$+c_4\int_{-1}^0 e^{c_1 M^{p-2}\zeta}\|v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\|^p_{L^p(\mathcal{O})}d\zeta + c_1^{-1}c_5 M^{2-p},$$

which together with Lemma 3.2 and Lemma 3.3 implies that there exist $c_6 = c_6(\tau,\omega) > 0$ and $T = T(\tau,\omega,D_1) \geq 2$ such that for all $t \geq T$,

$$c_3\int_{-1}^0 e^{c_2 M^{p-2}\zeta}\int_{\mathcal{O}}(v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)-M)^{2p-2}_+dxd\zeta$$

$$\leq c_6 e^{-c_1 M^{p-2}}+c_6\int_{-1}^0 e^{c_1 M^{p-2}\zeta}d\zeta+c_1^{-1}c_5 M^{2-p} \leq c_6 e^{-c_1 M^{p-2}}+c_1^{-1}(c_5+c_6)M^{2-p}. \qquad (72)$$

Since $p > 2$, we find that for every $\eta > 0$, there exists $M_0 = M_0(\tau,\omega,\eta) > 0$ such that for all $M \geq M_0$ and $t \geq T$,

$$\int_{-1}^0 e^{c_2 M^{p-2}\zeta}\int_{\mathcal{O}}(v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)-M)^{2p-2}_+dyd\zeta \leq \eta. \qquad (73)$$

Note that $|v| \leq 2(v-M)_+$ for $v \geq 2M$, which together with (73) yields that for all $M \geq M_0$ and $t \geq T$,

$$\int_{-1}^0 e^{c_2 M^{p-2}\zeta}\int_{\{y\in\mathcal{O}:\ v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)\geq 2M\}}|v^\varepsilon(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,\psi^\varepsilon)|^{2p-2}dyd\zeta$$

$$\leq 2^{2p-2} \int_{-1}^{0} e^{c_2 M^{p-2} \zeta} \int_{\mathcal{O}} (v^\varepsilon(\zeta + \tau, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon) - M)_+^{2p-2} dx d\zeta \leq 2^{2p-2}\eta. \quad (74)$$

Similarly, one can verify that there exist $M_1 = M_1(\tau, \omega, \eta) > 0$ and $T_1 = (\tau, \omega, D) \geq 2$ such that for all $M \geq M_1$ and $t \geq T_1$,

$$\int_{-1}^{0} e^{c_2 M^{p-2} \zeta} \int_{\{y \in \mathcal{O}: \; v^\varepsilon(\zeta + \tau, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon) \leq -2M\}} |v^\varepsilon(\zeta + \tau, \tau - t, \theta_{-\tau}\omega, \psi^\varepsilon)|^{2p-2} dy d\zeta \leq 2^{2p-2}\eta. \quad (75)$$

Then Lemma 3.4 follows from (3) and (75) immediately. $\qquad\square$

Note that A_ε is a family of linear operators in Y_ε, for $0 \leq \varepsilon \leq \varepsilon_0$, where $Y_\varepsilon = H_g(\mathcal{O})$, for $0 < \varepsilon \leq \varepsilon_0$, and $Y_0 = H_g(\mathcal{Q})$, is self-adjoint and has a compact resolvent. Then, $\sigma(A_\varepsilon)$ consists of only eigenvalues $\{\lambda_n^\varepsilon\}_{n=1}^\infty$ with finite multiplicity:

$$0 \leq \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \ldots \leq \lambda_n^\varepsilon \leq \cdots \to +\infty,$$

and their associated eigenfunctions $\{\varpi_n^\varepsilon\}_{n=1}^\infty$ form an orthonormal basis of Y_ε.

It follows from Corollary 9.7 in [8] that the eigenvalues and the eigenfunctions of A_ε are convergent with respect to ε.

Next, we introduce the spectral projections. We use P_m^ε to denote the projection from Y_ε onto the eigenspace span$\{\varpi_i^\varepsilon\}_{i=1}^m$ given by

$$P_n^\varepsilon(u) = \sum_{i=1}^{m} (u, \varpi_i^\varepsilon)_{Y_\varepsilon} \varpi_i^\varepsilon \quad \text{for } u \in Y_\varepsilon.$$

We use Q_m^ε to denote its orthogonal complement projection, i.e., $P_m^\varepsilon + Q_m^\varepsilon = I_\varepsilon$, where I_ε is the identity operators on Y_ε. It is clear that

$$a_\varepsilon(u, u) = (A_\varepsilon u, u)_{H_g(\mathcal{O})} \leq \lambda_n^\varepsilon (u, u)_{H_g(\mathcal{O})}, \quad \forall u \in P_n^\varepsilon D\left(A_\varepsilon^{1/2}\right). \quad (76)$$

and

$$a_\varepsilon(u, u) = (A_\varepsilon u, u)_{H_g(\mathcal{O})} \geq \lambda_{m+1}^\varepsilon (u, u)_{H_g(\mathcal{O})}, \quad u \in Q_m^\varepsilon D\left(A_\varepsilon^{1/2}\right). \quad (77)$$

Let $u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon$ and $v^\varepsilon = v_1^\varepsilon + v_2^\varepsilon$, where $u_1^\varepsilon = P_m^\varepsilon u^\varepsilon$, $u_2^\varepsilon = Q_m^\varepsilon u^\varepsilon$, $v_1^\varepsilon = P_m^\varepsilon v^\varepsilon$, and $v_2^\varepsilon = Q_m^\varepsilon v^\varepsilon$, respectively.

Lemma 3.5. *Assume that (8)-(11), (39) and (42) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\eta > 0$ and $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, there exists $T = T(\tau, \omega, D_1, \eta) \geq 2$, $m = m(\tau, \omega, D, \eta) \in \mathbb{N}$ and $0 < \varepsilon_1 = \varepsilon_1(n) < \varepsilon_0$ such that for all $t \geq T$, $0 < \varepsilon < \varepsilon_1$ and $\phi^\varepsilon \in D_1(\tau - t, \theta_{-t}\omega)$, the solution u^ε of (19)-(20) with ω replaced by $\theta_{-\tau}\omega$ satisfies*

$$\|u_2^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, \phi^\varepsilon)\|_{H^1(\mathcal{O})} \leq \eta.$$

Proof. Taking the inner product (36) with $A_\varepsilon v_2^\varepsilon$ in $H_g(\mathcal{O})$, we get

$$\frac{1}{2}\frac{d}{dt} a_\varepsilon(v_2^\varepsilon, v_2^\varepsilon) + \|A_\varepsilon v_2^\varepsilon\|^2 \leq (\delta(\theta_t\omega) v_2^\varepsilon, A_\varepsilon v_2^\varepsilon)$$
$$+ \left(Q_n^\varepsilon \mathcal{T}^{-1}(\theta_t\omega) h_\varepsilon(t, y, \mathcal{T}(\theta_t\omega) v^\varepsilon), A_\varepsilon v_2^\varepsilon\right)$$
$$+ \left(Q_n^\varepsilon \mathcal{T}^{-1}(\theta_t\omega) G_\varepsilon(t, y), A_\varepsilon v_2^\varepsilon\right). \quad (78)$$

For the first term on the right-hand side of (78), we have

$$(\delta(\theta_t\omega) v_2^\varepsilon, A_\varepsilon v_2^\varepsilon) \leq \frac{1}{8}\|A_\varepsilon v_2^\varepsilon\|^2 + 2|\delta(\theta_t\omega)|^2 \|v_2^\varepsilon\|^2. \quad (79)$$

For the superlinear term, we have from (9) that

$$
\left(Q_n^\varepsilon \mathcal{T}^{-1}\left(\theta_t\omega\right) h_\varepsilon\left(t,y,\mathcal{T}\left(\theta_t\omega\right)v^\varepsilon\right), A_\varepsilon v_2^\varepsilon\right)
$$

$$
\leq \frac{1}{8}\left\|A_\varepsilon v_2^\varepsilon\right\|^2 + 2\left\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\right\|^2 \int_{\mathcal{O}}|h_\varepsilon\left(t,y,\mathcal{T}\left(\theta_t\omega\right)v^\varepsilon\right)|^2 dy
$$

$$
\leq \frac{1}{8}\left\|A_\varepsilon v_2^\varepsilon\right\|^2 + 2\alpha_2\left\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\right\|^2 \int_{\mathcal{O}}\left(|\mathcal{T}\left(\theta_t\omega\right)v^\varepsilon|^{p-1} + \psi_2\left(t,y^*,\varepsilon g(y^*)y_{n+1}\right)\right)^2 dy
$$

$$
\leq \frac{1}{8}\left\|A_\varepsilon v_2^\varepsilon\right\|^2 + 4\alpha_2\left\|\mathcal{T}\left(\theta_t\omega\right)\right\|^{2p-4}\|v\|_{2p-2}^{2p-2} + 4\alpha_2|\mathcal{O}|\left\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\right\|^2 \|\psi_2(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2.
$$
(80)

For the last term on the right-hand side of (78), we have

$$
\left(Q_n^\varepsilon T^{-1}\left(\theta_t\omega\right) G_\varepsilon\left(t,y\right), A_\varepsilon v_2^\varepsilon\right) \leq \frac{1}{8}\left\|A_\varepsilon v_2^\varepsilon\right\|^2 + 2|\mathcal{O}|\left\|T^{-1}\left(\theta_t\omega\right)\right\|^2 \|G\left(t,\cdot\right)\|_{L^\infty(\widetilde{\mathcal{O}})}^2
$$
(81)

Noting that $\left\|A_\varepsilon v_2^\varepsilon\right\|^2 \geq \lambda_{n+1}^\varepsilon a_\varepsilon(v_2^\varepsilon, v_2^\varepsilon)$, we obtain from all above estimates that

$$
\begin{aligned}
\frac{d}{dt}a_\varepsilon(v_2^\varepsilon, v_2^\varepsilon) &+ \lambda_{n+1}^\varepsilon a_\varepsilon(v_2^\varepsilon, v_2^\varepsilon) \leq 4\delta^2\left(\theta_t\omega\right)\|v_2^\varepsilon\|^2 \\
&+ 8\alpha_2\left\|\mathcal{T}\left(\theta_t\omega\right)\right\|^{2p-4}\|v^\varepsilon\|_{2p-2}^{2p-2} \\
&+ c\left\|\mathcal{T}^{-1}\left(\theta_t\omega\right)\right\|^2 (\|\psi_2(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 + \|G\left(t,\cdot\right)\|_{L^\infty(\widetilde{\mathcal{O}})}^2).
\end{aligned}
$$
(82)

Taking $\xi \in (\tau-1, \tau)$, multiplying (82) by $e^{\lambda_{n+1}^\varepsilon t}$, first integrating with respect to t on (ξ, τ), integrating with respect to ξ on $(\tau-1, \tau)$, and then replacing ω by $\theta_{-\tau}\omega$, we get

$$
a_\varepsilon(v_2^\varepsilon\left(\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right), v_2^\varepsilon\left(\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right))
$$

$$
\leq \int_{\tau-1}^\tau e^{\lambda_{n+1}^\varepsilon(r-\tau)}a_\varepsilon(v_2^\varepsilon\left(r, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right), v_2^\varepsilon\left(r, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right))dr
$$

$$
+ 4\delta^2\int_{\tau-1}^\tau e^{\lambda_{n+1}^\varepsilon(r-\tau)}\delta^2\left(\theta_{r-\tau}\omega\right)a_\varepsilon(v_2^\varepsilon\left(r, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right), v_2^\varepsilon\left(r, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right))dr
$$

$$
+ 8\alpha_2\int_{\tau-1}^\tau e^{\lambda_{n+1}^\varepsilon(r-\tau)}\left\|\mathcal{T}\left(\theta_{r-\tau}\omega\right)\right\|^{2p-4}\|v^\varepsilon\left(r, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right)\|_{2p-2}^{2p-2}dr
$$

$$
+ c\int_{\tau-1}^\tau e^{\lambda_{n+1}^\varepsilon(r-\tau)}\left\|\mathcal{T}^{-1}\left(\theta_{r-\tau}\omega\right)\right\|^2 (\|\psi_2(r,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2)dr
$$

$$
+ c\int_{\tau-1}^\tau e^{\lambda_{n+1}^\varepsilon(r-\tau)}\left\|\mathcal{T}^{-1}\left(\theta_{r-\tau}\omega\right)\right\|^2 \|G\left(r,\cdot\right)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 dr.
$$
(83)

Since $\varphi_2, G \in L_{loc}^2(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$, $\|\mathcal{T}(\theta_t\omega)\|$ is continuous on $[-1,0]$ and $\lambda_{n+1}^\varepsilon$ approaches λ_{n+1}^0 as $\varepsilon \to 0$, we find that there exists $c = c(\omega) > 0$ and $0 < \varepsilon^* < \varepsilon_0$ such that for $0 < \varepsilon < \varepsilon^*$,

$$
a_\varepsilon(v_2^\varepsilon\left(\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right), v_2^\varepsilon\left(\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right))
$$

$$
\leq c\int_{-1}^0 e^{\lambda_{n+1}^\varepsilon r}\|v^\varepsilon\left(r+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right)\|_{2p-2}^{2p-2}dr
$$

$$
+ c\int_{-1}^0 e^{\lambda_{n+1}^\varepsilon r}a_\varepsilon(v^\varepsilon\left(r+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right), v^\varepsilon\left(r+\tau, \tau-t, \theta_{-\tau}\omega, \psi^\varepsilon\right))dr
$$

$$+ c \int_{-1}^{0} e^{\lambda_{n+1}^{\varepsilon} r} dr$$

$$\leq c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} \left\| v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right) \right\|_{2p-2}^{2p-2} dr$$

$$+ c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} a_{\varepsilon}(v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right), v^{\varepsilon} \left(r + \tau + s, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)) dr$$

$$+ c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} \left\| v^{\varepsilon} \left(r + \tau - \rho_0(r + \tau + s), \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right) \right\|^{2} dr$$

$$+ c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} dr. \tag{84}$$

Given $\eta > 0$, let $T = T(\tau, \omega, D_1) \geq 2, \gamma = \gamma(\omega) > 0$, $M = M(\tau, \omega, \eta) \geq 1$ and $0 < \varepsilon_1 < \varepsilon^*$ be the constants in Lemma 3.4. Choose $N_1 = N_1(\tau, \omega, \eta) \geq 1$ large enough such that $\lambda_{n+1}^{0} - 1 \geq \gamma M^{p-2}$ for all $n \geq N_1$. Then, by Lemma 3.4, we obtain, for all $t \geq T$, $n \geq N_1$ and $0 < \varepsilon < \varepsilon_1$,

$$c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} \left\| v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right) \right\|_{2p-2}^{2p-2} dr$$

$$\leq c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} \int_{\{y \in \mathcal{O}: |v^{\varepsilon}| \geq 2M\}} |v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)|^{2p-2} dy dr$$

$$+ c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} \int_{\{y \in \mathcal{O}: |v^{\varepsilon}| < 2M\}} |v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)|^{2p-2} dy dr$$

$$\leq c \int_{-1}^{0} e^{\gamma M^{p-2} r} \int_{\{y \in \mathcal{O}: |v^{\varepsilon}| \geq 2M\}} |v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)|^{2p-2} dy dr$$

$$+ c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} \int_{\{y \in \mathcal{O}: |v^{\varepsilon}| < 2M\}} |v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)|^{2p-2} dy dr$$

$$\leq \eta + c2^{2p-2} M^{2p-2} |\mathcal{O}| \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} dr \leq \eta + c2^{2p-2} M^{2p-2} |\mathcal{O}| \frac{1}{\lambda_{n+1}^{0} - 1}. \tag{85}$$

For the last three terms on the right-hand side of (84), by Lemma 3.1, we find that there exist $c_1 = c_1(\tau, \omega) > 0$ and $T_1 = T_1(\tau, \omega, D_1) \geq T$ such that for all $t \geq T_1$,

$$c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} a_{\varepsilon}(v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right), v^{\varepsilon} \left(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)) dr$$

$$+ c \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} dr \leq c_1 \int_{-1}^{0} e^{(\lambda_{n+1}^{0}-1)r} dr \leq c_1 \frac{1}{\lambda_{n+1}^{0} - 1}. \tag{86}$$

Since $\lambda_{n+1}^{0} \to \infty$ as $n \to \infty$, we obtain from (84)-(86) that there exists $N_2 = N_2(\tau, \omega, \eta) \geq N_1$ such that for all $n \geq N_2$, $t \geq T_1$ and $0 < \varepsilon < \varepsilon_1$,

$$a_{\varepsilon}(v_2^{\varepsilon} \left(\tau + s, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right), v_2^{\varepsilon} \left(\tau + s, \tau - t, \theta_{-\tau}\omega, \psi^{\varepsilon}\right)) \leq 2\eta,$$

which together $v^{\varepsilon}(t) = \mathcal{T}^{-1} \left(\theta_t \omega\right) u^{\varepsilon}(t)$ and (77) completes the proof. $\quad\square$

4. **Existence of pullback attractor.** In this subsection, we establish the existence of $\mathcal{D}_1$-pullback attractor for the cocycle Φ_{ε} associated with the stochastic problem (19)-(20). We first show that problem (19)-(20) has a tempered pullback absorbing set as stated below.

Lemma 4.1. *Suppose (8)-(11), (39) and (43) hold. Then the cocycle Φ_ε associated with problem (19)-(20) has a closed measurable $\mathcal{D}_1$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$.*

Proof. We first notice that, by Lemma 3.2, Φ_ε has a closed $\mathcal{D}_1$-pullback absorbing set K in $H^1(\mathcal{O})$. More precisely, given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let

$$K(\tau, \omega) = \left\{ u \in H^1(\mathcal{O}) : \|u\|_{H^1(\mathcal{O})}^2 \leq L(\tau, \omega) \right\}, \tag{87}$$

where $L(\tau, \omega)$ is the constant given by the right-hand side of (55). It is evident that, for each $\tau \in \mathbb{R}$, $L(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. In addition, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}_1$, there exists $T = T(\tau, \omega, D) \geq 2$ such that for all $t \geq T$,

$$\Phi_\varepsilon(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega).$$

Thus we find that $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable set and pullback-attracts all elements in $\mathcal{D}_1$. By the similar argument as in [15] we can obtain easily from (43) that $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered. Consequently, K is a closed measurable D_1-pullback absorbing set for Φ_ε in D_1. $\square$

Lemma 4.2. *Assume that (8)-(11), (39) and (43) hold. Then, the cocycle Φ_ε is $\mathcal{D}_1$-pullback asymptotically compact in $H^1(\mathcal{O})$; that is, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\{\Phi_\varepsilon(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^\infty$ has a convergent subsequence in $H^1(\mathcal{O})$ whenever $t_n \to \infty$ and $x_n \in D_1(\tau - t_n, \theta_{-t_n}\omega)$ with $\{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$.*

Proof. We will show that for every $\eta > 0$, the sequence $\{u^\varepsilon(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^\varepsilon)\}_{n=1}^\infty$ has a finite open cover of balls with radii less than η. By Lemma 3.5, we infer that there exists $N_1 = N_1(\tau, \omega, D_1, \eta) \geq 1$, $m_0 = m_0(\tau, \omega, D_1, \eta) \in \mathbb{N}$ and $0 < \varepsilon_1 = \varepsilon_1(m_0) < \varepsilon_0$ such that for all $n \geq N_1$ and $0 < \varepsilon < \varepsilon_1$,

$$\|u_2^\varepsilon(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^\varepsilon)\|_{H^1(\mathcal{O})} = \|Q_{m_0}u^\varepsilon(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^\varepsilon)\|_{H^1(\mathcal{O})} < \frac{\eta}{4}. \tag{88}$$

On the other hand, by Lemma 3.2 we find that the sequence $\{P_{m_0}u^\varepsilon(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^\varepsilon)\}_{n=1}^\infty$ is bounded in the finite-dimensional space $P_{m_0}H^1(\mathcal{O})$ and hence is precompact, which together with (88) shows that the sequence $u^\varepsilon(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^\varepsilon)$ has a finite open cover of balls with radii less than η in $H^1(\mathcal{O})$, as desired. $\square$

Theorem 4.3. *Assume that (8)-(11), (39) and (43) hold. Then, the cocycle Φ_ε has a unique $\mathcal{D}_1$-pullback $(\mathcal{N}, H^1(\mathcal{O}))$-attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$.*

Proof. First, we know from Lemma 4.1 that Φ_ε has a a closed measurable $\mathcal{D}_1$-pullback absorbing set $K(\tau, \omega)$. Second, it follows from Lemma 4.2 that Φ_ε is $\mathcal{D}_1$-pullback asymptotically compact from $\mathcal{N}$ to $H^1(\mathcal{O})$. Hence, the existence of a unique $\mathcal{D}_1$-pullback $(\mathcal{N}, H^1(\mathcal{O}))$-attractor for the cocycle Φ_ε follows from Proposition 2.5 in [7]. $\square$

Analogous results also hold for the solution of (4)-(5). In particular, we have:

Theorem 4.4. *Assume that (8)-(11), (39) and (43) hold. Then, the cocycle Φ_0 has a unique $\mathcal{D}_0$-pullback $(\mathcal{M}, H^1(\mathcal{Q}))$-attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$.*

5. **Upper semicontinuity of attractors.** The following estimates are needed when we derive the convergence of pullback attractors. By the similar proof of that of Theorem 5.1 in [14], we get the following lemma.

Lemma 5.1. *Assume that (8)-(11) and (39) hold. Then for every $0 < \varepsilon \leq \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, and $\lambda_1 > \lambda_0$, the solution v^ε of (36) satisfies, for all $t \in [\tau, \tau + T]$,*

$$\int_\tau^t \|v^\varepsilon(r, \tau, \omega, \psi^\varepsilon)\|_{H_\varepsilon^1(\mathcal{O})}^2 dr \leq c \|\psi^\varepsilon\|_{\mathcal{N}}^2$$

$$+ c \int_\tau^{\tau+T} \left(\|G(r, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 + \|\psi_1(r, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 \right) dr,$$

where c is a positive constant depending on τ, ω, λ_0 and T, but independent of ε.

Similarly, one can prove

Lemma 5.2. *Assume that (8)-(11) and (39) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, and $\lambda_1 > \lambda_0$, the solution v^0 of (37) satisfies, for all $t \in [\tau, \tau + T]$,*

$$\int_\tau^t \|v^0(r, \tau, \omega, \psi^0)\|_{H^1(\mathcal{Q})}^2 dr$$

$$\leq c \|\psi^0\|_{\mathcal{M}}^2 + c \int_\tau^{\tau+T} \left(\|G(r, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 + \|\psi_1(r, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}^2 \right) dr,$$

where c is a positive constant depending on τ, ω, λ_0 and T, but independent of ε.

In the sequel, we further assume the functions G and H satisfy that for all $t, s \in \mathbb{R}$,

$$\|G_\varepsilon(t, \cdot) - G_0(t, \cdot)\|_{L^2(\mathcal{O})} \leq \kappa_1(t)\varepsilon \tag{89}$$

and

$$\|H_\varepsilon(t, \cdot, s) - H_0(t, \cdot, s)\|_{L^2(\mathcal{O})} \leq \kappa_2(t)\varepsilon, \tag{90}$$

where $\kappa_1(t), \kappa_2(t) \in L_{loc}^2(\mathbb{R})$.

By (12) and (90) we have, for all $x \in \widetilde{\mathcal{O}}$ and $t, s \in \mathbb{R}$,

$$\|h_\varepsilon(t, \cdot, s) - h_0(t, \cdot, s)\|_{L^2(\mathcal{O})} \leq \kappa_2(t)\varepsilon. \tag{91}$$

Since $\mathcal{M}$ can be embedded naturally into $\mathcal{N}$ as the subspace of functions independent of y_{n+1}, we can consider the cocycle Φ_0 as a mapping from $\mathcal{M}$ into $\mathcal{N}$. Therefore we can compare Φ_0 with Φ_ε.

Theorem 5.3. *Suppose (8)-(11), (39), and (89)-(90) hold. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varepsilon_n \to 0$ and a positive number $L(\tau, \omega)$, if $\phi^{\varepsilon_n} \in H_{\varepsilon_n}^1(\mathcal{O}))$ such that $\|\phi^{\varepsilon_n}\|_{H_{\varepsilon_n}^1(\mathcal{O}))} \leq L(\tau, \omega)$, then there exists $\phi^0 \in \mathcal{M}$ such that, up to a subsequence, for $t \geq 0$,*

$$\lim_{n \to \infty} \left\| \Phi_{\varepsilon_n}(t, \tau, \omega, \phi^{\varepsilon_n}) - \Phi_0(t, \tau, \omega, \phi^0) \right\|_{\mathcal{N}} = 0.$$

Proof. Since $\phi^{\varepsilon_n} \in H_{\varepsilon_n}^1(\mathcal{O})$, there exists $\phi^0 \in \mathcal{M}$ such that $\phi^{\varepsilon_n} \to \phi^0$ in $\mathcal{N}$. By the similar proof of that of Theorem 5.4 in [14], for any $T > 0$, we have for $t \in [\tau, \tau + T]$

$$\left\| v^{\varepsilon_n}(t) - v^0(t) \right\|_{\mathcal{N}}^2 \leq c \left\| \phi^{\varepsilon_n} - \phi^0 \right\|_{\mathcal{N}}^2 + c \max_{\nu \in [\tau, t]} \xi(\theta_\nu \omega) \int_\tau^t \left\| v^{\varepsilon_n}(s) - v^0(s) \right\|_{\mathcal{N}}^2 ds$$

$$+ c\varepsilon_n \max_{\nu \in [\tau, t]} \left\| \mathcal{T}^{-1}(\theta_\nu \omega) \right\| \int_\tau^t \left(\left\| v^{\varepsilon_n}(s) \right\|^2_{H^1_{\varepsilon_n}(\mathcal{O})} + \left\| v^0(s) \right\|^2_{H^1(\mathcal{Q})} \right) ds$$

$$+ c\varepsilon_n \max_{\nu \in [\tau, t]} \left\| \mathcal{T}^{-1}(\theta_\nu \omega) \right\| \int_\tau^t \left(\kappa_1^2(s) + \kappa_2^2(s) \right) ds$$

$$+ c\varepsilon_n \int_\tau^t \left(\left\| v^{\varepsilon_n}(s) \right\|^2_{H^1_{\varepsilon_n}(\mathcal{O})} + \left\| v^0(s) \right\|^2_{H^1(\mathcal{Q})} \right) ds, \tag{92}$$

where $\xi(\theta_t \omega) = \beta + |\delta(\theta_t \omega)|$. By Lemma 5.1 and Lemma 5.2 we find that there exists a positive constant $\varrho = \varrho(\tau, \omega, T)$, independent of ε_n, such that for all $t \in [\tau, \tau + T]$,

$$\left\| v^{\varepsilon_n}(t) - v^0(t) \right\|^2_{\mathcal{N}} \leq e^{c(1 + \max_{\nu \in [\tau, \tau+T]} \xi(\theta_\nu \omega))T} \left\| \phi^{\varepsilon_n} - \phi^0 \right\|^2_{\mathcal{N}}$$

$$+ \varrho \varepsilon_n e^{c(1 + \max_{\nu \in [\tau, \tau+T]} \xi(\theta_\nu \omega))T} [\|\psi^0\|^2_{\mathcal{M}} + \|\psi^{\varepsilon_n}\|^2_{\mathcal{N}}$$

$$+ \int_\tau^{\tau+T} \left(\kappa_1^2(s) + \kappa_2^2(s) \right) ds$$

$$+ \int_\tau^{\tau+T} \left(\|G(s, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \right) ds]. \tag{93}$$

Notice that, for all $t \in [\tau, \tau + T]$,

$$\|u^{\varepsilon_n}(t, \tau, \omega, \phi^\varepsilon) - u^0(t, \tau, \omega, \phi^0)\|^2_{\mathcal{N}}$$

$$\leq \max_{\nu \in [\tau, \tau+T]} \|\mathcal{T}(\theta_\nu \omega)\|^2 \|v^{\varepsilon_n}(t, \tau, \omega, \mathcal{T}^{-1}(\theta_\tau \omega)\phi^\varepsilon) - v^0(t, \tau, \omega, \mathcal{T}^{-1}(\theta_\tau \omega)\phi^0)\|^2_{\mathcal{N}},$$

which together with (93) implies the desired results. $\qquad\square$

The next result is concerned with uniform compactness of attractors with respect to ε.

Lemma 5.4. *Assume that (8)-(11), (39) and (43) hold. If $\varepsilon_n \to 0$ and $u^{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}(\tau, \omega)$, then there exist a subsequence of $(u^{\varepsilon_n})_{n \in \mathbb{N}}$, again denoted by $(u^{\varepsilon_n})_{n \in \mathbb{N}}$, and $u \in H^1(\mathcal{Q})$ such that*

$$\lim_{n \to \infty} \|u^{\varepsilon_n} - u\|_{H^1(\mathcal{O})} = 0.$$

Proof. Take a sequence $t_n \to \infty$. By the invariance of $\mathcal{A}_{\varepsilon_n}$ there exists $\phi^{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_n, \theta_{-t_n}\omega)$ such that

$$u^{\varepsilon_n} = \Phi_{\varepsilon_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \phi^{\varepsilon_n}). \tag{94}$$

By Lemma 4.1, we have $\phi^{\varepsilon_n} \in K(\tau - t_n, \theta_{-t_n}\omega) \in D_1$. Since $\varepsilon_n \to 0$ and $t_n \to \infty$, By Lemma 3.5, for any $\eta > 0$, there exists a large enough $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\left\| Q^{\varepsilon_n}_{N_1} u^{\varepsilon_n}(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^{\varepsilon_n}) \right\|_{H^1(\mathcal{O})} \leq \eta. \tag{95}$$

By Lemma 3.2, we have

$$\|P^{\varepsilon_n}_{N_1} u^{\varepsilon_n}(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^{\varepsilon_n})\|_{H^1(\mathcal{O})} < M. \tag{96}$$

It follows from (95) and (96) that $(u^{\varepsilon_n}(\tau, \tau - t_n, \theta_{-\tau}\omega, \phi^{\varepsilon_n}))_{n \in \mathbb{N}}$ is precompact in $H^1(\mathcal{O})$. Since the estimate (55) holds, there exists u in $H^1(\mathcal{Q})$ and a subsequence of $(u^{\varepsilon_n})_{n \in \mathbb{N}}$, again denoted by $(u^{\varepsilon_n})_{n \in \mathbb{N}}$, such that

$$\lim_{n \to \infty} \|u^{\varepsilon_n} - u\|_{H^1(\mathcal{O})} = 0. \tag{97}$$

This completes the proof. $\qquad\square$

Now we are in a position to prove the main result of this paper.

Theorem 5.5. *Assume that (8)-(11), (39), (43), and (89)-(90) hold. The attractors $\mathcal{A}_\varepsilon$ are upper-semicontinuous at $\varepsilon = 0$, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\lim_{\varepsilon \to 0} dist_{H^1(\mathcal{O})}\left(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)\right) = 0.$$

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by the invariance of $\mathcal{A}_\varepsilon$ and (55) we find that there exists $\varepsilon_0 > 0$ such that

$$\|u\|_{H^1_\varepsilon(\mathcal{O})}^2 \leq L(\tau, \omega) \quad \text{for all } 0 < \varepsilon < \varepsilon_0 \text{ and } u \in \mathcal{A}_\varepsilon(\tau, \omega), \tag{98}$$

where $L(\tau, \omega)$ is the positive constant given by the right-hand side of (55) which is independent of ε. If the theorem is not true, there exist $\delta > 0$, a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positives constants, $\varepsilon_n \to 0$, and a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in \mathcal{A}_{\varepsilon_n}(\tau, \omega)$ for all $n \in \mathbb{N}$, such that

$$\mathrm{dist}_{H^1(\mathcal{O})}(z_n, \mathcal{A}_0(\tau, \omega)) \geq \delta \quad \text{for all} \quad n \in \mathbb{N}. \tag{99}$$

By Lemma 5.4 there exists z_* in $H^1(\mathcal{Q})$ and a subsequence of $(z_n)_{n\in\mathbb{N}}$, again denoted by $(z_n)_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} \|z_n - z_*\|_{H^1(\mathcal{O})} = 0. \tag{100}$$

By the invariance property of the attractor $\mathcal{A}_{\varepsilon_n}(\tau, \omega)$, for every $t > 0$ there exists $y_n^t \in \mathcal{A}_{\varepsilon_n}(\tau - t, \theta_{-t}\omega)$ such that

$$z_n = \Phi_{\varepsilon_n}\left(t, \tau - t, \theta_{-t}\omega, y_n^t\right). \tag{101}$$

By Lemma 5.4 again there exists y_*^t in $H^1(\mathcal{Q})$ and a subsequence of $(y_n^t)_{n\in\mathbb{N}}$, again denoted by $(y_n^t)_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} \left\|y_n^t - y_*^t\right\|_{H^1(\mathcal{O})} = 0. \tag{102}$$

It follows from Theorem 5.3 that for every $t > 0$,

$$\lim_{n\to\infty} \Phi_{\varepsilon_n}\left(t, \tau - t, \theta_{-t}\omega, y_n^t\right) = \Phi_0\left(t, \tau - t, \theta_{-t}\omega, y_*^t\right) \quad \text{in} \quad \mathcal{N}. \tag{103}$$

By (100), (101), (103) and uniqueness of limits we obtain

$$z_* = \Phi_0\left(t, \tau - t, \theta_{-t}\omega, y_*^t\right) \quad \text{in} \quad H^1(\mathcal{O}). \tag{104}$$

Notice that $\mathcal{A}_{\varepsilon_n}(\tau - t, \theta_{-t}\omega) \subseteq K(\tau - t, \theta_{-t}\omega)$ and $y_n^t \in \mathcal{A}_{\varepsilon_n}(\tau - t, \theta_{-t}\omega)$ for all $n \in \mathbb{N}$. Thus by (98) we have

$$\limsup_{n\to\infty} \left\|y_n^t\right\|_{H^1(\mathcal{O})} \leq \|K(\tau - t, \theta_{-t}\omega)\|_{H^1(\mathcal{O})} \leq L(\tau - t, \theta_{-t}\omega). \tag{105}$$

By (102) and (105) we get, for every $t > 0$,

$$\left\|y_*^t\right\|_{H^1(\mathcal{Q})} \leq L(\tau - t, \theta_{-t}\omega). \tag{106}$$

By $K_0 \in \mathcal{D}_0$ and the attraction property of $\mathcal{A}_0$ in $\mathcal{D}_0$, we obtain from (104) and (106) that

$$\begin{aligned}
&\mathrm{dist}_{H^1(\mathcal{Q})}(z_*, \mathcal{A}_0(\tau, \omega)) \\
=\ &\mathrm{dist}_{H^1(\mathcal{Q})}\left(\Phi_0\left(t, \tau - t, \theta_{-t}\omega, y_*^t\right), \mathcal{A}_0(\tau, \omega)\right) \\
\leq\ &\mathrm{dist}_{H^1(\mathcal{Q})}\left(\Phi_0\left(t, \tau - t, \theta_{-t}\omega, K_0(\tau - t, \theta_{-t}\omega)\right), \mathcal{A}_0(\tau, \omega)\right) \\
&\to 0, \quad \text{as } t \to \infty.
\end{aligned} \tag{107}$$

This implies that $z_* \in \mathcal{A}_0(\tau, \omega)$ since $\mathcal{A}_0(\tau, \omega)$ is compact. Therefore, we have

$$\mathrm{dist}_{H^1(\mathcal{O})}(z_n, \mathcal{A}_0(\tau, \omega)) \leq \mathrm{dist}_{H^1(\mathcal{O})}(z_n, z_*) \to 0,$$

a contradiction with (99). This completes the proof. $\qquad\square$

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REFERENCES

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

[2] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **246** (2009), 845–869.

[3] P. W. Bates, K. Lu and B. Wang, Tempered random attractors for parabolic equations in weighted spaces, *J. Math. Phy.*, **54** (2013), 081505, 26 pp.

[4] T. Caraballo, I. D. Chueshov and P. E. Kloeden, Synchronization of a stochastic reaction-diffusion system on a thin two-layer domain, *SIAM J. Math. Anal.*, **38** (2006/07), 1489–1507.

[5] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dynam. Differential Equations*, **9** (1997), 307–341.

[6] J. Duan, K. Lu and B. Schmalfuss, Invariant manifolds for stochastic partial differential equations, *Ann. Probab.*, **31** (2003), 2109–2135.

[7] A. Gu, D. Li, B. Wang and H. Yang, Regularity of random attractors for fractional stochastic reaction-diffusion equations on $\mathbb{R}^n$, *J. Differential Equations*, **264** (2018), 7094–7137.

[8] J. K. Hale and G. Raugel, A reaction-diffusion equation on a thin L-shaped domain, *Proc. Roy. Soc. Edinburgh Sect. A*, **125** (1995), 283–327.

[9] X. Han, P. E. Kloeden and B. Usman, Long term behavior of a random Hopfield neural lattice model, *Commun. Pure Appl. Anal.*, **18** (2019), 809–824.

[10] X. Han, W. Shen and S. Zhou, Random attractors for stochastic lattice dynamical systems in weighted spaces, *J. Differential Equations*, **250** (2011), 1235–1266.

[11] F. Li, Y. Li and R. Wang, Strong convergence of bi-spatial random attractors for parabolic equations on thin domains with rough noise, *Topol. Methods Nonlinear Anal.*, **53** (2019), 659–682.

[12] F. Li, Y. Li and R. Wang, Regular measurable dynamics for reaction-diffusion equations on narrow domains with rough noise, *Discrete Contin. Dyn. Syst.*, **38** (2018), 3663–3685.

[13] D. Li, K. Lu, B. Wang and X. Wang, Limiting behavior of dynamics for stochastic reaction-diffusion equations with additive noise on thin domains, *Discrete Contin. Dyn. Syst.*, **38** (2018), 187–208.

[14] D. Li, K. Lu, B. Wang and X. Wang, Limiting dynamics for non-autonomous stochastic retarded reaction-diffusion equations on thin domains, *Discrete Contin. Dyn. Syst.*, **39** (2019), 3717–3747.

[15] D. Li and L. Shi, Upper semicontinuity of attractors of stochastic delay reaction-diffusion equations in the delay, *J. Math. Phys.*, **59** (2018), 032703, 35 pp.

[16] D. Li and X. Wang, Asymptotic behavior of stochastic complex Ginzburg-Landau equations with deterministic non-autonomous forcing on thin domains, *Discrete Contin. Dyn. Syst. B*, **24** (2019), 449–465.

[17] D. Li, B. Wang and X. Wang, Limiting behavior of non-autonomous stochastic reaction-diffusion equations on thin domains, *J. Differential Equations*, **262** (2017), 1575–1602.

[18] W. Liu and B. Wang, Poisson-Nernst-Planck systems for narrow tubular-like membrane channels, *J. Dynam. Differential Equations*, **22** (2010), 413–437.

[19] Z. Shen, S. Zhou and W. Shen, One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation, *J. Differential Equations*, **248** (2010), 1432–1457.

[20] L. Shi, R. Wang, K. Lu and B. Wang, Asymptotic behavior of stochastic FitzHugh-Nagumo systems on unbounded thin domains, *J. Differential Equations*, **267** (2019), 4373–4409.

[21] B. Wang, Suffcient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012), 1544–1583.

[22] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **34** (2014), 269–300.

[23] Y. Wang and J. Wang, Pullback attractors for multi-valued non-compact random dynamical systems generated by reaction-diffusion equations on an unbounded domain, *J. Differential Equations*, **259** (2015), 728–776.

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