COMBINATORICS OF SOME FIFTH AND SIXTH ORDER
MOCK THETA FUNCTIONS

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Abstract. The goal of this paper is to provide a new combinatorial meaning
to two fifth order and four sixth order mock theta functions. Lattice paths of
Agarwal and Bressoud with certain modifications are used as a tool to study
these functions.

1. **Introduction.** Mock theta functions were introduced by Ramanujan in his Last
Letter with 17 mock theta functions of order 3, 5 and 7 as first examples. These
functions have been studied quite widely especially after the discovery of Lost Note-
book by Andrews. There is a substantial number of papers where these functions
have been studied analytically, a complete history of these studies can be found in
the survey paper of Gordon and McIntosh [18]. There is also a considerable number
of texts on the combinatorial connections of these functions. The first and second
Mock Theta Conjectures, given by Andrews and Garvan [11] and proved by Hick-
erson [19], are based on the rank of partitions. There are a number of mock theta
functions which have simple combinatorial interpretations in terms of partitions.
For example, Fine [17] interpreted a third order mock theta function $\psi(q)$ as the
generator for partitions into odd parts without gaps. Another third order mock
theta function $f(q)$ is a generating function for the difference of number of parti-
tions with even rank and number of partitions with odd rank. In recent times, there
has been a good amount of study on the combinatorics of the mock theta functions.
In 2009, Andrews [9] associated a seventh order mock theta function $\mathcal{F}_1(q)$ to the
partitions with initial repetitions. In [10], Andrews et al. showed that the third
order mock theta function $\omega(q)$ is a generating function for the partitions with each
odd part less than twice the smallest part. They also proved that another third
order mock theta function $\nu(q)$ is related to these partitions with distinct parts. In

[12], Andrews and Yee have given two variable generalizations of the results of [10] and Li and Yang [20] have further studied these generalized results combinatorially.

In addition to ordinary partitions, some authors have also studied mock theta functions in terms of $(n+t)$–color partitions, generalized Frobenius partitions and lattice paths. For a detailed study reader can refer to ([2], [3] [6], [7], [23], [24], [25], [26], [27]). In [5], Agarwal and Bressoud introduced the idea of interpreting basic hypergeometric series with lattice paths. In fact, it was a manifestation of the idea of Burge ([14], [15]) where he interpreted the summation side of generalized Rogers–Ramanujan identity in terms of weighted binary words. The first objective of this paper is to obtain the interpretations of the following two fifth order mock theta functions in terms of lattice paths:

$$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_n}, \tag{1}$$

$$\chi_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_{n+1}}. \tag{2}$$

Andrews [8], while providing enumerative proofs of certain q-identities, has proved that $\chi_0(q)$ is the generating function for partitions with unique smallest part and largest part at most twice the smallest part. In a similar manner, $q\chi_1(q)$ can be considered as the generating function for partitions with no part as large as twice the smallest part. Here, we study these functions in terms of lattice paths. However, we use lattice paths with certain modifications being discussed in detail in Section 2. In Section 3, a constructive method of obtaining the interpretations of these functions with the lattice paths is described.

Although Ramanujan's Last Letter contained mock theta functions of odd orders only, mock theta functions of sixth and tenth order exist in the Lost Notebook [22]. The following two sixth order mock theta functions are defined in the Lost Notebook:

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q; q)_{2n}}, \tag{3}$$

$$\psi(q) := \sum_{n=0}^{\infty} \frac{(-1)^{n-1} q^{n^2} (q; q^2)_{n-1}}{(-q; q)_{2n-1}}. \tag{4}$$

He provided certain identities relating these functions to the following q-series, which are also mock theta functions of order 6:

$$\rho(q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{\binom{n+1}{2}}}{(q; q^2)_{n+1}}, \tag{5}$$

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{\binom{n+2}{2}}}{(q; q^2)_{n+1}}, \tag{6}$$

$$\gamma(q) := \sum_{n=0}^{\infty} \frac{(q; q)_n q^{n^2}}{(q^3; q^3)_n}, \tag{7}$$

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q; q^2)_n}{(-q; q)_n}, \tag{8}$$

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n}. \tag{9}$$

The series defining $\mu(q)$ in (9) does not converge, however the sequences of its even and odd partial sums converge and $\mu(q)$ is considered to be the average of these two values. In fact, if we consider the sum of summands of $\rho(q)$ and $\sigma(q)$ over the negative integers, we obtain $\frac{\lambda(q)}{2}$ and $\frac{\mu(q)}{2}$ respectively. In a similar examination of the summands of $\phi(q)$ and $\psi(q)$ the following two mock theta functions are obtained:

$$\phi_-(q) := \sum_{n=1}^{\infty} \frac{q^n (-q;q)_{2n-1}}{(q;q^2)_n}, \tag{10}$$

$$\psi_-(q) := \sum_{n=1}^{\infty} \frac{q^n (-q;q)_{2n-2}}{(q;q^2)_n}. \tag{11}$$

These functions were independently discovered by Berndt and Chan [13], and McIntosh [21]. Choi [16] provided combinatorial interpretations of $\lambda(q)$ and $\psi_-(q)$ using n–color overpartitions along with some other third and sixth order mock theta functions. In Section 4, we study $\lambda(q)$, $2\mu(q)$, $\phi_-(q)$ and $\psi_-(q)$ by using lattice paths, where

$$2\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(q;q^2)_n}{(-q;q)_{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}(q;q^2)_n}{(-q;q)_{n+1}}. \tag{12}$$

2. **Preliminaries.** In this section, first of all we recapitulate the definition of $(n+t)$–color partitions introduced by Agarwal and Andrews [4].

Definition 2.1. An $(n+t)$–color partition, $t \geq 0$ (also called a partition with "$(n+t)$ copies of n"), is a partition in which a part of size n, $n \geq 0$, can occur in $(n+t)$ different colors denoted by $n_1, n_2, \cdots, n_{n+t}$. Note that for $t > 0$ one copy of zero can occur, but zeros cannot repeat.

Definition 2.2. The weighted difference of two parts $(a_i)_{b_i}$ and $(a_j)_{b_j}$ $(a_i \geq a_j)$ in an $(n+t)$–color partition $(a_1)_{b_1} + (a_2)_{b_2} + \cdots + (a_k)_{b_k}$ such that $(a_1)_{b_1} \geq (a_2)_{b_2} \geq \cdots \geq (a_k)_{b_k}$, is $a_i - b_i - a_j - b_j$ and denoted by $(w.d)_{i,j}$.

Now, we consider the terminology of lattice paths given by Agarwal and Bressoud [5].

Definition 2.3. Paths of finite length which lie in the first quadrant, begin on the y-axis and terminate on the x-axis with the following steps are considered:
North-East(NE): from (i,j) to $(i+1, j+1)$.
South-East(SE): from (i,j) to $(i+1, j-1)$, allowed for $j > 0$ only.
Horizontal(H): from $(i,0)$ to $(i+1, 0)$.
The following terminology arises with these paths:
Peak: either a vertex which is preceded by a north-east step and followed by a south-east step or a vertex on the y-axis followed by a south-east step.
Valley: a vertex which is preceded by a south-east step and followed by a north-east step.
Mountain: a section of the path that begins on the x-axis or y-axis and ends on the x-axis but does not contain a point $(x, 0)$ other than the end points.

Range: a section of the path that starts either on the y-axis or at a vertex preceded by a horizontal step and terminates at the end of the path or at a vertex followed by a horizontal step but contains no horizontal step in between.

Plain: a section of the path that comprises of only horizontal steps, starting either on the y-axis or at a vertex preceded by a south-east step and ending at a vertex followed by a north-east step.

The **Height** of a vertex is its y-coordinate. The **Weight** of a vertex is its x-coordinate. The **Weight** of a path is the sum of the weights of its peaks. We will denote weight of a path P by $|P|$.

Agarwal and Bressoud linked these paths with restricted $(n+t)$–color partitions by encoding each path with the sequence of weights of peaks subscripted by the height of the respective peak. Thus, we see that the path in FIGURE 1 corresponds to the n–color partition $(2_1, 5_2, 9_4, 16_2, 19_3)$. Agarwal and Bressoud [5] remarked
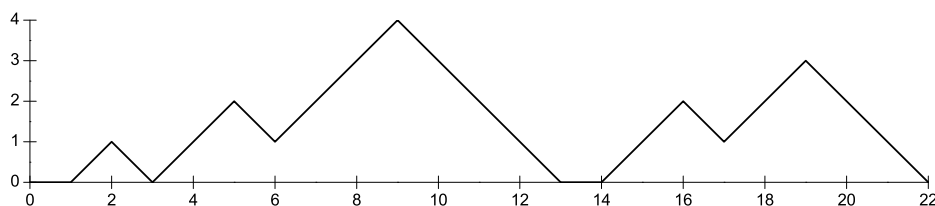

FIGURE 1.

that these paths correspond to certain restricted $(n+t)$–color partitions only. Agarwal initiated the interpretation of mock theta functions in terms of $(n + t)$–color partitions and the above defined lattice paths. The first set of mock theta functions which he interpreted in [2] and [3] contained the following mock theta functions:

$$\psi(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}, \tag{13}$$

$$F_0(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n}, \tag{14}$$

$$\phi_0(q) := \sum_{n=0}^{\infty} q^{n^2}(-q; q^2)_n, \tag{15}$$

$$\phi_1(q) := \sum_{n=0}^{\infty} q^{(n+1)^2}(-q; q^2)_n. \tag{16}$$

In the following four theorems, we recapitulate the results proved by Agarwal in [2] and [3]:

Theorem 2.4. *For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n–color partitions satisfying*

(i) $a_k = b_k$;
(ii) $a_i \equiv b_i \pmod{2}$, $1 \leq i \leq k$;
(iii) $(w.d)_{i,i+1} \geq 0$, $1 \leq i \leq k - 1$.

And, $B_1(\nu)$ denote the number of lattice paths of weight ν starting at $(0,0)$ with no valley above height 0 and no plain. Then

$$\sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_1(\nu)q^{\nu} = \psi(q).$$

Theorem 2.5. *For $\nu \geq 0$, let $A_2(\nu)$ denote the number of n–color partitions satisfying*

(i) $a_k = b_k$;

(ii) $a_i \equiv b_i \pmod 2$, $b_i > 1$, $1 \leq i \leq k$;

(iii) $(w.d)_{i,i+1} = 0$, $1 \leq i \leq k-1$.

And, $B_2(\nu)$ denote the number of lattice paths of weight ν starting at $(0,0)$ with no valley above height 0, no plain and height of each peak ≥ 2. Then

$$\sum_{\nu=0}^{\infty} A_2(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_2(\nu)q^{\nu} = F_0(q).$$

Theorem 2.6. *For $\nu \geq 0$, let $A_3(\nu)$ denote the number of n–color partitions satisfying*

(i) $a_k = 1$ or 2;

(ii) $b_i = \begin{cases} 1 & \text{if } a_i \text{ odd,} \\ 2 & \text{if } a_i \text{ even,} \end{cases}$ $1 \leq i \leq k$;

(iii) $(w.d)_{i,i+1} = 0$, $1 \leq i \leq k-1$.

And, $B_3(\nu)$ denote the number of lattice paths of weight ν starting at $(0,0)$ with no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2 and weight of the first peak is 1 or 2. Then

$$\sum_{\nu=0}^{\infty} A_3(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_3(\nu)q^{\nu} = \phi_0(q).$$

Theorem 2.7. *For $\nu \geq 1$, let $A_4(\nu)$ denote the number of n–color partitions satisfying*

(i) $a_k = 1$;

(ii) $b_i = \begin{cases} 1 & \text{if } a_i \text{ odd,} \\ 2 & \text{if } a_i \text{ even,} \end{cases}$ $1 \leq i \leq k$;

(iii) $(w.d)_{i,i+1} = 0$, $1 \leq i \leq k-1$.

And, $B_4(\nu)$ denote the number of lattice paths of weight ν starting at $(0,0)$ with no valley above height 0, no plain, the height of each peak of odd weight is 1 and that of even weight is 2, and weight of the first peak is 1. Then

$$\sum_{\nu=1}^{\infty} A_4(\nu)q^{\nu} = \sum_{\nu=1}^{\infty} B_4(\nu)q^{\nu} = \phi_1(q).$$

It is very obvious that the paths introduced by Agarwal and Andrews can not correspond to $(n+t)$–color partitions with repeating parts. Also, it is clear from above theorems that in the above $(n+t)$–color partition-theoretic interpretations of mock theta functions no part is repeating. Thus, it was possible to obtain the corresponding interpretations in terms of lattice paths. However, in the $(n+t)$–color partition-theoretic interpretations of the six mock theta functions to be considered in this manuscript parts can repeat. We have obtained these interpretation in a recently published manuscript [26]. It prompted us to make following modifications to the terminology of the lattice paths by introducing backward horizontal step:

Definition 2.8. Here we will be considering only those paths which have no valley above height zero.

Backward Horizontal Step: from $(i, 0)$ to $(i - 1, 0)$, allowed for $i > 0$ only.

Now a plain can be of the following two types:

Forward Plain: same as the plain described earlier.

Backward Plain: a section of path that consists of only backward horizontal steps, starts at a vertex preceded by a south-east step and terminates at a vertex followed by a north-east step. It will be considered as a plain of negative length.

Since we have introduced backward horizontal steps, so a peak can now repeat and a peak having lower weight can occur later in the path. For example, the path shown in FIGURE 2 corresponds to the sequence of peaks $(1_1, 4_1, 3_2, 5_1)$. However,
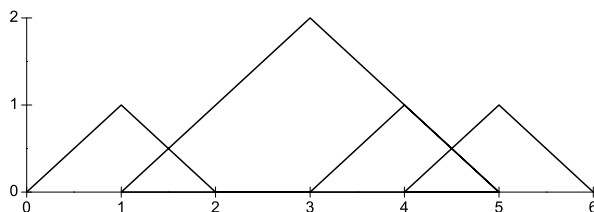

Figure 2.

the introduction of backward steps can lead to ambiguity while reading the paths. Thus, it becomes necessary to exhibit the sequence of steps taken to draw the path, as the sequence of steps for a particular path is unique. In this paper, we promptly use sequence of steps to describe these paths and denote the possible steps with the Greek letters in the following manner:

North-East-α, South-East-β, Forward Horizontal-γ, Backward Horizontal-δ.

Thus, the sequence of steps for the path of FIGURE 2 is $\alpha\beta\gamma\alpha\beta\delta\delta\delta\delta\alpha\alpha\beta\beta\delta\alpha\beta$. A α step followed by a β step or an initial β step constitute a peak. Since in our paths there is no valley above height zero, so whenever there exists a peak, α steps are followed by an equal number of β steps and the number of these steps gives the height of that peak. Weight of a peak is equal to the number of steps other than δ steps minus the number of δ steps before that peak. Length of forward plain is the number of γ steps and of the backward plain is the negative of the count of δ steps comprising the plain. Thus, the sequence $\alpha\beta\gamma\alpha\beta\delta\delta\delta\delta\alpha\alpha\beta\beta\delta\alpha\beta$ has its first peak with weight 1 and height 1, second peak with weight 4 and height 1, third peak with weight 3 and height 2, and fourth peak with weight 5 and height 1. There is a plain of length 1 between the first and second peak, a plain of length -4 between second and third peak, and a plain of length -1 between third and fourth peak.

In [26], we have obtained the interpretations through step by step construction of partitions. The most widely used method to prove such results proceed by obtaining recurrence relations and this method was introduced in [1]. In case of these six mock theta functions also, it is possible present a proof by establishing recurrence relations. However, there is a significant difference in transforming the classes to obtain recurrence relations, which makes it important to present these proofs which are included in Section 5.

3. **Fifth order mock theta functions.** To interpret mock theta functions $\chi_0(q)$ and $\chi_1(q)$, we will consider the following forms of these functions respectively:

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n(-q;q)_n}, \tag{17}$$

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_{n+1}(-q;q)_n}. \tag{18}$$

Remark 1. For interpreting the functions $\chi_0(q)$, $\chi_1(q)$, $\lambda(q)$ and $2\mu(q)$, we need to attach certain weight to the count of each relevant path which we denote by $w_{P_i}(P) = (-1)^{t_i}$, $i = 1, 2, 5 - 7$, for a path $P \in P_i$, where P_i is a set of paths satisfying certain specified conditions and the value of t_i depends upon the set of paths P_i. Also define

$$p_i(\nu) = \sum_{\substack{|P|=\nu \\ P \in P_i}} w_{P_i}(P).$$

Remark 2. In the remaining part of the paper, (x_i, y_i), $1 \le i \le k$, denotes the ith peak in the path, where x_i and y_i are the weight and height of the peak respectively.

Theorem 3.1. *Let P_1 denote the set of paths starting at $(0,0)$ with length of plain between any two consecutive peaks ≥ -2 and in front of the first peak ≥ 0. Then for $t_1 = x_k - y_k$,*

$$\chi_0(q) = \sum_{\nu=0}^{\infty} p_1(\nu) q^{\nu}.$$

Proof.   **Step I.:** Let us consider the kth term of the series

$$\frac{q^k}{(q;q^2)_k(q;q)_k}.$$

Begin the construction of the path with the factor q^k, which generates a path with k peaks at $(1,1)$ and a plain of length -2 between peaks. An example of a path with four peaks is as shown in FIGURE 3 and corresponding sequence of steps is $\alpha\beta\delta\delta\alpha\beta\delta\delta\alpha\beta\delta\delta\alpha\beta$.
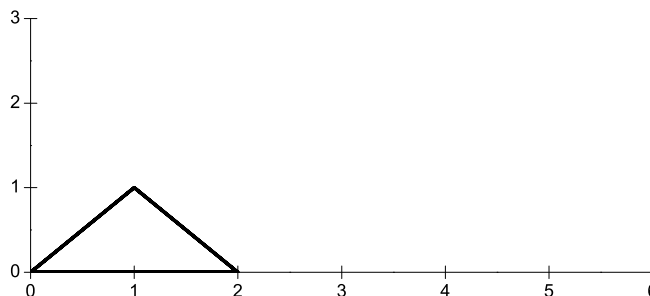

FIGURE 3.

Step II.: Factor $(q; q^2)_k^{-1}$ introduces nonnegative multiples of $1, 3, \cdots, 2k-1$, say, $f_1 \times 1, f_2 \times 3, \cdots, f_k \times (2k-1)$, $f_i \geq 0$. Transform the path by increasing the number of α as well as β steps of the ith peak by f_{k-i+1}. This increases the weight of each of the $(i+1)^{th}$ to k^{th} peak by $2f_{k-i+1}$ and of the ith peak by f_{k-i+1}. Thus for $f_1 = 2$, $f_2 = 1$, $f_3 = 0$ and $f_4 = 1$, the path of FIGURE 3 transforms to the path of FIGURE 4 and corresponding sequence of steps is given by $\alpha\alpha\beta\beta\delta\delta\alpha\beta\delta\delta\alpha\alpha\beta\beta\delta\delta\alpha\alpha\alpha\beta\beta\beta$.
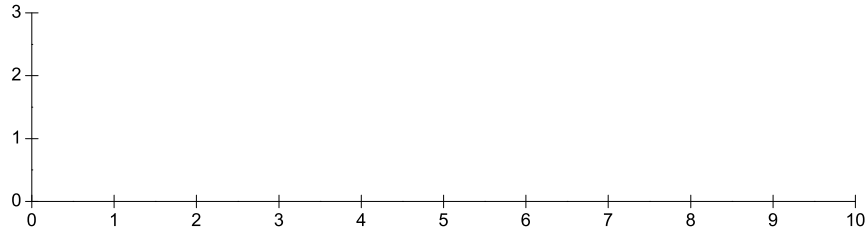

FIGURE 4.

Step III.: The factor $(q; q)_k^{-1}$ generates nonnegative multiples of $1, 2, \cdots, k-1$, say, $g_1 \times 1, g_2 \times 2, \cdots, g_k \times k$, $g_i \geq 0$. Introducing g_{k-i+1} γ steps in front of the ith peak increases the weight of each of the ith to k^{th} peaks by g_{k-i+1} (if some δ steps already exist before that peak then each delta step will cancel one γ step and remaining γ or δ steps will appear before that peak). For $g_1 = 1$, $g_2 = 0$, $g_3 = 2$ and $g_4 = 1$, the path of FIGURE 4 transforms to the path of FIGURE 5 with sequence of steps $\gamma\alpha\alpha\beta\beta\alpha\beta\delta\delta\alpha\alpha\beta\beta\delta\alpha\alpha\alpha\beta\beta\beta$.
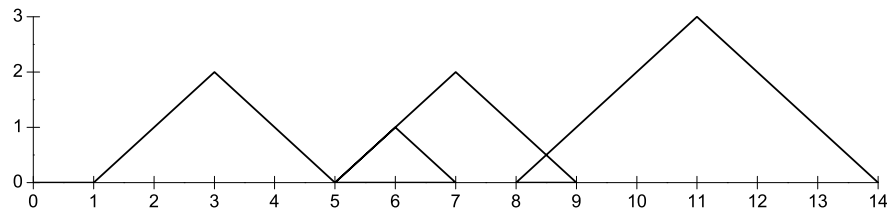

FIGURE 5.

Initially there was a plain of length -2 between any two peaks, so in the final path the length of plain between any two consecutive peaks will be ≥ -2 and in front of the first peak will be ≥ 0. Thus, the path belongs to P_1 and every path of P_1 is uniquely generated in this manner.

The last peak (x_k, y_k) is given by

$$x_k = 1 + 2\sum_{i=1}^{k-1} f_i + f_k + \sum_{j=1}^{k} g_j,$$
$$y_k = 1 + f_k. \tag{19}$$

The series on the right hand side of (17) contains the factor $(-q; q)_k^{-1}$ instead of $(q; q)_k^{-1}$, so a weight is to be attached to the count of each path P generated in the above manner and is given by

$$w_{P_1}(P) = (-1)^{t_1} = (-1)^{\sum_{j=1}^k g_j} = (-1)^{\sum_{j=1}^k g_j + 2\sum_{i=1}^{k-1} f_i}.$$

So $t_1 = x_k - y_k$. □

In the case of $\chi_1(q)$, an additional peak at $(0, 1)$ is considered. Therefore, initially there is only one backward step in between the first and second peak which corresponds to the weighted difference -1. This can be summarized in the following theorem:

Theorem 3.2. *Let P_2 denote the set of paths starting at $(0, 1)$ such that the length of plain between the first and second peak is ≥ -1 and in between the other consecutive peaks is ≥ -2. Then for*

$$t_2 = \begin{cases} 0 & \text{if } k = 1, \\ x_k - y_k & \text{if } k > 1, \end{cases}$$

$$\chi_1(q) = \sum_{\nu=0}^{\infty} p_2(\nu) q^\nu.$$

4. **Sixth order mock theta functions.** In this section, we interpret mock theta functions of order six. The first function which we consider here is $\phi_-(q)$.

Theorem 4.1. *For $\nu \geq 1$, let $P_3(\nu)$ denote the set of paths starting at $(0,0)$ with weight ν such that the length of plain in front of the first peak is 0 or 1 and between any two consecutive peaks is ≤ 2 and*

$$\geq \begin{cases} -2 & \text{if } x_{i+1} \equiv y_{i+1} \ (\text{mod } 2), x_i \equiv y_i \ (\text{mod } 2), \\ 0 & \text{if } x_{i+1} \not\equiv y_{i+1} \ (\text{mod } 2), x_i \not\equiv y_i \ (\text{mod } 2), \\ -1 & \text{otherwise.} \end{cases}$$

Then

$$\phi_-(q) = \sum_{\nu=1}^{\infty} P_3(\nu) q^\nu.$$

Proof. In the construction of path for $\phi_-(q)$, **Step I** and **Step II** proceed exactly in the same manner as in Theorem 3.1. In **Step III**, the factor $(-q; q)_k^{-1}$ is replaced by $(-q^2; q^2)_{k-1}$. This factor generates nonnegative multiples of $2, 4, \cdots, 2k-2$, say, $g_1 \times 2, g_2 \times 4, \cdots, g_{k-1} \times (2k-2)$, $g_i = 0$ or 1. Introducing $2g_{k-i+1} \gamma$ steps in front of the ith peak increases the weight of each of the ith to k^{th} peak by $2g_{k-i+1}$. Thus for $g_1 = 1$, $g_2 = 0$ and $g_3 = 2$, the path of FIGURE 4 transforms to path shown in FIGURE 6 with sequence of steps $\alpha\alpha\beta\beta\gamma\gamma\alpha\beta\delta\delta\alpha\alpha\beta\beta\alpha\alpha\alpha\beta\beta\beta$.

Step IV. The factor $(-q; q^2)_k$ generates either 0 or an odd number $\leq 2k-1$, say, $h_1 \times 1, h_2 \times 3, \cdots, h_k \times (2k-1)$, $h_i = 0$ or 1. Introducing $h_{k-i+1} + h_{k-i+2} \gamma$ steps in front of the ith peak increases the weight of each of the ith to k^{th} by $h_{k-i+1} + h_{k-i+2}$. Thus, the total increase in the weight of the path is $h_1 \times 1 + h_2 \times 3 + \cdots + h_k \times (2k-1)$. For $h_1 = 1$, $h_2 = 0$, $h_3 = 1$ and $h_4 = 1$, path of FIGURE 6 transform to the path shown in FIGURE 7 with sequence of steps $\gamma\alpha\alpha\beta\beta\gamma\gamma\gamma\gamma\alpha\beta\delta\alpha\alpha\beta\beta\gamma\alpha\alpha\alpha\beta\beta\beta$. It can be easily seen that the way these paths are generated each path of weight ν satisfies the conditions of $P_3(\nu)$. Every path enumerated by $P_3(\nu)$ is uniquely generated in this manner. □
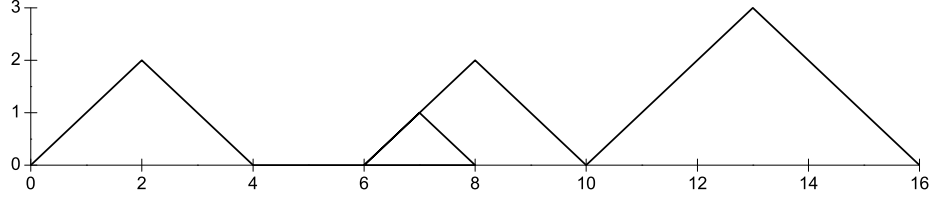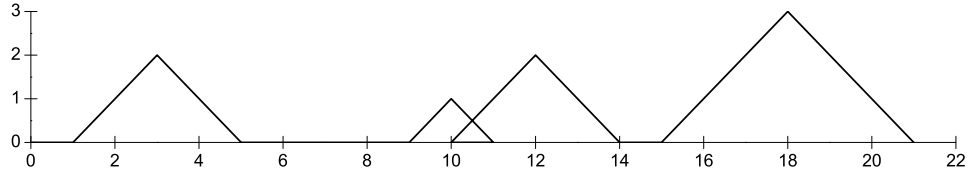
FIGURE 6.

FIGURE 7.

The results for the remaining three mock theta functions of order 6 are presented in the following theorems without proofs, as proofs follow essentially in the similar manner as the previous ones.

Theorem 4.2. *For $\nu \geq 1$, let $P_4(\nu)$ denote the set of paths from $P_3(\nu)$ which have no plain in front of the first peak and the length of plain in front of the second peak is ≤ 1. Then*

$$\psi_-(q) = \sum_{\nu=1}^{\infty} P_4(\nu)q^{\nu}.$$

Theorem 4.3. *Let P_5 denote the set of paths starting at $(0,0)$ such that height of each peak is ≤ 2, length of plain between consecutive peaks is ≥ -2 and in front of the first peak is ≥ 0. Then for $t_5 = x_k - \sum_{i=1}^{k-1} y_i + 2k - 2$,*

$$\lambda(q) = \sum_{\nu=0}^{\infty} p_5(\nu)q^{\nu}.$$

Theorem 4.4. *Let P_6 denote the set of paths starting at $(0,0)$ such that the height of first peak is 1 and rest of the peaks is ≤ 2. Also, the length of plain in front of the first peak is ≥ 0 and in front of rest of the peaks is ≥ -2. Let P_7 be the set of those paths of P_6 for which the length of plain between the first and second peak is ≥ -1. Then for*

$$t_6 = x_k + k - 2 - \sum_{i=2}^{k-1} y_i - 1 \text{ and } t_7 = \begin{cases} x_k - y_k & \text{if } k = 1, \\ x_k + k - 1 - \sum_{i=2}^{k-1} y_i - 1 & \text{if } k > 1, \end{cases}$$

$$2\mu(q) = \sum_{\nu=0}^{\infty} [p_6(\nu) + p_7(\nu)]q^{\nu}.$$

5. $(n+t)$–color partition-theoretic interpretations by establishing recurrence relations.

Remark 3. Throughout this section, if $C_i(\nu)$ denote the number of partitions of ν with certain specified conditions then $C_i(k,\nu)$ will denote the partitions of ν with same conditions into k parts and

$$f_i(z,q) = \sum_{\nu=0}^{\infty} \sum_{k=0}^{\infty} C_i(k,\nu) z^k q^\nu. \tag{20}$$

To obtain the interpretations of mock theta functions $\chi_0(q)$, $\chi_1(q)$, $\lambda(q)$ and $2\mu(q)$, certain weight is to be attached to the relevant partitions as in the case of lattice paths. The required weights can be simply obtained by identifying each peak (x_i, y_i) as a colored part $(x_i)_{y_i}$

Theorem 5.1. *Let $C_1(\nu)$ denote the number of n–color partitions of ν such that $(w.d)_{i,i+1} \geq -2$ for $1 \leq i \leq k-1$. Then*

$$\sum_{\nu=0}^{\infty} C_1(\nu) q^\nu = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n (q;q)_n}.$$

Proof. Split the partitions enumerated by $C_1(k,\nu)$ into three classes. First class contains the partitions for which $a_k \neq b_k$. Subtracting 1 from each a_i, where $1 \leq i \leq k$, we get a partition of $\nu - k$ into k parts with $(w.d)_{i,i+1} \geq -2$. These type of partitions are enumerated by $C_1(k, \nu - k)$. Partitions of the second class are signified by $a_k = b_k = 1$. Deleting the least part 1_1 from such partitions we get a partition of $\nu - 1$ into $k - 1$ parts such that $(w.d)_{i,i+1} \geq -2$. Therefore, these partitions are enumerated by $C_1(k-1, \nu-1)$. The partitions in the third class are the remaining partitions which satisfy $a_k = b_k > 1$. For these partitions, transform the least part $(a_k)_{b_k}$ to $(a_k-1)_{b_k-1}$ and subtract 2 from each a_i, where $1 \leq i \leq k-1$. These partitions are enumerated by $C_1(k, \nu - 2k + 1) - C_1(k, \nu - 3k + 1)$ since the least part $(a_k)_{b_k}$ of such partitions satisfy $a_k = b_k$. All the transformations implied here are completely reversible. Thus, we obtain the following recurrence relation:

$$C_1(k,\nu) = C_1(k,\nu-k) + C_1(k-1,\nu-1) + C_1(k,\nu-2k+1) - C_1(k,\nu-3k+1). \tag{21}$$

Now, substituting for $C_1(k,\nu)$ from (21) into (20) with $i = 1$ and then simplifying

$$f_1(z,q) = f_1(zq,q) + zq f_1(z,q) + q^{-1} f_1(zq^2,q) - q^{-1} f_1(zq^3,q). \tag{22}$$

Substituting $f_1(z,q) = \sum_{n=0}^{\infty} \tau_n(q) z^n$, and then comparing the coefficient of z^n on both sides of (22), we get

$$\tau_n(q) = \frac{q}{(1-q^n)(1-q^{2n-1})} \tau_{n-1}(q), \ n \geq 1. \tag{23}$$

Iterating (23) n times and considering the fact that $\tau_0(q) = 1$, we find that

$$\tau_n(q) = \frac{q^n}{(q;q)_n (q;q^2)_n}, \ n \geq 0. \tag{24}$$

Thus,

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (q;q^2)_n} z^n. \tag{25}$$

$$\sum_{\nu=0}^{\infty} C_1(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} \left(\sum_{k=0}^{\infty} C_1(k,\nu) \right) q^{\nu}$$

$$= f_1(1,q) \tag{26}$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n(q;q^2)_n}. \tag{27}$$

□

The proofs for the remaining theorems proceed in a similar manner. The only tricky part is to obtain recurrence relation, so we will omit the remaining details for the results that follow.

Theorem 5.2. *Let $C_2(\nu)$ denote the number of $(n+1)$–color partitions of ν such that*

(i) $a_k - b_k = -1$;
(ii) $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq k-2$;
(iii) $(w.d)_{k-1,k} \geq -1$.
Then

$$\sum_{\nu=0}^{\infty} C_2(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_{n+1}(q;q)_n}.$$

Proof. Split the partitions enumerated by $C_2(k,\nu)$ into two classes. First class contains the partitions for which $b_k = 1$. Deleting the least part 0_1 from such partitions, we get a partition of ν into $k-1$ parts such that $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq k-2$. Therefore, these partitions are enumerated by $C_1(k-1,\nu)$. The partitions in the second class are the remaining partitions which satisfy $b_k > 1$. For these partitions, transform the least part $(a_k)_{b_k}$ to $(a_k-1)_{b_k-1}$ and subtract 2 from each a_i, where $1 \leq i \leq k-1$. These partitions are enumerated by $C_2(k,\nu-2k+1)$. Thus, we obtain the following recurrence relation:

$$C_2(k,\nu) = C_1(k-1,\nu) + C_2(k,\nu-2k+1). \tag{28}$$

□

To obtain the interpretations of $\phi_-(q)$ and $\psi_-(q)$, we first need to prove the following result:

Theorem 5.3. *Let $C_3(\nu)$ denote the number of n–color partitions of ν such that*
(i) $0 \leq a_k - b_k \leq 3$;
(ii) $(w.d)_{i,i+1} \leq 2$ *and*

$$\geq \begin{cases} -2 & \text{if } a_i \equiv b_i, \ a_{i+1} \equiv b_{i+1} \ (\text{mod } 2), \\ 0 & \text{if } a_i \not\equiv b_i, \ a_{i+1} \not\equiv b_{i+1} \ (\text{mod } 2), \quad 1 \leq i \leq k-1. \\ -1 & \text{otherwise}, \end{cases}$$

Then

$$\sum_{\nu=0}^{\infty} C_3(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{q^n(-q;q)_{2n}}{(q;q^2)_n}.$$

Proof. Partitions enumerated by $C_3(k,\nu)$ split into the following five classes as signified by their least part:
(i) $a_k = b_k = 1$;
(ii) $a_k = b_k + 1 = 2$;

(iii) $a_k = b_k + 2 = 3$;
(iv) $a_k = b_k + 3 = 4$;
(v) $b_k > 1$.

By transforming the partitions of above classes, we obtain the following recurrence relation:

$$C_3(k,\nu) = C_3(k-1,\nu-1) + C_3(k-1,\nu-2k) + C_3(k-1,\nu-2k-1)+$$
$$+C_3(k-1,\nu-4k) + C_3(k,\nu-2k+1).$$

□

Theorem 5.4. *For $\nu \geq 1$, let $C_4(\nu)$ denote the number of n–color partitions of ν such that*

(i) $a_k - b_k = 0$ *or* 1;
(ii) $(w.d)_{i,i+1} \leq 2$ *and*

$$\geq \begin{cases} -2 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \ (\text{mod } 2), \\ 0 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \ (\text{mod } 2), \quad 1 \leq i \leq k-1. \\ -1 & \text{otherwise,} \end{cases}$$

Then

$$\sum_{\nu=1}^{\infty} C_4(\nu)q^\nu = \phi_-(q).$$

Proof. Here, we split the partitions enumerated by $C_4(k,\nu)$ into the following three classes as signified by their least part:

(i) $a_k = b_k = 1$;
(ii) $a_k = b_k + 1 = 2$;
(iii) $b_k > 1$.

And, the corresponding recurrence relation obtained by transforming the classes is given by

$$C_4(k,\nu) = C_3(k-1,\nu-1) + C_3(k-1,\nu-2k) + C_4(k,\nu-2k+1).$$

□

Theorem 5.5. *For $\nu \geq 1$, let $C_5(\nu)$ denote the number of n–color partitions of ν such that*

(i) $a_k = b_k$;
(ii) $(w.d)_{i,i+1} \leq 2$ *and*

$$\geq \begin{cases} -2 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \ (\text{mod } 2), \\ 0 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \ (\text{mod } 2), \quad 1 \leq i \leq k-1. \\ -1 & \text{otherwise,} \end{cases}$$

Then

$$\sum_{\nu=1}^{\infty} C_5(\nu)q^\nu = \psi_-(q).$$

Proof. Here, we split the partitions enumerated by $C_5(k,\nu)$ into the following two classes signified by their least part:

(i) $a_k = 1$;
(ii) $a_k > 1$.

And, the corresponding recurrence relation obtained by transforming the classes is given by

$$C_5(k, \nu) = C_3(k - 1, \nu - 1) + C_5(k, \nu - 2k + 1).$$

□

Theorem 5.6. *For $\nu \geq 0$, let $C_6(\nu)$ denote the number of n–color partitions of ν such that*

(i) $b_i = 1$ *or* 2, $1 \leq i \leq k$;
(ii) $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq k - 1$.

Then

$$\sum_{\nu=0}^{\infty} C_6(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^n(-q; q^2)_n}{(q; q)_n}.$$

Proof. The partitions enumerated by $C_6(k, \nu)$ split into the following three classes as signified by their least part:

(i) $a_k = b_k = 1$;
(ii) $a_k = b_k = 2$;
(iii) $a_k \neq b_k$.

By transforming these classes, we obtain the following recurrence relation:

$$C_6(k, \nu) = C_6(k - 1, \nu - 1) + C_6(k - 1, \nu - 2k) + C_6(k, \nu - k).$$

□

Theorem 5.7. *For $\nu \geq 1$, let $C_7(\nu)$ denote the number of n–color partitions of ν such that*

(i) $b_k = 1$;
(ii) $b_i = 1$ *or* 2, $1 \leq i \leq k$;
(iii) $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq k - 1$.

Then

$$\sum_{\nu=1}^{\infty} C_7(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q; q^2)_n}{(q; q)_{n+1}}.$$

Proof. The partitions enumerated by $C_7(k, \nu)$ split into the following two classes as signified by their least part:

(i) $a_k = 1$;
(ii) $a_k > 1$.

By transforming these classes, we obtain the following recurrence relation:

$$C_7(k, \nu) = C_6(k - 1, \nu - 1) + C_7(k, \nu - k).$$

□

Theorem 5.8. *For $\nu \geq 1$, let $C_8(\nu)$ denote the number of n–color partitions of ν such that*

(i) $b_k = 1$;
(ii) $b_i = 1$ *or* 2, $1 \leq i \leq k$;
(iii) $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq k - 1$;
(iv) $(w.d)_{k-1,k} \geq -1$.

Then

$$\sum_{\nu=1}^{\infty} C_8(\nu)q^\nu = \sum_{n=0}^{\infty} \frac{q^{2n+1}(-q; q^2)_n}{(q; q)_{n+1}}.$$

Proof. The partitions enumerated by $C_8(k, \nu)$ split into three classes. The first class satisfies $a_k = 1$ and $b_{k-1} = 1$. The partitions of this class are transformed by deleting the least part, subtracting 1 from a_{k-1} and keeping the remaining parts unaltered. The second class is signified by the partitions with $a_k = 1$ and $b_{k-1} = 2$. Here, the transformed partitions are obtained by deleting 1_1, subtracting 2 from a_{k-1}, 1 from b_{k-1} and subtracting 2 from all the remaining parts. Third class contains partitions with $a_k > 1$ and these partitions are transformed by simply deleting 1 from all the parts. Thus, we obtain the following recurrence relation:

$$C_8(k, \nu) = C_8(k - 1, \nu - 2) + C_8(k - 1, \nu - 2k + 1) + C_8(k, \nu - k).$$

$\square$

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