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SOME RECENT PROGRESS ON INVERSE SCATTERING PROBLEMS WITHIN GENERAL POLYHEDRAL GEOMETRY

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Abstract. Unique identifiability by finitely many far-field measurements in the inverse scattering theory is a highly challenging fundamental mathematical topic. In this paper, we survey some recent progress on the inverse obstacle scattering problems and the inverse medium scattering problems associated with time-harmonic waves within a certain polyhedral geometry, where one can establish the unique identifiability results by finitely many measurements. Some unique identifiability issues on the inverse diffraction grating problems are also considered. Furthermore, the geometrical structures of Laplacian and transmission eigenfunctions are reviewed, which have important applications in the unique determination for inverse obstacle and medium scattering problems with finitely many measurements. We discuss the mathematical techniques and methods developed in the literature. Finally, we raise some intriguing open problems for the future investigation.

1. **Introduction.** Inverse scattering problems are a central topic in applied mathematics, which have many applications of practical importance in modern technologies including radar, medical imaging, nondestructive testing, remote sensing, geophysical exploration and ultrasound tomography (cf. [27]). Inverse obstacle scattering problems and inverse medium scattering problems are two main themes in inverse scattering problems. Uniqueness issue in inverse scattering problems is concerned with the unique identifiability on the shape and/or the physical material parameters of the underlying obstacle and medium by the corresponding measurements through wave probing. In this paper, we provide an overview of some recent mathematical developments on the inverse scattering problems and the inverse

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medium scattering problems as well as the inverse diffraction grating problems regarding on the unique identifiability issue. Those are fundamental results in the inverse scattering theory.

In this paper, we focus on the inverse obstacle and medium scattering problems for time-harmonic acoustic waves from an impenetrable or penetrable scatterer in a homogeneous background medium. Considering the scattering of a time-harmonic acoustic wave by a bounded obstacle $\Omega \subset \mathbb{R}^n$ which is unknown or inaccessible, $n \geq 2$. Let $k = \omega/c \in \mathbb{R}_+$ be the wavenumber with $\omega \in \mathbb{R}_+$ and $c \in \mathbb{R}_+$, respectively, denoting the frequency and the sound speed. Assume that the incident wave is given by the plane wave of the form

$$u^i(\mathbf{x}; k, \mathbf{d}) = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where $\mathbf{d} \in \mathbb{S}^{n-1}$ signifies the incident direction and $\mathbb{S}^{n-1} := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ is the unit sphere in $\mathbb{R}^n$. Physically speaking, Ω can be a penetrable or an impenetrable obstacle in scattering phenomenon, which shall fulfill the following two systems, respectively. For a penetrable inhomogeneous medium, the forward medium scattering problem can be modeled by

$$\begin{cases} \Delta u + k^2 V u = 0 & \text{in} \quad \mathbb{R}^n, \\ u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x}), & \\ \lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0, & \end{cases} \tag{2}$$

where V denotes the refractive index satisfying $V(\mathbf{x}) \equiv 1$ for $\mathbf{x} \in \mathbb{R}^n \backslash \overline{\Omega}$, $r = |\mathbf{x}|$ and u^s is the scattered wave field generated by the presence of the obstacle $\Omega = \operatorname{supp}(1 - V)$. The limit equation in (2) is known as the Sommerfield radiation condition which ensures the outgoing nature of the scattered wave u^s. The unique solvability of the scattering problem (2) in $H^2_{loc}(\mathbb{R}^n)$ is well-known if $V \in L^\infty(\mathbb{R}^n)$ (see [27] for details).

If Ω is an impenetrable obstacle, the corresponding forward obstacle scattering problem can be formulated by the following Helmholtz system:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \mathbb{R}^n \backslash \overline{\Omega}, \\ \mathcal{B}u = 0 & \text{on} \quad \partial\Omega, \\ u = u^i + u^s & \text{in} \quad \mathbb{R}^n, \\ \lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0. & \end{cases} \tag{3}$$

The boundary operator relies on the physical property of the obstacle. Precisely speaking, $\mathcal{B}$ could be of Dirichlet type $\mathcal{B}u := u$ for the *sound-soft* obstacle or Neumann type $\mathcal{B}u := \partial_\nu u$ for the *sound-hard* obstacle or Robin type $\mathcal{B}u := \partial_\nu u + \eta u$ with ν being the exterior unit normal vector to $\partial\Omega$ and $\eta \in L^\infty(\partial\Omega)$ for the *impedance* obstacle. Here, η denotes the boundary impedance parameter fulfilling $\Re\eta \geq 0$ and $\Im\eta \geq 0$. In what follows, we formally take $u = 0$ on $\partial\Omega$ as $\partial_\nu u + \eta u = 0$ on $\partial\Omega$ with $\eta = +\infty$. In doing so, we can unify the three boundary conditions as the generalized impedance boundary condition:

$$\mathcal{B}(u) = \partial_\nu u + \eta u = 0 \quad \text{on} \quad \partial\Omega, \tag{4}$$

where η could be ∞, corresponding to a sound-soft obstacle. It is well-understood in [27] and [48] that there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^n \backslash \overline{\Omega})$ for Ω with Lipschitz boundary $\partial\Omega$.

The Sommerfeld radiation condition in (2) or (3) leads to the asymptotic expansion (cf. [27]) for u^s to (2) or (3) as follows

$$u^s(\mathbf{x}; k, \mathbf{d}) = \frac{e^{\mathrm{i}kr}}{r^{(n-1)/2}} u^\infty(\hat{\mathbf{x}}; k, \mathbf{d}) + \mathcal{O}\left(\frac{1}{r^{(n+1)/2}}\right) \quad \text{as} \quad r \to \infty \tag{5}$$

uniformly with respect to all directions $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^{n-1}$, where $u^\infty(\hat{\mathbf{x}}; k, \mathbf{d})$ is referred to as the far-field pattern or the scattering amplitude, and $\hat{\mathbf{x}}$ denotes the observation direction. Furthermore, $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^{n-1}$ is a real-analytic function on $\mathbb{S}^{n-1}$. By introducing an operator $\mathcal{F}$ which sends the obstacle Ω together with the physical parameter V (the refractive index) or η (the boundary parameter) to the corresponding far-field pattern through the Helmholtz system (2) or (3), the inverse medium or obstacle problem can be formulated as the following abstract operator equations:

$$\mathcal{F}_{\mathsf{P}}(\Omega, V) = u^\infty(\hat{\mathbf{x}}; k, \mathbf{d}), \tag{6}$$

and

$$\mathcal{F}_{\mathsf{I}}(\Omega, \eta) = u^\infty(\hat{\mathbf{x}}; k, \mathbf{d}), \tag{7}$$

where $\mathcal{F}_{\mathsf{P}}$ and $\mathcal{F}_{\mathsf{I}}$ are defined by the forward medium or obstacle scattering system respecitivley, and are nonlinear. That is, one intends to determine (Ω, V) or (Ω, η) from the knowledge of $u^\infty(\hat{\mathbf{x}}; k, \mathbf{d})$.

A primary issue for the inverse medium problem (6) and inverse obstacle problem (7) is the unique identifiability, which is concerned with the sufficient conditions such that the correspondence between Ω and u^∞ is one-to-one. There is a widespread belief that one can establish uniqueness for (6) and (7) by a single or at most finitely many far-field patterns. We remark that by a single far-field pattern we mean that $u^\infty(\hat{\mathbf{x}}, k, \mathbf{d})$ is collected for all $\hat{\mathbf{x}} \in \mathbb{S}^{n-1}$, but is associated with a fixed incident wave $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$. Indeed, it states that the analytic function u^∞ on the unit sphere associated with at most finitely many k and $\mathbf{d}$ can determine the shape of a generic domain Ω and the physical parameter V or η. This problem is known as the *Schiffer problem* in the inverse scattering community. It is named after M. Schiffer for his pioneering contribution around 1960 which was actually appeared as a private communication in the monograph by Lax and Phillips [41]. There is a long and colourful history on the study of the Schiffer's problem, and we refer to a recent survey paper by Colton and Kress [28] which contains an excellent account of the historical development of this problem.

By now, only under a-prior geometric assumptions on the size or the shape of the scatterer, unique identifiability results by using one incident plane wave have been established. Colton and Sleeman [29] proved that the shape of the obstacle could be uniquely determined with one incident wave when the size of the scatterer satisfies some generic conditions. The sound-soft ball can be uniquely determined by a single measurement (cf. [42]). Later, Alessandrani and Rondi in [2, 3], Cheng and Yamamoto in [25] and [26], Liu and Zou [46] studied the global uniqueness results with respect to a single far-field pattern for sound-soft, sound-hard obstacles within a certain polyhedral geometry. The proofs in [2, 3, 25, 46] mainly utilize the reflection principle for the Helmholtz equation with respect to a Dirichlet or Neumann hyperplane and the path argument, where the path argument is initially developed in [46]. Such kind of methodologies cannot tackle the uniqueness results for an impedance polygonal or polyhedral obstacle with a single far-field pattern. Please refer to Section 2 for more discussions related to the Schiffer's problem on the inverse obstacle problems.

The obstacle with the impedance boundary condition can be uniquely determined by the far-field pattern with infinite incident directions and one fixed wave number (cf. [40]). The Schiffer's problem for an impedance obstacle is mathematically challenging, which concerns with the unique determination for an impedance obstacle with a single far-field pattern. Recently, two of the authors establish the "local" unique identifiability results [23, 24] on the polygonal or polyhedral obstacle with the impedance boundary condition by at most two far-field patterns, where the polygonal or polyhedral obstacles satisfy some generic geometrical conditions. Furthermore, the local arguments for uniquely determining the shape and boundary parameters of the impedance obstacle by at most two far-field patterns are developed in [23, 24], which are quite different from the proofs in [2, 3, 25, 46] for tackling the uniqueness results for sound-soft and sound-hard obstacles with respect to a single far-filed pattern. A direct consequence of the uniqueness results in [23, 24] is that the shape of the convex hull of a concave polygonal or polyhedral obstacle can be uniquely determined by at most two far-field patterns. To our best knowledge, the findings in [23, 24] are the first results concerning the uniqueness results with finite many far-field measurements for general concave polygonal or polyhedral obstacles. The corresponding detailed discussions for Schiffer's problem with respect to the aforementioned results can be seen in Section 2.

For the inverse medium problem (6), Nachman [51], Novikov [52], and Ramm [53] established that the refractive index V in (2) could be uniquely determined from the far-field pattern with infinite incident directions and a fixed wave number. A uniqueness result in determining the refractive index V by using the Cauchy data for the inverse medium scattering problem (6) in the two-dimensional case was established by Bukhgeim [15]. Uniqueness and stability results using a single incident plane wave to determine the support of V under the assumption that the support is a convex polyhedron were presented in [10, 12, 35].

Consider the direct medium scattering problem (6) with the transmission condition

$$u^- = u^+, \quad \partial_\nu u^- = \partial_\nu u^+ \quad \text{on} \quad \partial\Omega, \quad \Omega = \operatorname{supp}(1 - V),$$

where u^- and u^+ are interior and exterior total wave fields with respect to Ω corresponding to (2) separately, and ν is the exterior normal vector to $\partial\Omega$. Blåsten and Liu [12] proved that a single far-field pattern of the direct medium scattering problem (6) with the transmission condition determines the refractive index V on the corners of its support under certain assumptions, where these assumptions can be satisfied for example in the low acoustic frequency regime. As a consequence, if the refractive index V is piecewise constant with either a polyhedral nest geometry or a known polyhedral cell geometry, such as a pixel or voxel array, the unique identifiability for V by a single far-field pattern was established, which is the first unique determination result of its type in the literature.

Recently, two of the authors [30] consider the unique determination of the shape of the reflective index's support Ω by a single far-field pattern corresponding to the direct medium scattering problem (6) with the conductive transmission condition

$$u^- = u^+, \quad \partial_\nu u^- = \partial_\nu u^+ + \eta u^+ \quad \text{on} \quad \partial\Omega,$$

where $\eta \in L^\infty(\partial\Omega)$. In the two-dimensional case, when Ω is a convex polygon, a single far-field pattern can uniquely determine the shape of Ω and the constant conductive parameter η under some generic assumptions. Please refer to Section 3 for more detailed discussions.

Very recently, the inverse problem of recovering a conductive medium body was considered in [22]. The conductive medium body arises from several applications of practical importance, including the modeling of an electromagnetic object coated with a thin layer of a highly conducting material and the magnetotellurics in geophysics. The inverse problem is concerned with the determination of the material parameters inside the body as well as on the conductive interface by the associated electromagnetic far-field measurements. Under the transverse-magnetic polarisation, two of the authors derived two novel unique identifiability results in determining a 2D piecewise conductive medium body associated with a polygonal-nest or a polygonal-cell geometry by a single active or passive far-field measurement. The detailed discussion can be found in Section 3.

Indeed, the unique identifiability [12,22,30] for the inverse medium problem (6) by a single far-field pattern relies heavily on the geometrical structures of the interior transmission eigenfunctions [21]. The study on the interior transmission eigenvalue problem has a long history and is of significant importance in scattering theory; see [27,37]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, and $V \in L^\infty(\Omega)$ be possibly complex-valued functions. The *interior transmission eigenvalue problem* can be formulated as

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in} \quad \Omega, \\ (\Delta + k^2(1+V))w = 0 & \text{in} \quad \Omega, \\ w = v, \quad \partial_\nu w = \partial_\nu v, & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

If there exits a nontrivial pair (v, w) and $k \in \mathbb{R}_+$ fulfill (8), then (v, w) is named as the transmission eigenfunctions and k is the corresponding interior transmission eigenvalue. One can refer [21] for a recent survey on the systematic discussions about the spectral properties of the transmission eigenvalues. In [11], Blåsten and Liu presented the first quantitive result on the intrinsic geometric properties of transmission eigenfunctions. We refer to Section 3 for more relevant discussions on recent progress on the study of transmission eigenfunctions and inverse medium scattering problems.

The study on scattering problems by periodic structures especially for diffraction grating has received a lot of attention (cf. [5]). The corresponding discussions are of practical applications in many areas such as radar imaging, micro-optics and nondestructive testing. The problem was first raised by Rayleigh on the scattering of plane waves from corrugated surface. In Section 4, we will give more explanations on the recent developments and the mathematical formulations towards the direct and inverse diffraction grating problems.

The rest of the paper is organized as follows. In Section 2, we discuss some recent progress on Schiffer's problem on inverse obstacle scattering problems including sound-soft, sound-hard and impedance obstacles within a certain polyhedral geometry. The study on the unique identifiability for impedance obstacles are important due to the novelty and challenge compared with the sound-soft and sound-hard cases. In Sections 3, we present some intriguing studies on the inverse medium scattering problem together with the geometrical structures of interior transmission eigenfunctions at the corner point. Section 4 is devoted to the mathematical developments of inverse diffraction grating problems. Similar to Section 2, we mainly present the unique determination for gratings with impedance boundary. In Section 5, we conclude our review paper on recent progress on inverse scattering problems and present some intriguing open problems.

2. **Inverse obstacle scattering problems.** In this section, we focus on the unique determination of the shape of the obstacle in (7) by finite many far-field patterns when the obstacle is a polygon or a polyhedron. As mentioned in the introduction, the aforementioned theory is related to Schiffer's problem in inverse scattering problem, which has a long and colorful history; see a recent survey paper [28] for related developments. Indeed, for sound-soft or sound-hard polyhedral scatterers, the corresponding unique identifiability results can be found in [2], [25], [44], [45] and [46], where the mathematical arguments adopt the reflection principle and path argument. In [47], the partial determination for impedance obstacle by finite many far-field patterns was also attained by Liu and Zou following the reflection principle argument. Recently, Cao, Diao, Liu and Zou in [23] and [24] developed a completely new approach on Schiffer's problem for impedance obstacles in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The methods therein can uniformly tackle Schiffer's problem for sound-soft, sound-hard and impedance obstacles and are completely local, which can be applicable to recover the obstacle and the surface impedance parameter simultaneously in a more general scenario. The technical arguments in [23] and [24] are heavily related to the geometrical structures of Laplacian eigenfunctions at the intersecting point of two line segments or plane cells when the underlying Laplacian eigenfunctions fulfill certain homogenous boundary conditions on the corresponding line segments or plane cells; see more discussions in Subsection 2.1.

In the following, we first review recent progress on the geometrical structures of Laplacian eigenfunctions (cf. [23,24]). Subsections 2.2 and 2.3 are devoted to review the unique inedibility results for sound-soft, sound-hard and impedance polyhedral scatterers.

2.1. **Recent progress on geometrical structures on Laplacian eigenfunctions.** The geometric structures of Laplacian eigenfunctions and their deep relationship to the quantitive behaviours of the underlying eigenfunctions in $\mathbb{R}^2$ was revealed in [23]. As an extension and continuation, in [24], two of the authors investigated the analytic behaviours of Laplacian eigenfunctions at corners in $\mathbb{R}^3$. These new findings and novel results are of significant importance in spectral theory and Schiffer's problem in the inverse obstacle scattering problems.

For $u \in L^2(\Omega)$ and $\lambda \in \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is an open subset, consider the Laplacian eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega. \tag{9}$$

Recall the following two definitions in [23].

Definition 2.1. [23, Definition 1.1] For a Laplacian eigenfunction u in (9), a line segment $\Gamma_h \subset \Omega$ is called a *nodal line* of u if $u = 0$ on Γ_h, where $h \in \mathbb{R}_+$ signifies the length of the line segment. For a given complex-valued function $\eta \in L^\infty(\Gamma_h)$, if it holds that

$$\partial_\nu u(\mathbf{x}) + \eta(\mathbf{x})u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_h, \tag{10}$$

then Γ_h is referred to as a *generalized singular line* of u. For the special case that $\eta \equiv 0$ in (10), a generalized singular line is also called a *singular line* of u in Ω. We use $\mathcal{N}_\Omega^\lambda$, $\mathcal{S}_\Omega^\lambda$ and $\mathcal{M}_\Omega^\lambda$ to denote the sets of nodal, singular and generalized singular lines, respectively, of an eigenfunction u in (9).

Definition 2.2. [23, Definition 1.2] Let u satisfy (9) and be a nontrivial eigenfunction. For a given point $\mathbf{x}_0 \in \Omega$, if there exits a number $N \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{r\to +0} \frac{1}{r^m} \int_{B(\mathbf{x}_0,r)} |u(\mathbf{x})| \,\mathrm{d}\mathbf{x} = 0 \quad \text{for} \quad m = 0, 1, \ldots, N+1, \tag{11}$$

where $B(\mathbf{x}_0, r)$ is a disk centered at $\mathbf{x}_0$ with radius $r \in \mathbb{R}_+$, we say that u vanishes at $\mathbf{x}_0$ up to the order N. The largest possible N such that (11) is fulfilled is called the vanishing order of u at $\mathbf{x}_0$, and we write

$$\mathrm{Vani}(u; \mathbf{x}_0) = N.$$

If (11) holds for any $N \in \mathbb{N}$, then we say that the vanishing order is infinity.

Combining with Definition 2.1 and Definition 2.2, under the mathematical setup in [23, Section 3], where we assume that

$$\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi, \; \alpha \in (0, 2), \quad \text{and} \quad \Gamma_h^+ \cap \Gamma_h^- = \mathbf{0} \in \Omega, \tag{12}$$

with $\Gamma_h^\pm$ being the intersecting two line segments, and $\angle(\Gamma_h^+, \Gamma_h^-)$ denoting the corresponding intersecting angle, we are provided with an accurate characterization of the relationship between the vanishing orders of Laplacian eigenfunctions and the intersecting angle of two nodal/generalized singular lines. Roughly speaking, it is known that the vanishing order is generically infinity if the intersecting angle is irrational and the vanishing order is finite if the intersecting angle is rational. The precise spectral results are as follows.

Theorem 2.3. *Let u be a Laplacian eigenfunction to (9). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (12) holds. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$, where C_1 and C_2 are two constants. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } u(\mathbf{0}) = 0 \text{ and } \alpha \neq \frac{q}{p}, \, p = 1, \ldots, n-1, \tag{13}$$

where $n \in \mathbb{N}$, $n \geq 3$ and for a fixed p, $q = 1, 2, \ldots, p-1$.

Theorem 2.4. *Let u be a Laplacian eigenfunction to (9). Suppose that there are two nodal lines Γ_h^+ and Γ_h^- from $\mathcal{N}_\Omega^\lambda$ such that (12) holds. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } \alpha \neq \frac{q}{p}, \, p = 1, \ldots, n-1, \tag{14}$$

where $n \in \mathbb{N}$, $n \geq 3$ and for a fixed p, $q = 1, 2, \ldots, p-1$.

Theorem 2.5. *Let u be a Laplacian eigenfunction to (9). Suppose that there are two singular lines Γ_h^+ and Γ_h^- from $\mathcal{S}_\Omega^\lambda$ such that (12) holds. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } u(\mathbf{0}) = 0 \text{ and } \alpha \neq \frac{q}{p}, \, p = 1, \ldots, n-1, \tag{15}$$

where $n \in \mathbb{N}$, $n \geq 3$ and for a fixed p, $q = 1, 2, \ldots, p-1$.

Next, we have vanishing orders of Laplacian eigenfunctions at the corner intersected by a generalized singular (singular) line and a nodal line.

Theorem 2.6. *Let u be a Laplacian eigenfunction to (9). Suppose that a generalized singular line $\Gamma_h^+ \in \mathcal{M}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^- \in \mathcal{N}_\Omega^\lambda$ at $\mathbf{0}$ with the angle*

$\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter $\eta_2 \equiv C_2$ on Γ_h^+ is a constant. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:

$$N \geq n, \quad \text{if } \alpha \neq \frac{2q+1}{2p},\ p = 1, \ldots, n-1, \tag{16}$$

where $n \in \mathbb{N}$, $n \geq 2$ and for a fixed p, $q = 0, 1, \ldots, p-1$.

Theorem 2.7. *Let u be a Laplacian eigenfunction to (9). Suppose that a singular line $\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^- \in \mathcal{N}_\Omega^\lambda$ at the origin with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } \alpha \neq \frac{2q+1}{2p},\ p = 1, \ldots, n-1, \tag{17}$$

where $n \in \mathbb{N}$, $n \geq 2$ and for a fixed p, $q = 0, 1, \ldots, p-1$.

Theorem 2.8. *Let u be a Laplacian eigenfunction to (9). Suppose that a singular line $\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda$ intersects with a generalized singular line $\Gamma_h^- \in \mathcal{M}_\Omega^\lambda$ at the origin with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter η_1 on Γ_h^- is a non-zero constant, i.e., $\eta_1 \equiv C_1 \neq 0$. Then the Laplacian eigenfunction u vanishes up to the order N at $\mathbf{0}$:*

$$N \geq n, \quad \text{if } u(\mathbf{0}) = 0 \text{ and } \alpha \neq \frac{q}{p},\ p = 1, \ldots, n-1, \tag{18}$$

where $n \in \mathbb{N}$, $n \geq 3$ and $q = 1, 2, \ldots, p-1$ for a fixed p.

Indeed, Theorem 2.8 is a direct corollary of Theorem 2.3 by taking $\eta_2 = 0$.

If α in (12) is irrational, which implies that the conditions (13), (14), (15), (16), (17) and (18) are automatically satified, we have the vanishing orders of Laplacian eigenfunctions for irrational intersection in the following few Theorems.

Theorem 2.9. *Let u be a Laplacian eigenfunction to (9). Suppose that there are two generalized singular lines Γ_h^+ and Γ_h^- from $\mathcal{M}_\Omega^\lambda$ such that (12) holds. Assume that $\eta_1 \equiv C_1$ and $\eta_2 \equiv C_2$, where C_1 and C_2 are two constants. If $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$ with $\alpha \in (0, 2)$ irrational, then there hold that*

$$\begin{aligned} &\mathrm{Vani}(u; \mathbf{0}) = 0, && \text{if } u(\mathbf{0}) \neq 0; \\ &\mathrm{Vani}(u; \mathbf{0}) = +\infty, && \text{if } u(\mathbf{0}) = 0. \end{aligned}$$

Theorem 2.10. *Let u be a Laplacian eigenfunction to (9). Suppose that a generalized singular line $\Gamma_h^- \in \mathcal{M}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^+ \in \mathcal{N}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter $\eta_1 \equiv C_1$ on Γ_h^- is a constant. If $\alpha \in (0, 2)$ is irrational, then there holds*

$$\mathrm{Vani}(u; \mathbf{0}) = +\infty.$$

Theorem 2.11. *Let u be a Laplacian eigenfunction to (9). Suppose that a singular line $\Gamma_h^+ \in \mathcal{S}_\Omega^\lambda$ intersects with a generalized singular line $\Gamma_h^- \in \mathcal{M}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. Assume that the boundary parameter $\eta_1 \equiv C_1$ on Γ_h^- is a constant. If $\alpha \in (0, 2)$ is irrational, then there hold that*

$$\begin{aligned} &\mathrm{Vani}(u; \mathbf{0}) = 0, && \text{if } u(\mathbf{0}) \neq 0; \\ &\mathrm{Vani}(u; \mathbf{0}) = +\infty, && \text{if } u(\mathbf{0}) = 0. \end{aligned}$$

Theorem 2.12. *Let u be a Laplacian eigenfunction to (9). Suppose that a singular line $\Gamma_h^- \in \mathcal{S}_\Omega^\lambda$ intersects with a nodal line $\Gamma_h^+ \in \mathcal{N}_\Omega^\lambda$ at $\mathbf{0}$ with the angle $\angle(\Gamma_h^+, \Gamma_h^-) = \alpha \cdot \pi$. If $\alpha \in (0, 2)$ is irrational, then there holds*

$$\mathrm{Vani}(u; \mathbf{0}) = +\infty.$$

We can know from above theorems that the eigenfunction is generically vanishing to infinity, namely u is identically zero in Ω for irrational intersection. Here, the generic condition $u(\mathbf{0}) = 0$ can be easily fulfilled in the inverse obstacle scattering problem by the superposition of two incident waves and one can refer [23, Section 7] for more detailed discussions.

In the subsequent studies, we only present some latest studies concerning the unique identifiability for sound-soft, sound hard and impedance polyhedral scatterers, respectively.

2.2. **Unique identifiability for sound-soft or sound-hard polyhedral scatterers.** In this subsection, we review the existing results on the unique identifiability for sound-soft or sound-hard polyhedral scatterers by finite many far-field patterns. In 1994, Liu and Nachman [43] investigated the unique determination results for the convex hull of a polyhedral obstacle by knowledge of the far-field pattern with the help of the reflection principle for solutions of the Helmholtz equation across a flat boundary. Later, Cheng and Yamamoto [25] proved that a polygonal obstacle in $\mathbb{R}^2$ could be uniquely determined by the far-field pattern under a certain geometrical condition, which was expressed by the absence of trapped rays in the exterior domain of the obstacle. Then, Alessandrini and Rondi [2] in 2005 studied the uniqueness of a sound-soft polyhedral scatterer by the far-field pattern corresponding to an incident wave at one given wavenumber and a given incident direction in $\mathbb{R}^n$, $n \geq 2$. For further illustration, we consider the obstacle scattering problem (3) with $\mathcal{B}u = u$ for time-harmonic acoustic waves with a sound-soft obstacle $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then the determination of Ω by a single far-field measurement can be stated as

Theorem 2.13. *[2, Theorem 2.2] Let us fix $\mathbf{d} \in \mathbb{S}^{n-1}$ and $k > 0$. A polyhedral scatterer Ω is uniquely determined by the far-field pattern u^∞.*

Theorem 2.13 can be proved by following a reflection argument discussed in [25]. Instead of examining the boundary behavior of the nodal set of u, Alessandrini and Rondi investigated the structure of the nodal set in the exterior of Ω. The detailed proof of Theorem 2.13 can be found in [2, Section 3].

By introducing the concept of Dirichlet/Neumann set and Dirichlet/Neumann hyperplane, Liu and Zou [46] in 2006 initiated a nowadays well-known path argument to establish the uniqueness for both the sound-soft and sound-hard cases. Following the same definition of a polyhedral scatterer in [2], Liu and Zou derived the unique identifiability result as follows.

Theorem 2.14. *[46, Theorem 2] The polyhedral scatterer Ω is uniquely determined by a single far-field pattern u^∞ corresponding to an incident wave $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ with $k > 0$ and $\mathbf{d} \in \mathbb{S}^{n-1}$ fixed.*

Very recently, Cao, Diao, Liu and Zou [23] established a completely novel approach in dealing with the Schiffer's problem for sound-soft, sound-hard and impedance obstacles. The uniqueness results can be regarded as direct applications of the new spectral findings by utilizing the critical connection between the intersecing angles of the nodal/generalized singular lines and the vanishing order of the Laplacian eigenfunctions in $\mathbb{R}^2$. The relevant study for the three-dimensional case is also concerned in [24]. The unique identifiabilities in [23] and [24] develop a new argument that can treat the unique identifiability issue for several inverse scattering problems in a unified manner, especially in terms of general materials properties. Indeed, most existing uniqueness results concerning Schiffer's problem require certain geometrical a-prior knowledge, which is also need in [23] and [24]. Please refer to Subsection 2.3 for more detailed discussions.

The existing studies on the unique identifiability for sound-hard obstacles with finitely many measurements depend heavily on the geometric setup of the scatterer. Cheng and Yamamoto in [25] presented the unique determination results for sound-hard polygons in $\mathbb{R}^2$ by two incident plane waves under a so called "non-trapping" condition. Later, Elschner and Yamamoto in 2005 demonstrated that a sound-hard polygonal obstacle could be uniquely determined by one single incident plane wave. However, the argument in [33] is only valid in $\mathbb{R}^2$. In 2006, Liu and Zou investigated the uniqueness for a very general sound-hard obstacle and the approaches they established work for any $\mathbb{R}^n$, $n \geq 1$. We can see the following few theorems for the mathematical illustrations in the aforementioned literatures.

Considering the scattering problem (3) with $\mathcal{B}u = \partial_\nu u$, the paper by Liu and Zou [46] significantly generalized the aforementioned results in two aspects. First the restriction on the geometry of the scatterer is relaxed to a very general sound-hard obstacle. Second, the uniqueness results in [46] are appliable to any dimension larger than 1. However, finitely many incident plane waves are still required therein. The main approach in the corresponding argument is reflection principle. We present the main theorem as follows without proof. One can refer [46, Section 2] for the rigorous analyses.

Theorem 2.15. *[46, Theorem 1] Let $\mathbf{d}_\ell \in \mathbb{S}^{n-1}$, $\ell = 1, 2, \cdots, n$, be n linearly independent direcions and $k > 0$ be fixed. A polyhedral scatterer Ω is uniquely determined by the far-field patterns $\mathcal{U}^\infty = \{u^{1,\infty}, u^{2,\infty}, \cdots, u^{n,\infty}\}$.*

We would like to remark that the scatterer Ω defined in Theorem 2.15 is supposed to fulfill the requirements introduced in [2].

Later, Liu, Petrini, Rondi and Xiao [44] also established an optimal stability estimate for the determination of sound-hard polyhedral scatterers in $\mathbb{R}^n$, $n \geq 2$, with the help of a minimal number of far-field measurements. They gave more discussions of several admissible scatterers with minimal regularity assumptions, which can be utilized on many occasions. The main idea follows similar to [54].

Very recently, as we mentioned earlier in Subsection 2.2, the unique determination for sound-hard obstacles in a certain polygonal setup without any further technical restrictions was derived in [23] for the two-dimiensional case and [24] for the three-dimensional case, respectively. Compared with the existing literatures, the results in [23] and [24] hold for the scatterers of more general material properties and can be obtained by at most two far-field patterns. To avoid repetation, we also refer to Subsection 2.3 for a brief introduction on the novel findings of Laplacian eigenfunctions as well as the Schiffer's problem for sound-hard obstacles, which are presented in a unified representation on the boundary condition in (21).

2.3. **Unique identifiability for impedance polyhedral scatterers.** As discussed in Subsection 2.2, there is a widespread consensus that the unique identifiability for a sound-soft obstacle can be derived by a single incident wave and for a sound-hard obstacle, the unique determination can be obtained by a single incident plane wave with some fixed $k \in \mathbb{R}_+$ and $\mathbf{d} \in \mathbb{S}^{n-1}$. However, the study on the Schiffer's problem for impedance obstacles still remains to be challenging since the reflection principle for Dirichlet and Neumann hyperplanes does not work for impedance hyperplanes. But the study for impedance obstacle is of more practical applications compared with sound-soft and sound-hard cases. In [17], [18], [19] and [20], reconstruction by linear sampling method was conducted for numerical studies. On the theoritical side, in 2007, Liu and Zou [47] addressed the unique determination for partially coated polyhedral scatterers in $\mathbb{R}^n$, $n \geq 2$. Precisely speaking, two kinds of partially coated structures are considered. One is the scatterer of mixed sound-soft and impedance type. It suffices to derive the uniqueness results by one single incident wave. The other one is the scatterer of mixed sound-soft, sound-hard and impedance type. For this case, n independent incident waves are needed in the recovery in $\mathbb{R}^n$ with $n \geq 3$ and one incident wave is sufficient to uniquely determine the scatterer in $\mathbb{R}^2$. Here, considering the scattering problem (3), we only present the main unique identifiability results for partially coated obstacles according to the cases stated above. One can refer [47] for more details.

For the scatterer Ω, assume that

$$\partial\Omega = \partial\Omega_D \cup U \cup \partial\Omega_I,$$

where $\partial\Omega_D \subset \partial\Omega$ and $\partial\Omega_I \subset \partial\Omega$ are two disjoint relatively open subsets having U as their common boundary on $\partial\Omega$. The total wave field in (3) fulfills

$$u = 0 \quad \text{on} \quad \partial\Omega_D, \quad \partial_\nu u + \mathrm{i}\lambda u = 0 \quad \text{on} \quad \partial\Omega_I, \tag{19}$$

where $\lambda(\mathbf{x}) \in C(\overline{\partial\Omega_I})$ satisfing $\lambda(\mathbf{x}) \geq \lambda_0 > 0$. Then we can know that for the scatterer of mixed sound-soft and impedance type, there holds

Theorem 2.16. *[47, Theorem 2.1] For any fixed $k_0 > 0$ and $\mathbf{d}_0 \in \mathbb{S}^{n-1}$, let Ω and $\widetilde{\Omega}$ be two polyhedral scatterers on which the condition (19) is enforced with respective surface impedance $\lambda(\mathbf{x})$ and $\widetilde{\lambda}(\mathbf{x})$. Then we have $\Omega = \widetilde{\Omega}$ and $\lambda = \widetilde{\lambda}$, as long as $u^\infty(\hat{\mathbf{x}}; \Omega, \lambda, k_0, \mathbf{d}_0) = u^\infty(\hat{\mathbf{x}}; \widetilde{\Omega}, \widetilde{\lambda}, k_0, \mathbf{d}_0)$ for $\hat{\mathbf{x}} \in \mathbb{S}^{n-1}$.*

And for the scatterer of mixed sound-soft, sound-hard and impedance type, the following boundary conditions are satisfied

$$u = 0 \text{ on } \quad \partial\Omega_I, \quad \partial_\nu u = 0 \text{ on } \quad \partial\Omega_N, \quad \partial_\nu u + \mathrm{i}\lambda u = 0 \text{ on } \quad \partial\Omega_I. \tag{20}$$

In (20), $\partial\Omega_D \subset \partial\Omega$, $\partial\Omega_N \subset \partial\Omega$ and $\partial\Omega_I \subset \partial\Omega$ are three disjoint relatively open subsets. Then the unique determination can be derived as follows.

Theorem 2.17. *[47, Theorem 2.2] For any fixed $k_0 > 0$ and n linearly independent directions $\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_n \in \mathbb{S}^{n-1}$, let $u_1(\mathbf{x}), \cdots, u_n(\mathbf{x})$ be the total fields of the scattering problem associated with the boundary condition (20) corresponding to the incident waves $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_1}, \cdots, e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_n}$, respectively. Let Ω and $\widetilde{\Omega}$ be two polyhedral scatterers on which (20) is enforced with respective surface impedance $\lambda(\mathbf{x})$ and $\widetilde{\lambda}(\mathbf{x})$. Then we have $\Omega = \widetilde{\Omega}$ and $\lambda = \widetilde{\lambda}$, as long as $u^\infty(\hat{\mathbf{x}}; \Omega, \lambda, k_0, \mathbf{d}_j) = u^\infty(\hat{\mathbf{x}}; \widetilde{\Omega}, \widetilde{\lambda}, k_0, \mathbf{d}_j)$ for $j = 1, \cdots, n$ and $\hat{\mathbf{x}} \in \mathbb{S}^{n-1}$.*

In particular, if $\partial\Omega_I = \emptyset$, which implies that the scatterer is of mixed sound-soft and sound-hard type, then the following theorem holds concerning more general results in $\mathbb{R}^2$ allowing all possible physical conditions.

Theorem 2.18. *[47, Theorem 2.4] For any fixed $\mathbf{d}_0 \in \mathbb{S}^1$ and $k_0 > 0$, let Ω and $\widetilde{\Omega}$ be two polygonal scatterers in $\mathbb{R}^2$ with unknown physical properties. Then we have $\Omega = \widetilde{\Omega}$, and both scatterers have the same physical properties, as long as $u^\infty(\hat{\mathbf{x}}; \Omega, k_0, \mathbf{d}_0) = u^\infty(\hat{\mathbf{x}}; \widetilde{\Omega}, k_0, \mathbf{d}_0)$ for $\hat{\mathbf{x}} \in \mathbb{S}^1$.*

As a direct application of the geometric structures of Laplacian eigenfunctions in Subsection 2.1, we have the unique identifiability results for Schiffer's problem for a certain type of admissible complex polygonal obstacle by at most two far-field patterns. Furthermore, the constant impedance boundary parameter η can be recovered simultaneously for impedance obstacle.

Definition 2.19. [23, Definition 8.1] Suppose that $\Omega \subset \mathbb{R}^2$ is an obstacle associated with the following boundary condition

$$\mathcal{B}u = \partial_\nu u + \eta u = 0 \quad \text{on} \quad \partial\Omega. \tag{21}$$

Ω is said to be an admissible polygonal obstacle if it is an open polygon and on each edge of Ω, η is a constant (possibly zero) or ∞. If all the angles of Ω are irrational, Ω is called an irrational obstacle. Otherwise, if there exists a corner angle of Ω being rational, then it is called a rational obstacle.

Definition 2.20. Ω is said to be an admissible complex polygonal obstacle if it fulfills for $N \in \mathbb{N}$

$$(\Omega, \eta) = \bigcup_{\ell=1}^{N} (\Omega_\ell, \eta_\ell), \quad \text{with} \quad \eta = \bigcup_{\ell=1}^{N} \eta_\ell \chi_{\partial\Omega_\ell \cap \partial\Omega}, \tag{22}$$

where each Ω_ℓ is an admissible polygonal obstacle. Moreover, if all Ω_ℓ are irrational, $\ell = 1, \cdots, N$, then Ω is said to be irrational, otherwise, it is said to be rational of degree p, where p is the smallest degree among all the degrees of rational components.

Now the unique determination results on Schiffer's problem with respect to an admissible complex irrational polygonal obstacle or an admissible complex rational polygonal obstacle of degree $p \geq 3$ by at most two far-field patterns can be stated as the following two theorems, respectively.

Theorem 2.21. *Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_j$, $j = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Let $\mathbf{G}$ denote the unbounded connected component of $\mathbb{R}^2 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. Let u^∞ and $\widetilde{u}^\infty$ be, respectively, the far-field patterns associated with (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$. If*

$$u^\infty(\hat{\mathbf{x}}, \mathbf{d}_j) = \widetilde{u}^\infty(\hat{\mathbf{x}}, \mathbf{d}_j), \quad \hat{\mathbf{x}} \in \mathbb{S}^1, j = 1, 2, \tag{23}$$

then one has that

$$\left(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}\right) \cup \left(\partial\widetilde{\Omega} \backslash \partial\overline{\Omega}\right)$$

cannot have a corner on $\partial\mathbf{G}$.

The precise proof of this theorem can be found in [23, Theorem 8.3]. The similar unique identifiability also holds for the convex hull of an admissible complex irrational obstacle and the impedance boundary parameter η can be recovered at the same time. Indeed, we have the following corollary.

Corollary 1. *Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_j$, $j = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Let $\mathbf{G}$ denote the unbounded connected component of $\mathbb{R}^2 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. Let u^∞ and $\widetilde{u}^\infty$ be, respectively, the far-field patterns associated with (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$. If*

$$u^\infty(\hat{\mathbf{x}}, \mathbf{d}_j) = \widetilde{u}^\infty(\hat{\mathbf{x}}, \mathbf{d}_j), \quad \hat{\mathbf{x}} \in \mathbb{S}^1, j = 1, 2, \tag{24}$$

then one has that

$$\mathcal{CH}(\Omega) = \mathcal{CH}(\widetilde{\Omega}) := \Sigma, \tag{25}$$

and

$$\eta = \widetilde{\eta} \quad on \quad \partial\Omega \cap \partial\widetilde{\Omega} \cap \partial\Sigma. \tag{26}$$

The uniqueness of the impedance boundary parameter η can be proved by contradiction with the help of Holmgren uniqueness property.

For an admissible complex rational obstacle of degree $p \geq 3$, we have

Theorem 2.22. *Let (Ω, η) be an admissible complex rational obstacle of degree $p \geq 3$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_j$, $j = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$. Set $u_j(\mathbf{x}) = u(\mathbf{x}; k, \mathbf{d}_j)$ to be the total wave fields associated with (Ω, η) and $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_j}$, $j = 1, 2$, respectively. Recall that $\mathbf{G}$ denotes the unbounded connected component of $\mathbb{R}^2 \backslash \overline{(\Omega \cup \widetilde{\Omega})}$. If the following condition is fulfilled,*

$$\mathcal{L}\left(u_2 \cdot \nabla u_1 - u_1 \cdot \nabla u_2\right)(\mathbf{x}_c) \neq 0, \tag{27}$$

where $\mathbf{x}_c$ is any vertex of Ω, then then one has that

$$\left(\partial\Omega \backslash \partial\overline{\widetilde{\Omega}}\right) \cup \left(\partial\widetilde{\Omega} \backslash \partial\overline{\Omega}\right)$$

cannot have a corner on $\partial\mathbf{G}$.

Theorem 2.22 is based on the fact that the rational degree of an admissible complex rational polygonal obstacle is at least 2, which is a direct conclusion of theorem 2.3 to theorem 2.12.

Remark 1. In (32), for a function $f \in L^2_{loc}(\mathbb{R}^2 \backslash \overline{\Omega})$, $\mathcal{L}(f)$ is defined as

$$\mathcal{L}(f)(\mathbf{x}_c) := \lim_{r \to +0} \frac{1}{|\Omega_r(\mathbf{x}_c)|} \int_{\Omega_r(\mathbf{x}_c)} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \tag{28}$$

if the limit exists, where $\Omega_r(\mathbf{x}_c) = B(\mathbf{x}_c, r) \cap \mathbb{R}^2 \backslash \overline{\Omega}$, $\quad r \in \mathbb{R}_+$, with $B(\mathbf{x}_c, r)$ being introduced in (11). Indeed, from a practical point of view, this condition (32) depends on the a-priori knowledge of the underlying obstacle as well as the choice of the incident waves. One can refer [23, Page 36] for more specific explanations.

Similar to Corollary 1, the unique determination of the convex hull of an admissible complex rational obstacle can also be derived, which is omitted.

The new approach developed in the proofs of Theorem 2.21 and Theorem 2.22 can uniformly tackle the unique determination for sound-soft, sound-hard and impedance obstacles by at most two far-field measurements and is completely local, which enables us to determine an impedance obstacle as well as its surface impedance by at most two far-field patterns.

It is the first time in the literature to present a systematic study of the intriguing connections between the vanishing orders of Laplacian eigenfunctions and the intersecting angles of their nodal/generalized singular lines. The unique identifiability for the impedance or generalized impedance cases in Theorem 2.21 and 2.22 has been an open problem for a long time. Therefore, these results should be truly original and of significant interest in the spectral theory of Laplacian eigenfunctions and also Schiffer's problem for inverse obstacle scattering problems.

As an extension of [23], two of the authors [24] further investigated in $\mathbb{R}^3$ the analytic behaviours of Laplacian eigenfunctions at places where nodal or generalized singular planes intersect. The relevant geometric setup is more complicated, where an edge corner intersected by two planes and a vertex corner intersected by at least three planes are considered respectively. Precisely speaking, the vanishing order of Laplacian eigenfunctions at an edge corner is related to the rationality of the intersecting angle similar to the two-dimensional case and the vertex corner case is related to the intersecting angle as well as the intrinsic properties of the Legendre polynomails. An important and direct application of these vanishing results is the unique identifiability for determining the scatterer and the surface impedance parameter (for impedance obstacle) by at most two far-field measurements. To avoid repetation, we skip the detailed analyses for the vanishing order of Laplacian eigenfunctions in [24, Section 2 and 3] at edge corners and vertex corners, respectively. But we present the unique determination results for sound-soft, sound-hard and impedance obstacles in a uniform way in the following theorem. In particular, for impedance obstacles, the corresponding surface parameter can be recovered simultaneously.

The definition for an admissible polyhedral obstacle is quite different from that of two-dimensional case due to the more complicated geometric formulation. We refer to [24, Definition 6.1] for the rigorous statements.

Similar to the two-dimensional case, we separately have the unique determination for an admissible complex irraitonal polyhedral obstacle and an admissible complex rational polyhedral obstacle of degree $p \geq 3$ in the following two therorems.

Theorem 2.23. *Consider a fixed $k \in \mathbb{R}_+$, and two distinct incident directions $\mathbf{d}_1$ and $\mathbf{d}_2$ from $\mathbb{S}^2$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex irrational obstacles, with u^∞ and $\widetilde{u}^\infty$ being their corresponding far-field patterns and $\mathbf{G}$ being the unbounded connected component of $\mathbb{R}^3\backslash\overline{(\Omega \cup \widetilde{\Omega})}$. If u^∞ and $\widetilde{u}^\infty$ are the same in the sense that*

$$u^\infty(\hat{\mathbf{x}}; k, \mathbf{d}_j) = \widetilde{u}^\infty(\hat{\mathbf{x}}; k, \mathbf{d}_j), \quad \text{for} \;\; j = 1, 2 \text{ and all } \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{29}$$

then $(\partial\Omega\backslash\partial\overline{\widetilde{\Omega}}) \bigcup (\partial\widetilde{\Omega}\backslash\partial\overline{\Omega})$ cannot possess a vertex corner on $\partial\mathbf{G}$. Moreover,

$$\eta = \widetilde{\eta} \quad \text{on} \quad \partial\Omega \cap \partial\widetilde{\Omega}. \tag{30}$$

Theorem 2.24. *Consider a fixed $k \in \mathbb{R}_+$, and two distinct incident directions $\mathbf{d}_1$ and $\mathbf{d}_2$ from $\mathbb{S}^2$. Let (Ω, η) and $(\widetilde{\Omega}, \widetilde{\eta})$ be two admissible complex rational obstacles of degree $p \geq 3$, with $u_j(\mathbf{x}) := u(\mathbf{x}; k, \mathbf{d}_j)$ and $\widetilde{u}_j := \widetilde{u}(\mathbf{x}; k, \mathbf{d}_j)$ being their corresponding total wave fields associated with the incident field $e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}_j}$, and $u^\infty(\hat{\mathbf{x}}; k, \mathbf{d}_j)$ and $\widetilde{u}^\infty(\hat{\mathbf{x}};, k, \mathbf{d}_j)$ being their corresponding far-field patterns for $j = 1, 2$. We further write $\mathbf{G}$ for the unbounded connected component of $\mathbb{R}^3\backslash\overline{(\Omega \cup \widetilde{\Omega})}$. Then the set $(\partial\Omega\backslash\partial\overline{\widetilde{\Omega}}) \cup (\partial\widetilde{\Omega}\backslash\partial\overline{\Omega})$ can not possess a vertex corner on $\partial\mathbf{G}$, if the following conditions are satisfied:*

$$u_j^\infty(\hat{\mathbf{x}}; k, \mathbf{d}_j) = \widetilde{u}_j^\infty(\hat{\mathbf{x}}; k, \mathbf{d}_j), \quad \hat{\mathbf{x}} \in \mathbb{S}^2, j = 1, 2, \tag{31}$$

$$\mathcal{L}\left(u_2 \cdot \nabla u_1 - u_1 \cdot \nabla u_2\right)(\mathbf{x}_c) \neq 0 \quad \text{and} \quad \mathcal{L}\left(\widetilde{u}_2 \cdot \nabla \widetilde{u}_1 - \widetilde{u}_1 \cdot \nabla \widetilde{u}_2\right)(\mathbf{x}_c) \neq 0 \tag{32}$$

for all vertices $\mathbf{x}_c$ of Ω, where $\mathcal{L}$ is defined in (28).

The main proofs of Theorem 2.23 and Theorem 2.24 are similar to the corresponding two-dimensional case, one can also refer [24, Section 6] for detailed analyses.

We would like to emphasize that the unique identifiability results as well as the corresponding argument in both $\mathbb{R}^2$ and $\mathbb{R}^3$ are "localized" around the corner by utilizing the spectral results for Laplacian eigenfunctions. This indeed provides a completely new and effective approach in studying inverse scattering problems.

3. **Inverse medium scattering problems.** As mentioned in the introduction, there are quite a lot of studies concerning the unique identifiability on the inverse medium scattering problems with the penetrable scatterers in the inhomogeneous medium, which shall be focused on the case that the support of the medium parameter is of polygonal or polyhedral geometry in this section. Before that, we first review some recent progress on geometrical structures of transmission eigenfunctions to (35), which have important applications in unique determinations by a single far-field pattern in the inverse medium problems.

3.1. **Recent progress on geometrical structures of transmission eigenfunctions.** The interior transmission eigenvalue problem was first introduced by A. Kirsch [37] in 1986. The theoretical studies on interior transmission eigenvalue problems are of significant interest in the inverse medium scattering problems. An important application is to the invisibility phenomenon. The first quantitive result on the intrinsic properties of transmission eigenfunctions was studied by Blåsten and Liu in [11]. They rigorously investigated the vanishing properties of the interior transmission eigenfunctions at a corner whose angle is less than π by establishing a quantitive lower bound associated with incident Herglotz waves. Compared with the study for eigenvalues, the analyses concerning eigenfunctions are more difficult and still remain to be a fascinating topic. Later, in [9], it was further numerically investigated that the transmission eigenfunctions vanish near a corner whose angle is less than π, whereas the transmission eigenfunctions would localize at a corner whose angle is larger than π. Under the H^2-regularity on the transmission eigenfunctions, Blåsten [8] proved that the the transmission eigenfunctions must vanish at the corner point if the corner is not degenerated.

The mathematical argument in [11] is indirect which connects the vanishing property of the interior transmission eigenfunctions with the stability of a certain wave scattering problem with respect to variation of the wave field at the corner point. The main results in [11] can be summarized by

Theorem 3.1. *[11, Theorem 3.2] Let $n \in \{2,3\}$ and $V = \phi\chi_\Omega$, where $\phi(\mathbf{x})$ is a Hölder-continuous function in $\mathbb{R}^n$ and Ω is a convex polygon in $\mathbb{R}^2$ or a cuboid in $\mathbb{R}^3$. Assume that $k > 0$ is a transmission eigenvalue and the corresponding transmission eigenfunctions $v, w \in L^2(\Omega)$ satisfying (8). If v can be approximated in the L^2-norm by a sequence of Herglotz waves v_j defined by*

$$v_j(\mathbf{x}) = \int_{\mathbb{S}^{n-1}} e^{\mathrm{i}k\xi\cdot\mathbf{x}} g_j(\xi)\mathrm{d}\sigma(\xi), \quad \xi \in \mathbb{S}^{n-1}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{33}$$

satisfying one of the following two assumptions

(a) the kernel $g_j \in L^2(\mathbb{S}^{n-1})$ is uniformly bounded,

(b) the Herglotz waves v_j and its kernel g_j fulfill that

$$\|v - v_j\|_{L^2(\Omega)} \le e^{-j}, \quad \|g_j\|_{L^2(\mathbb{S}^{n-1})} \le C(\ln j)^\beta, \tag{34}$$

where the constants $C > 0$ and $0 < \beta < 1/(2n+8)$, $(n = 2, 3)$, *then*

$$\lim_{\rho\to 0}\frac{1}{m(B(\mathbf{x}_c,r)\cap\Omega)}\int_{B(\mathbf{x}_c,r)\cap\Omega}|v(\mathbf{x})|\,\mathrm{d}\mathbf{x}=0,$$

where $m(B(\mathbf{x}_c,r)\cap\Omega)$ *is the measure of* $B(\mathbf{x}_c,r)\cap\Omega$*,* $\mathbf{x}_c$ *is any vertex of* Ω *such that* $\phi(\mathbf{x}_c)\neq 0$*.*

Blåsten [8] utilized an energy identity from the enclosure method and constructed a new type of planar complex geometrical optics solution whose logarithm is a branch of the square root to reveal that the transmission eigenfunction v to (35) must vanish at the corner point in $\mathbb{R}^2$ or an edge corner point $\mathbf{x}_c$ in $\mathbb{R}^3$ if v is Hölder-continuous at $\mathbf{x}_c$ and $V(\mathbf{x}_c)\neq 0$. Indeed, it is stated that

Theorem 3.2. *[8, Theorem 4.2] Let* $n\in\{2,3\}$ *and* Ω *be bounded domain in* $\mathbb{R}^n$*. Let* $V\in L^\infty(\Omega)$*. Assume that* $k>0$ *is a transmission eigenvalue and the corresponding transmission eigenfunctions* $v,w\in L^2(\Omega)$ *satisfying* (8)*. Let* $\mathbf{x}_c$ *be any vertex or edge point of* Ω *such that* V *is* C^α *smooth near* $\mathbf{x}_c$*. If* v *or* w *is* H^2*-smooth in a neighborhood of* $\mathbf{x}_c$ *in* Ω*, then* $v(\mathbf{x}_c)=w(\mathbf{x}_c)=0$ *if* $V(\mathbf{x}_c)\neq 0$*.*

In [13], Blåsten and Liu further extended their results on geometric structures of transmission eigenfunctions at corners intersected by line segments to the corners with curvature. Roughly speaking, they established a relationship among the value of transmission eigenfunctions, the diameter of the domain and the underlying refractive index, which yields that the interior transmission eigenfunctions must be nearly vanishing at a high-curvature point on the boundary. These new findings significantly relaxed the dependence on the geometry of the scatterer (smallness assumption) but focus on local structures, which are more practicle and interesting. The main theoretical results on the vanishing properties of transmission eigenfunctions at high-curvature point can be seen in [13, Section 3] and one can also refer to [13, Section 4] for the uniqueness results for the inverse scattering problem associated with the high curvature geometry of the underlying obstacle.

Consider the following interior transmission eigenvalue problem with a conductive boundary condition for v, $w\in H^1(\Omega)$,

$$\begin{cases}\Delta w+k^2(1+V)w=0 & \text{in }\Omega,\\ \Delta v+k^2v=0 & \text{in }\Omega,\\ w=v,\ \partial_\nu v+\eta v=\partial_\nu w & \text{on }\partial\Omega,\end{cases}\tag{35}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n=2,3$, $V\in L^\infty(\Omega)$ and $\eta\in L^\infty(\partial\Omega)$, $\nu\in\mathbb{S}^{n-1}$ signifies the exterior unit normal vector to $\partial\Omega$. If for a certain $k\in\mathbb{R}_+$, there exists a pair of nontrivial solutions $(v,w)\in H^1(\Omega)\times H^1(\Omega)$ to (35), then k is called a conductive transmission eigenvalue and (v,w) is referred to as the corresponding pair of conductive transmission eigenfunctions. Especially, when $\eta\equiv 0$, (35) degenerates to be (8).

The geometric properties studied in [11] are significantly generalized in a recent paper [30] concerning the geometric structures of conductive transmission eigenfunctions to (35). Roughly speaking, the results are extended in the following three aspects. First, the conductive transmission eigenfunctions include the interior transmission eigenfunctions as a special case. The geometric structures established for the conductive transmission eigenfunctions in [30] include the results in [11] as a special case. Second, the vanishing property of the conductive transmission eigenfunctions is established for any corner as long as the corner singularity is not degenerate. Third, the regularity requirements on the interior transmission eigenfunctions in [11] are significantly relaxed for the conductive transmission eigenfunctions. Furthermore, geometrical structures have practical and important applications in the inverse medium problems. In the following, we review the intriguing discoveries regarding the geometric properties of conductive transmission eigenfunctions in $\mathbb{R}^2$ as well as a unique identification for the polygonal conductive obstacle by a single far-field measurement. The argument for the three-dimensional case follows a similar approach to the two-dimensional case with the help of the dimension reduction operator, hence we skip the details. However, we would like to point out that the corresponding analyses in $\mathbb{R}^3$ are much more tedious and one can refer to [30, Section 3] for more discussions.

In order to present main results on the geometric properties of the conductive transmission eigenfunctions, which shall play a critical role in the unique identifiability for the conductive scatterer, we first introduce the following notations. Denote $B_h(\mathbf{x})$ by an open ball of radius $h \in \mathbb{R}_+$ and centered at $\mathbf{x}$ and $W_{\mathbf{x}_c}(\theta_W)$ is an open sector in $\mathbb{R}^2$ with the vertex $\mathbf{x}_c$ and the open angle θ_W.

Then the main theorems in [30] concerning the vanishing properties of conductive transmission eigenfunctions can be summarized as follows.

Theorem 3.3. *[30, Theorem 2.1] Let $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ be a pair of eigenfunctions to (35) associated with $k \in \mathbb{R}_+$. Assume that the Lipschitz domain $\Omega \subset \mathbb{R}^2$ contains a corner $\Omega \cap B_R(\mathbf{x}_c) = \Omega \cap W_{\mathbf{x}_c}(\theta_W)$, where $\mathbf{x}_c$ is the vertex of $\Omega \cap W_{\mathbf{x}_c}(\theta_W)$ and $R > 0$. Moreover, there exits a sufficiently small neighbourhood $S_h(\mathbf{x}_c) = \Omega \cap B_h(\mathbf{x}_c) = \Omega \cap W_{\mathbf{x}_c}(\theta_W)$ (i.e. $h > 0$ is sufficiently small) of $\mathbf{x}_c$ in Ω, where*

$$\begin{aligned} \Gamma_h^{\pm}(\mathbf{x}_c) &:= \partial W_{\mathbf{x}_c}(\theta_W) \cap B_h(\mathbf{x}_c), \quad \Sigma_{\Lambda_h}(\mathbf{x}_c) := S_h(\mathbf{x}_c) \backslash S_{h/2}(\mathbf{x}_c), \\ S_{h/2}(\mathbf{x}_c) &:= \Omega \cap B_{h/2}(\mathbf{x}_c) = \Omega \cap W_{\mathbf{x}_c}(\theta_W), \end{aligned} \tag{36}$$

such that $qw \in C^\alpha(\overline{S_h(\mathbf{x}_c)})$ with $q := 1 + V$ and $\eta \in C^\alpha\left(\overline{\Gamma_h^{\pm}(\mathbf{x}_c)}\right)$ for $0 < \alpha < 1$, and $v - w \in H^2(\Sigma_{\Lambda_h}(\mathbf{x}_c))$. If the following conditions are fulfilled:

(a) the transmission eigenfunction v can be approximated in $H^1(S_h)$ by the Herglotz functions v_j, $j = 1, 2, \ldots$, with kernels g_j satisfying

$$\|v - v_j\|_{H^1(S_h)} \le j^{-1-\Upsilon}, \quad \|g_j\|_{L^2(\mathbb{S}^1)} \le C j^{\varrho}, \tag{37}$$

for some constants $C > 0$, $\Upsilon > 0$ and $0 < \varrho < 1$, where v_j is defined in (33);

(b) the function $\eta(\mathbf{x})$ doest not vanish at the corner, i.e.,

$$\eta(\mathbf{x}_c) \neq 0, \tag{38}$$

(c) the open angle of the open sector $W_{\mathbf{x}_c}(\theta_W)$ satisfy

$$\theta_W \neq \pi, \tag{39}$$

then one has

$$\lim_{\rho \to +0} \frac{1}{m(B_\rho(\mathbf{x}_c) \cap \Omega)} \int_{B_\rho(\mathbf{x}_c) \cap \Omega} |v(\mathbf{x})| \mathrm{d}\mathbf{x} = 0, \tag{40}$$

where $m(B_\rho(\mathbf{x}_c) \cap \Omega)$ is the area of $B_\rho(\mathbf{x}_c) \cap \Omega$.

The proof of this theorem is based on microlocal analysis combining with the specific complex geometrical optics solutions introduced in [8].

If stronger regularity conditions can be fulfilled by the conductive transmission eigenfunction v to (35), it is apparent to have

Theorem 3.4. *[30, Theorem 2.2] Let $v \in H^2(\Omega)$ and $w \in H^1(\Omega)$ be eigenfunctions to (35). Assume that the Lipschitz domain $\Omega \subset \mathbb{R}^2$ contains a corner $\Omega \cap B_R(\mathbf{x}_c) = \Omega \cap W_{\mathbf{x}_c}(\theta_W)$, where $\mathbf{x}_c$ is the vertex of $\Omega \cap W_{\mathbf{x}_c}(\theta_W)$ and $R > 0$. Moreover, there exits a sufficiently small neighbourhood $S_h(\mathbf{x}_c) = \Omega \cap B_h(\mathbf{x}_c) = \Omega \cap W_{\mathbf{x}_c}(\theta_W)$ (i.e. $h > 0$ is sufficiently small) of $\mathbf{x}_c$ in Ω such that $qw \in C^\alpha(\overline{S_h(\mathbf{x}_c)})$ with $q := 1 + V$ and $\eta \in C^\alpha\left(\overline{\Gamma_h^\pm(\mathbf{x}_c)}\right)$ for $0 < \alpha < 1$, and $v - w \in H^2(\Sigma_{\Lambda_h}(\mathbf{x}_c))$, where $\Gamma_h^\pm(\mathbf{x}_c)$ and $\Sigma_{\Lambda_h}(\mathbf{x}_c)$ are defined in (36). Under the following assumptions:*

(a) the function $\eta(\mathbf{x})$ doest not vanish at the corner, i.e.,

$$\eta(\mathbf{x}_c) \neq 0, \tag{41}$$

(b) the open angle of the open sector $W_{\mathbf{x}_c}(\theta_W)$ satisfy

$$\theta_W \neq \pi,$$

then we have $v(\mathbf{x}_c) = w(\mathbf{x}_c) = 0$.

3.2. **Unique recovery results for inverse medium problems.** Consider the following medium scattering system for $u \in H^1_{loc}(\mathbb{R}^n)$ in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \Delta u^- + k^2(1+V)u^- = 0 & \text{in } \ \Omega, \\ \Delta u^+ + k^2 u^+ = 0 & \text{in } \ \mathbb{R}^n\backslash\Omega, \\ u^+ = u^-, \quad \partial_\nu u^+ = \partial_\nu u^- & \text{on } \ \partial\Omega, \\ u^+ = u^i + u^s & \text{in } \ \mathbb{R}^n\backslash\Omega, \\ \lim\limits_{r\to\infty} r^{(n-1)/2}\left(\partial_r u^s - \mathrm{i}ku^s\right) = 0, & r = |\mathbf{x}|, \end{cases} \tag{42}$$

where $u^i = u^i = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ is the incident wave, V denotes the material parameter or potential of an inhomogeneous acoustic medium supported in Ω, and the last equation of (42) is the Sommerfield radiation condition which ensures the outgoing nature of the scattered wave u^s.

The Sommerfeld radiation condition in (42) implies that the asymptotic expansion (5) still holds with the far-field pattern $u^\infty(\hat{\mathbf{x}}; u^i)$ for all $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^{n-1}$. The inverse problem associated with (42) is to recover V by the knowledge of $u^\infty(\hat{\mathbf{x}}; u^i)$. Blåsten and Liu first established the unique identifiability results by single far-field pattern for (42) when the support of V fulfills certain polyhedral cell and nest geometry; see [12, Section 2].

Consider the scattering problem (42) with a piecewise constant refractive index

$$V := \sum_\ell V_\ell \chi_{\Sigma_\ell}, \quad \overline{\Omega} = \bigcup_\ell \overline{\Sigma_\ell}, \quad \ell \in \mathbb{N}, \tag{43}$$

where $V_\ell \in \mathbb{C}$ are constants and $\Sigma_\ell \subset \mathbb{R}^n$ are mutually disjoint bounded open subsets.

Definition 3.5. [12, Definition 2.1] An admissible cell $\Sigma \subset \mathbb{R}^n$ is a bounded open convex polytope, i.e. a polygon in $\mathbb{R}^2$ and a polyhedron in $\mathbb{R}^3$.

Based on Definition 3.5, in the following two definitions, the polyheral cell and nest geometry are defined, respectively.

Definition 3.6. [12, Definition 2.2] For $\ell \in \mathbb{N}$ let each of $\Sigma_\ell \subset \mathbb{R}^n$ be an admissible cell or the empty set, $\cup_\ell \overline{\Sigma_\ell}$ is simply connected, bounded and $\Sigma_\ell \cap \Sigma_k = \emptyset$ if $\ell \neq k$.

A bounded potential $V \in L^\infty(\mathbb{R}^n)$ is said to be *piecewise constant with polyhedral cell geometry* if

1. there are constants $V_\ell \in \mathbb{C}$ such that
$$V(\mathbf{x}) = \sum_{\ell=1}^{\infty} V_\ell \chi_{\Sigma_\ell}(\mathbf{x}),$$
with $V_\ell = 0$ if $\Sigma_\ell = \emptyset$, and
2. each $\Sigma_\ell \neq \emptyset$ has a vertex that can be connected to infinity by a path that stays distance $d \geq d_0 > 0$ from any Σ_k with $k > \ell$.

Definition 3.7. [12, Definition 2.4] For $\ell \in \mathbb{N}$, let each of $\Sigma_\ell \subset \mathbb{R}^n$ be an admissible cell or the empty set, and
$$\Sigma_\ell \Supset \Sigma_{\ell+1}.$$
A bounded potential $V \in L^\infty(\mathbb{R}^n)$ is said to be *piecewise constant with polyhedral nested geometry* if there are constants $V_\ell \in \mathbb{C}$, with $V_1 \neq 0$, $V_{\ell+1} \neq V_\ell$ such that
$$V(\mathbf{x}) = \sum_{\ell=1}^{\infty} V_\ell \chi_{U_\ell}(\mathbf{x})$$
where $U_\ell := \Sigma_\ell \setminus \overline{\Sigma_{\ell+1}}$.

Blåsten and Liu also introduced a more general case that the potential V is of Hölder continuous at the corner. The corresponding definition for the admissible potential can be described by

Definition 3.8. [12, Definition 2.4] A potential $V \in L^\infty$ is *a non-constant admissible potential* if there is an admissible cell $\Sigma \subset \mathbb{R}^n$ and bounded function $\psi \in L^\infty(\mathbb{R}^n)$ such that $V = \chi_\Sigma \psi$. Moreover we require that ψ be Hölder C^α-continuous in a neighbourhood of each of the vertices of Σ with $\alpha > 0$ in 2D and $\alpha > 1/4$ in 3D. Finally, the function ψ must not vanish at any of the vertices.

Blåsten and Liu presented the main uniqueness results for the refractive index in a certain medium structure based on the assumption that for each $\ell \in \mathbb{N}$, the total wave field does not vanish at any vertex of Σ_ℓ. This assumption can be fulfilled for low-frequencies with incident plane waves, i.e. k is small enough.

Theorem 3.9. *[12, Theorem 2.6] Let $n \in \{2,3\}$ and $k > 0$. Let V and $\widetilde{V}$ be two piecewise constant potentials with common polyhedral cell geometry. Let u^i be an incident wave such that $u(\mathbf{x}_c) \neq 0$ or $\widetilde{u}(\mathbf{x}_c) \neq 0$ for each vertex $\mathbf{x}_c$ of the cells of V and $\widetilde{V}$. If $u^\infty(\hat{\mathbf{x}}; u^i) = \widetilde{u}^\infty(\hat{\mathbf{x}}; u^i)$, then $V = \widetilde{V}$.*

Theorem 3.10. *[12, Theorem 2.7] Let $n \in \{2,3\}$ and $k > 0$. Let V and $\widetilde{V}$ be two piecewise constant potentials with polyhedral nested geometry. Let u^i be an incident wave such that $u(\mathbf{x}_c) \neq 0$ or $\widetilde{u}(\mathbf{x}_c) \neq 0$ for each vertex $\mathbf{x}_c$ of the cells of V and $\widetilde{V}$. If $u^\infty(\hat{\mathbf{x}}; u^i) = \widetilde{u}^\infty(\hat{\mathbf{x}}; u^i)$ then $V = \widetilde{V}$.*

For more general mediums with potentials of C^α-continuity, it is known that

Theorem 3.11. *[12, Theorem 2.5] Let $n \in \{2,3\}$, $k > 0$ and $V = \chi_\Sigma \psi, \widetilde{V} = \chi_{\widetilde{\Sigma}} \widetilde{\psi}$ be two non-constant admissible potentials. Let u^i be an incident wave such that $u(\mathbf{x}_c) \neq 0$ or $\widetilde{u}(\mathbf{x}_c) \neq 0$ for the total waves $u, \widetilde{u}$ at each vertex $\mathbf{x}_c$ of Σ or $\widetilde{\Sigma}$. Assume that*
$$u^\infty(\hat{\mathbf{x}}; u^i) = \widetilde{u}^\infty(\hat{\mathbf{x}}; u^i)$$

for the far-field patterns arising from V and $\widetilde{V}$, respectively. Then $\Sigma = \widetilde{\Sigma}$ and $\psi(\mathbf{x}_c) = \widetilde{\psi}(\mathbf{x}_c)$ on each vertex $\mathbf{x}_c$ of $\Sigma = \widetilde{\Sigma}$.

Theorems 3.9, 3.10 and 3.11 were established by investigating the singular behaviors of the transmission eigenfunctions at the corner point, which can also be proved by using the geometrical structures of transmission eigenfunctions at the corner (cf. [8,11]).

In the following, we are concerned with the time-harmonic electromagnetic wave scattering from a conductive medium body. The conductive medium body arises in several applications of practical importance, including the modeling of an electromagnetic object coated with a thin layer of a highly conducting material and the magnetotellurics in geophysics. Indeed, the following conductive scattering problem (44) can be derived by the transverse-magnetic (TM) polarisation from the time-harmonic Maxwell system

$$\begin{cases} \Delta u^- + k^2 q u^- = 0 & \text{in } \Omega, \\ \Delta u^+ + k^2 u^+ = 0 & \text{in } \mathbb{R}^2\backslash\Omega, \\ u^+ = u^-, \quad \partial_\nu u^+ + \eta u^+ = \partial_\nu u^- & \text{on } \partial\Omega, \\ u^+ = u^i + u^s & \text{in } \mathbb{R}^2\backslash\Omega, \\ \lim\limits_{r\to\infty} r^{1/2}\left(\partial_r u^s - \mathrm{i}k u^s\right) = 0, & r = |\mathbf{x}|, \end{cases} \tag{44}$$

where $q = 1 + V$, $u^i = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ is the impinging incident wave and u^s is the corresponding scattered wave interrupt by Ω. The well-posedness of the direct problem (44) is known (cf. [14]). The detailed discussions on the aforementioned two specific applications and the mathematical formulations of the associated inverse problems can be found in [22, Section 1] and [30, Section 4], respectively.

Theorem 3.3 and Theorem 3.4 can be applied directly to establish the uniquely determination of the shape of an admissible conductive scatterer by a single far-field pattern.

Definition 3.12. Let $(\Omega; k, \mathbf{d}, q, \eta)$ be a conductive scatterer associated with the incident plane wave $u^i = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ with $\mathbf{d} \in \mathbb{S}^1$ and $k \in \mathbb{R}_+$. Consider the scattering problem (44) and u is the total wave fields therein. The scatterer is said to be admissible if it fulfills the following conditions:

(a) Ω is a bounded simply connected Lipschitz domain in $\mathbb{R}^2$, and $q \in L^\infty(\Omega)$, $\eta \in L^\infty(\partial\Omega)$.
(b) Following the notations in Theorem 3.3, if Ω possesses a corner $B_h(\mathbf{x}_c) \cap \Omega = \Omega \cap W_{\mathbf{x}_c}(\theta_W)$ where $\mathbf{x}_c$ the vertex of the sector $W_{\mathbf{x}_c}$ and the open angle θ_W of $W_{\mathbf{x}_c}(\theta_W)$ satisfies $\theta_W \neq \pi$, then $qu \in C^\alpha(\overline{S_h(\mathbf{x}_c)})$, $\eta \in C^\alpha(\overline{\Gamma_h^\pm(\mathbf{x}_c)})$, where $S_h(\mathbf{x}_c)$ and $\Gamma_h^\pm(\mathbf{x}_c)$ are defined in (36).
(c) The total wave field u is non-vanishing everywhere in the sense that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\lim_{\rho\to+0} \frac{1}{m(B(\mathbf{x},\rho))} \int_{B(\mathbf{x},\rho)} |u(\mathbf{x})| \mathrm{d}x \neq 0. \tag{45}$$

Theorem 3.13. *[30, Theorem 4.1] Consider the conductive scattering problem (44) associated with two conductive scatterers $(\Omega_j; k, \mathbf{d}, q_j, \eta_j)$, $j = 1, 2$, in $\mathbb{R}^2$. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the scatterer $(\Omega_j; q_j, \eta_j)$ and the* *incident field u^i. Suppose that $(\Omega_j; k, \mathbf{d}, q_j, \eta_j)$, $j = 1, 2$ are admissible and*

$$u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i) \tag{46}$$

for all $\hat{\mathbf{x}} \in \mathbb{S}^1$ and a fixed incident wave u^i. Then

$$\Omega_1 \Delta \Omega_2 := \big(\Omega_1 \backslash \Omega_2\big) \cup \big(\Omega_2 \backslash \Omega_1\big) \tag{47}$$

cannot possess a corner. Hence, if Ω_1 and Ω_2 are convex polygons in $\mathbb{R}^2$, one must have

$$\Omega_1 = \Omega_2. \tag{48}$$

If conductive parameter η is constant, then it can be recovered simultaneously once the admissible conductive scatter Ω is determined.

Theorem 3.14. *[30, Theorem 4.2] Consider the conductive scattering problem (44) associated with the admissible conductive scatters $(\Omega_j; k, \mathbf{d}, q, \eta_j)$, where $\Omega_j = \Omega$ for $j = 1, 2$ and $\eta_j \neq 0$, $j = 1, 2$, are two constants. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the scatter $(\Omega; k, \mathbf{d}, q, \eta_j)$ and the incident field u^i. Suppose that $(\Omega; k, \mathbf{d}, q, \eta_j)$, $j = 1, 2$, are admissible and*

$$u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i) \tag{49}$$

for all $\hat{\mathbf{x}} \in \mathbb{S}^1$ and a fixed incident wave u^i. Then if k is not an eigenvalue of the partial differential operator $\Delta + k^2 q$ in $H_0^1(\Omega)$, we have $\eta_1 = \eta_2$.

It is clear that Theorem 3.14 is established with a-prior knowledge that the medium parameter q is known. Indeed, in our recent study [22], we proved further that the medium parameter q as well as the conductive surface parameter η can be uniquely determined simultaneously by a single far-field measurement within a more general geometry. In [22, Section 2], two geometric setups called polygonal-nest geometry and polygonal-cell geometry for the conductive medium body were introduced.

Similar to the geometric setup proposed from Definition 3.5 to Definition 3.7, following the rigorous definitions for the polygonal-nest geometry in [22, Definition 2.3] and the polygonal-cell geometry in [22, Definition 2.4], an admissible conductive medium of polygonal-nest or polygonal-cell structure was established in [22, Definition 4.1]. Indeed, the admissibility condition in [22, Definition 4.1] indicates that the total field u is not vanishing at any vertex of the polygonal-nest partition or the boundary of the polygonal-cell conductive medium body. For compatibility, a local uniqueness regarding the shape of an admissible polygonal-nest or polygonal-cell conductive medium body by a single far-field measurement was first presented without knowing the medium parameter and the conductive surface parameter.

Before presenting the unique identification results, similar to (43), we are supposed to introduce some necessary notions first, which shall be utilized in the subsequent discussions. For a polygonal-nest conductive medium body $(\Omega; q, \eta)$ as described in [22, Definition 2.3], denote

$$(\Omega; q, \eta) = \bigcup_{\ell=1}^{N} (U_\ell; q_\ell, \eta_\ell) \tag{50}$$

and

$$\Omega = \bigcup_{\ell=1}^{N} U_\ell, \quad q = \sum_{\ell=1}^{N} q_\ell \chi_{U_\ell}, \quad \eta = \sum_{\ell=1}^{N} \eta_\ell \chi_{\partial \Sigma_\ell}, \tag{51}$$

where $U_\ell := \Sigma_\ell \backslash \overline{\Sigma_{\ell+1}}$ and each Σ_ℓ is a conductive medium body, $q := 1 + V \in L^\infty(\mathbb{R}^n)$ with V being introduced in (42).

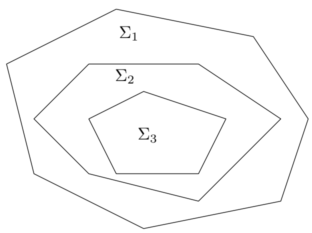
For a polygonal-cell conductive medium body $(\Omega; q, \eta)$ as described in [22, Definition 2.4], denote

$$(\Omega; q, \eta) = \bigcup_{\ell=1}^{N} (\Sigma_\ell; q_\ell, \eta^*) \tag{52}$$

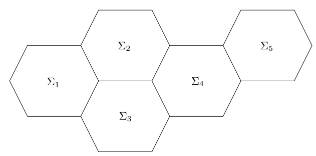
and

$$\Omega = \bigcup_{\ell=1}^{N} \Sigma_\ell, \quad q = \sum_{\ell=1}^{N} q_\ell \chi_{\Sigma_\ell}, \quad \eta = \sum_{\ell=1}^{N} \eta^* \chi_{\partial\Sigma_\ell}, \tag{53}$$

where each Σ_ℓ is a conductive medium body and η^* is the uniform conductive surface parameter.



(a) Polygonal-nest geometry



(b) Polygonal-cell geometry

FIGURE 1. Schematic illustration of the two polygonal geometries in $\mathbb{R}^2$ for a conductive medium body.

Fig. 1 (b) presents a typical polygonal-cell partition of Ω with five hexagonal cells. It is interesting to note that it is the honeycomb graphene structure. We would like to emphasize that for a polygonal-cell partition, each cell is not necessary to be convex.

Definition 3.15. Let $(\Omega; q, \lambda)$ be polygonal-nest or polygonal-cell conductive medium body as described in (50) and (52), respectively. The scatterer is said to be admissible if it fulfills the following condition: consider a polygonal-nest conductive medium body with the polygonal-nest partition $\{\Sigma_\ell\}_{\ell=1}^N$, for any vertex $\mathbf{x}_c \in \partial\Sigma_\ell$, $u(\mathbf{x}_c) \neq 0$; consider a polygonal-cell conductive medium body, for any vertex $\mathbf{x}_c \in \partial\Omega$, $u(\mathbf{x}_c) \neq 0$, where u is the solution to (44).

Theorem 3.16. *[22, Theorem 4.1] Consider the conductive scattering problem (44) associated with two admissible polygonal-nest or polygonal-cell conductive medium bodies $(\Omega_j; q_j, \eta_j)$, $j = 1, 2$, in $\mathbb{R}^2$. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the conductive medium body $(\Omega_j; q_j, \eta_j)$ and the incident field u^i, respectively. Suppose that*

$$u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i) \tag{54}$$

for all $\hat{\mathbf{x}} \in \mathbb{S}^1$ and a fixed incident wave u^i. Then

$$\Omega_1 \Delta \Omega_2 := \big(\Omega_1 \backslash \Omega_2\big) \cup \big(\Omega_2 \backslash \Omega_1\big) \tag{55}$$

cannot possess a corner. Furthermore, if Ω_1 and Ω_2 are two admissible polygonal-nest conductive medium bodies, one must have

$$\partial\Omega_1 = \partial\Omega_2. \tag{56}$$

It is easy to see from Theorem 3.16 that for a polygonal-nest medium body, the local uniqueness results readily imply the global uniqueness results. The corresponding proof for Theorem 3.16 is based on the geometrical structures of conductive transmission eigenfunctions established in Theorem 3.4.

Next, the simultaneous unique determination for the piecewise constant medium parameter as well as the conductive surface parameter associated with an admissible polygonal-nest or polygonal-cell conductive medium body can be achieved as the following two theorems. However, a-prior knowledge on the cell structure of an admissible polygonal-cell conductive medium body is assumed to be known in advance. The proof of Theorems 3.17 and 3.18 can be obtained by utilizing certain microlocal analysis and the complex geometrical optics solution introduced in [8].

Theorem 3.17. *[22, Theorem 4.2] Considering the conductive scattering problem (44) associated with two admissible polygonal-cell conductive medium bodies $(\Omega; q_j, \eta_j)$ in $\mathbb{R}^2$, $j = 1, 2$. For $j = 1, 2$, we let the material parameters q_j and η_j with a common polygonal-cell partition $\{\Sigma_\ell\}_{\ell=1}^N$ defined in (53) be characterized by*

$$q_j = \sum_{\ell=1}^{N} q_{\ell,j} \chi_{\Sigma_{\ell,j}}, \quad \eta_j = \sum_{\ell=1}^{N} \eta_j^* \chi_{\partial\Sigma_{\ell,j}}. \tag{57}$$

Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the corresponding far-field pattern associated with the incident wave u^i corresponding to $(\Omega; q_j, \eta_j)$, respectively. Suppose that $(\Omega; q_j, \eta_j)$, $j = 1, 2$, fulfill

$$u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i) \tag{58}$$

for all $\hat{\mathbf{x}} \in \mathbb{S}^1$ and a fixed incident wave u^i. Then we have $q_1 = q_2$ and $\eta_1 = \eta_2$.

Theorem 3.18. *Considering the conductive scattering problem (44) associated with two admissible polygonal-nest conductive medium bodies $(\Omega_j; q_j, \eta_j)$ in $\mathbb{R}^2$, $j = 1, 2$, where the associated polygonal-nest partitions $\{\Sigma_{\ell,1}\}_{\ell=1}^{N_1}$ and $\{\Sigma_{\ell,2}\}_{\ell=1}^{N_2}$ are defined in (51) with $\Omega_1 = \bigcup_{\ell=1}^{N_1} U_{\ell,1}$ and $\Omega_2 = \bigcup_{\ell=1}^{N_2} U_{\ell,2}$, where $U_{\ell,j} = \Sigma_{\ell,j} \backslash \overline{\Sigma_{\ell+1,j}}$, $\ell = 1, \ldots, N_j$. For $j = 1, 2$, we let the material parameters q_j and η_j be characterized by*

$$q_j = \sum_{\ell=1}^{N_j} q_{\ell,j} \chi_{U_{\ell,j}}, \quad \eta_j = \sum_{\ell=1}^{N_j} \eta_{\ell,j} \chi_{\partial\Sigma_{\ell,j}}. \tag{59}$$

Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the corresponding far-field pattern associated with the incident wave u^i corresponding to $(\Omega_j; q_j, \eta_j)$, respectively. Suppose that $(\Omega_j; q_j, \eta_j)$, $j = 1, 2$, fulfill

$$u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i) \tag{60}$$

for all $\hat{\mathbf{x}} \in \mathbb{S}^1$ and a fixed incident wave u^i. Then we have $N_1 = N_2 = N$, $\partial\Sigma_{\ell,1} = \partial\Sigma_{\ell,2}$ for $\ell = 1, \ldots, N$, $q_1 = q_2$ and $\eta_1 = \eta_2$.

4. **Inverse diffraction grating problems.** The study on scattering theory by periodic structures has received a lot of attention in recent years. It arises from the first work studied by Rayleigh on the scattering by plane waves from corrugated surfaces. In particular, if the corrugations are exact sinusoids, then the sinusoidally corrugated surface provides a model of a reflection grating. There are quite many applications on grating problems in spectroscopy and oceanography. For example, it can be utilized to study the structure of the ocean surface by measuring sound scattering from below or the scattering of light or radar from above. We refer to [5] and [55] for more historical discussions.

In this section, we focus on the existing studies on the uniqueness issue for inverse diffraction grating problems. It is known that a general periodic grating structure can be uniquely determined by one incident wave if the wave number k is a real number; see [5] and [34]. In [5], the researcher also presented the uniqueness results within C^2-smooth functions in $\mathbb{R}^2$ in the case of lossy medium. Kirsch in [39] and Hettlich, Kirsch in [34] also investigated the case for C^2-smooth functions by all quasi-periodic incident waves and for a sufficiently small wave number or grating height, respectively. In $\mathbb{R}^3$, Ammari [4], Bao and Zhou [7] as well as Bao, Zhang and Zou [6] discussed the more complicated doubly periodic structures. By making use of reflection principle, Elschner, Schmidt and Yamamoto (see [31] and [32]) obtained some results on global uniqueness for the particular piecewise linear periodic structures.

In the subsequent discussions, we just present some of the aforementioned results to illustrate the development on inverse diffraction grating problem. Consider the direct diffraction grating problem associated to Helmholtz system as:

$$\Delta u + k^2 u = 0 \quad \text{in } \ \Omega_f; \quad \mathcal{B}(u)\big|_{\Lambda_f} = 0 \quad \text{on } \ \Lambda_f, \tag{61}$$

with the generalized impedance boundary condition

$$\mathcal{B}(u) = \partial_\nu u + \eta u = 0, \tag{62}$$

where η can also be ∞ or 0, corresponding to a sound-soft or sound-hard grating, respectively.

In (61), for a periodic Lipschitz function f with period 2π,

$$\Omega_f := \{\mathbf{x} \in \mathbb{R}^2; x_2 > f(x_1), x_1 \in \mathbb{R}\}$$

is filled with a material whose refraction index (or wave number) k is a positive constant. Λ_f signifies a diffraction grating profile given by the curve

$$\Lambda_f = \{(x_1, x_2) \in \mathbb{R}^2;\ x_2 = f(x_1)\}. \tag{63}$$

The corresponding incident wave is defined as

$$u^i(\mathbf{x}; k, \mathbf{d}) = e^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}}, \quad \mathbf{d} = (\sin\theta, -\cos\theta)^\top, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

which propogates to Λ_f from the top.

In order to derive the uniqueness results for the inverse grating problem associated with (61), the total wave field u should be α-quasiperiodic in the x_1-direction, with $\alpha = k\sin\theta$, which means that

$$u(x_1 + 2\pi, x_2) = e^{2\mathrm{i}\alpha\pi} \cdot u(x_1, x_2),$$

and the scattered field u^s satisfies the Rayleigh expansion (cf. [49, 50]):

$$u^s(\mathbf{x}; k, \mathbf{d}) = \sum_{n=-\infty}^{+\infty} u_n e^{\mathrm{i}\xi_n(\theta)\cdot\mathbf{x}} \quad \text{for} \quad x_2 > \max_{x_1 \in [0, 2\pi]} f(x_1), \tag{64}$$

where $u_n \in \mathbb{C}(n \in \mathbb{Z})$ are called the Rayleigh coefficient of u^s, and

$$\begin{aligned} \xi_n(\theta) &= (\alpha_n(\theta), \beta_n(\theta))^\top, \quad \alpha_n(\theta) = n + k\sin\theta, \\ \beta_n(\theta) &= \begin{cases} \sqrt{k^2 - \alpha_n^2(\theta)}, & \text{if } |\alpha_n(\theta)| \le k \\ \mathrm{i}\sqrt{\alpha_n^2(\theta) - k^2}, & \text{if } |\alpha_n(\theta)| > k \end{cases}. \end{aligned} \tag{65}$$

The existence and uniqueness of the α-quasiperiodic solution to (61) for the sound-soft or impedance boundary condition with $\eta \in \mathbb{C}$ being a constant satisfying $\Im(\eta) > 0$ can be found in [1,16,38,39]. It should be pointed out that the uniqueness of the direct scattering problem associated with the sound-hard condition is not always true (see [36]). In our subsequent study, we assume the well-posedness of the forward scattering problem and focus on the study of the inverse grating problem.

Introduce a measurement boundary as

$$\Gamma_b := \{(x_1, b) \in \mathbb{R}^2;\ 0 \leq x_1 \leq 2\pi,\ b > \max_{x_1 \in [0,2\pi]} |f(x_1)|\}.$$

The inverse diffraction grating problem is to determine (Λ_f, η) from the knowledge of $u(\mathbf{x}|_{\Gamma_b}; k, \mathbf{d})$, and can be formulated as the operator equation:

$$\mathcal{F}(\Lambda_f, \eta) = u(\mathbf{x}; k, \mathbf{d}), \quad \mathbf{x} \in \Gamma_b,$$

where $\mathcal{F}$ is defined by the forward diffraction scattering system, and is nonlinear.

The pioneer work of Kirsch in [39] determined the unknown grating structure by the knowledge of incident waves and the scattered wave fields with the help of associated expansion of incident wave as

$$u^i(\mathbf{x}) = u^i(x_1, x_2) = \sum_{n \in \mathbb{Z}} \psi_n e^{\mathrm{i}\alpha_n x_1 + \mathrm{i}\beta_n (b - x_2)}, \quad x_2 \leq b, \tag{66}$$

where $\mathbf{x} = (x_1, x_2)$, ψ_n are the Fourier coefficients of any $\psi \in L^2(0, 2\pi)$ such that $\mathcal{F}_{\Lambda_f} \psi(x_1) = u(x_1, b)$. Instead of the notations introduced in [39], we follow the uniform notations in this section to present the main uniqueness result.

Theorem 4.1. *[39, Theorem 3.1] Assume that $\beta_n \neq 0$ in (66) for all $n \in \mathbb{N}$. Let $f, \widetilde{f} \in C^2(\mathbb{R})$ be 2π-periodic and $b > \max\{\max_{x_1 \in \mathbb{R}} f, \max_{x_1 \in \mathbb{R}} \widetilde{f}\}$. If $\mathcal{F}_{\Lambda_f} = \mathcal{F}_{\Lambda_{\widetilde{f}}}$ then $\Lambda_f = \Lambda_{\widetilde{f}}$, i.e. if the outputs are the same for f and $\widetilde{f}$ for all incident waves (66), then Λ_f and $\Lambda_{\widetilde{f}}$ have to coincide.*

All the literatures mentioned at the beginning of this subsections are concerning the unique recovery results on the inverse diffraction grating problem with the sound-soft (namely $\eta = \infty$ in (62)) or sound-hard (namely $\eta = 0$ in (62), partially solved) boundary condition by a finite number of incident plane waves. However, the unique identifiability has still remained to be open for the impedance or generalized impedance cases until recently in [23], a completely new approach was proposed by utilizing the spectral properties of Laplacian eigenfunctions in Subsection 2.1 to address this problem in $\mathbb{R}^2$. Indeed, the unique determination for the inverse diffraction grating problem in [23] is also referred to as a direct application of the geometric structures of Laplacian eigenfunctions presented in Subsection 2.1. We only provide the main unique determination results and some relating discussions here.

First, we are supposed to give the precise definition of admissible polygonal gratings associated with the inverse diffraction grating problem.

Definition 4.2. [23, Definition 8.1] Let (Λ_f, η) be a periodic grating as described in (63). Suppose there is a partition, $[0, 2\pi] = \cup_{i=1}^{\ell} [a_i, a_{i+1}]$ with $a_i < a_{i+1}$, $a_1 = 0$ and $a_{\ell+1} = 2\pi$. If on each piece $[a_i, a_{i+1}]$, $1 \leq i \leq \ell$, f is a linear polynomial and η is either a constant (possibly zero) or ∞, then (Λ_f, η) is said to be an admissible polygonal grating.

Definition 4.3. Let (Λ_f, η) be an admissible polygonal grating. Let Γ^+ and Γ^- be two adjacent pieces of Λ_f. The intersecting point of Γ^+ and Γ^- is called a corner point of Λ_f, and $\angle(\Gamma^+, \Gamma^-)$ is called a corner angle. If all the corner angles of Λ_f are irrational, then it is said to be an *irrational polygonal grating.* If a corner angle of Λ_f is rational, it is called a *rational polygonal grating.* The smallest degree of the rational corner angles of Λ_f is referred to as the *rational degree* of Λ_f.

Similar to Theorem 2.21 and Theorem 2.22 for the study on admissible irrational and rational complex polygonal obstacles, we have the unique determination results for rational and irrational polygonal gratings, respectively. In particular, for the rational case, we are concerned with the admissible rational polygonal grating of degree p, $p \geq 3$, due to the fact that the rational degree of Λ_f is at least 2.

Theorem 4.4. *[23, Theorem 8.13] Let (Λ_f, η) and $(\Lambda_{\widetilde{f}}, \widetilde{\eta})$ be two admissible irrational polygonal gratings, and $\mathbf{G}$ be the unbounded connected component of $\Omega_f \cap \Omega_{\widetilde{f}}$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d}_\ell$, $\ell = 1, 2$ be two distinct incident directions from $\mathbb{S}^1$, with*

$$\mathbf{d}_\ell = (\sin\theta_\ell, -\cos\theta_\ell)^\top, \quad \theta_\ell \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Let $u(\mathbf{x}; k, \mathbf{d}_\ell)$ and $\widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell)$ denote the total fields associated with (Λ_f, η) and $(\Lambda_{\widetilde{f}}, \widetilde{\eta})$ respectively and let Γ_b be a measurement boundary given by

$$\Gamma_b := \left\{ (x_1, b) \in \mathbb{R}^2;\ 0 \leq x_1 \leq 2\pi,\ b > \max\left\{ \max_{x_1 \in [0, 2\pi]} |f(x_1)|, \max_{x_1 \in [0, 2\pi]} |\widetilde{f}(x_1)| \right\} \right\},$$

If it holds that

$$u(\mathbf{x}; k, \mathbf{d}_\ell) = \widetilde{u}(\mathbf{x}; k, \mathbf{d}_\ell), \quad \ell = 1, 2, \quad \mathbf{x} = (x_1, b) \in \Gamma_b, \tag{67}$$

then it cannot be true that there exists a corner point of Λ_f lying on $\partial\mathbf{G} \backslash \partial\Lambda_{\widetilde{f}}$, or a corner point of $\Lambda_{\widetilde{f}}$ lying on $\partial\mathbf{G} \backslash \partial\Lambda_f$.

Combining with the fact that $\{e^{\mathrm{i}\xi_\ell \cdot \mathbf{x}}; \mathbf{x} \in U, \ell = 1, 2, \cdots, n\}$ are linearly independent with n distinct vectors $\xi_\ell, \ell = 1, 2, \cdots, n$, for any open subset $U \subset \mathbb{R}^2$, the proof of this theorem can be formulated similarly to the argument in Theorem 2.21. One can refer [23, Section 8] for the rigorous analyses.

As the result in Theorem 2.22 for the inverse obstacle problem, the unique determination of an admissible rational polygonal grating of degree p, $p \geq 3$, can also be derived by two measurements if a similar condition to (32) is introduced in this new setup. In such a case, one can establish the local unique recovery result, similar to Theorem 4.4.

5. **Discussions and some open problems.** In this paper, we present a review on some recent progress on inverse scattering problems including inverse obstacle problems, inverse medium scattering problems and inverse diffraction grating problems.

For Schiffer' problem in inverse obstacle scattering in Section 2, there has been a colorful and long history for the relevant study. We investigate the unique identification of sound-soft, sound-hard and impedance obstacles with certain polyhedral geometry, respectively. The existing literatures are mainly concerned with the uniqueness discussions for sound-soft and sound-hard obstacles, where the reflection principle is utilized for establishing the corresponding results. It has been an open problem for a long time on the study about impedance case. In [47], Liu and

Zou investigated the unique determination for certain partially coated structures of the obstacle following reflection principle. And very recently, in [23] and [24], by applying the geometric structures of Laplacian eigenfunctions in Subsection 2.1, the unique identifiability for a more general structure of impedance obstacles were considered in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The theoretical findings and the corresponding applications are completely novel and inspiring.

For inverse medium scattering problems in Section 3, we first review thegeometrical structure of interior transmission eigenfunctions and the more generalized case with the conductive boundary condition. Those theoretical novel findings play an important role in the corresponding applications for the unique identifiability of the inverse medium scattering problems within the polyhedral geometry by a single far-field measurement under certain assumptions. Recently, some simultaneous recovery results regarding the scatterer as well as the conductive surface parameter and the medium parameter were obtained in [22]. The study therein are more generalized with respect to the medium structure and the regularity assumptions.

In the study on the inverse diffraction grating problem in Section 4, we first formulate the mathematical setup with the generalized impedance boundary which is more practical. By introducing the existing uniqueness results which are mainly concerning gratings with sound-soft or sound-hard boundary, we emphasize the developments on impedance case therein. The corresponding argument is similar to the unique identifiability for impedance obstacles by making use of the spectral properties of Laplacian eigenfunctions.

At the end of our paper, we propose some interesting open problems as follows.

- Establish the global unique identifiability result for an impedance polyhedral obstacle by a single far-field measurement by relaxing the generic condition in Definition 2.19. A more challenging problem is to establish the unique identification for an impedance obstacle of a non-polyhedral geometry by a single far-field measurement.
- Develop a uniform approach to tackle the unique identification result for the Schiffer's problem associated with an impedance polyhedral obstacle in $\mathbb{R}^n$, $n > 4$.
- Establish the global unique identifiability result for a medium polyhedral scatterer by a single far-field measurement by relaxing the generic condition in Definitions 3.8 and 3.12. A more challenging problem is to establish the unique identification for a medium scatterer within more general geometries by a single far-field measurement.
- Generalize the geometrical structures of interior transmission eigenfunctions at the corner to a more general interior transmission eigenvalue problem with respect to the anisotropic metric which can be formulated as

$$\begin{cases} \Delta_g u + k^2(1+V)u = 0 & \text{in} \quad \Omega, \\ \Delta v + k^2 v = 0 & \text{in} \quad \Omega, \\ u = v, \quad \partial_{\nu_g} u = \partial_{\nu_g} v & \text{on} \quad \partial\Omega, \end{cases}$$

  where (Ω, g) is a Riemannian manifold and Δ_g is the Laplacian-Beltrami operator.

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