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ON A FINAL VALUE PROBLEM FOR A NONLINEAR FRACTIONAL PSEUDO-PARABOLIC EQUATION

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Abstract. In this paper, we investigate a final boundary value problem for a class of fractional with parameter β pseudo-parabolic partial differential equations with nonlinear reaction term. For $0 < \beta < 1$, the solution is regularity-loss, we establish the well-posedness of solutions. In the case that $\beta > 1$, it has a feature of regularity-gain. Then, the instability of a mild solution is proved. We introduce two methods to regularize the problem. With the help of the modified Lavrentiev regularization method and Fourier truncated regularization method, we propose the regularized solutions in the cases of globally or locally Lipschitzian source term. Moreover, the error estimates is established.

1. **Introduction.** We consider the final value problem:

$$\begin{cases} u_t - m\Delta u_t + (-\Delta)^\beta u &= f(t,x;u), \quad \text{in} \quad (0,T]\times\Omega, \\ u(t,x) &= 0, \quad \text{on} \quad (0,T]\times\partial\Omega, \\ u(T,x) &= \varphi(x), \quad \text{in} \quad \Omega, \end{cases} \tag{$\mathbb{P}_T$}$$

where $m > 0$, and $\Omega \subset \mathbb{R}^d, (d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, the operator $(-\Delta)^\beta$ with is the fractional Laplace operator with $0 < \beta \neq 1$ and the

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final data $\varphi \in L^2(\Omega)$. Pseudo-parabolic equations have many applications in science and technology, especially in physical phenomena such as seepage of homogeneous fluids through a fissured rock, aggregation of populations,... see e.g. Ting [24], R. Xu [17, 31–33] and references therein.

For $\beta = 1$, the direct problem is

$$u_t - m\Delta u_t - \Delta u = f(t, x; u), \quad \text{in} \quad (0, T] \times \Omega, \tag{1.1}$$

with conditions $u(0, x) = u_0(x),\ x \in \Omega$ and $u(t, x) = 0,\ (t, x) \in (0, T] \times \partial\Omega$. Problem (1.1) has been studied by many authors. Specifically,

1. $f = 0$, see e.g. [11, 22, 24], the existence and uniqueness of solutions is established. Moreover, the asymptotic behavior and regularity are investigated.
2. $f(u) = u^p,\ p \geq 1$, in [5], the authors investigate large time behavior of solutions. R. Xu et al. [32] proved the invariance of some sets, global existence, nonexistence and asymptotic behavior of solutions with initial energy $J(u_0) \leq d$ and finite time blow-up with high initial energy $J(u_0) > d$ and some related works [18, 34]. For the case of $f(u) = |u|^{p-2}u$, there are other results on the large time behavior of solutions of the pseudo-parabolic see [7, 28–30, 35, 36] and their references.
3. When the source term is a logarithmic nonlinearity $f(u) = |u|^p u \log |u|$, very recently, the work [10] focus on the initial conditions, which ensure the solutions to exist globally, blow up in finite time and blow up at infinite time. The asymptotic behaviour for the solutions has been considered in [4, 6, 8, 12, 34] and the references therein.
4. For nonlocal source, $f(u) = |u|^p \displaystyle\int_\Omega G(x, y)|u|^{p+1}(y)\mathrm{d}y,\ y \in \Omega$, the authors of [19] considered blow-up time for solutions, obtained a lower bound as well as an upper bound for the blow-up time under different conditions, respectively. Also, they investigated a nonblow-up criterion and compute an exact exponential decay, see also [9, 23].

For $0 < \beta \neq 1$, [14] considered the Cauchy problem

$$u_t - m\Delta u_t + (-\Delta)^\beta u = u^{p+1}, \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \tag{1.2}$$

supplemented with initial condition $u(0, x) = u_0(x),\ x \in \Omega$ and Dirichlet boundary condition $u(t, x) = 0,\ (t, x) \in \mathbb{R}_+ \times \partial\Omega$. The paper established the global existence and time-decay rates for small-amplitude solutions.

As mention above, initial value problems of nonlinear pseudo-parabolic equations have been considered in many papers see [1, 4–15, 19, 21–24, 32, 35–37]. However, there are not many results devoted to Problem ($\mathbb{P}_T$). Our approach includes as special cases all previously on the reaction terms. In this work, we consider two cases; first, the source f is globally Lipschitz and in the second case, we consider f is general locally Lipschitz function (a coercive-type condition). At this point, we remark that there exists some locally Lipschitz functions, but we cannot determine its specific Lipschitz coefficient e.g. $f = u(a - bu^2), b > 0$ of the Ginzburg-Landau equation. Hence, we have to find another method to study the problem with the locally Lipschitz source which is similar to the Ginzburg-Landau equation, etc. (see Subsection 4.2.2 for more details).

The solution of Problem ($\mathbb{P}_T$) is of the regularity-loss structure for $0 < \beta < 1$, $x \in \Omega, t \in (0, T]$, we consider the existence and regularity of Problem ($\mathbb{P}_T$). In the case $\beta > 1$, the regularity-gain type for $x \in \Omega, t \in (0, T]$ and the Problem ($\mathbb{P}_T$) is ill-posed. So the regularization methods are required. As we know, there are many regularization methods to suit each problem [2,3,16,20,25–27]. For problem ($\mathbb{P}_T$), we propose two methods to regularize solution: Modified Lavrentiev regularization (MLR) method and Fourier truncated regularization (FTR) method.

The plan of the paper is as follows. Section 2 contains notations and formulation of a solution of Problem ($\mathbb{P}_T$) and the proof of its instability. In Section 3, the case $0 < \beta < 1$, well-posedness of solutions of Problem ($\mathbb{P}_T$) is established. In Section 4, the case $\beta > 1$, the proof that Problem ($\mathbb{P}_T$) is ill-posed and the well-posedness of the regularized problem are presented. We also propose two regularization methods: MLR method and FTR method for the globally Lipschitz or locally Lipschitz reaction terms, respectively.

2. Preliminaries.

2.1. Relevant notations.

Let us recall that the spectral problem

$$\begin{cases} (-\Delta)^{\beta} e_j(x) = \lambda_j^{\beta} e_j(x), & \text{in } \Omega,\ \beta > 0, \\ e_j(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

admits a family of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_j \le ... \nearrow \infty.$$

The notation $\|\cdot\|_B$ stands for the norm in the Banach space B. We denote by $L^q(0,T;B), 1 \le q \le \infty$, the Banach space of real-valued measurable functions $w : (0;T) \to B$ with norm

$$\|w\|_{L^q(0,T;B)} = \left(\int_0^T \|w(t)\|_B^q \mathrm{d}t \right)^{\frac{1}{q}}, \quad \text{for } 1 \le q < \infty,$$

$$\|w\|_{L^\infty(0,T;B)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|w(t)\|_B, \quad \text{for } q = \infty.$$

The norm of the function space $C^k([0,T];B), 0 \le k \le \infty$ is denoted by

$$\|w\|_{C^k([0,T];B)} = \sum_{i=0}^{k} \sup_{t\in[0,T]} \|w^{(i)}(t)\|_B < \infty.$$

For any $\nu \ge 0$, we define the space

$$H^\nu(\Omega) = \left\{ w \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2\nu} \langle w, e_j \rangle_{L^2(\Omega)}^2 < \infty \right\},$$

where $\langle \cdot,\cdot \rangle_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$; $H^\nu(\Omega))$ is a Hilbert space with the norm

$$\|w\|_{H^\nu(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2\nu} \langle w, e_j \rangle_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

The Gevrey of order β class of functions with index $\eta_1, \eta_2 > 0$, defined by the spectrum of the Laplacian is denoted by

$$\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega) := \left\{ w \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{\eta_1\beta} \exp\left(\eta_2 \lambda_j^{\beta}\right) \langle w, e_j \rangle_{L^2(\Omega)}^2 < \infty \right\},$$

and its norm given by

$$\|w\|_{\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{\eta_1\beta} \exp\left(\eta_2 \lambda_j^\beta\right) \langle w, e_j \rangle_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Next, we give the formulation of solution of the problem ($\mathbb{P}_T$).

2.2. **Mild solution of the Problem ($\mathbb{P}_T$).** Now, assume that Problem ($\mathbb{P}_T$) has a unique solution, then we find its the form. Let $u(t,x) = \sum_{j=1}^{\infty} u_j(t) e_j(x)$ be the decomposition of u in $L^2(\Omega)$ with $u_j(t) = \langle u(t,\cdot), e_j \rangle_{L^2(\Omega)}$. From ($\mathbb{P}_T$), taking the inner product of both sides of ($\mathbb{P}_T$) with $e_j(x)$, we obtain the ordinary differential equation

$$(1 + m\lambda_j) u_j'(t) + \lambda_j^\beta u_j(t) = f_j(u)(t),$$

where $u_j'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} \langle u(t,\cdot), e_j \rangle_{L^2(\Omega)}$, $f_j(u)(t) = \langle f(t,\cdot;u), e_j \rangle_{L^2(\Omega)}$, whose solution is

$$u_j(t) = \exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j} \right) \varphi_j - \frac{1}{1+m\lambda_j} \int_t^T \exp\left(\frac{(\tau - t)\lambda_j^\beta}{1+m\lambda_j} \right) f_j(u)(\tau) \mathrm{d}\tau,$$

where $\varphi_j = \langle \varphi, e_j \rangle_{L^2(\Omega)}$.

Definition 2.1 (Mild solution of Problem ($\mathbb{P}_T$))**.** A function u is a mild solution of ($\mathbb{P}_T$) if $u \in C([0,T]; L^2(\Omega))$ and satisfies the following integral equation

$$u(t,x) = \sum_{j=1}^{\infty} \left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j} \right) \varphi_j - \frac{1}{1+m\lambda_j} \int_t^T \exp\left(\frac{(\tau - t)\lambda_j^\beta}{1+m\lambda_j} \right) f_j(u)(\tau) \mathrm{d}\tau \right] e_j(x). \tag{2.2}$$

for all $(t,x) \in (0,T) \times \Omega$, and $\beta > 0$.

Now we introduce the main results in this paper.

3. The case $0 < \beta < 1$: Well-posedness of solutions to the Problem ($\mathbb{P}_T$).

In this section, we prove that the Problem ($\mathbb{P}_T$) is well-posed. First prove that for the Problem ($\mathbb{P}_T$) exists a unique mild solution, then the regularity of the solution is established.

We will make the following assumptions:

($\mathbf{A_1}$) Assume that f satisfy the global Lipschitz condition:

$$\|f(t,\cdot;w_1) - f(t,\cdot;w_2)\|_{L^2(\Omega)} \le K \|w_1 - w_2\|_{L^2(\Omega)}, \tag{3.1}$$

with $K > 0$ independent of t, x, w_1, w_2, and $(t,x) \in [0,T) \times \Omega, w_i \in C([0,T]; L^2(\Omega)), i = 1,2$.

($\mathbf{A_2}$) We set $f(0) := f(t,x;0) = 0,\ (t,x) \in [0,T) \times \Omega$ and

$$\|f(t,\cdot;w)\|_{L^2(\Omega)} \le K \|w\|_{L^2(\Omega)}, \quad w \in C([0,T]; L^2(\Omega)). \tag{3.2}$$

Theorem 3.1 (L^∞-Existence)**.** *Let $0 < \beta < 1$, assume that f satisfy the assumption ($\mathbf{A}_1$). Then, the integral equation (2.2) has a unique mild solution in $L^\infty(0,T; L^2(\Omega))$.*

Proof. We prove the existence of the solution $u \in L^\infty(0,T;L^2(\Omega))$ of the integral equation (2.2). For $w \in L^\infty(0,T;L^2(\Omega))$, we consider the following operator:

$$\mathscr{H}(w)(t,x) = \sum_{j=1}^{\infty}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j - \frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_j(w)(\tau)\mathrm{d}\tau\right] e_j(x), \tag{3.3}$$

and we aim to show that $\mathscr{H}^k$ is a contraction mapping on the space $L^\infty(0,T;L^2(\Omega))$. In fact, we will prove that for every $w_1, w_2 \in L^\infty(0,T;L^2(\Omega))$, it holds

$$\left\|\mathscr{H}^k(w_1)(t,\cdot) - \mathscr{H}^k(w_2)(t,\cdot)\right\|_{L^2(\Omega)} \le \frac{\left(\frac{K}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)(T-t)\right)^k}{k!}\|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))}. \tag{3.4}$$

We will prove (3.4) by induction. For $k=1$, using Parseval's relation and assumption $(\mathbf{A_1})$, one obtains

$$\begin{aligned}
&\|\mathscr{H}(w_1)(t,\cdot) - \mathscr{H}(w_2)(t,\cdot)\|_{L^2(\Omega)} \\
&\le \int_t^T \left\|\sum_{j=1}^{\infty}\frac{1}{1+m\lambda_j}\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)(f_j(w_1)(\tau) - f_j(w_2)(\tau))e_j\right\|_{L^2(\Omega)} \mathrm{d}\tau \\
&\le \int_t^T \sqrt{\sum_{j=1}^{\infty}\frac{1}{(1+m\lambda_j)^2}\exp\left(\frac{2(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)|f_j(w_1)(\tau) - f_j(w_2)(\tau)|^2}\mathrm{d}\tau \\
&\le \frac{1}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\int_t^T \|f(\tau,\cdot,w_1) - f(\tau,\cdot,w_2)\|_{L^2(\Omega)}\mathrm{d}\tau \\
&\le \frac{K}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\int_t^T \|w_1(\tau,\cdot) - w_2(\tau,\cdot)\|_{L^2(\Omega)}\mathrm{d}\tau \\
&\le \frac{K}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)(T-t)\|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned} \tag{3.5}$$

Assume now that (3.4) is satisfied for $k = k_0$, let us prove that it is satisfied for $k = k_0 + 1$. It holds

$$\begin{aligned}
&\left\|\mathscr{H}^{k_0+1}(w_1)(t,\cdot) - \mathscr{H}^{k_0+1}(w_2)(t,\cdot)\right\|_{L^2(\Omega)} \\
&\le \frac{K}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\int_t^T \left\|\mathscr{H}^{k_0}(w_1)(\tau,\cdot) - \mathscr{H}^{k_0}(w_2)(\tau,\cdot)\right\|_{L^2(\Omega)}\mathrm{d}\tau \\
&\le \frac{1}{k_0!}\left(\frac{K}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\right)^{k_0+1}\|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))}\int_t^T (T-\tau)^{k_0}\mathrm{d}\tau \\
&\le \frac{1}{(k_0+1)!}\left(\frac{K}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\right)^{k_0+1}(T-t)^{k_0+1}\|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))} \\
&\le \frac{1}{(k_0+1)!}\left(\frac{KT}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\right)^{k_0+1}\|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}$$

Therefore, by the induction principle we get (3.4). Since the right hand side of (3.5) is independent of t, we deduce that

$$\left\|\mathscr{H}^k(w_1) - \mathscr{H}^k(w_2)\right\|_{L^\infty(0,T;L^2(\Omega))} \le \frac{1}{k!}\left(\frac{KT}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\right)^k\|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))}.$$

When k is large enough, we have $\frac{1}{k_1!}\left(\frac{KT}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\right)^{k_1} \to 0$. Hence there exists k_1 such that

$$0 < \frac{1}{k_1!}\left(\frac{KT}{m\lambda_1}\exp\left(\frac{T}{m\lambda_1}\right)\right)^{k_1} < 1.$$

We claim that the mapping $\mathscr{H}^{k_1}$ is a contraction, i.e. $\mathscr{H}^{k_1}(u) = u$. We have

$$\mathscr{H}^{k_1}(\mathscr{H}(u)) = \mathscr{H}(\mathscr{H}^{k_1}(u)) = \mathscr{H}(u).$$

Due to the uniqueness of the fixed point of $\mathscr{H}^{k_1}$, it holds $\mathscr{H}(u) = u$. We conclude that the integral equation (3.3) has a unique solution in $L^\infty(0,T;L^2(\Omega))$. $\square$

Theorem 3.2 (Regularity)**.** *Let $0 < \beta < 1$, and f only satisfy the assumption $(\mathbf{A_1})$, we have the following:*
a) If $\varphi \in L^2(\Omega)$ and $f(0) \in L^1(0,T;L^2(\Omega))$, then there exists $C(T,m,\lambda_1) > 0$ such that

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \le C(T,m,\lambda_1)\left(\|\varphi\|_{L^2(\Omega)} + \frac{\|f(0)\|_{L^1(0,T;L^2(\Omega))}}{m\lambda_1}\right). \tag{3.6}$$

b) If $\varphi \in H^\beta(\Omega)$, $f(0) \in L^\infty(0,T;L^2(\Omega))$ then there exists $C(T,m,\lambda_1,\beta) > 0$ such that

$$\|u\|_{L^\infty(0,T;H^\beta(\Omega))} \le C(T,m,\lambda_1,\beta)\left(\|\varphi\|_{H^\beta(\Omega)} + \frac{T\,\|f(0)\|_{L^\infty(0,T;L^2(\Omega))}}{m\lambda_1^{1-\beta}}\right). \tag{3.7}$$

Here, we recall $f(0) := f(t,x;0)$, $\forall (t,x) \in [0,T) \times \Omega$.

Proof. First, we set

$$\mathcal{M}_1(\varphi)(t,x) := \sum_{j=1}^{\infty} \exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j e_j(x),$$
$$\mathcal{M}_2(u)(t,x) := -\sum_{j=1}^{\infty}\left[\frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_j(u)(\tau)\mathrm{d}\tau\right] e_j(x).$$

a) First, using the Parseval's relation, we infer that

$$\|\mathcal{M}_1(\varphi)(t,\cdot)\|_{L^2(\Omega)} = \left(\sum_{j=1}^{\infty}\exp\left(\frac{2(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^2\right)^{1/2} \le \exp\left(\frac{T}{m\lambda_1^{1-\beta}}\right)\|\varphi\|_{L^2(\Omega)}, \tag{3.8}$$

where for $\beta \in (0,1)$, we have $\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right) \le \exp\left(\frac{T}{m\lambda_1^{1-\beta}}\right), \forall t \in [0,T), j \in \mathbb{N}^*$. We also obtain

$$\|\mathcal{M}_2(u)(t,\cdot)\|_{L^2(\Omega)} \le \int_t^T \left\|\sum_{j=1}^{\infty}\frac{1}{1+m\lambda_j}\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_j(u)(\tau)e_j\right\|_{L^2(\Omega)} \mathrm{d}\tau$$

$$= \int_t^T \left(\sum_{j=1}^{\infty} \frac{1}{(1+m\lambda_j)^2} \exp\left(\frac{2(\tau-t)\lambda_j^\beta}{1+m\lambda_j} \right) |f_j(u)(\tau)|^2 \right)^{1/2} \mathrm{d}\tau$$

$$\leq \frac{1}{m\lambda_1} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \|f(\tau,\cdot;u)\|_{L^2(\Omega)} \,\mathrm{d}\tau, \tag{3.9}$$

where we have used $\frac{1}{1+m\lambda_j} \leq \frac{1}{m\lambda_1}$, $\forall j \in \mathbb{N}^*$ and for $\beta \in (0,1)$ we implies that

$$\exp\left(\frac{2(\tau-t)\lambda_j^\beta}{1+m\lambda_j} \right) \leq \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right), \quad 0 \leq t \leq \tau \leq T.$$

Using (3.1), we obtain

$$\|f(t,\cdot;u) - f(t,\cdot;0)\|_{L^2(\Omega)} \leq K\|u(t,\cdot)\|_{L^2(\Omega)},$$

then

$$\|f(t,\cdot;u)\|_{L^2(\Omega)} \leq K\|u(t,\cdot)\|_{L^2(\Omega)} + \|f(t,\cdot;0)\|_{L^2(\Omega)}, \tag{3.10}$$

Combining (3.10) with (3.9), leads to

$$\begin{aligned}\|\mathcal{M}_2(u)(t,\cdot)\|_{L^2(\Omega)} \leq \frac{1}{m\lambda_1} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \|f(0)\|_{L^1(0,T;L^2(\Omega))} \\ + \frac{K}{m\lambda_1} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \|u(\tau,\cdot)\|_{L^2(\Omega)} \,\mathrm{d}\tau.\end{aligned} \tag{3.11}$$

From (3.8) and (3.11) yields

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(\Omega)} \leq \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \left(\|\varphi\|_{L^2(\Omega)} + \frac{1}{m\lambda_1} \|f(0)\|_{L^1(0,T;L^2(\Omega))} \right) \\ + \frac{K}{m\lambda_1} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \|u(\tau,\cdot)\|_{L^2(\Omega)} \,\mathrm{d}\tau.\end{aligned}$$

Thanks to Grönall's inequality

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(\Omega)} \leq \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \left(\|\varphi\|_{L^2(\Omega)} + \frac{1}{m\lambda_1} \|f(0)\|_{L^1(0,T;L^2(\Omega))} \right) \\ \times \exp\left(\frac{K}{m\lambda_1} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) (T-t) \right),\end{aligned}$$

this imlpies (3.6).
b) We observe that for $\beta \in (0,1)$

$$\begin{aligned}\|\mathcal{M}_1(\varphi)(t,\cdot)\|_{H^\beta(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2\beta} \exp\left(\frac{2(T-t)\lambda_j^\beta}{1+m\lambda_j} \right) \varphi_j^2 \right)^{1/2} \\ \leq \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \|\varphi\|_{H^\beta(\Omega)}.\end{aligned} \tag{3.12}$$

We conclude that

$$\begin{aligned}
\|\mathcal{M}_2(u)(t,\cdot)\|_{H^\beta(\Omega)} &\le \int_t^T \left\| \sum_{j=1}^{\infty} \frac{1}{1+m\lambda_j} \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j} \right) f_j(u)(\tau) e_j \right\|_{H^\beta(\Omega)} \mathrm{d}\tau \qquad (3.13)\\
&= \int_t^T \sqrt{ \sum_{j=1}^{\infty} \frac{\lambda_j^{2\beta}}{(1+m\lambda_j)^2} \exp\left(\frac{2(\tau-t)\lambda_j^\beta}{1+m\lambda_j} \right) |f_j(u)(\tau)|^2 \mathrm{d}\tau } \\
&= \int_t^T \sqrt{ \sum_{j=1}^{\infty} \frac{1}{m^2\lambda_j^{2-2\beta}} \exp\left(\frac{2(\tau-t)\lambda_j^\beta}{1+m\lambda_j} \right) |f_j(u)(\tau)|^2 \mathrm{d}\tau } \\
&= \frac{1}{m\lambda_1^{1-\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \sqrt{ \sum_{j=1}^{\infty} |f_j(u)(\tau)|^2 \mathrm{d}\tau } \\
&\le \frac{1}{m\lambda_1^{1-\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \|f(\tau,\cdot;u)\|_{L^2(\Omega)} \mathrm{d}\tau.
\end{aligned}$$

Inequality (3.10) associated with (3.13) leads to

$$\begin{aligned}
&\|\mathcal{M}_2(u)(t,\cdot)\|_{H^\beta(\Omega)} \\
&\le \int_t^T \frac{1}{m\lambda_1^{1-\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \Big(K\|u(\tau,\cdot)\|_{L^2(\Omega)} + \|f(\tau,\cdot;0)\|_{L^2(\Omega)} \Big) \mathrm{d}\tau \\
&\le \frac{T}{m\lambda_1^{1-\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \|f(0)\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \frac{K}{m\lambda_1^{1-\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \|u(\tau,\cdot)\|_{L^2(\Omega)} \mathrm{d}\tau. \qquad (3.14)
\end{aligned}$$

Estimates (3.12) and (3.14) lead to

$$\begin{aligned}
\|u(t,\cdot)\|_{H^\beta(\Omega)} &\le \left\| \sum_{j=1}^{\infty} \exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j} \right) \varphi_j e_j \right\|_{H^\beta(\Omega)} \\
&+ \int_t^T \left\| \sum_{j=1}^{\infty} \frac{1}{1+m\lambda_j} \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j} \right) f_j(u)(\tau) e_j \right\|_{H^\beta(\Omega)} \mathrm{d}\tau \\
&\le \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \left(\|\varphi\|_{H^\beta(\Omega)} + \frac{T}{m\lambda_1^{1-\beta}} \|f(0)\|_{L^\infty(0,T;L^2(\Omega))} \right) \\
&+ \frac{K}{m\lambda_1^{1-2\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}} \right) \int_t^T \|u(\tau,\cdot)\|_{H^\beta(\Omega)} \mathrm{d}\tau, \qquad (3.15)
\end{aligned}$$

where we have used that $\|w\|_{H^\beta(\Omega)} \ge \lambda_1^\beta \|w\|_{L^2(\Omega)}$ for $\beta \in (0,1)$. Grönwall's inequality allows to obtain

$$\|u(t,\cdot)\|_{H^\beta(\Omega)} \le \left(\|\varphi\|_{H^\beta(\Omega)} + \frac{T}{m\lambda_1^{1-\beta}} \|f(0)\|_{L^\infty(0,T;L^2(\Omega))} \right)$$

$$\times \exp\left(\frac{KT}{m\lambda_1^{1-2\beta}} \exp\left(\frac{T}{m\lambda_1^{1-\beta}}\right) + \frac{T}{m\lambda_1^{1-\beta}}\right),$$

we implies (3.7). The proof is complete. □

4. The case $\beta > 1$: Ill-posedness and regularization methods for Problem ($\mathbb{P}_T$).

4.1. Ill-posedness of the solution of Problem ($\mathbb{P}_T$).

Definition 4.1 (Ill-posed)**.** The well-posed problem in the sense of Hadamard is to satisfy the following properties:
i) a solution exists;
ii) the solution is unique;
iii) the solution's behaviour changes continuously with the initial conditions.
If at least one of the three properties above does not satisfy, the problem is ill-posed.

Next, we give an example which shows that the solution $\widetilde{u}^{(k)}(t,x)$ (for any $k \in \mathbb{N}^*$) of Problem ($\mathbb{P}_T$) is not stable (property iii) is unsatisfied). For $\beta > 1$, let us set

$$\widetilde{\varphi}^{(k)}(x) = \lambda_k^{-1} e_k(x), \quad \forall k \in \mathbb{N}^*, \tag{4.1}$$

$$\widetilde{f}(t,x;w) = \sum_{j=1}^{\infty} \lambda_j T^{-1} \exp\left(\frac{-T\lambda_j^{\beta}}{1+m\lambda_j}\right) \langle w(t,\cdot), e_j\rangle_{L^2(\Omega)} e_j(x),\ m > 1, \tag{4.2}$$

for $(t,x) \in (0,T) \times \Omega$ and $w \in C([0,T];L^2(\Omega))$. The solution $\widetilde{u}^{(k)}(t,x)$ satisfies the integral equation

$$\begin{aligned}\widetilde{u}^{(k)}(t,x) = &\sum_{j=1}^{\infty} \left[\exp\left(\frac{(T-t)\lambda_j^{\beta}}{1+m\lambda_j}\right) \widetilde{\varphi}_j^{(k)}\right] e_j(x) \\ &- \sum_{j=1}^{\infty} \left[\frac{1}{1+m\lambda_j} \int_t^T \exp\left(\frac{(\tau-t)\lambda_j^{\beta}}{1+m\lambda_j}\right) \widetilde{f}_j(u^{(k)})(\tau) \mathrm{d}\tau\right] e_j(x),\end{aligned} \tag{4.3}$$

where

$$\widetilde{\varphi}_j^{(k)} = \left\langle \widetilde{\varphi}^{(k)}, e_j \right\rangle_{L^2(\Omega)},$$

and

$$\widetilde{f}_j(\widetilde{u}^{(k)})(t) = \left\langle \widetilde{f}(t,\cdot;\widetilde{u}^{(k)}), e_j \right\rangle_{L^2(\Omega)}.$$

• *Step 1. We show that* (4.3) *has a unique solution* $\widetilde{u}^{(k)} \in C([0,T];L^2(\Omega))$.
Indeed, we consider the function

$$\begin{aligned}\mathscr{G}(w)(t,x) = &\sum_{j=1}^{\infty} \left[\exp\left(\frac{(T-t)\lambda_j^{\beta}}{1+m\lambda_j}\right) \widetilde{\varphi}_j^{(k)}\right] e_j(x) \\ &- \sum_{j=1}^{\infty} \left[\frac{1}{1+m\lambda_j} \int_t^T \exp\left(\frac{(\tau-t)\lambda_j^{\beta}}{1+m\lambda_j}\right) \widetilde{f}_j(w)(\tau) \mathrm{d}\tau\right] e_j(x).\end{aligned}$$

Then for any $w_1, w_2 \in C([0,T]; L^2(\Omega))$, we obtain

$$\begin{aligned}
&\|\mathscr{G}(w_1)(t,\cdot) - \mathscr{G}(w_2)(t,\cdot)\|_{L^2(\Omega)} \\
&\le \int_t^T \left\| \sum_{j=1}^{\infty} \left[\frac{1}{1+m\lambda_j} \exp\left(\frac{(\tau - t)\lambda_j^\beta}{1+m\lambda_j} \right) \left(\widetilde{f}_j(w_1)(\tau) - \widetilde{f}_j(w_2)(\tau) \right) \right] e_j \right\|_{L^2(\Omega)} \mathrm{d}\tau \\
&= \int_t^T \left[\sum_{j=1}^{\infty} \left(\frac{1}{1+m\lambda_j} \exp\left(\frac{(\tau - t)\lambda_j^\beta}{1+m\lambda_j} \right) \right)^2 \lambda_j^2 T^{-2} \exp\left(\frac{-T\lambda_j^\beta}{1+m\lambda_j} \right) |w_{1,j}(\tau) - w_{2,j}(\tau)|^2 \right]^{1/2} \mathrm{d}\tau \\
&= \int_t^T \left[\sum_{j=1}^{\infty} \frac{1}{m^2T^2} \exp\left(\frac{(\tau - t - T)\lambda_j^\beta}{1+m\lambda_j} \right) |w_{1,j}(\tau) - w_{2,j}(\tau)|^2 \right]^{1/2} \mathrm{d}\tau \\
&\le \int_t^T \frac{1}{mT} \|w_1(\tau,\cdot) - w_2(\tau,\cdot)\|_{L^2(\Omega)} \mathrm{d}\tau \le \frac{1}{m} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))},
\end{aligned}$$

where we denote $w_{i,j}(t) := \langle w_i(t,\cdot), e_j \rangle_{L^2(\Omega)}, \ i = 1, 2$.

This implies that

$$\|\mathscr{G}(w_1) - \mathscr{G}(w_2)\|_{C([0,T];L^2(\Omega))} \le \frac{1}{m} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}.$$

Hence $\mathscr{G}$ is a contraction because $m > 1$. Using the Banach fixed-point theorem, we conclude that $\mathscr{G}(w) = w$ has a unique solution $\widetilde{u}^{(k)} \in C([0,T]; L^2(\Omega))$.

• Step 2. *The solution of Problem (4.3) is instable.* We have

$$\begin{aligned}
\|\widetilde{u}^{(k)}(t,\cdot)\|_{L^2(\Omega)} \ge & \underbrace{\left\| \sum_{j=1}^{\infty} \exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j} \right) \widetilde{\varphi}_j^{(k)} e_j \right\|_{L^2(\Omega)}}_{=:\mathcal{M}_3\left(\widetilde{\varphi}^{(k)}\right)(t)} \\
& - \underbrace{\left\| \sum_{j=1}^{\infty} \left[\frac{1}{1+m\lambda_j} \int_t^T \exp\left(\frac{(\tau - t)\lambda_j^\beta}{1+m\lambda_j} \right) \widetilde{f}_j(\widetilde{u}^{(k)})(\tau) \mathrm{d}\tau \right] e_j \right\|_{L^2(\Omega)}}_{=:\mathcal{M}_4\left(\widetilde{u}^{(k)}\right)(t)}.
\end{aligned}$$

It is easy to see that (here, noting that from (4.2), we have $\widetilde{f}_j(0) = 0$)

$$\begin{aligned}
\left| \mathcal{M}_4\left(\widetilde{u}^{(k)}\right)(t) \right| &= \|\mathscr{G}(\widetilde{u}^{(k)})(t) - \mathscr{G}(0)(t)\|_{L^2(\Omega)} \\
&\le \frac{1}{m} \|\widetilde{u}^{(k)}\|_{C([0,T];L^2(\Omega))}.
\end{aligned}$$

Hence

$$\|\widetilde{u}^{(k)}(t,\cdot)\|_{L^2(\Omega)} \ge \left| \mathcal{M}_3(\widetilde{\varphi}^{(k)})(t) \right| - \frac{1}{m} \|\widetilde{u}^{(k)}\|_{C([0,T];L^2(\Omega))}.$$

This leads to

$$\|\widetilde{u}^{(k)}\|_{C([0,T];L^2(\Omega))} \ge \frac{m}{m+1} \sup_{0 \le t \le T} \left| \mathcal{M}_3(\widetilde{\varphi}^{(k)})(t) \right|. \tag{4.4}$$

We continue to estimate the right hand side of the latter inequality. Indeed, since $\{e_j(x)\}_{j\ge1}$ is a basis of $L^2(\Omega)$, i.e.,

$$\begin{cases} \langle e_k, e_j \rangle_{L^2(\Omega)} = 1, & k = j, \\ \langle e_k, e_j \rangle_{L^2(\Omega)} = 0, & k \ne j, \end{cases}$$

we have

$$\left|\mathcal{M}_3\left(\widetilde{\varphi}^{(k)}\right)(t)\right| = \sqrt{\sum_{j=1}^{\infty} \exp\left(\frac{2(T-t)\lambda_j^{\beta}}{1+m\lambda_j}\right)\left\langle \widetilde{\varphi}_j^{(k)}, e_j\right\rangle_{L^2(\Omega)}^2}$$
$$= \sqrt{\sum_{j=1}^{\infty} \exp\left(\frac{2(T-t)\lambda_j^{\beta}}{1+m\lambda_j}\right)\left\langle \lambda_k^{-1} e_k, e_j\right\rangle_{L^2(\Omega)}^2}$$
$$= \frac{1}{\lambda_k}\exp\left(\frac{(T-t)\lambda_k^{\beta}}{1+m\lambda_k}\right).$$

Since the function $\chi(t) := \exp\left(\dfrac{(T-t)\lambda_k^{\beta}}{1+m\lambda_k}\right)$ is a decreasing function with respect to variable $t \in [0,T]$ and $\beta > 1$, we deduce that

$$\sup_{0\le t\le T}\left|\mathcal{M}_3\left(\widetilde{\varphi}^{(k)}\right)(t)\right| \geq \sup_{0\le t\le T}\left(\frac{1}{\lambda_k}\exp\left(\frac{(T-t)\lambda_k^{\beta}}{1+m\lambda_k}\right)\right) = \exp\left(\frac{T\lambda_k^{\beta}}{1+m\lambda_k}\right)\frac{1}{\lambda_k}. \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$\left\|\widetilde{u}^{(k)}\right\|_{C([0,T];L^2(\Omega))} \geq \frac{m}{m+1}\exp\left(\frac{T\lambda_k^{\beta}}{1+m\lambda_k}\right)\frac{1}{\lambda_k}.$$

As $k \nearrow \infty$, we see that

$$\lim_{k\nearrow\infty}\left\|\widetilde{\varphi}^{(k)}\right\|_{L^2(\Omega)} = \lim_{k\nearrow\infty}\frac{\|e_k\|_{L^2(\Omega)}}{\lambda_k} = \lim_{k\nearrow\infty}\frac{1}{\lambda_k} = 0,$$

but

$$\lim_{k\nearrow\infty}\left\|\widetilde{u}^{(k)}\right\|_{C([0,T];L^2(\Omega))} \geq \lim_{k\nearrow\infty}\frac{m}{m+1}\exp\left(\frac{T\lambda_k^{\beta}}{1+m\lambda_k}\right)\frac{1}{\lambda_k} = \infty.$$

Thus, it is shown that Problem ($\mathbb{P}_T$)is ill-posed in the Hadamard sense in L^2-norm for $\beta > 1$.

4.2. **Regularization and error estimate.** In order to obtain the stable numerical solutions, we propose two regularization methods to solve the Problem ($\mathbb{P}_T$) in two cases of f:

- f is globally Lipschitz: MLR method.
- f is locally Lipschitz: FTR method.

4.2.1. *MLR method: Globally Lipschitz source term.* In this subsection, the functions $f(t,x;u)$ is satisfy the globally Lipschitz ($\mathbf{A_1}$). To approximate u, we introduce the regularized solutions $\mathcal{U}_\alpha^\varepsilon$ given by MLR method as follows

$$\mathcal{U}_\alpha^\varepsilon(t,x) = \sum_{j=1}^{\infty}\left[\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^{\beta}}{1+m\lambda_j}\right)\varphi_j^\varepsilon\right]e_j(x)$$
$$-\sum_{j=1}^{\infty}\frac{1}{1+m\lambda_j}\left[\int_t^T \mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^{\beta}}{1+m\lambda_j}\right)f_j(\mathcal{U}_\alpha^\varepsilon)(\tau)\mathrm{d}\tau\right]e_j(x), \tag{4.6}$$

where we set

$$\mathcal{L}_j(\alpha,T)=\frac{\exp\left(\dfrac{-T\lambda_j^\beta}{1+m\lambda_j}\right)}{\alpha\lambda_j^\beta+\exp\left(\dfrac{-T\lambda_j^\beta}{1+m\lambda_j}\right)},\tag{4.7}$$

and the coefficient $\alpha:=\alpha(\varepsilon)$ satifies $\lim\limits_{\varepsilon\to 0^+}\alpha=0$; it plays the role of a regularization parameter.

The following technical lemma plays the key role in our analysis.

Lemma 4.2. *For $0\le t\le\tau\le T$, we have*

$$(a)\qquad \left|\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\right|\le\alpha^{\frac{t}{T}-1}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}},\tag{4.8}$$

$$(b)\qquad \left|\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\right|\le\alpha^{\frac{t-\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{\tau-t}{T}}.\tag{4.9}$$

Proof. *a*). First, from (4.7), it holds

$$\begin{aligned}
\left|\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\right|&=\frac{\exp\left(\frac{-t\lambda_j^\beta}{1+m\lambda_j}\right)}{\alpha\lambda_j^\beta+\exp\left(\frac{-T\lambda_j^\beta}{1+m\lambda_j}\right)}\\
=\frac{\exp\left(\frac{-t\lambda_j^\beta}{1+m\lambda_j}\right)}{\left(\alpha\lambda_j^\beta+\exp\left(\frac{-T\lambda_j^\beta}{1+m\lambda_j}\right)\right)^{\frac{t}{T}}\left(\alpha\lambda_j^\beta+\exp\left(\frac{-T\lambda_j^\beta}{1+m\lambda_j}\right)\right)^{1-\frac{t}{T}}}&\\
\le\frac{1}{\left(\alpha\lambda_j^\beta+\exp\left(\frac{-T\lambda_j^\beta}{1+m\lambda_j}\right)\right)^{1-\frac{t}{T}}}\le\frac{1}{\left(\alpha\lambda_j^\beta+\exp\left(-T\lambda_j^\beta\right)\right)^{1-\frac{t}{T}}}.&
\end{aligned}\tag{4.10}$$

Using the inequality

$$\frac{1}{\nu_1\zeta+\exp\left(-\zeta\nu_2\right)}\le\frac{\frac{\nu_2}{\nu_1}}{\log(\frac{\nu_2}{\nu_1})},\tag{4.11}$$

for $\nu_i>0,\ i=1,2$ and $0<\zeta<e\nu_2$, if $\alpha<eT$, we obtain

$$\frac{1}{\left(\alpha\lambda_j^\beta+\exp\left(-T\lambda_j^\beta\right)\right)^{1-\frac{t}{T}}}\le\left(\frac{\alpha^{-1}T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}.$$

Whereupon

$$\left|\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\right|\le\left(\frac{\alpha^{-1}T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}.\tag{4.12}$$

Using (4.12) into (4.10), we obtain (4.8).

With the same argument as in the proof of (4.8), we obtain (4.9). This concludes the proof of the lemma. □

A. The well-posedness of the regularized solution (4.6). In this subsection, we will obtain the existence and regularity results for the regularized solution (4.6).

Theorem 4.3 (Existence-uniqueness)**.** *Suppose that f satisfying the assumption $(\mathbf{A_1})$ then the regularized solution (4.6) has unique weak solution $\mathcal{U}_\alpha^\varepsilon \in C\big([0,T]; L^2(\Omega)\big)$.*

Proof. For any $\mathscr{F} \in C([0,T]; L^2(\Omega))$, we define the function

$$\mathscr{F} : C([0,T]; L^2(\Omega)) \to C([0,T]; L^2(\Omega))$$

by

$$\begin{aligned}\mathscr{F}(\vartheta)(t,x) := &\sum_{j=1}^{\infty}\left[\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^\varepsilon\right]e_j(x)\\ &-\sum_{j=1}^{\infty}\left[\frac{1}{1+m\lambda_j}\int_t^T \mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)f_j(\vartheta)(\tau)\mathrm{d}\tau\right]e_j(x).\end{aligned} \tag{4.13}$$

We also define $\mathscr{F}^k$ as follows

$$\mathscr{F}^k(\vartheta) = \underbrace{\mathscr{F}\cdots\mathscr{F}\big(\mathscr{F}(\vartheta)\big)}_{k-\text{times}}.$$

We shall prove by induction, for any couple $\vartheta_1, \vartheta_2 \in C([0,T]; L^2(\Omega))$, that

$$\left\|\mathscr{F}^k(\vartheta_1)(t,\cdot) - \mathscr{F}^k(\vartheta_2)(t,\cdot)\right\|_{H^2(\Omega)} \le \frac{\left(K\frac{\alpha^{-1}T}{m\log\left(\alpha^{-1}T\right)}(T-t)\right)^k}{k!}\|\vartheta_1-\vartheta_2\|_{C([0,T];H^2(\Omega))}. \tag{4.14}$$

For $k=1$, by using $(\mathbf{A_1})$ and Lemma 4.2 and noting that $\frac{\alpha^{-1}T}{\log(\alpha^{-1}T)} \ge 1$, it holds

$$\alpha^{\frac{t-\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{\tau-t}{T}} \le \frac{\alpha^{-1}T}{\log\left(\alpha^{-1}T\right)}, \quad \text{for all } 0\le t\le\tau\le T,$$

then we have

$$\begin{aligned}&\|\mathscr{F}(\vartheta_1)(t,\cdot)-\mathscr{F}(\vartheta_2)(t,\cdot)\|_{L^2(\Omega)}\\ &\le \int_t^T\left\|\sum_{j=1}^{\infty}\left[\frac{1}{1+m\lambda_j}\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)(f_j(\vartheta_1)(\tau)-f_j(\vartheta_2)(\tau))\right]e_j\right\|_{L^2(\Omega)}\mathrm{d}\tau\\ &\le \int_t^T\left[\sum_{j=1}^{\infty}\frac{1}{(1+m\lambda_j)^2}\left|\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\right|^2|f_j(\vartheta_1)(\tau)-f_j(\vartheta_2)(\tau)|^2\right]^{1/2}\mathrm{d}\tau\\ &\le K\frac{\alpha^{-1}T}{m\lambda_1\log\left(\alpha^{-1}T\right)}\int_t^T\left[\sum_{j=1}^{\infty}|\vartheta_{1,j}(\tau)-\vartheta_{2,j}(\tau)|^2\right]^{1/2}\mathrm{d}\tau\\ &\le K\frac{\alpha^{-1}T}{m\lambda_1\log\left(\alpha^{-1}T\right)}\int_t^T\|\vartheta_1(\tau,\cdot)-\vartheta_2(\tau,\cdot)\|_{L^2(\Omega)}\mathrm{d}\tau\\ &\le K\frac{\alpha^{-1}T(T-t)}{m\lambda_1\log\left(\alpha^{-1}T\right)}\|\vartheta_1-\vartheta_2\|_{C([0,T];L^2(\Omega))}, \qquad \vartheta_{i,j} = \langle\vartheta_i, e_j\rangle_{L^2(\Omega)}, i=1,2.\end{aligned}$$

Hence, (4.14) holds for $k=1$.

Assume that (4.14) holds for $k = N$. We show now that (4.14) holds for $k = N+1$. In fact, we have

$$\begin{aligned}
&\left\|\mathscr{F}^{N+1}(\vartheta_1)(t,\cdot) - \mathscr{F}^{N+1}(\vartheta_2)(t,\cdot)\right\|_{H^\nu(\Omega)} \\
&\le K\frac{\alpha^{-1}T}{m\log(\alpha^{-1}T)}\int_t^T \left\|\mathscr{F}^{N}(\vartheta_1)(\tau,\cdot) - \mathscr{F}^{N}(\vartheta_2)(\tau,\cdot)\right\|_{L^2(\Omega)} \mathrm{d}\tau \\
&\le \left(K\frac{\alpha^{-1}T}{m\log(\alpha^{-1}T)}\right)^{N+1}\int_t^T \frac{(T-\tau)^N}{N!}\mathrm{d}\tau \left\|\vartheta_1 - \vartheta_2\right\|_{C([0,T];L^2(\Omega))} \\
&\le \frac{1}{(N+1)!}\left(K\frac{\alpha^{-1}T}{m\log(\alpha^{-1}T)}(T-t)\right)^{N+1} \left\|\vartheta_1 - \vartheta_2\right\|_{C([0,T];L^2(\Omega))}.
\end{aligned}$$

By the induction principle, we deduce that (4.14) holds for all $k \in \mathbb{N}^*$. Notice that, as α is fixed, then $\frac{1}{k!}\left(K\frac{\alpha^{-1}T}{m\log(\alpha^{-1}T)}(T-t)\right)^k$ tends to 0 when $k \nearrow \infty$, so there exists a positive integer k_0 such that

$$\frac{1}{k_0!}\left(K\frac{\alpha^{-1}T}{m\log(\alpha^{-1}T)}(T-t)\right)^{k_0} < 1.$$

It means that $\mathscr{F}^{k_0}$ is a contraction. Finally, it follows the desired conclusion that the problem (4.6) has a unique solution $\mathcal{U}_\alpha^\varepsilon \in C([0,T];L^2(\Omega))$. The proof is complete. $\square$

Given a constant $\alpha \in (0,1)$ (which will be assumed from now on) and a function $w \in C([0,T];L^2(\Omega))$, we denote the scaling with α as follows:

$$\|w\|_{\alpha,\infty} = \sup_{0\le t\le T}\left(\alpha^{-\frac{t}{T}}\|w(t,\cdot)\|_{L^2(\Omega)}\right). \tag{4.15}$$

Theorem 4.4 (Regularity)**.** *Assume that f satisfy the assumptions* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$). *We have the following results:*
a) If $\varphi^\varepsilon \in L^2(\Omega)$, then $\mathcal{U}_\alpha^\varepsilon$ given by (4.6) satisfies

$$\|\mathcal{U}_\alpha^\varepsilon\|_{\alpha,\infty} \le \frac{\alpha^{-1}T}{\log(T)}\exp\left(\frac{KT^2}{m\lambda_1\log(T)}\right)\|\varphi^\varepsilon\|_{L^2(\Omega)}. \tag{4.16}$$

b) If $\varphi^\varepsilon \in H^1(\Omega)$, then

$$\|\mathcal{U}_\alpha^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le \frac{\alpha^{-1}T}{\log(T)}\exp\left(\frac{KT^2\alpha^{-1}}{m\lambda_1\log(T)}\right)\|\varphi^\varepsilon\|_{H^1(\Omega)}. \tag{4.17}$$

Proof. We rewrite (4.6) as

$$\begin{aligned}
\mathcal{U}_\alpha^\varepsilon(t,x) = &\underbrace{\sum_{j=1}^\infty\left[\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^\varepsilon\right]e_j(x)}_{=:\mathcal{M}_5(\varphi^\varepsilon)(t,x)} \\
&\underbrace{-\sum_{j=1}^\infty\left[\frac{1}{1+m\lambda_j}\int_t^T\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)f_j(\mathcal{U}_\alpha^\varepsilon)(\tau)\mathrm{d}\tau\right]e_j(x)}_{=:\mathcal{M}_6(\mathcal{U}_\alpha^\varepsilon)(t,x)}.
\end{aligned} \tag{4.18}$$

a) Using (3.10) (noting that $f(t,x;0)=0$, for all $(t,x)\in(0,T)\times\Omega$) and Lemma 4.2b), we first observe that

$$\begin{aligned}
&\|\mathcal{M}_6(\mathcal{U}_\alpha^\varepsilon)(t,\cdot)\|_{L^2(\Omega)}\\
&\le \int_t^T \sqrt{\sum_{j=1}^\infty \left[\frac{1}{1+m\lambda_j}\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_j(\mathcal{U}_\alpha^\varepsilon)(\tau)\right]^2}\,\mathrm{d}\tau\\
&\le \frac{1}{m\lambda_1}\int_t^T \alpha^{\frac{t-\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{\tau-t}{T}} \|f(\tau,\cdot;\mathcal{U}_\alpha^\varepsilon)\|_{L^2(\Omega)}\,\mathrm{d}\tau\\
&\le \frac{KT}{m\lambda_1\log(T)}\int_t^T \alpha^{\frac{t-\tau}{T}}\|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)\|_{L^2(\Omega)}\,\mathrm{d}\tau,
\end{aligned}\tag{4.19}$$

where we have used $\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{\tau-t}{T}}\le\frac{T}{\log(T)}$, for $\alpha\in(0,1), 0\le t\le\tau\le T$.

The next observation, from Lemma 4.2a), is that

$$\begin{aligned}
&\|\mathcal{M}_5(\varphi^\varepsilon)(t,\cdot)\|_{L^2(\Omega)}\\
&\le \sum_{j=1}^\infty\left[\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^\varepsilon\right]^2\\
&\le \alpha^{\frac{t}{T}-1}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}\|\varphi^\varepsilon\|_{L^2(\Omega)} \le \alpha^{\frac{t}{T}-1}\frac{T}{\log(T)}\|\varphi^\varepsilon\|_{L^2(\Omega)},
\end{aligned}\tag{4.20}$$

noting that $\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{\tau-t}{T}}\le\frac{T}{\log(T)}$, for all $0\le t\le T, 0<\alpha<1$.

Combining (4.18)-(4.20) and multiplying the two sides of of the result obtained by $\alpha^{-\frac{t}{T}}$, we deduce that

$$\begin{aligned}
&\alpha^{-\frac{t}{T}}\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)\|_{L^2(\Omega)}\\
&\le \|\mathcal{M}_5(\varphi^\varepsilon)(t,\cdot)\|_{L^2(\Omega)}+\|\mathcal{M}_6(\mathcal{U}_\alpha^\varepsilon)(t,\cdot)\|_{L^2(\Omega)}\\
&\le \alpha^{-1}\frac{T}{\log(T)}\|\varphi^\varepsilon\|_{L^2(\Omega)}+\frac{KT}{m\lambda_1\log(T)}\int_t^T\alpha^{-\frac{\tau}{T}}\|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)\|_{L^2(\Omega)}\,\mathrm{d}\tau.
\end{aligned}$$

Using Grönwall's inequality, we get

$$\alpha^{-\frac{t}{T}}\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)\|_{L^2(\Omega)}\le\frac{\alpha^{-1}T}{\log(T)}\|\varphi^\varepsilon\|_{L^2(\Omega)}\exp\left(\frac{KT^2}{m\lambda_1\log(T)}\right).\tag{4.21}$$

Since the right side of (4.21) does not depend on t, we have

$$\|\mathcal{U}_\alpha^\varepsilon\|_{\alpha,\infty}\le\frac{\alpha^{-1}T}{\log(T)}\exp\left(\frac{KT^2}{m\lambda_1\log(T)}\right)\|\varphi^\varepsilon\|_{L^2(\Omega)}.$$

This leads to (4.16).

b) From Lemma 4.2a), we first observe that

$$\begin{aligned}
\|\mathcal{M}_5(\varphi^\varepsilon)(t,\cdot)\|_{H^1(\Omega)} &\le \sum_{j=1}^\infty\left[\lambda_j\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^\varepsilon\right]^2\\
&\le \alpha^{\frac{t}{T}-1}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}\|\varphi^\varepsilon\|_{H^1(\Omega)}\le\frac{\alpha^{-1}T}{\log(T)}\|\varphi^\varepsilon\|_{H^1(\Omega)}.
\end{aligned}\tag{4.22}$$

The next observation, using Lemma 4.2b), is that

$$
\begin{aligned}
&\|\mathcal{M}_6(\mathcal{U}_\alpha^\varepsilon)(t,\cdot)\|_{H^1(\Omega)} \\
&\le \int_t^T \sqrt{\sum_{j=1}^\infty \left[\frac{\lambda_j}{1+m\lambda_j}\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_j(\mathcal{U}_\alpha^\varepsilon)(\tau)\right]^2}\, \mathrm{d}\tau \\
&\le \frac{1}{m}\int_t^T \alpha^{\frac{t-\tau}{T}} \left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{\tau-t}{T}} \|f(\tau,\cdot;\mathcal{U}_\alpha^\varepsilon)\|_{L^2(\Omega)}\, \mathrm{d}\tau \\
&\le \frac{KT\alpha^{-1}}{m\log(T)}\int_t^T \|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)\|_{L^2(\Omega)}\, \mathrm{d}\tau,
\end{aligned} \tag{4.23}
$$

where we have used $\alpha^{\frac{t-\tau}{T}}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{\tau-t}{T}} \le \frac{\alpha^{-1}T}{\log(T)}$, $0\le t\le\tau\le T$ for $0<\alpha<1$.

From (4.22) and (4.23), we deduce that

$$
\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)\|_{H^1(\Omega)} \le \frac{\alpha^{-1}T}{\log(T)}\|\varphi^\varepsilon\|_{H^1(\Omega)} + \frac{KT\alpha^{-1}}{m\lambda_1\log(T)}\int_t^T \|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)\|_{H^1(\Omega)}\, \mathrm{d}\tau,
$$

where $\|w\|_{L^2(\Omega)} \le \frac{\|w\|_{H^1(\Omega)}}{\lambda_1}$. Thanks to Grönwall's inequality, we get

$$
\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)\|_{H^1(\Omega)} \le \frac{\alpha^{-1}T}{\log(T)}\|\varphi^\varepsilon\|_{H^1(\Omega)} \exp\left(\frac{KT\alpha^{-1}}{m\lambda_1\log(T)}(T-t)\right),
$$

which implies (4.17). This concludes the proof. $\square$

B. Error estimate In this subsection, by using MLR method, the error between the exact solution and the regularized solution is obtained. Now, we can formulate the main theorem.

Theorem 4.5 (Error estimate)**.** *Let $\alpha := \alpha(\varepsilon)$ satisfy*

$$
\begin{cases}
\lim\limits_{\varepsilon\to 0^+} \alpha = 0, \\
\lim\limits_{\varepsilon\to 0^+} \varepsilon\alpha^{-1} = M_0 < \infty.
\end{cases} \tag{4.24}
$$

Assume that f satisfy the assumptions $(\mathbf{A_1})-(\mathbf{A_2})$ and Problem $(\mathbb{P}_T)$ has a unique solution u satisfying

$$
u \in L^\infty\left(0,T;\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)\right) \quad \text{and} \ \|u\|_{L^\infty(0,T;\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega))} \le P_0, \tag{4.25}
$$

for some known positive constants P_0 and $\eta_1\ge 2, \eta_2 \ge \frac{2T}{m\lambda_1}$. Then (for all $t\in[0,T]$)

$$
\|\mathcal{U}_\alpha^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} \le \sqrt{2}\,(M_0+P_0)\exp\left(\frac{K^2T}{m^2\lambda_1^2}(T-t)\right)\alpha^{\frac{t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}. \tag{4.26}
$$

Remark 1. The error estimate in (4.26) is of order $\mathcal{O}(\varepsilon) = \alpha^{\frac{t}{T}}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{1-\frac{t}{T}}$, $t\in[0,T]$.

- If $t\approx T$, then $\mathcal{O}(\varepsilon)=\alpha$ tends to zero according to (4.24).
- If $t\approx 0$, then $\mathcal{O}(\varepsilon) = T\log^{-1}\left(\alpha^{-1}T\right)$ tends to zero as $\varepsilon\to 0^+$.

Remark 2. Choose $\alpha = \varepsilon^r \xrightarrow{\varepsilon\to 0^+} 0$, for some $0<r<1$, then the estimate in (4.26) is of order

$$
\varepsilon^{r\frac{t}{T}}\left[T/\log\left(T/\varepsilon^{-r}\right)\right]^{\frac{t-T}{T}}.
$$

Proof. For all $t \in (0,T)$, the triangle inequality gives

$$\|\mathcal{U}_\alpha^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} \le \underbrace{\|\mathcal{U}_\alpha^\varepsilon(t,\cdot) - \mathcal{U}_\alpha(t,\cdot)\|_{L^2(\Omega)}}_{=:\mathcal{M}_7^{\alpha,\varepsilon}(t,x)} + \underbrace{\|\mathcal{U}_\alpha(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)}}_{=:\mathcal{M}_8^{\alpha,\varepsilon}(t,x)}, \tag{4.27}$$

where $\mathcal{U}_\alpha(t,x)$ is the solution of Problem (4.6) corresponding to exact datum $\varphi(x)$. In order to estimate $\mathcal{M}_7^{\alpha,\varepsilon}(t,x)$ and $\mathcal{M}_8^{\alpha,\varepsilon}(t,x)$, we need to divide the proof in two steps.

Step 1. Estimate $\mathcal{M}_7^{\alpha,\varepsilon}(t,x)$.

Thanks to Parseval's relation and basic inequality $(a+b)^2 \le 2a^2 + 2b^2$, $\forall a,b \ge 0$, we obtain

$$\begin{aligned}\|\mathcal{U}_\alpha^\varepsilon(t,\cdot) - \mathcal{U}_\alpha(t,\cdot)\|^2_{L^2(\Omega)} &\le \underbrace{2\sum_{j=1}^\infty \left[\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)(\varphi_j^\varepsilon - \varphi_j)\right]^2}_{=:\mathcal{M}_{7a}^{\alpha,\varepsilon}(t)} \\ &+ \underbrace{2\sum_{j=1}^\infty \left[\frac{1}{1+m\lambda_j}\int_t^T \mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)(f_j(\mathcal{U}_\alpha^\varepsilon)(\tau) - f_j(\mathcal{U}_\alpha)(\tau))\,\mathrm{d}\tau\right]^2}_{=:\mathcal{M}_{7b}^{\alpha,\varepsilon}(t)}.\end{aligned} \tag{4.28}$$

To estimate $\mathcal{M}_{7a}^{\alpha,\varepsilon}(t)$, let us use Lemma 4.2a) yields

$$\begin{aligned}&|\mathcal{M}_{7a}^{\alpha,\varepsilon}(t)| \\ &\le 2\sum_{j=1}^\infty \alpha^{\frac{2t}{T}-2}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{2-\frac{2t}{T}} \left|\varphi_j^\varepsilon - \varphi_j\right|^2 \\ &\le 2\alpha^{\frac{2t}{T}-2}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{2-\frac{2t}{T}} \|\varphi^\varepsilon - \varphi\|^2_{L^2(\Omega)} \le 2\alpha^{\frac{2t}{T}-2}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{2-\frac{2t}{T}} \varepsilon^2.\end{aligned} \tag{4.29}$$

Next, we estimate $M_{7b}^{\alpha,\varepsilon}(t)$. Using Hölder's inequality, $(\mathbf{A_1})$ and Lemma 4.2b) , one obtains

$$\begin{aligned}&|M_{7b}^{\alpha,\varepsilon}(t)| \\ &= 2\sum_{j=1}^\infty \left[\frac{1}{1+m\lambda_j}\int_t^T \mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)(f_j(\mathcal{U}_\alpha^\varepsilon)(\tau) - f_j(\mathcal{U}_\alpha)(\tau))\,\mathrm{d}\tau\right]^2 \\ &\le 2T\sum_{j=1}^\infty \frac{1}{(1+m\lambda_j)^2}\int_t^T \left|\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\right|^2 |f_j(\mathcal{U}_\alpha^\varepsilon)(\tau) - f_j(\mathcal{U}_\alpha)(\tau)|^2\,\mathrm{d}\tau \\ &\le \frac{2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{2t-2\tau}{T}}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{2\tau-2t}{T}} \sum_{j=1}^\infty |f_j(\mathcal{U}_\alpha^\varepsilon)(\tau) - f_j(\mathcal{U}_\alpha)(\tau)|^2\,\mathrm{d}\tau \\ &\le \frac{2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{2t-2\tau}{T}}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{2\tau-2t}{T}} \|f(\mathcal{U}_\alpha^\varepsilon)(\tau) - f(\mathcal{U}_\alpha)(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\end{aligned}$$

$$\leq \frac{2K^2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{2t-2\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2\tau-2t}{T}}\|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)-\mathcal{U}_\alpha(\tau,\cdot)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau. \tag{4.30}$$

Combining the results in (4.28)-(4.30), we get

$$\begin{aligned}
&\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)-\mathcal{U}_\alpha(t,\cdot)\|_{L^2(\Omega)}^2\\
&\leq 2\alpha^{\frac{2t}{T}-2}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2-\frac{2t}{T}}\varepsilon^2\\
&+\frac{2K^2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{2t-2\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2\tau-2t}{T}}\|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)-\mathcal{U}_\alpha(\tau,\cdot)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau.
\end{aligned}$$

Multiplying by $\alpha^{\frac{-2t}{T}}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{2t}{T}}$ both sides, we deduce that

$$\begin{aligned}
&\alpha^{\frac{-2t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2t}{T}}\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)-\mathcal{U}_\alpha(t,\cdot)\|_{L^2(\Omega)}^2\\
&\leq 2\alpha^{-2}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2}\varepsilon^2\\
&\quad+\frac{2K^2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{-2\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2\tau}{T}}\|\mathcal{U}_\alpha^\varepsilon(\tau,\cdot)-\mathcal{U}_\alpha(\tau,\cdot)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau.
\end{aligned}$$

Grönwall's inequality allows to obtain

$$\begin{aligned}
&\alpha^{\frac{-2t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2t}{T}}\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)-\mathcal{U}_\alpha(t,\cdot)\|_{L^2(\Omega)}^2\\
&\leq 2\alpha^{-2}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2}\varepsilon^2\exp\left(\frac{2K^2T}{m^2\lambda_1^2}(T-t)\right).
\end{aligned}$$

Similar calculations yield

$$\|\mathcal{U}_\alpha^\varepsilon(t,\cdot)-\mathcal{U}_\alpha(t,\cdot)\|_{L^2(\Omega)}\leq\sqrt{2}\alpha^{-1}\varepsilon\exp\left(\frac{K^2T}{m^2\lambda_1^2}(T-t)\right)\alpha^{\frac{t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}. \tag{4.31}$$

Step 2. Estimate $\mathcal{M}_8^{\alpha,\varepsilon}(t)$. Let us define an operator

$$\Theta w(t,x)=\sum_{j=1}^{\infty}\left[\mathcal{L}_j(\alpha,T)\left\langle w(t,\cdot),e_j\right\rangle_{L^2(\Omega)}\right]e_j(x),\quad \text{for } w\in C([0,T];L^2(\Omega)).$$

It clearly follows that

$$\begin{aligned}
\Theta u(t,x)&=\sum_{j=1}^{\infty}\left[\mathcal{L}_j(\alpha,T)\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j\right]e_j(x)\\
&\quad-\sum_{j=1}^{\infty}\left[\frac{1}{1+m\lambda_j}\int_t^T\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)f_j(u)(\tau)\mathrm{d}\tau\right]e_j(x).
\end{aligned}$$

The triangle inequality allows to write

$$\begin{aligned}
\|\mathcal{U}_\alpha(t,\cdot)-u(t,\cdot)\|_{L^2(\Omega)}^2&\leq 2\left\|\mathcal{U}_\alpha(t,\cdot)-\Theta u(t,\cdot)\right\|_{L^2(\Omega)}^2+2\left\|\Theta u(t,\cdot)-u(t,\cdot)\right\|_{L^2(\Omega)}^2\\
&=:\mathcal{M}_{8a}^\alpha(t)+\mathcal{M}_{8b}^\alpha(t).
\end{aligned}$$

We estimate for $\mathcal{M}_{8a}^{\alpha}(t)$ as follows (similarly as in (4.30)):

$$
\begin{aligned}
&|\mathcal{M}_{8a}^{\alpha}(t)| \\
&= 2\sum_{j=1}^{\infty}\left[\int_t^T \frac{1}{1+m\lambda_j}\mathcal{L}_j(\alpha,T)\exp\left(\frac{(\tau-t)\lambda_j^{\beta}}{1+m\lambda_j}\right)(f_j(\mathcal{U}_\alpha)(\tau)-f_j(u)(\tau))\,\mathrm{d}\tau\right]^2 \\
&\le \frac{2K^2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{2t-2\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2\tau-2t}{T}}\|\mathcal{U}_\alpha(\tau,\cdot)-u(\tau,\cdot)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau.
\end{aligned}
$$

The term $\mathcal{M}_{8b}^{\alpha}(t)$ is estimated by using (4.8) as

$$
\begin{aligned}
|\mathcal{M}_{8b}^{\alpha}(t)| &= \sum_{j=1}^{\infty}\left(\mathcal{L}_j(\alpha,T)-1\right)^2|u_j(t)|^2 \\
&= 2\sum_{j=1}^{\infty}\left(\frac{\exp\left(\frac{-T\lambda_j^{\beta}}{1+m\lambda_j}\right)}{\alpha\lambda_j^{\beta}+\exp\left(\frac{-T\lambda_j^{\beta}}{1+m\lambda_j}\right)}-1\right)^2|u_j(t)|^2 \\
&= 2\sum_{j=1}^{\infty}\left(\frac{\alpha\lambda_j^{\beta}}{\alpha\lambda_j^{\beta}+\exp\left(\frac{-T\lambda_j^{\beta}}{1+m\lambda_j}\right)}\right)^2|u_j(t)|^2 \\
&\le 2\alpha^2\sum_{j=1}^{\infty}\left(\frac{\exp\left(\frac{-t\lambda_j^{\beta}}{1+m\lambda_j}\right)}{\alpha\lambda_j^{\beta}+\exp\left(\frac{-T\lambda_j^{\beta}}{1+m\lambda_j}\right)}\right)^2\lambda_j^{2\beta}\exp\left(\frac{2T\lambda_j^{\beta}}{1+m\lambda_j}\right)|u_j(t)|^2 \\
&\le 2\alpha^2\sum_{j=1}^{\infty}\alpha^{\frac{2t}{T}-2}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2-\frac{2t}{T}}\lambda_j^{2\beta}\exp\left(\frac{2T}{m\lambda_1}\lambda_j^{\beta}\right)|u_j(t)|^2 \\
&\le 2\alpha^{\frac{2t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2-\frac{2t}{T}}\|u(t,\cdot)\|_{\mathbf{G}_{\beta}(\eta_1,\eta_2)(\Omega)}^2,
\end{aligned}
$$

where we have used $\dfrac{\lambda_j^{\beta}}{1+m\lambda_j}\le\dfrac{\lambda_j^{\beta}}{m\lambda_1}$, for all $j\in\mathbb{N}^*$ and $\eta_1\ge 2,\eta_2\ge\frac{2T}{m\lambda_1}$.

Combining all these inequalities, we deduce

$$
\begin{aligned}
&\|\mathcal{U}_\alpha(t,\cdot)-u(t,\cdot)\|_{L^2(\Omega)}^2 \\
&\le 2\alpha^{\frac{2t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2-\frac{2t}{T}}\|u(t,\cdot)\|_{\mathbf{G}_{\beta}(\eta_1,\eta_2)(\Omega)}^2 \\
&+\frac{2K^2T}{m^2\lambda_1^2}\int_t^T \alpha^{\frac{2t-2\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2\tau-2t}{T}}\|\mathcal{U}_\alpha(\tau,\cdot)-u(\tau,\cdot)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau.
\end{aligned}
$$

Multiplying by $\alpha^{\frac{-2t}{T}}\left(\frac{T}{\log(\alpha^{-1}T)}\right)^{\frac{2t}{T}}$, we obtain

$$
\alpha^{\frac{-2t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2t}{T}}\|\mathcal{U}_\alpha(t,\cdot)-u(t,\cdot)\|_{L^2(\Omega)}^2
$$

$$
\begin{aligned}
&\leq 2\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{2}\|u(t,\cdot)\|^2_{\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)}\\
&+\frac{2K^2T}{m^2\lambda_1^2}\int_t^T\alpha^{\frac{-2\tau}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{\frac{2\tau}{T}}\|\mathcal{U}_\alpha(\tau,\cdot)-u(\tau,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau.
\end{aligned}
$$

Using Grönwall's inequality, we thus obtain

$$
\begin{aligned}
&\|\mathcal{U}_\alpha(t,\cdot)-u(t,\cdot)\|_{L^2(\Omega)}\\
&\leq\sqrt{2}\alpha^{\frac{t}{T}}\left(\frac{T}{\log\left(\alpha^{-1}T\right)}\right)^{1-\frac{t}{T}}\|u(t,\cdot)\|_{\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)}\exp\left(\frac{K^2T}{m^2\lambda_1^2}(T-t)\right).
\end{aligned}\tag{4.32}
$$

Combining (4.27), (4.31) and (4.32), we complete the proof of Theorem 4.5. □

4.2.2. *FTR method: Locally Lipschitz source term.* In Subsection 4.2.1 has addressed the Problem ($\mathbb{P}_T$) in which f is a globally Lipschitz function, in the rest of this paper, we extend the analysis to a locally Lipschitz function f. Up to the present, the results of Problem ($\mathbb{P}_T$) for the locally Lipschitz cases are still very scarce. Hence, we have to find another regularization method to study the problem with the locally Lipschitz source. Thus, the FTR method is a very effective method for this case.

We begin by establishing the locally Lipschitz properties of f by the following assumption:

($\mathbf{A_3}$) Assume that for each $\varrho>0$, there exists $K_\varrho>0$ such that

$$
|f(t,x;u)-f(t,x;v)|\leq K_\varrho|u-v|,\quad \text{if } \max\{|u|,|v|\}\leq\varrho,\tag{4.33}
$$

and

$$
K_\varrho=\sup\left\{\left|\frac{f(t,x;u)-f(t,x;v)}{u-v}\right|,\quad |u|,|v|\leq\varrho,\quad u\neq v,\ (t,x)\in[0,T)\times\Omega\right\},
$$

K_ϱ is a non-decreasing function of ϱ. We assume that $\lim\limits_{\varrho\to\infty}K_\varrho=\infty$.

Example 1. Let $f_1(u)=u|u|^2$. Easy calculations show that

$$
\begin{aligned}
|f_1(u)-f_1(v)|&=\left|u|u|^2-v|v|^2\right|\\
&=\left|(u-v)|u|^2+v(|u|^2-|v|^2)\right|\\
&=\left(|u|^2+|v||u|+|v|^2\right)|u-v|\,.
\end{aligned}
$$

Clearly, f_1 is not globally Lipschitz. For $\varrho\geq\max\{|u|,|v|\}$, from (4.33), one can compute explicitely the coefficient $K_\varrho=3\varrho^2$.

Example 2. Let $f_2(u)=u(a-bu^2)$ (Ginzburg-Landau type), with $a\in\mathbb{R},b>0$. It can be easily seen that f_2 is locally Lipschitz source. Moreover, we are possible to verify computationally the coefficient $K_\varrho=3\varrho^2\max\{|a|,b\}$. In order to solve the problem with the locally Lipschitz sources as above (and some other types), we outline our ideas to construct an approximation of the function f. For all $\varrho>0$, we define

$$
f_\varrho(t,x;u):=f(t,x;\widetilde{u}),\quad \text{where}\quad \widetilde{u}:=\begin{cases}-\varrho, & \text{if}\quad u\in(-\infty,-\varrho),\\ u, & \text{if}\quad u\in[-\varrho,\varrho],\\ \varrho, & \text{if}\quad u\in(\varrho,\infty).\end{cases}\tag{4.34}
$$

With this definition, we claim that f_ϱ is global Lipschitz function. Before stating the main theorem, we first consider the following lemma.

Lemma 4.6. *Let $f_{\varrho^\varepsilon} \in L^\infty([0,T]\times\Omega\times\mathbb{R})$ given as (4.34). Then we have*

$$|f_{\varrho^\varepsilon}(t,x;u) - f_{\varrho^\varepsilon}(t,x;v)| \le K_{\varrho^\varepsilon}|u-v|, \quad \forall (t,x)\in[0,T)\times\Omega,\ u,v\in\mathbb{R}, \tag{4.35}$$

where $\lim_{\varepsilon\to 0^+}\varrho^\varepsilon = \lim_{\varepsilon\to 0^+} K_{\varrho^\varepsilon} = \infty$.

Proof. The proof can be found in [2]. □

Remark 3. For $\varrho^\varepsilon > 0$ satisfying $\lim\limits_{\varepsilon\to 0^+}\varrho^\varepsilon = \infty$, we implies that

$$\widetilde{u} \equiv u, \qquad \text{almost everywhere in } [0,T)\times\Omega,$$
$$f_{\varrho^\varepsilon}(t,x;u) \equiv f(t,x;\widetilde{u}), \quad \text{almost everywhere in } [0,T)\times\Omega.$$

Based on the above analysis, we propose the regularized solution by using FTR method as follows

$$\begin{aligned}&\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(t,x)\\ =&\sum_{j=1}^{[\Lambda^\varepsilon]}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^\varepsilon - \frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_{\varrho^\varepsilon,j}(\mathcal{V}^\varepsilon_{\Lambda^\varepsilon})(\tau)\mathrm{d}\tau\right] e_j(x),\end{aligned} \tag{4.36}$$

where $f_{\varrho^\varepsilon,j}(\mathcal{V}^\varepsilon_{\Lambda^\varepsilon})(\tau) = \langle f_{\varrho^\varepsilon}(\tau,\cdot;\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}), e_j\rangle_{L^2(\Omega)}$.

The regularity estimates of the solution $\mathcal{V}^\varepsilon_{\Pi^\alpha}$ given by (4.36) is possible that but we will not develop this point here because it is an argument analogous to the previous one. We continue with the error estimate result.

Theorem 4.7 (Error estimate)**.** *Suppose that we can choose $\varrho^\varepsilon, \Lambda^\varepsilon > 0$ such that $\lim_{\varepsilon\to 0^+}\Lambda^\varepsilon = \lim_{\varepsilon\to 0^+}\varrho^\varepsilon = \infty$ and satisfying*

$$\lim_{\varepsilon\to 0^+}\exp\left(\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}T\right)\varepsilon = N_0 < \infty, \tag{4.37}$$

and let us choose

$$K_{\varrho^\varepsilon} = \left|\log\left(\log\left(\varepsilon^{-r}\right)\right)^{\frac{m\lambda_1}{T}}\right|^{1/2} \longrightarrow \infty, \text{ as } \varepsilon \text{ tends to } 0^+.$$

Let f_{ϱ^ε} satisfy Lemma 4.6. Then nonlinear integral system (4.36) has a unique solution $\mathcal{V}^\varepsilon_{\Lambda^\varepsilon} \in C([0,T];L^2(\Omega))$. Assuming further that equation (4.36) has a unique exact solution u satisfying

$$u \in L^\infty\left(0,T;\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)\right), \quad \text{with } \eta_1 \ge 2+2\delta,\ \eta_2 \ge \frac{2T}{m\lambda_1},$$

and

$$\|u\|_{L^\infty(0,T;\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega))} \le \widetilde{P_0}, \text{ for some known constant } \widetilde{P_0} \ge 0.$$

Then the following stability estimate holds for any $t\in[0,T],\ \delta>0$,

$$\begin{aligned}&\|\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)}\\ \le&\sqrt{2}\left(\frac{\widetilde{P_0}}{(\Lambda^\varepsilon)^{\beta(1+\delta)}} + N_0\right)\left(\log\left(\frac{1}{\varepsilon^{-r}}\right)\right)^{\frac{T-t}{T}}\exp\left(-t\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right).\end{aligned} \tag{4.38}$$

Remark 4. We can choose $\Lambda^\varepsilon = \frac{m^{1/\beta}\lambda_1^{1/\beta}\sqrt[\beta]{\log(\varepsilon^{-r})}}{T^{1/\beta}} \xrightarrow{\varepsilon\to 0^+} \infty$, for some $r\in(0,1)$, then the condition in (4.37) is fulfiled. Then the estimate in (4.38) is of order

$$\varepsilon^{\frac{rt}{T}}\left(\log(\varepsilon^{-r})\right)^{1-\frac{t}{T}} \to 0 \text{ as } \varepsilon\to 0^+, \quad \text{for all } t\in[0,T).$$

Remark 5. Obviously, from (4.26), one can raise the question about I_0 in (4.25) is large, then the estimate in (4.26) is not exist naturally. By using FTR method, this problem is improved as in (4.38) by the term $\frac{I_0}{(\varLambda^\varepsilon)^{\beta+\delta}} \to 0$ as ε goes to 0^+.

Proof. We divide the proof into two parts. In Part a, we prove that the integral equation (4.36) has a unique solution $\mathcal{V}^\varepsilon_{\varLambda^\varepsilon} \in C([0,T];L^2(\Omega))$. In Part b, the error between the exact solution and regularized solution is obtained.

Part a. *The existence and uniqueness of a solution of the integral equation* (4.36).

For $\zeta \in C([0,T];L^2(\Omega))$, we consider the following function

$$\begin{aligned}&\mathscr{L}(\zeta)(t,x)\\ =&\sum_{j=1}^{[\varLambda^\varepsilon]}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j^\varepsilon - \frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_{\varrho,j}(\zeta)(\tau)\mathrm{d}\tau\right] e_j(x).\end{aligned} \tag{4.39}$$

Let us define $\mathscr{L}^k$ as follows

$$\mathscr{L}^k(\zeta) = \underbrace{\mathscr{L}\cdots\mathscr{L}(\mathscr{L}(\zeta))}_{k-\text{times}}.$$

To explore the proof, we only need to prove that there exists $k_0 \in \mathbb{N}^*$ such that the operator $\mathscr{L}^{k_0}$ which also maps $C([0,T];L^2(\Omega))$ into itself, is a contraction. In fact, we prove by mathematical induction that for any couples $\zeta_1,\zeta_2 \in C([0,T];L^2(\Omega))$ then

$$\begin{aligned}&\left\|\mathscr{L}^k(\zeta_1)-\mathscr{L}^k(\zeta_2)\right\|_{C([0,T];L^2(\Omega))}\\ &\le \frac{\left[K_{\varrho^\varepsilon}\exp\left(T\frac{(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)(T-t)\right]^k}{k!}\|\zeta_1-\zeta_2\|_{C([0,T];L^2(\Omega))},\end{aligned}$$

for $\zeta_1,\zeta_2 \in C([0,T];L^2(\Omega))$. The proof of the latter inequality is similar to that of Theorem 4.3 and thus it is omitted.

Part b. *The error estimate between the exact solution u and the regularized solution* $\mathcal{V}^\varepsilon_{\varLambda^\varepsilon}$. Using the triangle inequality, we have

$$\|\mathcal{V}^\varepsilon_{\varLambda^\varepsilon}(t,\cdot)-u(t,\cdot)\|_{[L^2(\Omega)]^2} \le \underbrace{\|\mathcal{V}^\varepsilon_{\varLambda^\varepsilon}(t,\cdot)-\mathcal{W}_{\varLambda^\varepsilon}(t,\cdot)\|_{L^2(\Omega)}}_{=:\mathcal{M}_9^{\varLambda^\varepsilon,\varepsilon}(t)} + \underbrace{\|\mathcal{W}_{\varLambda^\varepsilon}(t,\cdot)-u(t,\cdot)\|_{L^2(\Omega)}}_{=:\mathcal{M}_{10}^{\varLambda^\varepsilon}(t)}, \tag{4.40}$$

where

$$\begin{aligned}&\mathcal{W}_{\varLambda^\varepsilon}(t,x)\\ =&\sum_{j=1}^{[\varLambda^\varepsilon]}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j - \frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right) f_{\varrho^\varepsilon,j}(\mathcal{W}_{\varLambda^\varepsilon})(\tau)\mathrm{d}\tau\right] e_j(x).\end{aligned}$$

To estimate (4.40), we divide the right-hand side of (4.40) into two steps

- *Step 1. Estimate* $\mathcal{M}_9^{\varLambda^\varepsilon,\varepsilon}(t)$. One has

$$\mathcal{V}^\varepsilon_{\varLambda^\varepsilon}(t,x)-\mathcal{W}_{\varLambda^\varepsilon}(t,x) = \sum_{j=1}^{[\varLambda^\varepsilon]}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\left(\varphi_j^\varepsilon-\varphi_j\right)\right]e_j(x)$$

$$+\sum_{j=1}^{[\Lambda^\varepsilon]}\left[\frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\left(f_{\varrho^\varepsilon,j}(\mathcal{W}_{\Lambda^\varepsilon})(\tau)-f_{\varrho^\varepsilon,j}(\mathcal{V}^\varepsilon_{\Lambda^\varepsilon})(\tau)\right)\mathrm{d}\tau\right]e_j(x). \tag{4.41}$$

Using the Hölder's inequality with Parseval's relation, we obtain

$$\begin{aligned}
&|\mathcal{M}_9^{\Lambda^\varepsilon,\varepsilon}(t)|^2\\
&=2\sum_{j=1}^{[\Lambda^\varepsilon]}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)(\varphi_j^\varepsilon-\varphi_j)\right]^2\\
&+2\sum_{j=1}^{[\Lambda^\varepsilon]}\left[\frac{1}{1+m\lambda_j}\int_t^T\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\left(f_{\varrho^\varepsilon,j}(\mathcal{W}_{\Lambda^\varepsilon})(\tau)-f_{\varrho^\varepsilon,j}(\mathcal{V}^\varepsilon_{\Lambda^\varepsilon})(\tau)\right)\mathrm{d}\tau\right]^2\\
&\le 2\exp\left(\frac{2(T-t)(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\sum_{j=1}^{[\Lambda^\varepsilon]}|\varphi_j^\varepsilon-\varphi_j|^2\\
&+\frac{2T}{m^2\lambda_1^2}\int_t^T\exp\left(\frac{2(\tau-t)(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\sum_{j=1}^{[\Lambda^\varepsilon]}\left|f_{\varrho^\varepsilon,j}(\mathcal{W}_{\Lambda^\varepsilon})(\tau)-f_{\varrho^\varepsilon,j}(\mathcal{V}^\varepsilon_{\Lambda^\varepsilon})(\tau)\right|^2\mathrm{d}\tau\\
&\le 2\exp\left(\frac{2(T-t)(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\varepsilon^2\\
&+\frac{2TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}\int_t^T\exp\left(\frac{2(\tau-t)(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\|\mathcal{W}_{\Lambda^\varepsilon}(\tau,\cdot)-\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau.
\end{aligned}\tag{4.42}$$

Multiplying both sides (4.42) by $\exp\left(\frac{2(\Lambda^\varepsilon)^\beta}{m\lambda_1}t\right)$, we get

$$\begin{aligned}
&\exp\left(2t\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\|\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(t,\cdot)-\mathcal{W}_{\Lambda^\varepsilon}(t,\cdot)\|^2_{L^2(\Omega)}\\
&\le 2\exp\left(2T\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\varepsilon^2+\frac{2TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}\int_t^T\exp\left(2\tau\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\|\mathcal{W}_{\Lambda^\varepsilon}(\tau,\cdot)-\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau.
\end{aligned}$$

Applying Grönwall's inequality yields

$$\begin{aligned}
&\exp\left(2t\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}\right)\|\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(t,\cdot)-\mathcal{W}_{\Lambda^\varepsilon}(t,\cdot)\|^2_{L^2(\Omega)}\\
&\le 2\exp\left(\frac{2(\Lambda^\varepsilon)^\beta}{m\lambda_1}T\right)\varepsilon^2\exp\left(\frac{2TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}(T-t)\right).
\end{aligned}$$

Consequently,

$$\|\mathcal{V}^\varepsilon_{\Lambda^\varepsilon}(t,\cdot)-\mathcal{W}_{\Lambda^\varepsilon}(t,\cdot)\|_{L^2(\Omega)}\le\sqrt{2}\exp\left(\frac{(\Lambda^\varepsilon)^\beta}{m\lambda_1}(T-t)\right)\varepsilon\exp\left(\frac{2TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}(T-t)\right). \tag{4.43}$$

• *Step 2. Estimate of* $\mathcal{M}_{10}^{\Lambda^\varepsilon}(t)$. One has

$$\begin{aligned}
&\left|\mathcal{M}_{10}^{\Lambda^\varepsilon}(t)\right|^2\\
&=2\sum_{j=[\Lambda^\varepsilon]+1}^{\infty}\left[\exp\left(\frac{(T-t)\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j-\frac{1}{1+m\lambda_j}\int_t^T\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)f_{\varrho^\varepsilon,j}(\mathcal{W}_{\Lambda^\varepsilon})(\tau)\mathrm{d}\tau\right]^2
\end{aligned}$$

$$+2\sum_{j=1}^{[\varLambda^\varepsilon]}\left[\frac{1}{1+m\lambda_j}\int_t^T \exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\left(f_j(u)(\tau)-f_{\varrho^\varepsilon,j}(\mathcal{W}_{\varLambda^\varepsilon})(\tau)\right)\mathrm{d}\tau\right]^2$$
$$=: \mathcal{M}_{10a}^{\varLambda^\varepsilon}(t)+\mathcal{M}_{10b}^{\varLambda^\varepsilon}(t). \tag{4.44}$$

To estimate $\mathcal{M}_{10a}^{\varLambda^\varepsilon}(t)$, we proceed as follows for $\delta>0$ be arbitrary

$$\left|\mathcal{M}_{10a}^{\varLambda^\varepsilon}(t)\right|$$
$$=2\sum_{j=[\varLambda^\varepsilon]+1}^{\infty}\exp\left(\frac{-2t\lambda_j^\beta}{1+m\lambda_j}\right)\left[\exp\left(\frac{T\lambda_j^\beta}{1+m\lambda_j}\right)\varphi_j-\right.$$
$$\left.\frac{1}{1+m\lambda_j}\int_t^T\exp\left(\frac{-\tau\lambda_j^\beta}{1+m\lambda_j}\right)f_{\varrho^\varepsilon,j}(\mathcal{W}_{\varLambda^\varepsilon})(\tau)\mathrm{d}\tau\right]^2$$
$$\le 2(\varLambda^\varepsilon)^{-2\beta-2\beta\delta}\exp\left(-\frac{2t(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)\sum_{j=[\varLambda^\varepsilon]+1}^{\infty}\lambda_j^{2(\beta+\beta\delta)}\exp\left(\frac{2T\lambda_j^\beta}{m\lambda_1}\right)\langle u(t,\cdot),e_j(\cdot)\rangle^2_{L^2(\Omega)}$$
$$\le 2(\varLambda^\varepsilon)^{-2\beta-2\beta\delta}\exp\left(-\frac{2t(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)\|u(t,\cdot)\|^2_{\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)},\qquad \text{for } 2\le\eta_1,\eta_2\ge\frac{2T}{m\lambda_1}. \tag{4.45}$$

Since $\lim_{\varepsilon\to0^+}\varrho^\varepsilon=\infty$, for a sufficiently small $\varepsilon>0$, there is an $\varrho^\varepsilon>0$ such that $\varrho^\varepsilon\ge\|u\|_{L^\infty(0,T;L^2(\Omega))}$. For this value of ϱ^ε (from (3.54)) we have $f_{\varrho^\varepsilon}(t,x;u)\approx f(t,x;u)$. We estimate of $\mathcal{M}_{10b}^{\varLambda^\varepsilon}(t)$ as follows

$$\left|\mathcal{M}_{10b}^{\varLambda^\varepsilon}(t)\right|$$
$$=2\sum_{j=1}^{[\varLambda^\varepsilon]}\left[\frac{1}{1+m\lambda_j}\int_t^T\exp\left(\frac{(\tau-t)\lambda_j^\beta}{1+m\lambda_j}\right)\left(f_{\varrho^\varepsilon,j}(u)(\tau)-f_{\varrho^\varepsilon,j}(\mathcal{W}_{\varLambda^\varepsilon})(\tau)\right)\mathrm{d}\tau\right]^2$$
$$\le\frac{2T}{m^2\lambda_1^2}\int_t^T\exp\left(\frac{2(\tau-t)(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)\sum_{j=1}^{[\varLambda^\varepsilon]}\left|f_{\varrho^\varepsilon,j}(u)(\tau)-f_{\varrho^\varepsilon,j}(\mathcal{W}_{\varLambda^\varepsilon})(\tau)\right|^2\mathrm{d}\tau$$
$$\le\frac{2TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}\int_t^T\exp\left(\frac{2(\tau-t)(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)\|\mathcal{W}_{\varLambda^\varepsilon}(\tau,\cdot)-u(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau. \tag{4.46}$$

Combining (4.45) and (4.46), we get

$$\|\mathcal{W}_{\varLambda^\varepsilon}(t,\cdot)-u(t,\cdot)\|^2_{L^2(\Omega)}$$
$$\le 2(\varLambda^\varepsilon)^{-2\beta-2\beta\delta}\exp\left(-2t\frac{(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)\|u(t,\cdot)\|^2_{\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)}$$
$$+\frac{2TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}\int_t^T\exp\left(\frac{2(\tau-t)(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)\|\mathcal{W}_{\varLambda^\varepsilon}(\tau,\cdot)-u(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau. \tag{4.47}$$

Multiplying by $\exp\left(2t\frac{(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)$ both sides and using Grönwall's inequality, we obtain

$$\|\mathcal{W}_{\varLambda^\varepsilon}(t,\cdot)-u(t,\cdot)\|_{L^2(\Omega)}$$
$$\le\frac{\sqrt{2}\exp\left(-\frac{t(\varLambda^\varepsilon)^\beta}{m\lambda_1}\right)}{(\varLambda^\varepsilon)^{\beta+\beta\delta}}\|u(t,\cdot)\|_{\mathbf{G}_\beta(\eta_1,\eta_2)(\Omega)}\exp\left(\frac{TK^2_{\varrho^\varepsilon}}{m^2\lambda_1^2}(T-t)\right). \tag{4.48}$$

Combining (4.40), (4.43) and (4.48), we obtain (4.38). This completes the proof of Theorem 4.7. □

5. **Conclusion.** The paper investigate a final boundary value problem for a class of pseudo-parabolic partial differential equations with nonlinear reaction term. For $0 < \beta < 1$, the well-posedness of solution is established. For $\beta > 1$, the problem is ill-posed. Thus, we propose two methods to regularize the problem. The error estimates are given in the cases of globally or locally Lipschitzian source term.

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