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A WEAK GALERKIN FINITE ELEMENT METHOD FOR NONLINEAR CONSERVATION LAWS

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Abstract. A weak Galerkin (WG) finite element method is presented for nonlinear conservation laws. There are two built-in parameters in this WG framework. Different choices of the parameters will lead to different approaches for solving hyperbolic conservation laws. The convergence analysis is obtained for the forward Euler time discrete and the third order explicit TVDRK time discrete WG schemes respectively. The theoretical results are verified by numerical experiments.

1. **Introduction.** The nonlinear hyperbolic equation of conservation laws is considered: seeking an unknown function u satisfying

$$u_t + f(u)_x = 0, \quad (x,t) \in I \times (0,T], \tag{1.1}$$

$$u(x,0) = \phi(x), \quad x \in I, \tag{1.2}$$

with $I = (0,1)$ and a periodic boundary condition. For simplicity, the nonlinear flux function $f(u)$ is assumed to be smooth enough.

The Runge-Kutta discontinuous Galerkin (RKDG) method has been developed for solving time-dependent nonlinear conservation laws [1, 2]. The RKDG method, by name, uses DG method for spacial discretization and explicit high order Runge-Kutta method for time discretizations. The stability and error analysis of the RKDG method has been studied in [22, 23] for the second and the third order explicit total variation diminishing Runge-Kutta method. A discontinuous Galerkin method with Lagrange multiplier (DGLM) has been developed in [6, 7] for nonlinear conservation laws with backward Euler method for time discretization. Lagrange multipliers are introduced on each element so that they are the only globally coupled variables in

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the resulting system. The final global system of the DGLM has fewer numbers of coupled unknowns than the usual DG methods.

The most finite element methods for conservation law employ purely upwind or general monotone fluxes. In [11], discontinuous Galerkin methods using more general upwind-biased numerical fluxes have been investigated for time-dependent linear conservation laws. Optimal order of convergence rate has been obtained. As pointed out in [11], purely upwind fluxes may be difficult to construct for complex systems.

Weak Galerkin methods refer to general finite element techniques for partial differential equations and were first introduced in [18, 19] for second order elliptic equations. Weak Galerkin methods make use of discontinuous piecewise polynomials on finite element partitions with arbitrary shape of polygons and polyhedrons. The weak Galerkin methods have been applied to solve various PDEs such as second order elliptic equations, biharmonic equations, Stokes equations, parabolic equations, second order hyperbolic equations, Maxwell's equations and singularly perturbed convection-diffusion-reaction problems [8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 20]. A least-squares based weak Galerkin method is presented for stationary linear hyperbolic equations [21].

The objective of this work is to develop a weak Galerkin finite element method for the time-dependent nonlinear conservation laws (1.1)-(1.2), with the explicit first order Euler method and the third order explicit TVD Runge-Kutta method for time discretization. Similar to [11], this new WG formulation provides a class of finite element methods featuring two built-in parameters λ_1 and λ_2. By tuning these parameters, different schemes can be obtained for solving the problem (1.1)-(1.2) including purely upwinding scheme. However, unlike the method in [11], our new WG method can be used for the time-dependent nonlinear conservation laws. The stability is derived for the semi-discretized WG method. For the forward Euler time discrete WG method, the L^2 error estimate of $O(h^{k+\frac{1}{2}}+\tau)$ is derived in general and convergence rate of $O(h^{k+1}+\tau)$ is obtained for some special combination of the parameters. The temporal-spatial CFL condition $\tau < Ch^2$ is necessary in the error analysis for the first order forward Euler method. If the third order explicit TVDRK time discrete scheme [3] is used, the L^2 error estimate of $O(h^{k+\frac{1}{2}}+\tau^3)$ is proved under the CFL condition $\tau < Ch$.

The rest of the paper is organized as follows. In Section 2, a WG finite element is proposed for spatial discretization. The stability is derived for semi-discretized WG method. The forward Euler time discretized WG scheme and its error analysis are presented in Section 3. Error analysis of the WG method with third order explicit TVDRK time discretization can be found in Section 4. Numerical experiments are presented in Section 5 to support the theoretical results. We end the paper with a conclusion.

The usual notation of norms in Sobolev spaces will be used. For any integer $s \geq 0$, let $H^s(D)$ represent the well-known Sobolev space equipped with the norm $\|\cdot\|_{s,D}$, which consists of functions with (distributional) derivatives of order no more than s in $L^2(\Omega)$. Next, denote by $(\cdot,\cdot)_D$ the scalar inner product on $L^2(D)$ and $\|\cdot\|_D$ denotes the associated L^2 norm. Furthermore, let $\|\cdot\|_{\infty,D}$ be the norm on $L^\infty(D)$. If $D = I$, we omit this subscript.

2. Semi-discrete weak Galerkin scheme and its stability. In this section, we introduce a weak Galerkin finite element method for solving the model problem (1.1)-(1.2).

2.1. Semi-discrete WG scheme. Let $\mathcal{T}_h = \cup_{i=1}^N I_i$ with $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ for $1 \le i \le N$ where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1.$$

Define

$$h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}},\ \ 1 \le i \le N;\ \ h = \max_{1\le i\le N} h_i,\ \ \partial I_i = \{x_{i-\frac{1}{2}}\} \cup \{x_{i+\frac{1}{2}}\}.$$

The weak Galerkin methods introduce a new way to define a function v, called weak function, which allows v taking different forms in the interior and on the boundary of the element:

$$v = \begin{cases} v_0, & \text{in } {I_i}^0, \\ v_b, & \text{on } \partial I_i, \end{cases}$$

where ${I_i}^0$ is the interior of I_i, where I_i is an element in $\mathcal{T}_h$. Since a weak function v is formed by two parts v_0 and v_b, we write v as $v = \{v_0, v_b\}$ in short without confusion.

Denote by $P_k(I_i)$ the set of polynomials on I_i with degree no more than k. Let V_h be a weak Galerkin finite element space consisting of weak function $v = \{v_0, v_b\}$ defined as follows for $k \ge 1$,

$$\begin{aligned} V_h = \{v = \{v_0, v_b\} :\ & v_0|_{I_i} \in P_k(I_i),\ v_b|_{\partial I_i} \in P_0(\partial I_i), \\ & v_b(0) = v_b(1), i = 1, \cdots, N\}. \end{aligned} \tag{2.1}$$

Denote by $v(x^-_{i+\frac{1}{2}})$ and $v(x^+_{i+\frac{1}{2}})$ the values of v at $x_{i+\frac{1}{2}}$ from the left element I_i and the right element I_{i+1}, respectively. Further, the jump of v at $x_{i+\frac{1}{2}}$ is denoted as $[\![v]\!]_{i+\frac{1}{2}} = v(x^+_{i+\frac{1}{2}}) - v(x^-_{i+\frac{1}{2}})$.

For $v, w \in V_h$, we introduce some notations,

$$\begin{aligned} \langle v,\ w\rangle_{\partial I_i} &= v(x^-_{i+\frac{1}{2}})w(x^-_{i+\frac{1}{2}}) + v(x^+_{i-\frac{1}{2}})w(x^+_{i-\frac{1}{2}}), \\ \langle v, w\rangle_{\partial \mathcal{T}_h} &= \sum_{i=1}^N \langle v, w\rangle_{\partial I_i}, \\ (v, w)_{\mathcal{T}_h} &= \sum_{i=1}^N (v, w)_{I_i} = \sum_{i=1}^N \int_{I_i} vw dx. \end{aligned}$$

For any $v = \{v_0, v_b\} \in V_h$, a weak derivative $D_w f(v) \in P_k(I_i)$ for $i = 1, \cdots, N$ satisfies

$$(D_w f(v), w)_{I_i} = -(f(v_0),\ Dw)_{I_i} + \langle f(v_b),\ wn\rangle_{\partial I_i}, \qquad \forall w \in P_k(I_i), \tag{2.2}$$

where $n = -1$ at $x_{i-\frac{1}{2}}$ and $n = 1$ at $x_{i+\frac{1}{2}}$. Here Dw means the first order derivative of w, i.e. $Dw = w'$.

Then we introduce a stabilizer in V_h as follows:

$$s_h(v, w) = \sum_{i=1}^N (\langle \lambda_1(v_0 - v_b), w_0 - w_b\rangle_{\partial_+ I_i} + \langle \lambda_2(v_0 - v_b), w_0 - w_b\rangle_{\partial_- I_i}), \tag{2.3}$$

where $\partial_+ I_i = x_{i+\frac{1}{2}}$, $\partial_- I_i = x_{i-\frac{1}{2}}$, and λ_1 and λ_2 are two parameters. Here $v_0(x_{i+\frac{1}{2}})$ and $w_0(x_{i+\frac{1}{2}})$ are the left limit of $v_0(x)$ and $w_0(x)$ at $x_{i+\frac{1}{2}}$ respectively, and $v_0(x_{i-\frac{1}{2}})$ and $w_0(x_{i-\frac{1}{2}})$ are the right limit of $v_0(x)$ and $w_0(x)$ at $x_{i-\frac{1}{2}}$ respectively.

The following is the semi-discretized weak Galerkin method.

Algorithm 1 (SD-WG method)**.** A numerical approximation for (1.1)-(1.2) can be obtained by seeking $u_h(t) = \{u_0(t), u_b(t)\} \in V_h$ satisfying $u_h(0) = Q_0\phi$ and the following equation,

$$(\partial_t u_0, v_0) + (D_w f(u_h), v_0)_{\mathcal{T}_h} + s_h(u_h, v) = 0, \quad \forall\ v = \{v_0, v_b\} \in V_h, \tag{2.4}$$

where Q_0 is the L^2 projection onto $P_k(I_i)$ on each element I_i.

Let I_i and I_{i+1} be the two intervals sharing $x_{i+\frac{1}{2}}$. Define the average $\{\!\{v\}\!\}$ on $x_{i+\frac{1}{2}}$ by

$$\{\!\{v\}\!\}_i = \frac{\lambda_1}{\lambda_1+\lambda_2} v(x^-_{i+\frac{1}{2}}) + \frac{\lambda_2}{\lambda_1+\lambda_2} v(x^+_{i+\frac{1}{2}}),$$

and

$$\{\!\{v\}\!\}_0 = v(0), \quad \{\!\{v\}\!\}_N = v(1).$$

Please note that the definition of average $\{\!\{\cdot\}\!\}$ above is different from the standard definition of average which is when $\lambda_1 = \lambda_2$.

Testing (2.4) by $v = \{v_0, v_b\}$ such that $v_0 = 0$ and $v_b = 1$ at $x_{i+\frac{1}{2}}$ and $v_b = 0$ otherwise, we can easily obtain that at $x_{i+\frac{1}{2}}$

$$u_b(t) = \{\!\{u_0(t)\}\!\}. \tag{2.5}$$

Remark 1 (Relation to the upwinding-type DG method)**.** If $f'(u) > 0$ and taking $\lambda_2 = 0$, then $u_b = u_0^-$ and $s_h(u_h, v_h) = 0$. Thus, the WG scheme (2.4) reduces to

$$(\partial_t u_0, v_0)_{I_i} - (f(u_0), v_0')_{I_i} + \langle f(u_0^-), v_0 n\rangle_{\partial I_i} = 0, \quad 1 \le i \le N,$$

which is the classical upwinding type discontinuous Galerkin method for nonlinear conservation law.

Remark 2 (Relation to the upwinding-biased DG method)**.** Considering a special case $f(u) = u$. Taking $\lambda_1 = \lambda_2 = \lambda$ in (2.3), and denote by

$$\widehat{u}_0 = (\frac{1}{2} + \frac{1}{4}\lambda)u_0^- + (\frac{1}{2} - \frac{1}{4}\lambda)u_0^+,$$

then the WG scheme (2.4) reduces to

$$(\partial_t u_0, v_0)_{I_i} - (u_0, v_0')_{I_i} + (\widehat{u}_0)_{i+1/2}(v_0)^-_{i+1/2} - (\widehat{u}_0)_{i-1/2}(v_0)^+_{i-1/2} = 0, \quad 1 \le i \le N,$$

which is the upwinding-biased DG method discussed in [11].

2.2. **Stability of the semi-discrete WG scheme.** In this subsection, we will study the stability of the semi-discrete WG scheme (2.4).

Lemma 2.1. *Let $\alpha_2 \le f'(s) \le \alpha_1$. If $\lambda_1 > \alpha_1/2$ and $\lambda_2 > -\alpha_2/2$, then for $v = \{v_0, v_b\} \in V_h$, there holds*

$$(D_w f(v), v_0)_{\mathcal{T}_h} + s_h(v, v) \ge 0. \tag{2.6}$$

Proof. As [5], we introduce $g'(s) = f(s)$, then

$$\begin{aligned}(f(v_0), Dv_0)_{\mathcal{T}_h} &= \sum_{i=1}^{N}\int_{I_i} f(v_0)Dv_0dx = \sum_{i=1}^{N}\int_{I_i} Dg(v_0)dx \\ &= \sum_{i=1}^{N}(g(v_0(x^-_{i+\frac{1}{2}})) - g(v_0(x^+_{i-\frac{1}{2}}))) \\ &= \sum_{i=1}^{N}\int_{v_b(x_{i+\frac{1}{2}})}^{v_0(x^-_{i+\frac{1}{2}})} f(s)ds - \sum_{i=1}^{N}\int_{v_b(x_{i-\frac{1}{2}})}^{v_0(x^+_{i-\frac{1}{2}})} f(s)ds.\end{aligned}$$

Since $f(v_b)$ and v_b take single value on ∂I_i, the periodic boundary condition implies

$$\sum_{i=1}^{N}\langle f(v_b), v_b n\rangle_{\partial I_i} = 0. \tag{2.7}$$

Using the definition of the weak derivative (2.2), the mean value theory and the periodic boundary condition that

$$\begin{aligned}(D_w f(v), v_0)_{\mathcal{T}_h} &= -(f(v_0), Dv_0)_{\mathcal{T}_h} + \langle f(v_b), v_0 n\rangle_{\partial\mathcal{T}_h} \\ &= -\sum_{i=1}^{N}\int_{v_b(x_{i+\frac{1}{2}})}^{v_0(x^-_{i+\frac{1}{2}})}(f(s) - f(v_b))ds + \sum_{i=1}^{N}\int_{v_b(x_{i-\frac{1}{2}})}^{v_0(x^+_{i-\frac{1}{2}})}(f(s) - f(v_b))ds \\ &= -\sum_{i=1}^{N}\int_{v_b(x_{i+\frac{1}{2}})}^{v_0(x^-_{i+\frac{1}{2}})} f'(\xi_1)(s - v_b)ds + \sum_{i=1}^{N}\int_{v_b(x_{i-\frac{1}{2}})}^{v_0(x^+_{i-\frac{1}{2}})} f'(\xi_2)(s - v_b)ds\end{aligned}$$

which implies

$$\begin{aligned}(D_w f(v), v_0)_{\mathcal{T}_h} &\geq -\frac{\alpha_1}{2}\sum_{i=1}^{N}(v_0(x^-_{i+\frac{1}{2}}) - v_b(x_{i+\frac{1}{2}}))^2 \\ &+ \frac{\alpha_2}{2}\sum_{i=1}^{N}(v_0(x^+_{i-\frac{1}{2}}) - v_b(x_{i-\frac{1}{2}}))^2,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}(D_w f(v), v_0)_{\mathcal{T}_h} &\leq \frac{\alpha}{2}(\sum_{i=1}^{N}(v_0(x^-_{i+\frac{1}{2}}) - v_b(x_{i+\frac{1}{2}}))^2 \\ &+ \sum_{i=1}^{N}(v_0(x^+_{i-\frac{1}{2}}) - v_b(x_{i-\frac{1}{2}}))^2),\end{aligned} \tag{2.9}$$

where $\alpha = \max\{|\alpha_1|, |\alpha_2|\}$. The equation (2.8) gives

$$\begin{aligned}(D_w f(v), v_0)_{\mathcal{T}_h} + s_h(v, v) &\geq \sum_{i=1}^{N}(\lambda_1 - \frac{\alpha_1}{2})(v_0(x^-_{i+\frac{1}{2}}) - v_b(x_{i+\frac{1}{2}}))^2 \\ &\quad + \sum_{i=1}^{N}(\lambda_2 + \frac{\alpha_2}{2})(v_0(x^+_{i-\frac{1}{2}}) - v_b(x_{i-\frac{1}{2}}))^2 \\ &\geq 0.\end{aligned}$$

We complete the proof. □

Define $\|v\|^2 = \int_I v^2 dx$. Then we have the following stability result.

Lemma 2.2 (Stability of the SD-WG method)**.** *Let $u_h = \{u_0, u_b\} \in V_h$ be the solution of the semi-discrete WG scheme (2.4), then*

$$\|u_0(T)\| \le \|u_0(0)\|. \tag{2.10}$$

Proof. Let $v = u_h$ in the semi-discrete WG scheme (2.4). From (2.6), we have

$$\frac{1}{2}\frac{d}{dt}\|u_0\|^2 \le ((u_0)_t, u_0) + (D_w f(u_h), u_0)_{\mathcal{T}_h} + s_h(u_h, u_h) = 0.$$

Integrating the above inequality with respect to time between 0 and T completes the proof. □

3. **Forward Euler WG method and its error analysis.**

We use forward Euler method for time discretization to obtain a full discrete WG finite element method.

3.1. **Forward Euler time discrete WG scheme.**

Let τ be a time step and $t_n = n\tau$.

Algorithm 2 (FE-WG method)**.** Find $u_h^{n+1} = \{u_0^{n+1}, u_b^{n+1}\} \in V_h$ satisfying $u_h^0 = \{Q_0\phi, \{\!\{Q_0\phi\}\!\}\}$ and

$$(u_0^{n+1} - u_0^n, v_0) + \tau(D_w f(u_h^n), v_0)_{\mathcal{T}_h} + \tau\, s_h(u_h^n, v) \quad = \quad 0, \quad \forall v \in V_h. \tag{3.1}$$

Testing (3.1) by $v = \{v_0, v_b\}$ such that $v_0 = 0$ and $v_b = 1$ at $x_{i+\frac{1}{2}}$ and $v_b = 0$ otherwise, we can easily obtain that at $x_{i+\frac{1}{2}}$

$$u_b^n = \{\!\{u_0^n\}\!\}. \tag{3.2}$$

We define a projection operator $Q_h u = \{Q_0 u, Q_b u\} \in V_h$, where Q_0 is the L^2 projection onto $P_k(I_i)$ on each element I_i and $Q_b u = \{\!\{Q_0 u\}\!\}$ on ∂I_i.

Define

$$e_h^n = \{e_0^n, e_b^n\} = Q_h u(t_n) - u_h^n = \{Q_0 u(t_n) - u_0^n, Q_b u(t_n) - u_b^n\}, \tag{3.3}$$

$$\rho_h^n = \{\rho_0^n, \rho_b^n\} = u(t_n) - Q_h u(t_n) = \{u(t_n) - Q_0 u(t_n), u(t_n) - Q_b u(t_n)\} \tag{3.4}$$

and

$$\ell_1(v) = (u_t(t_n) - \frac{u(t_{n+1}) - u(t_n)}{\tau}, v_0),$$

$$\ell_2(v) = (D_w f(u(t_n)), v_0)_{\mathcal{T}_h} - (D_w f(u_h^n), v_0)_{\mathcal{T}_h}.$$

Lemma 3.1. *The error function e_h^n defined in (3.3) satisfies the following equation,*

$$\frac{1}{2}\|e_0^{n+1}\|^2 + \frac{1}{2}\|e_0^{n+1} - e_0^n\|^2 - \frac{1}{2}\|e_0^n\|^2 + \tau\, s_h(e_h^n, e_h^{n+1} - e_h^n) + \tau\, s_h(e_h^n, e_h^n)$$
$$+ \tau\ell_2(e_h^{n+1}) = -\tau\ell_1(e_h^{n+1}) + \tau s_h(Q_h u(t_n), e_h^{n+1}). \tag{3.5}$$

Proof. Testing (1.1) by v_0 of $v = \{v_0, v_b\} \in V_h$ we arrive at

$$(u_t, v_0) + (Df(u), v_0)_{\mathcal{T}_h} = 0.$$

The definition of D_w implies,

$$(Df(u(t_n)), v_0)_{\mathcal{T}_h} = (D_w f(u(t_n)), v_0)_{\mathcal{T}_h}.$$

It follows from the definition of Q_0 and the above equation,

$$(Q_0 u(t_{n+1}) - Q_0 u(t_n), v_0) + \tau(D_w f(u(t_n)), v_0)_{\mathcal{T}_h} = -\tau\ell_1(v).$$

Adding $\tau s_h(Q_h u(t_n), v)$ to the both sides of the above equation implies

$$(Q_0 u(t_{n+1}) - Q_0 u(t_n), v_0) + \tau(D_w f(u(t_n)), v_0)_{\mathcal{T}_h} + \tau s_h(Q_h u(t_n), v) \\ = \tau s_h(Q_h u(t_n), v) - \tau \ell_1(v).$$

The difference of (3.1) and the equation above yields

$$(e_0^{n+1} - e_0^n, v_0) + \tau\ s_h(e_h^n, v) + \tau(D_w f(u(t_n)), v_0)_{\mathcal{T}_h} - \tau(D_w f(u_h^n), v_0)_{\mathcal{T}_h} \\ = \tau s_h(Q_h u(t_n), v) - \tau \ell_1(v). \tag{3.6}$$

Using the fact $2p(p-q) = p^2 + (p-q)^2 - q^2$ and letting $v = e_h^{n+1}$, (3.6) becomes

$$\frac{1}{2}\|e_0^{n+1}\|^2 + \frac{1}{2}\|e_0^{n+1} - e_0^n\|^2 - \frac{1}{2}\|e_0^n\|^2 + \tau s_h(e_h^n, e_h^{n+1} - e_h^n) + \tau\ s_h(e_h^n, e_h^n) \\ + \tau \ell_2(e_h^{n+1}) = -\tau \ell_1(e_h^{n+1}) + \tau s_h(Q_h u(t_n), e_h^{n+1}).$$

We have proved the lemma. □

3.2. **Error analysis of forward Euler WG method.** In this subsection we carry out an a priori error estimate for the fully discrete WG scheme with forward Euler time marching for smooth solutions. We will assume the nonlinear flux function $f(u)$ is smooth enough for simplicity.

For any function $\varphi \in H^1(I_i)$, the following trace inequality holds true,

$$\|\varphi\|_{\partial I_i}^2 \le C\left(h_i^{-1}\|\varphi\|_{I_i}^2 + h_i\|\nabla\varphi\|_{I_i}^2\right). \tag{3.7}$$

Lemma 3.2. *Let ρ_h^n and e_h^n be defined in (3.4) and (3.3) respectively. Then we have*

$$\|\rho_0^n\|_{I_i} \le Ch^{k+1}|u|_{k+1,I_i}, \quad \|\rho_b^n\|_{\partial I_i} \le Ch^{k+\frac{1}{2}}|u|_{k+1,I_i}, \tag{3.8}$$

$$\|e_b^n\|_{\partial I_i} \le Ch^{-1/2}\|e_0^n\|_{I_i}. \tag{3.9}$$

Proof. The first estimate in (3.8) is a direct result of the approximation property of the L^2 projection Q_0. The second estimate in (3.8) follows from the trace inequality (3.7) and the definitions of ρ_b^n and Q_h,

$$\|\rho_b^n\|_{\partial I_i} = \|u(t_n) - Q_b u(t_n)\|_{\partial I_i} = \|\{\!\{u(t_n) - Q_0 u(t_n)\}\!\}\|_{\partial I_i} \le Ch^{k+\frac{1}{2}}|u|_{k+1,I_i}.$$

Similarly, we can prove (3.9). □

Lemma 3.3. *Let $\tau \le ch^2$. Then we have*

$$\tau s_h(Q_h u(t_n), e_h^{n+1}) \le C\tau h^{2k+1}|u|_{k+1}^2 + \epsilon_1\|e_0^{n+1} - e_0^n\|^2 + \epsilon_2 \tau s_h(e_h^n, e_h^n), \tag{3.10}$$

$$\tau \ell_1(e_h^{n+1}) \le C\tau^3\|u_{tt}\|^2 + \epsilon_1\|e_0^{n+1} - e_0^n\|^2 + \tau\|e_0^n\|^2, \tag{3.11}$$

$$\tau s_h(e_h^n, e_h^{n+1} - e_h^n) \le C\tau\|e_0^n\|^2 + \epsilon_1\|e_0^{n+1} - e_0^n\|^2. \tag{3.12}$$

Proof. It follows from (3.7), the Cauchy-Schwarz inequality and the definition of Q_b that

$$\begin{aligned} s_h(Q_hu(t_n),e_h^{n+1}) &= s_h(Q_hu(t_n),e_h^{n+1}-e_h^n)+s_h(Q_hu(t_n),e_h^n)\\ &= \sum_{i=1}^N(\langle\lambda_1(Q_0u(t_n)-u(t_n)),(e_0^{n+1}-e_0^n)-(e_b^{n+1}-e_b^n)\rangle_{\partial_+I_i}\\ &+ \langle\lambda_2(Q_0u(t_n)-u(t_n)),(e_0^{n+1}-e_0^n)-(e_b^{n+1}-e_b^n)\rangle_{\partial_-I_i})\\ &+ \sum_{i=1}^N(\langle\lambda_1(Q_0u(t_n)-u(t_n)),e_0^n-e_b^n\rangle_{\partial_+I_i}\\ &+ \langle\lambda_2(Q_0u(t_n)-u(t_n)),e_0^n-e_b^n\rangle_{\partial_-I_i})\\ &\le C(h^{-1}h^{k+1}|u|_{k+1}\|e_0^{n+1}-e_0^n\|+h^{k+\frac12}|u|_{k+1}s_h^{1/2}(e_h^n,e_h^n)). \end{aligned}$$

Using the assumption $\tau\le ch^2$, we have

$$\begin{aligned} \tau s_h(Q_hu(t_n),e_h^{n+1}) &\le C\tau(h^{-1}h^{k+1}|u|_{k+1}\|e_0^{n+1}-e_0^n\|+h^{k+\frac12}|u|_{k+1}s_h^{1/2}(e_h^n,e_h^n))\\ &\le C(\tau\frac{\tau}{h^2}h^{2k+2}|u|_{k+1}^2+\tau h^{2k+1}|u|_{k+1}^2)\\ &\quad +\epsilon_1\|e_0^{n+1}-e_0^n\|^2+\epsilon_2\tau s_h(e_h^n,e_h^n)\\ &\le C\tau h^{2k+1}|u|_{k+1}^2+\epsilon_1\|e_0^{n+1}-e_0^n\|^2+\epsilon_2\tau s_h(e_h^n,e_h^n), \end{aligned}$$

which proves (3.10).

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \tau\ell_1(e_h^{n+1}) &\le C\tau^2\|u_{tt}\|\|e_0^{n+1}\|\\ &\le C\tau^2\|u_{tt}\|(\|e_0^{n+1}-e_0^n\|+\|e_0^n\|)\\ &\le C\tau^3\|u_{tt}\|^2+\epsilon_1\|e_0^{n+1}-e_0^n\|^2+\tau\|e_0^n\|^2. \end{aligned}$$

Using (3.7), Cauchy-Schwarz inequality, Lemma 3.2 and the assumption $\tau\le ch^2$, we have

$$\begin{aligned} \tau s_h(e_h^n,e_h^{n+1}-e_h^n) &\le C\frac{\tau}{h}\|e_0^n\|\|e_0^{n+1}-e_0^n\|\\ &\le C\tau\|e_0^n\|^2+\epsilon_1\|e_0^{n+1}-e_0^n\|^2. \end{aligned}$$

We have proved the lemma. □

Lemma 3.4. *For $\tau\le ch^2$ and $\lambda_1,\lambda_2\ge\alpha$. Then we have*

$$\begin{aligned} \tau\ell_2(e_h^{n+1}) \le& C\tau h^{2k+1}|u|_{k+1}^2+C\tau\|e_0^n\|^2\\ &+\epsilon_1\|e_0^{n+1}-e_0^n\|^2+\tau(\frac12+\epsilon_2)s_h(e_h^n,e_h^n). \end{aligned}\tag{3.13}$$

Proof. It follows from definition of the weak derivative D_w and the Taylor theory,

$$\begin{aligned} \ell_2(e_h^{n+1}) =&(D_wf(u(t_n)),e_0^{n+1})_{\mathcal{T}_h}-(D_wf(u_h^n),e_0^{n+1})_{\mathcal{T}_h}\\ =&-(f(u(t_n))-f(u_0^n),De_0^{n+1})_{\mathcal{T}_h}+\langle f(u(t_n))-f(u_b^n),e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}\\ =&-(f'(u(t_n))(\rho_0^n+e_0^n),De_0^{n+1})_{\mathcal{T}_h}+\frac12(f''(\xi_1)(\rho_0^n+e_0^n)^2,De_0^{n+1})_{\mathcal{T}_h}\\ &+\langle f'(u(t_n))(\rho_b^n+e_b^n),e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}-\frac12\langle f''(\xi_2)(\rho_b^n+e_b^n)^2,e_0^{n+1}n\rangle_{\partial\mathcal{T}_h} \end{aligned}$$

$$
\begin{aligned}
&= -\,(f'(u(t_n))\rho_0^n, De_0^{n+1})_{\mathcal{T}_h} + \langle f'(u(t_n))\rho_b^n, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}\\
&+(-(f'(u(t_n))e_0^n, De_0^{n+1})_{\mathcal{T}_h} + \langle f'(u(t_n))e_b^n, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h})\\
&+\frac{1}{2}(f''(\xi_1)(\rho_0^n+e_0^n)^2, De_0^{n+1})_{\mathcal{T}_h} - \frac{1}{2}\langle f''(\xi_2)(\rho_b^n+e_b^n)^2, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}\\
&=\sum_{i=1}^{5} M_i.
\end{aligned}
$$

Next, we will bound all the terms above. Define $\overline{f'(u)} = \frac{1}{|I_i|}\int_{I_i} f'(u)dx$. Using the definitions of ρ_0^n and Q_0, (3.8) and the inverse inequality, we arrive

$$
\begin{aligned}
M_1 &= (f'(u(t_n))\rho_0^n, De_0^{n+1})_{\mathcal{T}_h}\\
&= ((f'(u(t_n)) - \overline{f'(u(t_n))})\rho_0^n, De_0^{n+1})_{\mathcal{T}_h}\\
&\le Ch^{k+1}|u|_{k+1}\|e_0^{n+1}\|\\
&\le Ch^{k+1}|u|_{k+1}(\|e_0^{n+1}-e_0^n\| + \|e_0^n\|).
\end{aligned}
$$

It follows from the fact $\langle f'(u(t_n))\rho_b^n, e_b^n n\rangle_{\partial\mathcal{T}_h} = 0$, (3.7) and the inverse inequality,

$$
\begin{aligned}
M_2 &= \langle f'(u(t_n))\rho_b^n, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}\\
&= \langle f'(u(t_n))\rho_b^n, (e_0^{n+1}-e_0^n)n\rangle_{\partial\mathcal{T}_h} + \langle f'(u(t_n))\rho_b^n, (e_0^n-e_b^n)n\rangle_{\partial\mathcal{T}_h}\\
&\le C(h^{-1}h^{k+1}|u|_{k+1}\|e_0^{n+1}-e_0^n\| + h^{k+\frac{1}{2}}|u|_{k+1}s_h^{\frac{1}{2}}(e_h^n, e_h^n)).
\end{aligned}
$$

Using the inverse inequality, the trace inequality (3.7), (3.9) and (2.9), we obtain

$$
\begin{aligned}
M_3 &= -(f'(u(t_n))e_0^n, De_0^{n+1})_{\mathcal{T}_h} + \langle f'(u(t_n))e_b^n, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}\\
&= -((f'(u(t_n)) - \overline{f'(u(t_n))})e_0^n, De_0^{n+1})_{\mathcal{T}_h} + \langle (f'(u(t_n)) - \overline{f'(u(t_n))})e_b^n, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h}\\
&- (\overline{f'(u(t_n))}e_0^n, D(e_0^{n+1}-e_0^n))_{\mathcal{T}_h} + \langle \overline{f'(u(t_n))}e_b^n, (e_0^{n+1}-e_0^n)n\rangle_{\partial\mathcal{T}_h}\\
&- (\overline{f'(u(t_n))}e_0^n, De_0^n)_{\mathcal{T}_h} + \langle \overline{f'(u(t_n))}e_b^n, e_0^n n\rangle_{\partial\mathcal{T}_h}\\
&\le C(\|e_0^n\|^2 + h^{-1}\|e_0^n\|\|e_0^{n+1}-e_0^n\| + \frac{1}{2}s_h(e_h^n, e_h^n)).
\end{aligned}
$$

As [22, 23], we adopt the following a priori assumption to deal with the nonlinearity of the flux $f(u)$

$$
\|e_0^n\| \le Ch^{3/2}. \tag{3.14}
$$

And the justification of such assumption will be given in Remark 3. Then the above estimate with (3.2) gives

$$
\|e_0^n\|_\infty \le Ch, \quad \|e_b^n\|_\infty \le Ch. \tag{3.15}
$$

Using (3.8), (3.15) and the inverse inequality, we have

$$
\begin{aligned}
M_4 &= \frac{1}{2}(f''(\xi)(\rho_0^n+e_0^n)^2, De_0^{n+1})_{\mathcal{T}_h} \le C(h^{k+1}|u|_{k+1} + \|e_0^n\|)(\|e_0^{n+1}-e_0^n\| + \|e_0^n\|),\\
M_5 &= \frac{1}{2}\langle f''(\xi)(\rho_b^n+e_b^n)^2, e_0^{n+1}n\rangle_{\partial\mathcal{T}_h} \le C(h^{k+1}|u|_{k+1} + \|e_0^n\|)(\|e_0^{n+1}-e_0^n\| + \|e_0^n\|).
\end{aligned}
$$

Combining all the estimates above gives

$$
\tau\ell_2(e_h^{n+1}) \le C\tau h^{2k+1}|u|_{k+1}^2 + C\tau\|e_0^n\|^2 + \epsilon_1\|e_0^{n+1}-e_0^n\|^2 + \tau(\frac{1}{2}+\epsilon_2)s_h(e_h^n, e_h^n),
$$

which proves the lemma. □

Theorem 3.5. *Let $u_h^{n+1} \in V_h$ be the WG finite element solution of the problem (1.1)-(1.2) arising from (3.1). Then there exists a constant C such that*

$$\|e_0^{n+1}\| \le C(h^{k+\frac{1}{2}}|u|_{k+1} + \tau\|u_{tt}\|).$$

Proof. It follows from (3.5) that

$$\frac{1}{2}\|e_0^{n+1}\|^2 + \frac{1}{2}\|e_0^{n+1} - e_0^n\|^2 - \frac{1}{2}\|e_0^n\|^2 + \tau\, s_h(e_h^n, e_h^{n+1} - e_h^n) + \tau\, s_h(e_h^n, e_h^n) + \tau\ell_2(e_h^{n+1}) = -\tau\ell_1(e_h^{n+1}) + \tau s_h(Q_h u(t_n), e_h^{n+1}).$$

Then using Lemma 3.3 and Lemma 3.4, we have

$$\frac{1}{2}\|e_0^{n+1}\|^2 + (\frac{1}{2} - 4\epsilon_1)\|e_0^{n+1} - e_0^n\|^2 - \frac{1}{2}\|e_0^n\|^2 + (1 - \frac{1}{2} - 2\epsilon_2)s_h(e_h^n, e_h^n) \le C\tau h^{2k+1}|u|_{k+1}^2 + C\tau\|e_0^n\|^2 + C\tau^3\|u_{tt}\|^2.$$

Letting ϵ_1 and ϵ_2 small enough gives

$$\frac{1}{2}\|e_0^{n+1}\|^2 - \frac{1}{2}\|e_0^n\|^2 \le C\tau h^{2k+1}|u|_{k+1}^2 + C\tau\|e_0^n\|^2 + C\tau^3\|u_{tt}\|^2.$$

Summing the above equation over $n+1$, we have

$$\|e_0^{n+1}\|^2 \le \|e_0^0\|^2 + Ch^{2k+1}|u|_{k+1}^2 + C\tau^2\|u_{tt}\|^2 + \sum_{i=0}^{n} C\tau\|e_0^i\|^2.$$

The discrete Gronwall's inequality implies

$$\|e_0^{n+1}\|^2 \le C(h^{2k+1}|u|_{k+1}^2 + \tau^2\|u_{tt}\|^2),$$

which proves the theorem. □

Remark 3. The assumption (3.14) is obviously satisfied for $n = 0$ since $u_0^0 = Q_0\phi$. If (3.14) holds for a certain n, then it follows from the conclusion of Theorem 3.5 and $\tau \le ch^2$ that

$$\|e_0^{n+1}\| \le C(h^{k+1/2} + \tau) \le Ch^{3/2},$$

for $k \ge 1$. Thus the given a priori (3.14) is verified.

The result in the following theorem is a special case of Theorem 3.5. We omit the proof of the theorem since it is similar to the proof of Theorem 3.5.

Theorem 3.6. *Let $u_h^{n+1} \in V_h$ be the WG finite element solution of the problem (1.1)-(1.2) arising from (3.1). Assume $f'(u) > 0$ (or $f'(u) < 0$) and let $\lambda_2 = 0$ (or $\lambda_1 = 0$). Then there exists a constant C such that*

$$\|e_0^{n+1}\| \le C(h^{k+1}|u|_{k+1} + \tau\|u_{tt}\|).$$

4. **TVDRK3 WG method and its error analysis.**

This section discusses the fully discrete WG method coupled with the explicit TVDRK3 time-marching.

First, we set the initial value $u_h^0 = \{Q_0\phi, \{\!\{Q_0\phi\}\!\}\}$. Then for each $n \ge 0$, the approximate solution from the time $n\tau$ to the next time $(n+1)\tau$ is defined as follows:

Algorithm 3 (RK3-WG method)**.** Find $u_h^{n,1} = \{u_0^{n,1}, u_b^{n,1}\}, u_h^{n,2} = \{u_0^{n,2}, u_b^{n,2}\}$ and $u_h^{n+1} = \{u_0^{n+1}, u_b^{n+1}\}$ in the finite element space V_h such that, for any $v \in V_h$

$$(u_0^{n,1} - u_0^n, v_0) + \tau(D_w f(u_h^n), v_0)_{\mathcal{T}_h} + \tau\, s_h(u_h^n, v) = 0, \tag{4.1}$$

$$(u_0^{n,2} - \frac{1}{4}u_0^{n,1} - \frac{3}{4}u_0^n, v_0) + \frac{1}{4}\tau(D_w f(u_h^{n,1}), v_0)_{\mathcal{T}_h} + \frac{1}{4}\tau\, s_h(u_h^{n,1}, v) = 0, \tag{4.2}$$

$$(u_0^{n+1} - \frac{2}{3}u_0^{n,2} - \frac{1}{3}u_0^n, v_0) + \frac{2}{3}\tau(D_w f(u_h^{n,2}), v_0)_{\mathcal{T}_h} + \frac{2}{3}\tau\, s_h(u_h^{n,2}, v) = 0. \tag{4.3}$$

Similar as (3.2), we obtain the time update for u_b as

$$u_b^{n,1} = \{\!\{u_0^{n,1}\}\!\}, \quad u_b^{n,2} = \{\!\{u_0^{n,2}\}\!\}, \quad u_b^{n+1} = \{\!\{u_0^{n+1}\}\!\}. \tag{4.4}$$

Following [22], two reference functions are defined in parallel to the TVDRK3 time discretization stages for the exact solution of the conservation law (1.1). Let $u^{(0)}(x,t) = u(x,t)$ and

$$u^{(1)}(x,t) - u^{(0)}(x,t) + \tau[f(u^0(x,t))]_x = 0, \tag{4.5}$$

$$u^{(2)}(x,t) - \frac{1}{4}u^{(1)}(x,t) - \frac{3}{4}u^{(0)}(x,t) + \frac{1}{4}\tau[f(u^{(1)})(x,t)]_x = 0. \tag{4.6}$$

Lemma 4.1 ([23])**.** *If $\|u_{tttt}\|$ is bounded uniformly for any $t \in [0,T]$, we have*

$$u(x,t+\tau) - \frac{2}{3}u^{(2)}(x,t) - \frac{1}{3}u^{(0)}(x,t) + \frac{2}{3}\tau[f(u^{(2)})(x,t)]_x = \mathcal{E}(x,t), \tag{4.7}$$

where $\mathcal{E}(x,t)$ is the local truncation error in time and $\|\mathcal{E}(x,t)\| = \mathcal{O}(\tau^4)$ uniformly for any time $t \in [0,T]$.

Denote $u^{n,i} = u^{(i)}(x,t^n)$ for any time level n and $i = 0,1,2$. The error at each stage is denote by

$$e_h^{n,i} = Q_h u^{n,i} - u_h^{n,i}, \quad \rho_h^{n,i} = u^{n,i} - Q_h u^{n,i}$$

for any n and inner state $i = 0,1,2$, where $u_h^{n,0} = u_h^n$.

Lemma 4.2 (Error equation)**.** *For any $v \in V_h$, the error functions $e_h^{n,1}, e_h^{n,2}$ and e_h^{n+1} satisfy the following equations*

$$(e_0^{n,1}, v_0) = (e_0^n, v_0) + \tau\mathcal{H}(u^n, u_h^n; v), \tag{4.8}$$

$$(e_0^{n,2}, v_0) = \frac{3}{4}(e_0^n, v_0) + \frac{1}{4}(e_0^{n,1}, v_0) + \frac{1}{4}\tau\mathcal{H}(u^{n,1}, u_h^{n,1}; v), \tag{4.9}$$

$$(e_0^{n+1}, v_0) = \frac{1}{3}(e_0^n, v_0) + \frac{2}{3}(e_0^{n,2}, v_0) + \frac{2}{3}\tau\mathcal{H}(u^{n,2}, u_h^{n,2}; v) + (\mathcal{E}, v_0), \tag{4.10}$$

where

$$\mathcal{H}(u, u_h; v) := (D_w f(u_h) - D_w f(u), v_0)_{\mathcal{T}_h} + s_h(u_h, v_h). \tag{4.11}$$

Proof. Let $t = t^n$ and test (4.5) by v_0 of $v = \{v_0, v_b\}$ in V_h, we arrive at

$$(u^{n,1} - u^n, v_0) + \tau(Df(u^n), v_0)_{\mathcal{T}_h} = 0.$$

The definition of D_w implies

$$(Df(u^n), v_0)_{\mathcal{T}_h} = (D_w f(u^n), v_0)_{\mathcal{T}_h}.$$

It follows from the definition of Q_0 and the above equation that

$$(Q_0 u^{n,1} - Q_0 u^n, v_0) + \tau(D_w f(u^n), v_0)_{\mathcal{T}_h} = 0.$$

The difference of (4.1) and the equation above yields (4.8).

The error equations (4.9) and (4.10) can be obtained similarly. The details are therefore omitted. □

4.1. **Some basic estimates.** For any $u_h = \{u_0, u_b\}$ and $v_h = \{v_0, v_b\}$ in V_h, it follows from the definition of weak derivative D_w that

$$\begin{aligned}
&(D_w f(u_h) - D_w f(u), v_0)_{\mathcal{T}_h} \\
&= (f(u) - f(u_0), Dv_0)_{\mathcal{T}_h} - \langle f(u) - f(u_b), v_0 n\rangle_{\partial\mathcal{T}_h} \\
&= (\mathcal{R}_i(u, u_h), Dv_0)_{\mathcal{T}_h} - \langle \mathcal{R}_b(u, u_h), v_0 n\rangle_{\partial\mathcal{T}_h} \\
&\quad + (f'(u)(u - u_0), Dv_0)_{\mathcal{T}_h} - \langle f'(u)(u - u_b), v_0 n\rangle_{\partial\mathcal{T}_h},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_i(u, u_h) &:= f(u) - f(u_0) - f'(u)(u - u_0), \\
\mathcal{R}_b(u, u_h) &:= f(u) - f(u_b) - f'(u)(u - u_b).
\end{aligned}$$

Since $\mathcal{R}_b(u, u_0)$ and $f'(u)(u - u_b)$ take single value on ∂I_i, the periodic boundary condition implies

$$\begin{aligned}
-\langle \mathcal{R}_b(u, u_h), v_0 n\rangle_{\partial\mathcal{T}_h} &= \sum_{j=1}^{N} \mathcal{R}_b(u, u_h)_{j+1/2} [\![v_0]\!]_{j+1/2}, \\
-\langle f'(u)(u - u_b), v_0 n\rangle_{\partial\mathcal{T}_h} &= \sum_{j=1}^{N} (f'(u)(u - u_b))_{j+1/2} [\![v_0]\!]_{j+1/2}.
\end{aligned}$$

Therefore, the operator $\mathcal{H}(u, u_h; v_h)$ defined in (4.11) can be rewritten as

$$\mathcal{H}(u, u_h; v_h) = \mathcal{H}^{lin}(f'(u); u - u_0, v_0) + \mathcal{H}^{nls}(u, u_h; v_h) + s_h(u_h, v_h), \tag{4.12}$$

for $u_h, v_h \in V_h$, where

$$\begin{aligned}
\mathcal{H}^{lin}(\mathcal{Z}; v, w) &= (\mathcal{Z} v, Dw)_{\mathcal{T}_h} + \sum_{j=1}^{N} (\mathcal{Z} \{\!\{v\}\!\} [\![w]\!])_{j+1/2}, \\
\mathcal{H}^{nls}(u, u_h; v_h) &= (\mathcal{R}_i(u, u_h), Dv_0)_{\mathcal{T}_h} + \sum_{j=1}^{N} \mathcal{R}_b(u, u_h)_{j+1/2} [\![v_0]\!]_{j+1/2}.
\end{aligned}$$

Lemma 4.3. *For any continuous and differentiable function $\mathcal{Z}$, there exists a positive constant C, independent of n, h, τ, and u_h, such that*

$$|\mathcal{H}^{lin}(\mathcal{Z}; v, w)| \le Ch^{-1} \|\mathcal{Z}\|_\infty \|v\| \|w\|, \quad \forall v, w \in V_h, \tag{4.13}$$

$$|\mathcal{H}^{lin}(\mathcal{Z}; v, v)| \le C\|v\|^2, \quad \forall v \in V_h, \tag{4.14}$$

$$|\mathcal{H}^{lin}(\mathcal{Z}; u - Q_0 u, v)| \le \varepsilon |\mathcal{Z}| [\![v]\!]^2 + \varepsilon \|v\|^2 + Ch^{2k+1}, \quad \forall v \in V_h, \tag{4.15}$$

where ε is any positive constant, and $|\mathcal{Z}| [\![v]\!]^2 = \sum_{j=1}^{N} |\mathcal{Z}_{j+1/2}| [\![v]\!]^2_{j+1/2}$.

Proof. For the proof of (4.13), it follows from Cauchy-Schwartz inequality and the inverse inequality that

$$|\mathcal{H}^{lin}(\mathcal{Z}; v, w)| \le \sum_{j=1}^{N} (|(\mathcal{Z} v, Dw)_{I_j}| + |\mathcal{Z}|_{j+1/2} |\{\!\{v\}\!\}|_{j+1/2} |[\![w]\!]|_{j+1/2})$$

$$\leq \sum_{j=1}^{N} \|\mathcal{Z}\|_\infty (\|v\|_{I_j}\|Dw\|_{I_j} + \|v\|_{\partial I_j}\|w\|_{\partial I_j})$$
$$\leq Ch^{-1}\|\mathcal{Z}\|_\infty \|v\|\|w\|.$$

A simple manipulation indicates that

$$\begin{aligned}\mathcal{H}^{lin}(\mathcal{Z}; v, v) &= (\mathcal{Z}v, Dv)_{\mathcal{T}_h} + \sum_{j=1}^{N}(\mathcal{Z}\{\!\{v\}\!\}[\![v]\!])_{j+1/2} \\ &= \sum_{j=1}^{N}(\mathcal{Z}\{\!\{v\}\!\}[\![v]\!] - \frac{1}{2}\mathcal{Z}[\![v]\!]^2)_{j+1/2} - \int_{\mathcal{T}_h} \mathcal{Z}' v^2 \mathrm{d}x \\ &= -\int_{\mathcal{T}_h} \mathcal{Z}' v^2 \mathrm{d}x,\end{aligned}$$

which implies the second conclusion (4.14).

Let $\overline{\mathcal{Z}}_j = \frac{1}{|I_j|}\int_{I_j} \mathcal{Z}\mathrm{d}x$. It follows from Cauchy-Schwartz inequality and Young's inequality that

$$\begin{aligned}\mathcal{H}^{lin}(\mathcal{Z}; u - Q_0u, v) &= (\mathcal{Z}(u-Q_0u), Dv)_{\mathcal{T}_h} + \sum_{j=1}^{N}(\mathcal{Z}\{\!\{u-Q_0u\}\!\}[\![v]\!])_{j+1/2} \\ &= \sum_{j=1}^{N}((\mathcal{Z}-\overline{\mathcal{Z}}_j)(u-Q_0u), Dv)_{I_j} + \sum_{j=1}^{N}(\mathcal{Z}\{\!\{u-Q_0u\}\!\}[\![v]\!])_{j+1/2} \\ &\leq \sum_{j=1}^{N}\|\mathcal{Z}-\overline{\mathcal{Z}}_j\|_{L^\infty(I_j)}\|u-Q_0u\|_{I_j}\|Dv\|_{I_j} \\ &\quad + \sum_{j=1}^{N}|\mathcal{Z}_{j+1/2}|\|u-Q_0u\|_{L^\infty(I_j)}|[\![v]\!]|_{j+1/2} \\ &\leq \varepsilon\|v\|^2 + \varepsilon\sum_{j=1}^{N}|\mathcal{Z}_{j+1/2}|[\![v]\!]^2_{j+1/2} + Ch^{2k+1}.\end{aligned}$$

The proof is completed. ☐

Lemma 4.4. *Let $u \in H^{k+1}(I)$ be the exact solution of (1.1). For any $u_h = \{u_0, \{\!\{u_0\}\!\}\}, v_h = \{v_0, \{\!\{v_0\}\!\}\} \in V_h$, there holds*

$$|\mathcal{H}^{nls}(u, u_h; v_h)| \leq C[\|v_0\|^2 + h^{-2}\|u-u_0\|^2_\infty(\|Q_0u - u_0\|^2 + h^{2(k+1)})].$$

Proof. By Taylor expansion up to the second order derivative term, we obtain

$$\mathcal{R}_i(u, u_h) = -\frac{1}{2}f''(\xi_1)(u-u_0)^2, \quad \mathcal{R}_b(u, u_h) = -\frac{1}{2}f''(\xi_2)\{\!\{u-u_0\}\!\}^2,$$

where $f''(\xi_1)$ and $f''(\xi_2)$ are the second order derivative of f in the two expansion, which are both bounded.

Thus, by the triangle inequality and the inverse inequality, we have

$$|\mathcal{H}^{nls}(u, u_h; v_h)| \leq \frac{1}{2}\sum_{j=1}^{N}\left|\int_{I_j} f''(\xi_1)(u-u_0)^2 Dv_0 \mathrm{d}x\right|$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{j=1}^{N}\left|f''(\xi_2)\{\!\{u-u_0\}\!\}^2\right|_{j+1/2}|[\![v_0]\!]|_{j+1/2}\\
\le& C\|u-u_0\|_\infty\sum_{j=1}^{N}(\|u-u_0\|_{I_j}\|Dv_0\|_{I_j}+\|u-u_0\|_{\partial I_j}\|v_0\|_{\partial I_j})\\
\le& C\|u-u_0\|_\infty\sum_{j=1}^{N}[(\|u-Q_0u\|_{I_j}+\|Q_0u-u_0\|_{I_j})\cdot h^{-1}\|v_0\|_{I_j}\\
&+(\|u-Q_0u\|_{L^\infty(I_j)}+h^{-1/2}\|Q_0u-u_0\|_{I_j})\cdot h^{-1/2}\|v_0\|_{I_j}]\\
\le& Ch^{-1}\|u-u_0\|_\infty(\|Q_0u-u_0\|+h^{k+1})\|v_0\|,
\end{aligned}
$$

which together with Young's inequality completes the proof. □

Lemma 4.5. *Let $u \in H^{k+1}(I)$ be the exact solution of (1.1). For any $u_h = \{u_0, \{\!\{u_0\}\!\}\}, v_h = \{v_0, \{\!\{v_0\}\!\}\} \in V_h$, there hold*

$$|s_h(u_h, v_h)| \le Ch^{-1}(h^{k+1} + \|Q_0u - u_0\|)\|v_0\|, \tag{4.16}$$

$$s_h(u_h, e_h) \le Ch^{2k+1} - (1-\varepsilon)\vartheta[\![e_0]\!]^2, \tag{4.17}$$

where ε is any positive constant, $[\![e_0]\!]^2 := \sum_{j=1}^{N}[\![e_0]\!]^2_{j+\frac{1}{2}}$, and $\vartheta = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$.

Proof. Since u is continuous function, there holds $[\![u]\!] = 0$. It follows from (2.3) and (4.4) that

$$s_h(u_h, v_h) = \sum_{j=1}^{N}\vartheta[\![u_0]\!]_{j+1/2}[\![v_0]\!]_{j+1/2} = \sum_{j=1}^{N}\vartheta[\![u_0 - u]\!]_{j+1/2}[\![v_0]\!]_{j+1/2}. \tag{4.18}$$

Hence, by the triangle inequality and the inverse inequality, we have

$$
\begin{aligned}
|s_h(u_h,v_h)| \le& C\sum_{j=1}^{N}|[\![u_0-u]\!]|_{j+1/2}|[\![v_0]\!]|_{j+1/2}\\
\le& C\sum_{j=1}^{N}(\|u-Q_0u\|_{L^\infty(I_j)}+h^{-1/2}\|Q_0u-u_0\|_{I_j})\cdot h^{-1/2}\|v_0\|_{I_j}\\
\le& Ch^{-1}(h^{k+1}+\|Q_0u-u_0\|)\|v_0\|,
\end{aligned}
$$

which completed the proof of (4.16).

From (4.18) and Young's inequality, we have

$$
\begin{aligned}
s_h(u_h,e_h) =& \sum_{j=1}^{N}\vartheta[\![Q_0u-u]\!]_{j+1/2}[\![e_0]\!]_{j+1/2} - \sum_{j=1}^{N}\vartheta[\![e_0]\!]^2_{j+1/2}\\
\le& \frac{1}{4\varepsilon}\sum_{j=1}^{N}\vartheta[\![Q_0u-u]\!]^2_{j+1/2} - (1-\varepsilon)\sum_{j=1}^{N}\vartheta[\![e_0]\!]^2_{j+1/2}\\
\le& Ch^{2k+1} - (1-\varepsilon)\sum_{j=1}^{N}\vartheta[\![e_0]\!]^2_{j+1/2}.
\end{aligned}
$$

The proof is completed. □

Lemma 4.6. *If the time step satisfies $\tau = \mathcal{O}(h)$, then we have*

$$\|e_0^{n,1}\|^2 \le C(\|e_0^n\|^2 + h^{2k+2}), \tag{4.19}$$

$$\|e_0^{n,2}\|^2 \le C(\|e_0^n\|^2 + \|e_0^{n,1}\|^2 + h^{2k+2}), \tag{4.20}$$

where C is a positive constant independent of n, h, τ and u_h.

Proof. Taking $v = e_h^{n,1}$ in the error equation (4.8), we get

$$\|e_0^{n,1}\|^2 = (e_0^n, e_0^{n,1}) + \tau\mathcal{H}(u^n, u_h^n; e_h^{n,1}). \tag{4.21}$$

Firstly, we consider the first term of $\mathcal{H}(u^n, u_h^n; e_h^{n,1})$. It follows from the definition of the weak derivative D_w that

$$\begin{aligned}
&(D_w f(u_h^n) - D_w f(u^n), e_0^{n,1})_{\mathcal{T}_h} \\
&= (f(u^n) - f(u_0^n), De_0^{n,1})_{\mathcal{T}_h} - \langle f(u^n) - f(\{\!\{u_0^n\}\!\}), e_0^{n,1} n\rangle_{\partial\mathcal{T}_h} \\
&=: T_1 + T_2.
\end{aligned} \tag{4.22}$$

From Cauchy-Schwarz inequality, the inverse inequality and the approximation property of Q_0, we have

$$\begin{aligned}
|T_1| &\le \sum_{j=1}^N \|f(u^n) - f(u_0^n)\|_{I_j} \|De_0^{n,1}\|_{I_j} \\
&\le C\sum_{j=1}^N (\|u^n - Q_0u^n\|_{I_j} + \|Q_0u^n - u_0^n\|_{I_j}) \cdot h^{-1}\|e_0^{n,1}\|_{I_j} \\
&\le Ch^{-1}(h^{k+1} + \|e_0^n\|)\|e_0^{n,1}\|,
\end{aligned} \tag{4.23}$$

where the assumption $f'(u)$ is bounded has been used in the second inequality above.

Now we turn to the term T_2. Since periodic boundary condition is considered, T_2 can be rewritten as

$$T_2 = \sum_{j=1}^N (f(u^n) - f(\{\!\{u_0^n\}\!\}))_{j+1/2} [\![e_0^{n,1}]\!]_{j+1/2}.$$

Then, by the triangle inequality and the inverse inequality, we obtain

$$\begin{aligned}
|T_2| &\le C\sum_{j=1}^N (|u^n - Q_0u^n| + |Q_0u^n - \{\!\{u_0^n\}\!\}|)_{j+1/2} |[\![e_0^{n,1}]\!]|_{j+1/2} \\
&\le C\sum_{j=1}^N (\|u^n - Q_0u^n\|_{L^\infty(I_j)} + h^{-1/2}\|Q_0u^n - u_0^n\|_{I_j}) \cdot h^{-1/2}\|e_0^{n,1}\|_{I_j} \\
&\le Ch^{-1}(h^{k+1} + \|e_0^n\|)\|e_0^{n,1}\|.
\end{aligned} \tag{4.24}$$

Collecting (4.22), (4.23) and (4.24), we can conclude

$$|(D_w f(u_h^n) - D_w f(u^n), e_0^n)_{\mathcal{T}_h}| \le Ch^{-1}(h^{k+1} + \|e_0^n\|)\|e_0^{n,1}\|,$$

which together with the inequality (4.16) of Lemma 4.5 and (4.21) yields

$$\|e_0^{n,1}\|^2 \le \|e_0^n\|\|e_0^{n,1}\| + C\tau h^{-1}(h^{k+1} + \|e_0^n\|)\|e_0^{n,1}\|.$$

Cancelling $\|e_0^{n,1}\|$ on both sides of the above equation, and noting that $\tau = \mathcal{O}(h)$, we have

$$\|e_0^{n,1}\| \le C(\|e_0^n\| + h^{k+1}),$$

which implies (4.19).

In a similar way, we can obtain the conclusion (4.20). The proof is completed. □

By taking the test function $v = e_h^n, 4e_h^{n,1}$, and $6e_h^{n,2}$ in the error equations (4.8), (4.9) and (4.10), respectively, we obtain the energy equation for e_h^n in the form

$$3\|e_0^{n+1}\|^2 - 3\|e_0^n\|^2 = \Xi_1 + \Xi_2, \tag{4.25}$$

where

$$\begin{aligned}\Xi_1 :=& \tau\mathcal{H}(u^n, u_h^n; e_h^n) + \tau\mathcal{H}(u^{n,1}, u_h^{n,1}; e_h^{n,1}) + 4\tau\mathcal{H}(u^{n,2}, u_h^{n,2}; e_h^{n,2}) + 6(\mathcal{E}, e_0^{n,2}),\\ \Xi_2 :=& \|2e_0^{n,2} - e_0^{n,1} - e_0^n\|^2 + 3(e_0^{n+1} - e_0^n, e_0^{n+1} - 2e_0^{n,2} + e_0^n).\end{aligned}$$

4.2. **The estimate for Ξ_1.** It follows from (4.14) and (4.15) of Lemma 4.3 that

$$\begin{aligned}\mathcal{H}^{lin}(f'(u^n); u^n - u_h^n, e_0^n) &= \mathcal{H}^{lin}(f'(u^n); u^n - Q_0u^n, e_0^n) + \mathcal{H}^{lin}(f'(u^n); e_0^n, e_0^n)\\ &\le \varepsilon|f'(u^n)|[\![e_0^n]\!]^2 + (\varepsilon + C)\|e_0^n\|^2 + Ch^{2k+1}.\end{aligned}$$

Denote by $\mathbb{B} = \max_{s\in\mathbb{R}} |f'(s)|$. In the above inequality, taking ε small enough such that $\varepsilon\mathbb{B} \le 0.1\vartheta$, then we have

$$\mathcal{H}^{lin}(f'(u^n); u^n - u_h^n, e_0^n) \le 0.1\vartheta[\![e_0^n]\!]^2 + C\|e_0^n\|^2 + Ch^{2k+1}. \tag{4.26}$$

From Lemma 4.4, we have

$$\mathcal{H}^{nls}(u^n, u_h^n; e_h^n) \le C[\|e_0^n\|^2 + h^{-2}\|u^n - u_0^n\|_\infty^2(\|e_0^n\|^2 + h^{2(k+1)})]. \tag{4.27}$$

Using (4.17) of Lemma 4.5 with $\varepsilon = 0.1$, we have

$$s_h(u_h^n, e_h^n) \le Ch^{2k+1} - 0.9\vartheta[\![e_0^n]\!]^2. \tag{4.28}$$

Denote by

$$\mathbb{C}(u^{n,i}, u_0^{n,i}) = C(1 + h^{-2}\|u^{n,i} - u_0^{n,i}\|_\infty^2), \quad i = 0, 1, 2,$$

where C is a general positive constant independent of h, τ, u and u_h. Collecting (4.26), (4.27) and (4.28), we obtain

$$\mathcal{H}(u^n, u_h^n; e_h^n) \le \mathbb{C}(u^n, u_0^n)(\|e_0^n\|^2 + h^{2k+1}) - 0.8\vartheta[\![e_0^n]\!]^2.$$

Similarly, we have

$$\mathcal{H}(u^{n,i}, u_h^{n,i}; e_h^{n,i}) \le \mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2k+1}) - 0.8\vartheta[\![e_0^{n,i}]\!]^2,$$

for $i = 1, 2$.

Since $\mathcal{E} = \mathcal{O}(\tau^4)$, by Cauchy-Schwartz inequality, we have

$$(\mathcal{E}, e_0^{n,2}) \le \|\mathcal{E}\|\|e_0^{n,2}\| \le C\tau^4\|e_0^{n,2}\| \le C(\tau^7 + \tau\|e_0^{n,2}\|^2).$$

Collecting the three above estimates, we obtain the estimate of Ξ_1 as follows

$$\Xi_1 \le \tau\sum_{i=0}^{2}\{\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2k+1}) - 0.8\vartheta[\![e_0^{n,i}]\!]^2\} + C\tau^7. \tag{4.29}$$

4.3. **The estimate for Ξ_2.** Following [23], we introduce the following notations

$$E_h^n = e_h^{n,1} - e_h^n, \quad F_h^n = 2e_h^{n,2} - e_h^{n,1} - e_h^n, \quad G_h^n = e_h^{n+1} - 2e_h^{n,2} + e_h^n.$$

Obviously,

$$e_0^{n,2} - \frac{3}{4}e_0^n - \frac{1}{4}e_0^{n,1} = \frac{1}{4}E_0^n + \frac{1}{2}F_0^n,$$
$$e_0^{n+1} - \frac{1}{3}e_0^n - \frac{2}{3}e_0^{n,2} = \frac{2}{3}E_0^n + \frac{2}{3}F_0^n + G_0^n.$$

From the above equalities and the error equations (4.8)-(4.10), we easily obtain the following results.

Lemma 4.7. *For the fully discrete WG method (4.1)-(4.3) with the explicit TV-DRK3 time marching, we have the following equations*

$$(E_0^n, v_0) = \tau\mathcal{H}(u^n, u_h^n; v_h), \tag{4.30}$$

$$(F_0^n, v_0) = \frac{\tau}{2}[\mathcal{H}(u^{n,1}, u_h^{n,1}; v_h) - \mathcal{H}(u^n, u_h^n; v_h)], \tag{4.31}$$

$$(G_0^n, v_0) = \frac{\tau}{3}[2\mathcal{H}(u^{n,2}, u_h^{n,2}; v_h) - \mathcal{H}(u^{n,1}, u_h^{n,1}; v_h) - \mathcal{H}(u^n, u_h^n; v_h)] + (\mathcal{E}, v_0), \tag{4.32}$$

for any $v_h = \{v_0, v_b\} \in V_h$.

Note that $e_0^{n+1} - e_0^n = E_0^n + F_0^n + G_0^n$, there holds

$$\begin{aligned}\Xi_2 =&(F_0^n, F_0^n) + 3(E_0^n, G_0^n) + 3(F_0^n, G_0^n) + 3(G_0^n, G_0^n)\\ :=&J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{4.33}$$

Let $v_h = F_h^n$ in (4.31) and $v_h = E_h^n$ in (4.32), and by (4.12), we have

$$\begin{aligned}J_1 + J_2 =& - (F_0^n, F_0^n) + 2(F_0^n, F_0^n) + 3(E_0^n, G_0^n)\\ =& - \|F_0^n\|^2 + \tau[\mathcal{H}(u^{n,1}, u_h^{n,1}; F_h^n) - \mathcal{H}(u^n, u_h^n; F_h^n)]\\ &+ \tau[2\mathcal{H}(u^{n,2}, u_h^{n,2}; E_h^n) - \mathcal{H}(u^{n,1}, u_h^{n,1}; E_h^n) - \mathcal{H}(u^n, u_h^n; E_h^n)] + 3(\mathcal{E}, E_0^n)\\ =& - \|F_0^n\|^2 + \sum_{i=1}^{3}\mathcal{M}_i + 3(\mathcal{E}, E_0^n),\end{aligned} \tag{4.34}$$

where

$$\begin{aligned}\mathcal{M}_1 :=&\tau[\mathcal{H}^{lin}(f'(u^{n,1}); u^{n,1} - u_0^{n,1}, F_0^n) - \mathcal{H}^{lin}(f'(u^n); u^n - u_0^n, F_0^n)]\\ &+ \tau[2\mathcal{H}^{lin}(f'(u^{n,2}); u^{n,2} - u_0^{n,2}, E_0^n) - \mathcal{H}^{lin}(f'(u^{n,1}); u^{n,1} - u_0^{n,1}, E_0^n)\\ &- \mathcal{H}^{lin}(f'(u^n); u^n - u_0^n, E_0^n)],\\ \mathcal{M}_2 :=&\tau[\mathcal{H}^{nls}(u^{n,1}, u_h^{n,1}; F_h^n) - \mathcal{H}^{nls}(u^n, u_h^n; F_h^n)]\\ &+ \tau[2\mathcal{H}^{nls}(u^{n,2}, u_h^{n,2}; E_h^n) - \mathcal{H}^{nls}(u^{n,1}, u_h^{n,1}; E_h^n) - \mathcal{H}^{nls}(u^n, u_h^n; E_h^n)],\\ \mathcal{M}_3 :=&\tau[s_h(u_h^{n,1} - u_h^n, F_h^n) + s_h(2u_h^{n,2} - u_h^{n,1} - u_h^n, E_h^n)].\end{aligned}$$

Lemma 4.8 (Estimate of $\mathcal{M}_1$)**.** *If the time step satisfies $\tau = \mathcal{O}(h)$, then we have*

$$|\mathcal{M}_1| \le C\tau(\|e_0^{n,2}\|^2 + \|e_0^{n,1}\|^2 + \|e_0^n\|^2 + h^{2(k+1)}).$$

Proof. Since $u^{n,i} - u_h^{n,i} = \rho_h^{n,i} + e_h^{n,i}(i = 0, 1, 2)$, then $\mathcal{M}_1$ can be rewritten as

$$\mathcal{M}_1 = \tau[\mathcal{L}(\rho_h) + \mathcal{L}(e_h)], \tag{4.35}$$

where

$$\begin{aligned}\mathcal{L}(w) =&\mathcal{H}^{lin}(f'(u^{n,1}); w^{n,1}, F_0^n) - \mathcal{H}^{lin}(f'(u^n); w^n, F_0^n)\\&+ 2\mathcal{H}^{lin}(f'(u^{n,2}); w^{n,2}, E_0^n) - \mathcal{H}^{lin}(f'(u^{n,1}); w^{n,1}, E_0^n)\\&- \mathcal{H}^{lin}(f'(u^n); w^n, E_0^n),\end{aligned}$$

for $w = \rho_h$ or e_h.

Denote by

$$z^{n,i} = f'(u^{n,i}) - f'(u^n), i = 1, 2,$$

and collecting the terms with the same speed $f'(u^n)$, the operator $\mathcal{L}(w)$ can be expressed as

$$\begin{aligned}\mathcal{L}(w) =&\mathcal{H}^{lin}(f'(u^n); w^{n,1} - w^n, F_0^n) + \mathcal{H}^{lin}(f'(u^n); 2w^{n,2} - w^{n,1} - w^n, E_0^n)\\&+ \mathcal{H}^{lin}(z^{n,1}; w^{n,1}, F_0^n) + 2\mathcal{H}^{lin}(z^{n,2}; w^{n,2}, E_0^n) - \mathcal{H}^{lin}(z^{n,1}; w^{n,1}, E_0^n)\\:=&\sum_{i=1}^{5} \mathcal{L}_i(w).\end{aligned}$$

Now we estimate $\mathcal{L}_i(e_h), i = 1, \cdots, 5$ firstly. Due to periodic boundary condition, and using integration by parts, we have

$$\begin{aligned}\mathcal{L}_1(e_h) + \mathcal{L}_2(e_h) &= \mathcal{H}^{lin}(f'(u^n); e_h^{n,1}, F_0^n) + \mathcal{H}^{lin}(f'(u^n); e_h^{n,2}, E_0^n)\\&= \sum_{j=1}^{N} [\int_{I_j} f'(u^n)(E_0^n F_0^n)_x \mathrm{d}x + f'(u^n)_{j+1/2} [\![E_0^n F_0^n]\!]_{j+1/2}]\\&= \sum_{j=1}^{N} \int_{I_j} -[f'(u^n)]_x E_0^n F_0^n \mathrm{d}x.\end{aligned}$$

Consequently, by Cauchy-Schwartz inequality, we can conclude that

$$|\mathcal{L}_1(e_h) + \mathcal{L}_2(e_h)| \le C\|E_0^n\|\|F_0^n\|.$$

Since $|z^{n,1}| = \mathcal{O}(\tau) = \mathcal{O}(h)$, it follows from the equation (4.13) of Lemma 4.3 that

$$\begin{aligned}|\mathcal{L}_3(e_h)| &= |\mathcal{H}^{lin}(z^{n,1}; e_h^{n,1}, F_0^n)|\\&\le Ch^{-1}\|z^{n,1}\|_\infty \|e_0^{n,1}\|\|F_0^n\|\\&\le C\|e_0^{n,1}\|\|F_0^n\|.\end{aligned}$$

In a similar way, we obtain

$$|\mathcal{L}_4(e_h)| + |\mathcal{L}_5(e_h)| \le C(\|e_0^{n,2}\| + \|e_0^{n,1}\|)\|E_0^n\|.$$

Collecting the estimate of $\mathcal{L}_i(e_h), i = 1, \cdots, 5$ and using Cauchy-Schwarz inequality yields

$$|\mathcal{L}(e_h)| \le C(\|E_0^n\|^2 + \|F_0^n\|^2 + \|e_0^{n,1}\|^2 + \|e_0^{n,2}\|^2). \tag{4.36}$$

Next, we turn to the term $\mathcal{L}_i(\rho_h), i = 1, \cdots, 5$. Note that $f'(u)$ is bounded, it follows from the Cauchy-Schwartz inequality and inverse equality that

$$\begin{aligned}|\mathcal{L}_1(\rho_h)| &= |\mathcal{H}^{lin}(f'(u^n); \rho_0^{n,1} - \rho_0^n, F_0^n)| \\ &\le C\sum_{j=1}^{N} \|f'(u^n)\|_\infty (\|\rho_0^{n,1} - \rho_0^n\|_{I_j} \|DF_0^n\|_{I_j} + \|\rho_0^{n,1} - \rho_0^n\|_{\partial I_j} \|F_0^n\|_{\partial I_j}) \\ &\le C(h^{-1}\|\rho_0^{n,1} - \rho_0^n\| + h^{-1/2}\|\rho_0^{n,1} - \rho_0^n\|_\infty)\|F_0^n\|. \qquad (4.37)\end{aligned}$$

Since

$$u^{n,1} - u^n = -\tau[f(u^n)]_x,$$

it follows from the approximation property of L^2 projector Q_0 that

$$\begin{aligned}\|\rho_0^{n,1} - \rho_0^n\| &= \|(u^{n,1} - u^n) - Q_0(u^{n,1} - u^n)\| \le Ch^{k+1}\tau, \\ \|\rho_0^{n,1} - \rho_0^n\|_\infty &= \|(u^{n,1} - u^n) - Q_0(u^{n,1} - u^n)\|_\infty \le Ch^{k+1/2}\tau,\end{aligned}$$

which together with (4.37) yields

$$|\mathcal{L}_1(\rho_h)| \le Ch^k\tau\|F_0^n\| \le C(\|F_0^n\|^2 + h^{2k}\tau^2).$$

Similarly we can estimate the second term as $|\mathcal{L}_2(\rho_h)| \le C(\|E_0^n\|^2 + h^{2k}\tau^2)$.

Using $|z^{n,i}| = \mathcal{O}(\tau) = \mathcal{O}(h)$ and similar as the proof of $\mathcal{L}_1(\rho_h)$, it is easy to obtain

$$|\mathcal{L}_3(\rho_h)| + |\mathcal{L}_4(\rho_h)| + |\mathcal{L}_5(\rho_h)| \le C(\|E_0^n\|^2 + \|F_0^n\|^2 + h^{2(k+1)}).$$

Since $\tau = \mathcal{O}(h)$, finally we have

$$|\mathcal{L}(\rho_h)| \le C(\|E_0^n\|^2 + \|F_0^n\|^2 + h^{2(k+1)}). \qquad (4.38)$$

By the triangle inequality, we have

$$\|E_0^n\|^2 + \|F_0^n\|^2 \le C(\|e_0^{n,2}\|^2 + \|e_0^{n,1}\|^2 + \|e_0^n\|^2),$$

which together with (4.36) and (4.38) completes the proof. □

Lemma 4.9 (Estimate of $\mathcal{M}_2$)**.** *There exists a positive constant C, independent of n, h, τ, and u_h, such that*

$$|\mathcal{M}_2| \le C\tau\sum_{i=0}^{2}[\|e_0^{n,i}\|^2 + h^{-2}\|u^{n,i} - u_0^{n,i}\|_\infty^2(\|e_0^{n,i}\|^2 + h^{2(k+1)})].$$

Proof. It follows from Lemma 4.4 that

$$|\mathcal{H}^{nls}(u^{n,i}, u_h^{n,i}; w_h)| \le C[\|w_0\|^2 + h^{-2}\|u^{n,i} - u_0^{n,i}\|_\infty^2(\|e_0^{n,i}\|^2 + h^{2(k+1)})],$$

for $w_h \in V_h$ and $i = 0, 1, 2$.

Let $w_h = E_h^n$ and F_h^n in the above inequality, respectively. By the triangle inequality and the definition of $\mathcal{M}_2$, we have

$$\begin{aligned}|\mathcal{M}_2| \le&\tau[|\mathcal{H}^{nls}(u^{n,1}, u_h^{n,1}; F_h^n)| + |\mathcal{H}^{nls}(u^n, u_h^n; F_h^n)|] \\ &+ \tau[2|\mathcal{H}^{nls}(u^{n,2}, u_h^{n,2}; E_h^n)| + |\mathcal{H}^{nls}(u^{n,1}, u_h^{n,1}; E_h^n)| + |\mathcal{H}^{nls}(u^n, u_h^n; E_h^n)|] \\ \le&C\tau[\|E_0^n\|^2 + \|F_0^n\|^2 + \sum_{i=0}^{2} h^{-2}\|u^{n,i} - u_0^{n,i}\|_\infty^2(\|e_0^{n,i}\|^2 + h^{2(k+1)})],\end{aligned}$$

which completes the proof. □

Lemma 4.10 (Estimate of $\mathcal{M}_3$). *There exists a positive constant C, independent of n, h, τ, and u_h, such that*

$$|\mathcal{M}_3| \le Ch^{2k+1}\tau - 0.9\vartheta([\![E_0^n]\!]^2 + [\![F_0^n]\!]^2)\tau.$$

Proof. Using (4.17) of Lemma 4.5 with $\varepsilon = 0.1$, we have

$$s_h(u_h^{n,1} - u_h^n, F_h^n) \le Ch^{2k+1} - 0.9\vartheta[\![F_0^n]\!]^2,$$
$$s_h(2u_h^{n,2} - u_h^{n,1} - u_h^n, E_h^n) \le Ch^{2k+1} - 0.9\vartheta[\![E_0^n]\!]^2.$$

Combining the above two estimates completes the proof. ☐

Since $\mathcal{E} = \mathcal{O}(\tau^4)$, by Cauchy-Schwartz inequality, we have

$$(\mathcal{E}, E_0^n) \le C(\tau^7 + \tau\|E_0^n\|^2) \le C(\tau^7 + \tau\|e_0^n\|^2 + \tau\|e_0^{n,1}\|^2),$$

which together with (4.34) and Lemma 4.8, 4.9 and 4.10 yields

$$\begin{aligned} J_1 + J_2 \le -\|F_0^n\|^2 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2k+1}) \\ +C\tau^7 - 0.9\vartheta([\![E_0^n]\!]^2 + [\![F_0^n]\!]^2)\tau. \end{aligned} \tag{4.39}$$

Now we consider the upper bound of J_3 and J_4. Since $J_3 = 3(G_0^n, F_0^n)$ and $J_4 = 3(G_0^n, G_0^n)$, we estimate $3(G_0^n, v_0)$ in general. From (4.12) and (4.32), we have

$$\begin{aligned} 3(G_0^n, v_0) =& \tau[2\mathcal{H}(u^{n,2}, u_h^{n,2}; v_h) - \mathcal{H}(u^{n,1}, u_h^{n,1}; v_h) - \mathcal{H}(u^n, u_h^n; v_h)] + 3(\mathcal{E}, v_0) \\ :=& \tau[\mathcal{I}_1(v_0) + \mathcal{I}_2(v_0) + \mathcal{I}_3(v_0)] + 3(\mathcal{E}, v_0), \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} \mathcal{I}_1(v_0) :=& 2\mathcal{H}^{lin}(f'(u^{n,2}); u^{n,2} - u_0^{n,2}, v_0) - \mathcal{H}^{lin}(f'(u^{n,1}); u^{n,1} - u_0^{n,1}, v_0) \\ &- \mathcal{H}^{lin}(f'(u^n); u^n - u_0^n, v_0), \\ \mathcal{I}_2(v_0) :=& 2\mathcal{H}^{nls}(u^{n,2}, u_h^{n,2}; v_h) - \mathcal{H}^{nls}(u^{n,1}, u_h^{n,1}; v_h) - \mathcal{H}^{nls}(u^n, u_h^n; v_h), \\ \mathcal{I}_3(v_0) :=& 2s_h(u_h^{n,2}, v_h) - s_h(u_h^{n,1}, v_h) - s_h(u_h^n, v_h). \end{aligned}$$

Denote by

$$\widetilde{F}_0^n := 2\rho_0^{n,2} - \rho_0^{n,1} - \rho_0^n,$$

and collecting the terms with the same speed $f'(u^n)$, we rewrite the operator $\mathcal{I}_1(v_0)$ as

$$\begin{aligned} \mathcal{I}_1(v_0) :=& \mathcal{H}^{lin}(f'(u^n); F_0^n, v_0) + 2\mathcal{H}^{lin}(z^{n,2}; e_0^{n,2}, v_0) - \mathcal{H}^{lin}(z^{n,1}; e_0^{n,1}, v_0) \\ &+ \mathcal{H}^{lin}(f'(u^n); \widetilde{F}_0^n, v_0) + 2\mathcal{H}^{lin}(z^{n,2}; \rho_0^{n,2}, v_0) - \mathcal{H}^{lin}(z^{n,1}; \rho_0^{n,1}, v_0) \end{aligned} \tag{4.41}$$

It follows from (4.13) of Lemma 4.3, Cauchy-Schwartz inequality and $|z^{n,i}| = \mathcal{O}(\tau) = \mathcal{O}(h)$ that

$$|\mathcal{H}^{lin}(f'(u^n); F_0^n, v_0)| \le Ch^{-1}\|f'(u^n)\|_\infty\|F_0^n\|\|v_0\| \le Ch^{-1}(\|F_0^n\|^2 + \|v_0\|^2), \tag{4.42}$$

$$|\mathcal{H}^{lin}(z^{n,i}; e_0^{n,i}, v_0)| \le Ch^{-1}\|z^{n,i}\|_\infty\|e_0^{n,i}\|\|v_0\| \le C(\|e_0^{n,i}\|^2 + \|v_0\|^2),\ i = 1, 2. \tag{4.43}$$

Note that $f'(u)$ is bounded, it follows from the Cauchy-Schwartz inequality and inverse equality that

$$
\begin{aligned}
|\mathcal{H}^{lin}(f'(u^n);\widetilde{F}_0^n,v_0)| &\le \sum_{j=1}^N \|f'(u^n)\|_\infty(\|\widetilde{F}_0^n\|_{I_j}\|Dv_0\|_{I_j}+\|\widetilde{F}_0^n\|_{\partial I_j}\|v_0\|_{\partial I_j})\\
&\le C\sum_{j=1}^N(h^{-1}\|\widetilde{F}_0^n\|_{I_j}+h^{-1/2}\|\widetilde{F}_0^n\|_{\partial I_j})\|v_0\|_{I_j}. \qquad (4.44)
\end{aligned}
$$

Since

$$\widetilde{F}_0^n=(2u^{n,2}-u^{n,1}-u^n)-Q_0(2u^{n,2}-u^{n,1}-u^n)$$

and

$$2u^{n,2}-u^{n,1}-u^n=\frac{\tau}{2}([f(u^n)]_x-[f(u^{n,1})]_x),$$

by the approximation property of L^2 projector Q_0, we have

$$\|\widetilde{F}_0^n\|_{I_j}\le Ch^{k+1}\tau,\quad \|\widetilde{F}_0^n\|_{\partial I_j}\le Ch^{k+1/2}\tau. \qquad (4.45)$$

Plugging (4.45) into (4.44), and using $2ab\le a^2+b^2$, we get

$$|\mathcal{H}^{lin}(f'(u^n);\widetilde{F}_0^n,v_0)|\le C(h^{2k}\tau^2+\|v_0\|^2). \qquad (4.46)$$

Similar to the proof of (4.44), and note that $|z^{n,i}|=\mathcal{O}(\tau)$, we have

$$|\mathcal{H}^{lin}(z^{n,i};\rho_0^{n,i},v_0)|\le C(h^{2k}\tau^2+\|v_0\|^2),\ \ i=1,2. \qquad (4.47)$$

Combining (4.41), (4.42), (4.43), (4.46) and (4.47), and assume $\tau=\mathcal{O}(h)$, we have

$$\mathcal{I}_1(v_0)\le Ch^{-1}(\|F_0^n\|^2+\|v_0\|^2)+C(\|e_0^{n,1}\|^2+\|e_0^{n,2}\|^2+\|v_0\|^2+h^{2k+2}). \qquad (4.48)$$

Thanks to Lemma 4.4, we have

$$\mathcal{I}_2(v_0)\le C\|v_0\|^2+C\sum_{i=0}^2 h^{-2}\|u^{n,i}-u_0^{n,i}\|_\infty^2(\|e_0^{n,i}\|^2+h^{2(k+1)}). \qquad (4.49)$$

It follows from (4.18) and Young's inequality that

$$
\begin{aligned}
\mathcal{I}_3(v_0)&=s_h(2(u_h^{n,2}-u^{n,2})-(u_h^{n,1}-u^{n,1})-(u_h^n-u^n),v_h)\\
&=-\sum_{j=1}^N\vartheta[\![\widetilde{F}_0^n]\!]_{j+1/2}[\![v_0]\!]_{j+1/2}-\sum_{j=1}^N\vartheta[\![F_0^n]\!]_{j+1/2}[\![v_0]\!]_{j+1/2}\\
&\le\frac14\sum_{j=1}^N\vartheta[\![\widetilde{F}_0^n]\!]_{j+1/2}^2+2\sum_{j=1}^N\vartheta[\![v_0]\!]_{j+1/2}^2+\frac14\vartheta[\![F_0^n]\!]^2,
\end{aligned}
$$

which together with (4.45) and inverse inequality yields

$$\mathcal{I}_3(v_0)\le C(h^{2k+1}\tau+h^{-1}\|v_0\|^2)+\frac14\vartheta[\![F_0^n]\!]^2. \qquad (4.50)$$

Using the fact $\mathcal{E}=\mathcal{O}(\tau^4)$ and Cauchy-Schwartz inequality, we get

$$3(\mathcal{E},v_0)\le C(\tau^7+\tau\|v_0\|^2),$$

which combining with (4.48), (4.49), (4.50) and (4.40) yields

$$3(G_0^n, v_0) \le C\tau h^{-1}(\|F_0^n\|^2 + \|v_0\|^2) + C\tau\|v_0\|^2 + \frac{\tau}{4}\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}). \tag{4.51}$$

Taking $v_0 = F_0^n$ in (4.51) and using the assumption $\tau \le \gamma h$, we have

$$J_3 \le 2C\gamma\|F_0^n\|^2 + \frac{\tau}{4}\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}). \tag{4.52}$$

Taking $v_0 = G_0^n$ in (4.51), we have

$$3\|G_0^n\|^2 \le C\gamma\|F_0^n\|^2 + 2C\gamma\|G_0^n\|^2 + \frac{\tau}{4}\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}),$$

then

$$(3 - 2C\gamma)\|G_0^n\|^2 \le C\gamma\|F_0^n\|^2 + \frac{\tau}{4}\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}).$$

Therefore,

$$J_4 \le \frac{3}{3 - 2C\gamma}[C\gamma\|F_0^n\|^2 + \frac{\tau}{4}\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)})],$$

which combining with (4.52) yields

$$J_3 + J_4 \le (2C\gamma + \frac{3C\gamma}{3 - 2C\gamma})\|F_0^n\|^2 + \frac{1}{4}(1 + \frac{3}{3 - 2C\gamma})\tau\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}).$$

In the above inequality, taking τ small enough such that $C\gamma = 1/4$, then we get

$$J_3 + J_4 \le 0.8\|F_0^n\|^2 + 0.55\tau\vartheta[\![F_0^n]\!]^2 + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}),$$

which together with (4.39) and (4.33) yields

$$\Xi_2 \le -0.2\|F_0^n\|^2 - \tau\vartheta(0.9[\![E_0^n]\!]^2 + 0.35[\![F_0^n]\!]^2) + C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2(k+1)}). \tag{4.53}$$

4.4. **Main results.** In this subsection, we will give the L^2 norm error estimate between the exact solution $u(t^n)$ and the WG solution u_0^n.

Theorem 4.11. *Let u_h be the numerical solution of the fully discrete WG scheme (4.1)-(4.4) with the explicit TVDRK3 time marching. Let u be the exact solution of problem (1.1)-(1.2), where $f(u)$ is smooth enough. If u and its spatial derivatives up to the second order are all continuous in $I = (0,1)$, and $\|u\|_{k+1}$, $\|u_t\|_{k+1}$, $\|u_{tt}\|_{k+1}$ and $\|u_{ttt}\|_{k+1}$ are bounded uniformly for any time $t \in (0,T]$, then the following error estimate holds*

$$\|u(t^n) - u_0^n\| \le C(h^{k+1/2} + \tau^3) \tag{4.54}$$

under the condition $\tau \le \gamma h$ with a fixed constant $\gamma > 0$. Here C is a positive constant independent of h, τ and u_h.

Proof. Denote by

$$\Delta = 0.2\|F_0^n\|^2 + \tau\vartheta(0.9[\![E_0^n]\!]^2 + 0.35[\![F_0^n]\!]^2 + 0.8\sum_{i=0}^{2}[\![e_0^{n,i}]\!]^2),$$

then from (4.25), (4.29) and (4.53), we obtain

$$3\|e_0^{n+1}\|^2 - 3\|e_0^n\|^2 + \Delta \le C\tau^7 + \tau\sum_{i=0}^{2}\mathbb{C}(u^{n,i}, u_0^{n,i})(\|e_0^{n,i}\|^2 + h^{2k+1}). \tag{4.55}$$

As [22, 23], we adopt the following a priori assumption for $m\,(m\tau < T)$ to deal with the nonlinearity of the flux $f(u)$

$$\|e_0^{n,i}\| \le Ch^{3/2}, \quad \text{for } n \le m, i = 0,1,2. \tag{4.56}$$

The justification of such assumption will be given later.

It follows from the assumption (4.56) and the approximation property of L^2 operator Q_0

$$\|\rho_0^{n,i}\|_\infty = \|u^{n,i} - Q_0u^{n,i}\|_\infty \le Ch^{k+1/2}, \quad i = 0,1,2$$

implies that

$$\|u^{n,i} - u_0^{n,i}\|_\infty \le \|\rho_0^{n,i}\|_\infty + Ch^{-1/2}\|e_0^{n,i}\| \le Ch.$$

Therefore, we have

$$\mathbb{C}(u^{n,i}, u_0^{n,i}) \le C, \quad \text{for } n \le m, i = 0,1,2,$$

which together with (4.55), (4.19) and (4.20) yields

$$3\|e_0^{n+1}\|^2 - 3\|e_0^n\|^2 \le C\tau^7 + C(\|e_0^n\|^2 + h^{2k+1})\tau,$$

where C is a positive constant independent of m,n,h and τ. Then an application of the discrete Gronwall inequality yields

$$\|e_0^{n+1}\|^2 \le C(\tau^6 + h^{2k+1}), \quad n \le m, \tag{4.57}$$

which together with

$$\|\rho_0^{n+1}\| = \|u^{n+1} - Q_0u^{n+1}\| \le Ch^{k+1}$$

yields that

$$\|u^{n+1} - u_0^{n+1}\| \le C(\tau^3 + h^{k+1/2}).$$

Now we turn to verify the reasonableness of the a priori assumption (4.56). Since $e_0^0 = 0$, by (4.19) and (4.20), we easily obtain

$$\|e_0^{0,i}\| \le Ch^{k+1} \le Ch^{3/2}$$

for $i = 0, 1, 2$ and $k \ge 1$. Supposing (4.56) holds for m, we can show that this assumption is also true for $m + 1$. Inequality (4.57) as well as (4.19) and (4.20) imply that

$$\|e_0^{m+1,i}\| \le C(\|e_0^{m+1}\| + h^{k+1}) \le C(\tau^3 + h^{k+1/2}) \le Ch^{3/2}$$

for $k \ge 1$ and $i = 1, 2$. Thus the assumption (4.56) is reasonable, and hence the above error estimate holds for any $m\,(m\tau < T)$. The proof is completed. □

5. **Numerical examples.** In this section, we present the numerical examples to verify our theoretical findings. In our numerical experiments, we shall use piecewise uniform meshes which are constructed by equally dividing spatial domain into N subintervals. The main purpose of this paper is to investigate how to use WG method to discretize spacial variables in conservation law. In order to reduce the time errors, we use the third order explicit TVDRK time discrete scheme in time, and take time step $\tau = CFL \cdot h$, where CFL is a constant independent of the mesh size h. WG scheme with $P_k(k = 1, 2, 3)$ element is used for spatial discretization.

Example 1 (**A smooth solution of a linear equation).** We solve the following linear problem (1.1) with periodic boundary condition,

$$u_t + u_x = 0, \qquad (x, t) \in (0, 2\pi) \times (0, T],$$
$$u(x, 0) = \sin x, \qquad x \in (0, 2\pi).$$

The exact solution to this problem is

$$u(x, t) = \sin(x - t).$$

Let time step size $\tau = 0.05h$ in this example. WG scheme with $P_k(k = 1, 2, 3)$ element is used for spatial discretization. The L^2 errors and the order of convergence, at $T = 2\pi$, are reported in Table 1. It is observed that the order of convergence of the L^2 error achieves $(k + 1)$-th order of accuracy.

TABLE 1. L^2 errors and corresponding convergence rates of Example 1. $T = 2\pi$, $\lambda_1 = \lambda_2 = 1$.

	P_1 element		P_2 element		P_3 element	
N	$\|u - u_0\|$	Rate	$\|u - u_0\|$	Rate	$\|u - u_0\|$	Rate
8	1.29E-01		3.36E-03		2.66E-04	
16	3.02E-02	2.10	3.99E-04	3.08	1.94E-05	3.78
32	7.22E-03	2.06	4.93E-05	3.02	1.27E-06	3.93
64	1.78E-03	2.02	6.14E-06	3.00	8.06E-08	3.98
128	4.42E-04	2.01	7.67E-07	3.00	5.06E-09	4.00

Example 2 (**A blow-up solution of a nonlinear equation).** We solve the following nonlinear problem (1.1) with periodic boundary condition,

$$u_t + (u^2/2)_x = 0, \qquad (x, t) \in (0, 1) \times (0, T],$$
$$u(x, 0) = \phi(x), \qquad x \in (0, 1),$$

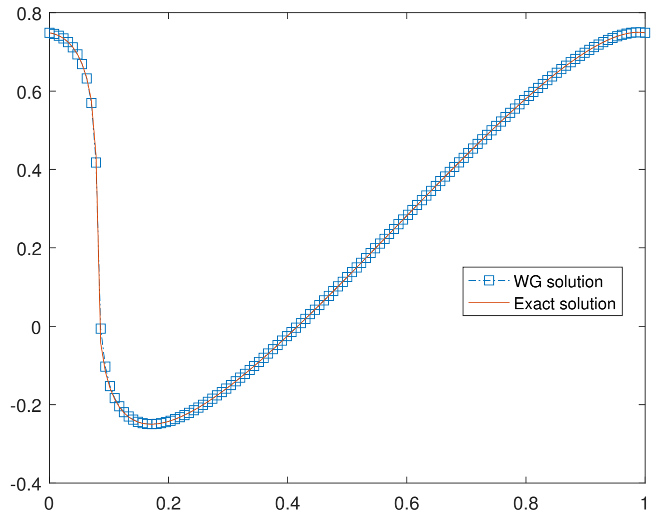


FIGURE 1. WG solution for Example 2, $T = 1/\pi, \lambda_1 = 2, \lambda_2 = 1, N = 128$.

with initial value function

$$\phi(x) = \frac{1}{4} + \frac{1}{2}\sin(\pi(2x-1)).$$

The exact solution $u(x,t)$ is obtained by solving characteristic relation $u = \phi(x - u \cdot t)$, which is smooth up to $t = 1/\pi$. Please find more details in [4].

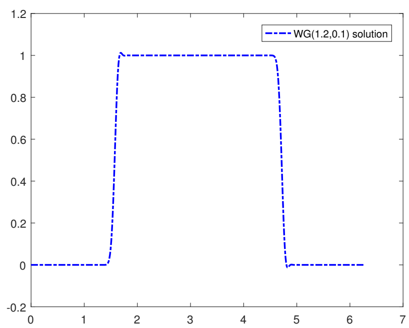
Let time step size $\tau = 0.1h$ in this example. In Table 2 we present the results at $T = 0.2$ when the solution is still smooth. Observe that there is at least $(k+1/2)$-th order of convergence rate in L^2 norm. u_b of WG solution $u_h = \{u_0, u_b\}$ at $T = 1/\pi$ is plotted in Figure 1, where P_2 element is used.

TABLE 2. L^2 errors and corresponding convergence rates of Example 2. $T = 0.2$, $\lambda_1 = \lambda_2 = 2.5$.

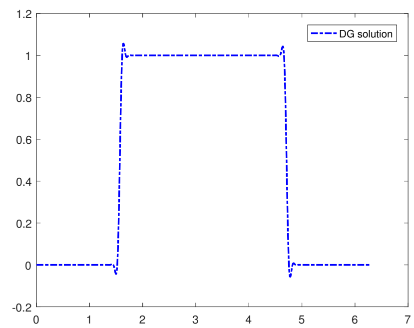
	P_1 element		P_2 element		P_3 element	
N	$\|u-u_0\|$	Rate	$\|u-u_0\|$	Rate	$\|u-u_0\|$	Rate
8	1.68E-02		6.60E-03		1.89E-03	
16	6.11E-03	1.46	7.86E-04	3.07	2.22E-04	3.09
32	1.42E-03	2.10	1.63E-04	2.27	9.96E-06	4.48
64	3.49E-04	2.03	2.85E-05	2.51	8.19E-07	3.60
128	8.67E-05	2.01	4.98E-06	2.51	5.81E-08	3.82

Example 3 (**A discontinuous solution of a linear equation).** We solve the following linear equation (1.1) with periodic boundary condition,

$$u_t + u_x = 0, \qquad (x,t) \in (0, 2\pi) \times (0, T],$$
$$u(x,0) = \phi(x), \qquad x \in (0, 2\pi),$$



(A) WG solution with $(\lambda_1, \lambda_2) = (1.2, 0.1)$



(B) DG solution

FIGURE 2. The P_1 WG solution and DG solution for Example 3, $T = 2\pi, N = 512$.

with initial value function

$$\phi(x) = \begin{cases} 1, & \pi/2 < x \le 3\pi/2, \\ 0, & \text{elsewhere.} \end{cases}$$

The exact solution is $u(x,t) = \phi(x - t)$.

In this problem the exact solution is non-smooth. It aims to show our WG method is also stable and efficient to the problems with non-smooth solutions. Let time step size $\tau = 0.1h$ in this example. The errors and the order of convergence are reported in Table 3. Observe that the method is stable. The WG solution using P_1 element with $(\lambda_1, \lambda_2) = (1.2, 0.1)$ and the DG solution with P_1 element are plotted in Figure 2 (a) and (b) respectively.

TABLE 3. L^2 errors and corresponding convergence rates of Example 3. $T = 2\pi$, $\lambda_1 = 2, \lambda_2 = 1$.

	P_1 element		P_2 element	
N	$\|u - u_0\|$	Rate	$\|u - u_0\|$	Rate
8	5.93E-01		4.23E-01	
16	5.01E-01	0.24	3.25E-01	0.38
32	3.93E-01	0.35	2.52E-01	0.37
64	3.26E-01	0.27	1.98E-01	0.35
128	2.72E-01	0.26	1.58E-01	0.33
256	2.26E-01	0.27	1.27E-01	0.32
512	1.89E-01	0.26	1.03E-01	0.30

6. **Conclusion.** A weak Galerkin finite element method is proposed for nonlinear conservation laws. This method provides a class of finite element schemes by tuning the built-in parameters including purely upwind scheme. This makes the WG method highly flexible. Compared with the DG method [11], the present method can be applied for solving nonlinear conservation laws. Error estimations are given. The convenience of the proposed method is validated by the numerical examples.

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