

GLOBAL CONSERVATIVE SOLUTIONS FOR A MODIFIED PERIODIC COUPLED CAMASSA-HOLM SYSTEM

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ABSTRACT. In present paper, we deal with the behavior of a solution beyond the occurrence of wave breaking for a modified periodic Coupled Camassa-Holm system. By introducing a new set of independent and dependent variables, which resolve all singularities due to possible wave breaking, this evolution system is rewritten as a closed semilinear system. The local existence of the semilinear system is obtained as fixed points of a contractive transformation. Moreover, this formulation allows us to continue the solution after wave breaking, and gives a global conservative solution where the energy is conserved for almost all times. Returning to the original variables. We finally obtain a semigroup of global conservative solutions, which depend continuously on the initial data. Additionally, our results repair some gaps in the pervious work.

1. Introduction. In this paper, we investigate the Cauchy problem of the modified periodic coupled Camassa-Holm system that has the following form:

$$\begin{cases} m_t + 2mu_x + m_xu + (mv)_x + nv_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ n_t + 2nv_x + n_xv + (nu)_x + mu_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ m(0, x) = m_0(x), \quad n(0, x) = n_0(x), & t = 0, \quad x \in \mathbb{R}, \\ m(t, x) = m(t, x + 1), \quad n(t, x) = n(t, x + 1), & t > 0, \quad x \in \mathbb{R}, \end{cases} \quad (1)$$

where $m = u - u_{xx}$ and $n = v - v_{xx}$ are periodic function respect to x . The system (1) is a generalization of the Camassa-Holm equation with peakon solitons in the form of a superposition of multipeakons which was firstly proposed by Fu and Qu in [13]. The well-posedness and blow-up solutions of periodic case for the system (1) were discussed by Fu et al. in [14]. In [30], the authors established the

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local well posedness and blow-up solutions in Besov spaces. The attractor, non-uniform dependence and persistence properties of the system (1) were discussed in [29, 33]. Furthermore, Tian et al. in [28] investigated the global conservative and dissipative solutions of system (1) by transforming it into an equivalent semilinear ODE system. However, there is gap in [28, 31], i.e., the second derivative terms u_{xx} and v_{xx} appeared in the equivalent ODE system cannot be controlled by the $H^1(\mathbb{R})$ conservation law. This paper repair the gap in the following sections.

It is known to us, when $u = v$ and let $\tilde{t} = 2t$, system (1) can be degenerated to the famous Camassa-Holm equation:

$$m_t + 2mu_x + m_x u = 0, \quad m = u - u_{xx}. \quad (2)$$

The Camassa-Holm equation (2) was first implicitly contained in a bi-Hamiltonian generalization of the Korteweg-de Vries equation by Fuchssteiner and Fokas [15], and later deduced as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [4]. Similar to the KdV equation, the Camassa-Holm equation has a bi-Hamilton structure [15, 24], and is completely integrable [4, 5, 23]. The equation (2) not only holds an infinity of conservative laws, but also can be solved by its corresponding inverse scattering transform [2, 9]. The solitary waves of equation (2) are solitons (i.e., it can keep their shape and velocity after interacted by the same type nonlinear wave). Compared to the KdV equation, the Camassa-Holm equation has many advantages, such as, it has both the finite time wave-breaking solutions (i.e. the solution keeps bounded but the slope becomes unbounded in finite time) and the global strong solutions [6, 7, 11, 25]. The solitary wave solutions are peaked waves and a specific case was given in [4], it is not a classical solution because it has a peak at their crest. The local well-posedness of equation (2) with the inial data $u_0 \in H^s, s > \frac{3}{2}$ was studied in [1, 7, 25]. Many papers have investigated the weak solutions for the equation (2). Especially, in the non-periodic case, Bressan and Constantin in [3] has done many works, they developed a new method to investigate a conservative solution's semigroup. In the periodic case, Holden and Raynaud in [22] applied the semigroup theory to investigate the periodic Camassa-Holm equation, and proved its conservative solutions, depending continuously on the initial data, which can also construct a semigroup.

Of course, the Camassa-Holm equation has many generalizations such as the modified two-component Camassa-Holm equation (M2CH) and the coupled Camassa-Holm equation. The M2CH equation was firstly introduced by Holm et al. in [21] as a modified version of the two-component Camassa-Holm equation (2CH) that was proposed by Constantin and Ivanov [10] in the context of shallow water theory. The M2CH equation is integrable and its form as follow;

$$\begin{cases} m_t + 2mu_x + m_x u + \rho \bar{\rho}_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \quad (3)$$

where $m = u - u_{xx}$. The equations (3) was testified that admits singular solutions in both of the variables m and ρ in [21]. The well-posedness, blow-up phenomena, Lipschitz metric and the global weak solution for the equations (3) were studied by Guan et al. in [17, 18, 19, 20]. Tan et al. investigated the global conservative solutions for both of the periodic case and the non-periodic case in [26, 27]. Guan proved the Cauchy problem of the M2CH with the initial data $z_0 = (u_0, \rho_0) \in H^1(\mathbb{R}) \times (H^1 \cap W^{1,\infty})$ has a unique global conservative weak solution in [16].

Inspired by [3, 22, 26, 34, 35], this paper mainly discusses the global conservative solutions of the modified periodic Camassa-Holm system. As is known to

us, equations (1) is a system, so it's more difficult than the single one. Moreover, the interactions between u and v greatly increases the complexity of the research. To overcome these difficulties, we set the characteristics that completely different from the one used in [3, 22, 26]. Thus, the calculation is largely reduced. By introducing new variables, we transform the system (1) into a equivalent semilinear ODE system. Firstly, we get the global solutions of the equivalent semilinear ODE system. Then we get the global conservative solutions for the system (1) from the global solutions of the equivalent ODE system. Finally, we obtain a semigroup of the solutions depending continuously on inial data for the original system.

The rest of this paper is organized as follows. Section 2 is the basic equation. In section 3, we get a equivalent semilinear system and the global solutions of the semilinear system. In section 4, we obtain the global conservative solution of the system (1) and construct a solution semigroup.

2. The basic equations. Now, we reformulate the system (1). Let $m = u - u_{xx}$ and $n = v - v_{xx}$. Note that $G := \frac{1}{2 \sinh \frac{1}{2}} \cosh(x - [x] - \frac{1}{2})$, $x \in \mathbb{R}$, and $(1 - \partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{R})$. Thus, we can rewrite system (1) as follow:

$$\begin{cases} u_t + (u + v)u_x + P_1 + P_{2,x} = 0, & t > 0, x \in \mathbb{R}, \\ v_t + (u + v)v_x + Q_1 + Q_{2,x} = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), v(0, x) = v_0(x), & t = 0, x \in \mathbb{R}, \\ u(t, x) = u(t, x + 1), v(t, x) = v(t, x + 1), & t > 0, x \in \mathbb{R}, \end{cases} \tag{4}$$

where P_1, P_2, Q_1, Q_2 have the following form:

$$\begin{cases} P_1 = G * (uv_x), \\ P_2 = G * (u^2 + \frac{1}{2}u_x^2 + u_xv_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2), \\ Q_1 = G * (vu_x), \\ Q_2 = G * (v^2 + \frac{1}{2}v_x^2 + u_xv_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2). \end{cases}$$

According to the fact that the above representations, P_1 and Q_1 are symmetrical, P_2 and Q_2 are symmetrical, For convenience, we can set $P_1(u, v) = Q_1(v, u) = G*(uv_x)$. So do P_2 and Q_2 .

In fact, for smooth solutions, differentiating the first and the second equations in (4) with respect to x , we get

$$\begin{cases} u_{xt} + u_x^2 + u_xv_x + uu_{xx} + vu_{xx} + P_{1,x} + P_2 - (u^2 + \frac{1}{2}u_x^2 + u_xv_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2)_x = 0, \\ v_{xt} + v_x^2 + u_xv_x + vv_{xx} + uv_{xx} + Q_{1,x} + Q_2 - (v^2 + \frac{1}{2}v_x^2 + u_xv_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2)_x = 0. \end{cases} \tag{5}$$

Multiplying the first and the second equations in (4) by u and v , and multiplying the first and the second equations in (5) by u_x and v_x , respectively, we get following conservation laws

$$\begin{cases} (\frac{u^2}{2})_t + (\frac{u_x^2}{2})_t + (\frac{u^2v + uu_x^2 + vu_x^2 + 2uP_2}{2})_x \\ - \frac{u_x(v^2 + v_x^2) + v_x(u^2 - u_x^2)}{2} + uP_1 + u_xP_{1,x} = 0, \\ (\frac{v^2}{2})_t + (\frac{v_x^2}{2})_t + (\frac{v^2u + vv_x^2 + uv_x^2 + 2vQ_2}{2})_x \\ - \frac{v_x(u^2 + u_x^2) + u_x(v^2 - v_x^2)}{2} + vQ_1 + v_xQ_{1,x} = 0. \end{cases} \tag{6}$$

For regular solutions, using (6) and integrating by parts, it is clear that the total energy

$$E(t) = \int_{\mathbb{S}} (u^2 + u_x^2 + v^2 + v_x^2) dx$$

is a constant in time. If u and v are smooth, from (4)-(6), it is not very hard to check that

$$\begin{aligned} (u^2 + u_x^2 + v^2 + v_x^2)_t + ((u + v)(u^2 + u_x^2 + v^2 + v_x^2))_x \\ = (u^3 - 2uP_2 - 2uP_{1,x} + v^3 - 2vQ_2 - 2vQ_{1,x})_x. \end{aligned} \tag{7}$$

3. A equivalent semilinear system. Firstly, we introduce the space E_1 :

$$E_1 = \{f \in H^1_{loc}(\mathbb{R}) | f(\theta + 1) = f(\theta) + 1\},$$

and define the characteristics $y : \mathbb{R} \mapsto E_1, t \mapsto y(t, \cdot)$ as the solution of

$$y_t(t, \theta) = (u + v)(t, y(t, \theta)). \tag{8}$$

In addition, we denote

$$\begin{cases} U(t, \theta) = u(t, y(t, \theta)), \\ V(t, \theta) = v(t, y(t, \theta)), \\ M(t, \theta) = u_x(t, y(t, \theta)), \\ N(t, \theta) = v_x(t, y(t, \theta)), \end{cases} \tag{9}$$

and

$$H(t, \theta) = \int_{y(t,0)}^{y(t,\theta)} (u^2 + u_x^2 + v^2 + v_x^2) dx. \tag{10}$$

By (7) and (8), we get

$$\begin{cases} \frac{dH}{dt} = [(u^3 - 2uP_{1,x} - 2uP_2 + v^3 - 2vQ_{1,x} - 2vQ_2) \circ y]_0^\theta, \\ H_\theta = [(u^2 + u_x^2 + v^2 + v_x^2) \circ y] y_\theta. \end{cases} \tag{11}$$

Using (10), the periodicity of u, v and $y \in E_1$, we obtain

$$H(t, \theta + 1) - H(t, \theta) = H(t, 1) - H(t, 0).$$

According to (11), it is very easy to prove that $H(t, 1) - H(t, 0)$ is a constant in time. Thus, we get that $H(t, 1) - H(t, 0) = H(0, 1) - H(0, 0)$. For every $t > 0$, H belongs to the vector space E defined as follow

$$E = \{f \in H^1_{loc}(\mathbb{R}) | f(\theta + 1) - f(\theta) = f(1) - f(0)\}.$$

We define the norm $\|f\|_E = \|f\|_{H^1_{[0,1]}}$ for E . For convenience, we will replace $H^1_{[0,1]}$ by H^1 . Later, we will verify that E is complete.

We derive formally a system equivalent to system (4). From the definition of the characteristics, it follows that

$$\begin{cases} U_t(t, \theta) = (-P_1 - P_{2,x}) \circ y(t, \theta), \\ V_t(t, \theta) = (-Q_1 - Q_{2,x}) \circ y(t, \theta), \\ M_t(t, \theta) = (-\frac{M^2}{2} - \frac{N^2}{2} + U^2 + \frac{V^2}{2} - P_{1,x} - P_2) \circ y(t, \theta), \\ N_t(t, \theta) = (-\frac{N^2}{2} - \frac{M^2}{2} + V^2 + \frac{U^2}{2} - Q_{1,x} - Q_2) \circ y(t, \theta). \end{cases} \tag{12}$$

Then, we get the explicit expression for $P_i, Q_i, P_{i,x}, Q_{i,x}$ ($i = 1, 2$):

$$\left\{ \begin{array}{l} P_1(u, v) = \frac{1}{2 \sinh \frac{1}{2}} \int_0^1 \cosh \left(x - y - [x - y] - \frac{1}{2} \right) (u(t, y)v_x(t, y)) dy, \\ P_{1,x}(u, v) = \frac{1}{2 \sinh \frac{1}{2}} \int_0^1 \sinh \left(x - y - [x - y] - \frac{1}{2} \right) (u(t, y)v_x(t, y)) dy, \\ P_2(u, v) = \frac{1}{2 \sinh \frac{1}{2}} \int_0^1 \cosh \left(x - y - [x - y] - \frac{1}{2} \right) \\ \quad \times \left(u^2(t, y) + \frac{1}{2} u_x^2(t, y) + u_x(t, y)v_x(t, y) + \frac{1}{2} v^2(t, y) - \frac{1}{2} v_x^2(t, y) \right) dy, \\ P_{2,x}(u, v) = \frac{1}{2 \sinh \frac{1}{2}} \int_0^1 \sinh \left(x - y - [x - y] - \frac{1}{2} \right) \\ \quad \times \left(u^2(t, y) + \frac{1}{2} u_x^2(t, y) + u_x(t, y)v_x(t, y) + \frac{1}{2} v^2(t, y) - \frac{1}{2} v_x^2(t, y) \right) dy, \end{array} \right.$$

and $Q_1(u, v) = P_1(v, u), Q_{1,x}(u, v) = P_{1,x}(v, u), Q_2(u, v) = P_2(v, u), Q_{2,x} = P_{2,x}(v, u)$.

In the above formulae, we can perform the change of variables $y = y(t, \theta')$, and rewrite the convolution respect to θ' . From (9), we get new expressions of $P_i, Q_i, P_{i,x}, Q_{i,x}$ ($i = 1, 2$) with the new variable θ

$$\left\{ \begin{array}{l} P_1(t, \theta) = \frac{1}{2(e-1)} \int_0^1 \cosh (y(t, \theta) - y(t, \theta')) (UNy_\theta)(t, \theta') d\theta' \\ \quad + \frac{1}{4} \int_0^1 \exp (-\operatorname{sgn}(\theta - \theta')(y(\theta) - y(\theta'))) (UNy_\theta)(t, \theta') d\theta', \\ P_{1,x}(t, \theta) = \frac{1}{2(e-1)} \int_0^1 \sinh (y(t, \theta) - y(t, \theta')) (UNy_\theta)(t, \theta') d\theta' \\ \quad - \frac{1}{4} \int_0^1 \operatorname{sgn}(\theta - \theta') \exp (-\operatorname{sgn}(\theta - \theta')(y(\theta) - y(\theta'))) (UNy_\theta)(t, \theta') d\theta', \\ P_2(t, \theta) = \frac{1}{2(e-1)} \int_0^1 \cosh (y(t, \theta) - y(t, \theta')) (H_\theta + (U^2 + 2MN - N^2)y_\theta)(t, \theta') d\theta' \\ \quad + \frac{1}{4} \int_0^1 \exp (-\operatorname{sgn}(\theta - \theta')(y(\theta) - y(\theta'))) \\ \quad \quad \times (H_\theta + (U^2 + 2MN - N^2)y_\theta)(t, \theta') d\theta', \\ P_{2,x}(t, \theta) = \frac{1}{2(e-1)} \int_0^1 \sinh (y(t, \theta) - y(t, \theta')) (H_\theta + (U^2 + 2MN - N^2)y_\theta)(t, \theta') d\theta' \\ \quad - \frac{1}{4} \int_0^1 \operatorname{sgn}(\theta - \theta') \exp (-\operatorname{sgn}(\theta - \theta')(y(\theta) - y(\theta'))) \\ \quad \quad \times (H_\theta + (U^2 + 2MN - N^2)y_\theta)(t, \theta') d\theta', \end{array} \right. \tag{13}$$

and

$$\left\{ \begin{array}{l} Q_1(u(t, \theta), v(t, \theta)) = P_1(v(t, \theta), u(t, \theta)), \\ Q_{1,x}(u(t, \theta), v(t, \theta)) = P_{1,x}(v(t, \theta), u(t, \theta)), \\ Q_2(u(t, \theta), v(t, \theta)) = P_2(v(t, \theta), u(t, \theta)), \\ Q_{2,x}(u(t, \theta), v(t, \theta)) = P_{2,x}(v(t, \theta), u(t, \theta)). \end{array} \right. \tag{14}$$

Straight computation shows that

$$\begin{cases} P_{1,\theta} = P_{1,x}y_\theta, P_{1,x\theta} = -UNy_\theta + P_1y_\theta, \\ P_{2,\theta} = P_{2,x}y_\theta, P_{2,x\theta} = -[H_\theta + (U^2 + 2MN - N^2)y_\theta] + P_2y_\theta, \\ Q_{1,\theta} = Q_{1,x}y_\theta, Q_{1,x\theta} = -VMy_\theta + Q_1y_\theta, \\ Q_{2,\theta} = Q_{2,x}y_\theta, Q_{2,x\theta} = -[H_\theta + (V^2 + 2MN - M^2)y_\theta] + Q_2y_\theta. \end{cases} \tag{15}$$

From (8), (11)-(12) and (13)-(15), we obtain a new system which is equivalent to system (4). And the Cauchy problem of the new system can be rewritten with respect to the variables (y, U, V, M, N, H) in the following form

$$\begin{cases} \frac{\partial y}{\partial t} = U + V, \\ \frac{\partial U}{\partial t} = -P_1 - P_{2,x}, \\ \frac{\partial V}{\partial t} = -Q_1 - Q_{2,x}, \\ \frac{\partial M}{\partial t} = -\frac{M^2}{2} - \frac{N^2}{2} + U^2 + \frac{V^2}{2} - P_{1,x} - P_2, \\ \frac{\partial N}{\partial t} = -\frac{N^2}{2} - \frac{M^2}{2} + V^2 + \frac{U^2}{2} - Q_{1,x} - Q_2, \\ \frac{\partial H}{\partial t} = (u^3 - 2uP_{1,x} - 2uP_2 + v^3 - 2vQ_{1,x} - 2vQ_2)|_0^\theta. \end{cases} \tag{16}$$

Differentiating (16) with respect to θ and utilizing (15), we have

$$\begin{cases} \frac{\partial y_\theta}{\partial t} = U_\theta + V_\theta, \\ \frac{\partial U_\theta}{\partial t} = \frac{H_\theta}{2} + \left(\frac{U^2}{2} + MN - N^2 - P_{1,x} - P_2\right)y_\theta, \\ \frac{\partial V_\theta}{\partial t} = \frac{H_\theta}{2} + \left(\frac{V^2}{2} + MN - M^2 - Q_{1,x} - Q_2\right)y_\theta, \\ \frac{\partial H_\theta}{\partial t} = (3U^2 - 2P_2)U_\theta + (3V^2 - 2Q_2)V_\theta - 2(UP_{2,x} + VQ_{2,x})y_\theta \\ \quad - 2(MP_{1,x} + UP_1 - U^2N + NQ_{1,x} + VQ_1 - V^2M)y_\theta. \end{cases} \tag{17}$$

The system (17) is semilinear for the variables $y_\theta, U_\theta, V_\theta$ and H_θ . By introducing the space H_{per}^1

$$H_{per}^1 = \{f \in H_{loc}^1(\mathbb{R}) | f(\theta + 1) = f(\theta)\},$$

with the norm $\|f\|_{H_{per}^1} = \|f\|_{H_{[0,1]}^1}$, We define a linear map $\Phi: \|f\|_{H_{per}^1} \times \mathbb{R} \mapsto E$ as $\Phi : (\sigma, h) \mapsto f = \sigma + hId$.

Lemma 3.1. *The map Φ defined above is homeomorphism from $H_{per}^1 \times \mathbb{R}$ to E .*

It is clear that the space E is a Banach space, because the space $H_{per}^1 \times \mathbb{R}$ is a Banach space. Let's introduce $\eta = y - Id$ and $(\sigma, h) = \sigma + hId$, i.e. $h =$

$H(t, 1) - H(t, 0)$ and $\sigma = H - hId$. Therefore, the system (16) is equivalent to

$$\begin{cases} \frac{\partial \eta}{\partial t} = U + V, \\ \frac{\partial U}{\partial t} = -P_1 - P_{2,x}, \\ \frac{\partial V}{\partial t} = -Q_1 - Q_{2,x}, \\ \frac{\partial M}{\partial t} = -\frac{M^2}{2} - \frac{N^2}{2} + U^2 + \frac{V^2}{2} - P_{1,x} - P_2, \\ \frac{\partial N}{\partial t} = -\frac{N^2}{2} - \frac{M^2}{2} + V^2 + \frac{U^2}{2} - Q_{1,x} - Q_2, \\ \frac{\partial \sigma}{\partial t} = (u^3 - 2uP_{1,x} - 2uP_2 + v^3 - 2vQ_{1,x} - 2vQ_2)|_0^\theta, \\ \frac{\partial h}{\partial t} = 0. \end{cases} \tag{18}$$

In the next section, the well-posedness of system (18) will be proved as an ordinary differential equations in the Banach space W . Note that

$$W = H_{per}^1 \times H_{per}^1 \times H_{per}^1 \times L_{per}^\infty \times L_{per}^\infty \times H_{per}^1 \times \mathbb{R}.$$

We have a bijection $(\eta, U, V, M, N, \sigma, h) \mapsto (y, U, V, M, N, H)$ from W to $E_1 \times H_{per}^1 \times H_{per}^1 \times L_{per}^\infty \times L_{per}^\infty \times E$ with $y = \eta + Id$ and $H = \sigma + hId$.

Theorem 3.2. *Let $\bar{X} = (\bar{\eta}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{\rho}, \bar{h}) \in W$, there exists a $T > 0$ depending only on $\|\bar{X}\|_W$ such that the system (18) has a unique solution in $C^1([0, T], E)$ with initial data \bar{X} .*

Proof. To prove the this theorem, the key step is to prove the Lipchitz continuity of the right side of system (18). We define the map $\psi: W \mapsto W$

$$\begin{aligned} \psi(X) = & (U + V, -P_1 - P_{2,x}, -Q_1 - Q_{2,x}, -\frac{M^2}{2} - \frac{N^2}{2} + U^2 + \frac{V^2}{2} - P_{1,x} - P_2, \\ & -\frac{N^2}{2} - \frac{M^2}{2} + V^2 + \frac{U^2}{2} - Q_{1,x} - Q_2, (u^3 - 2uP_{1,x} - 2uP_2 + v^3 - 2vQ_{1,x} - 2vQ_2)|_0^\theta, 0). \end{aligned}$$

Firstly, we testify that $P_i, Q_i, P_{i,x}, Q_{i,x}$ are local Lipchitz continuous. Note that $B_M = \{X = (\eta, U, V, M, N, \sigma, h) \in W \mid \|X\|_W \leq M\}$. Let $X = (\eta, U, V, M, N, \sigma, h)$ and $\bar{X} = (\bar{\eta}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{\sigma}, \bar{h})$ belong to B_M . Then, we get that

$$\|y\|_{L^\infty} = \|Id + \eta\|_{L^\infty} \leq 1 + C\|\eta\|_{H^1} \leq 1 + CM$$

and $\|\bar{y}\|_{L^\infty} \leq 1 + CM$. It is clear that $\cosh x$ and $\sinh x$ are local Lipchitz continuous on $\{X \in \mathbb{R} \mid |X| \leq 1 + CM\}$ where C is a constant that depends only on M , we yield

$$\begin{aligned} |\cosh(y(\theta) - y(\theta')) - \cosh(\bar{y}(\theta) - \bar{y}(\theta'))| & \leq C|y(\theta) - y(\theta') - \bar{y}(\theta) + \bar{y}(\theta')| \\ & \leq C\|\eta - \bar{\eta}\|_{L^\infty}, \end{aligned}$$

and

$$|\exp(-sgn(\theta - \theta')(y(\theta) - y(\theta'))) - \exp(-sgn(\theta - \theta')(\bar{y}(\theta) - \bar{y}(\theta')))| \leq C\|\eta - \bar{\eta}\|_{L^\infty},$$

for all θ, θ' in $[0, 1]$. Then, we have

$$\begin{aligned} & \| \cosh(y(\theta) - y(\theta'))((H_\theta + (U^2 + 2MN - N^2)y_\theta)) - \cosh(\bar{y}(\theta) - \bar{y}(\theta'))(\bar{H}_\theta + (\bar{U}^2 + 2\bar{M}\bar{N} - \bar{N}^2)\bar{y}_\theta) \|_{L^\infty} \\ & \leq C(\|\eta - \bar{\eta}\|_{L^\infty} + \|U - \bar{U}\|_{L^\infty} + \|V - \bar{V}\|_{L^\infty} + \|M - \bar{M}\|_{L^2} \\ & \quad + \|N - \bar{N}\|_{L^2} + \|\eta_\theta - \bar{\eta}_\theta\|_{L^2} + \|\sigma_\theta - \bar{\sigma}_\theta\|_{L^2} + |h - \bar{h}|) \\ & \leq C\|X - \bar{X}\|_W. \end{aligned}$$

So does $\exp(-sgn(\theta - \theta')(y(\theta) - y(\theta')))[H_\theta + (U^2 + 2MN - N^2)y_\theta]$. Thus, we obtain that

$$\|P_2 - \bar{P}_2\|_{L^\infty} \leq C\|X - \bar{X}\|_W.$$

The similar computation shows that

$$\|P_{2,x} - \bar{P}_{2,x}\|_{L^\infty} \leq C\|X - \bar{X}\|_W.$$

Utilizing (15), we have

$$\|P_{2,x\theta} - \bar{P}_{2,x\theta}\|_{L^2} \leq C(\|X - \bar{X}\|_W + \|P_2 - \bar{P}_2\|_{L^\infty}) \leq C\|X - \bar{X}\|_W,$$

and

$$\|P_2 - \bar{P}_2\|_{L^2} \leq C(\|X - \bar{X}\|_W).$$

In conclusion, the local Lipchitz continuity from W to H^1_{per} of P_2 and $P_{2,x}$ has been proven, and so do $P_1, P_{1,x}, Q_1, Q_{1,x}, Q_2$ and $Q_{2,x}$. For above certify, it is easy to prove that ψ is locally Lipchitz continuous from W to W . Therefore, the theorem follows the standard theory of ordinary differential equations on Banach spaces. \square

Now, we prove the existence of a global solution of system (18). As we all know that initial data is very significant to system (18), but, here, we will only consider a particular initial data that belong to

$$W_1 = W^{1,\infty}_{per} \times W^{1,\infty}_{per} \times W^{1,\infty}_{per} \times L^\infty_{per} \times W^{1,\infty}_{per} \times \mathbb{R}.$$

W_1 is complete subspace of W . Let $\bar{X} \in W_1$. We investigate the short time solution $X = (\eta, U, V, M, N, \sigma, h) \in ([0, T], E)$ of system (18) given by Theorem 3.2. Because $X \in C([0, T], W)$, $P_i, Q_i, P_{i,x}, Q_{i,x} \in C([0, T], W)$, we now consider $U, V, P_i, Q_i, P_{i,x}, Q_{i,x}$ as functions in $C([0, T], H^1_{per})$ and M, N in $C([0, T], L^\infty_{per})$. Then, for any fixed θ in \mathbb{R} , we can solve the following system of ordinary differential equations in \mathbb{R}^4 given by

$$\begin{cases} \frac{\partial}{\partial t} \alpha(t, \theta) = \beta(t, \theta) + \gamma(t, \theta), \\ \frac{\partial}{\partial t} \beta(t, \theta) = \frac{\bar{h} + \delta(t, \theta)}{2} + (\frac{U^2}{2} + MN - N^2 - P_{1,x} - P_2)(1 + \alpha(t, \theta)), \\ \frac{\partial}{\partial t} \gamma(t, \theta) = \frac{\bar{h} + \delta(t, \theta)}{2} + (\frac{V^2}{2} + MN - M^2 - Q_{1,x} - Q_2)(1 + \alpha(t, \theta)), \\ \frac{\partial}{\partial t} \delta(t, \theta) = (3U^2 - 2P_2 - 2P_{1,x} + 2V^2)\beta(t, \theta) + (3V^2 - 2Q_2 - 2Q_{1,x} + 2U^2)\gamma(t, \theta) \\ \quad - 2(U P_{2,x} + U P_1 + V Q_{2,x} + V Q_1)(1 + \delta(t, \theta)). \end{cases} \tag{19}$$

which is obtained by substituting $\eta_\theta, U_\theta, V_\theta$ and σ_θ in system (17) by α, β, γ and δ , respectively. We also replaced $h(t)$ by h , hence $h(t) = h$ for all t . Let

$$S = \{ \theta \in \mathbb{R} \mid |\bar{U}_\theta(\theta)| \leq \|\bar{U}_\theta(\theta)\|_{L^\infty}, |\bar{V}_\theta(\theta)| \leq \|\bar{v}_\theta(\theta)\|_{L^\infty}, |\bar{\eta}_\theta(\theta)| \leq \|\bar{\eta}_\theta(\theta)\|_{L^\infty}, |\bar{\sigma}_\theta(\theta)| \leq \|\bar{\sigma}_\theta(\theta)\|_{L^\infty} \}.$$

It is not very hard to check that $meas(S^c) = 0$. For all $\theta \in S$, let

$$(\alpha(0, \theta), \beta(0, \theta), \gamma(0, \theta), \delta(0, \theta)) = (\bar{\eta}_\theta(\theta), \bar{U}_\theta(\theta), \bar{v}_\theta(\theta), \bar{\sigma}_\theta(\theta)).$$

For $\theta \in S^c$, we take $(\alpha(0, \theta), \beta(0, \theta), \gamma(0, \theta), \delta(0, \theta)) = (0, 0, 0, 0)$.

Lemma 3.3. *Given initial data*

$$\bar{X} = (\bar{\eta}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{\sigma}, \bar{h}) \in [W_{per}^{1,\infty}]^3 \times [L_{per}^\infty]^2 \times W_{per}^{1,\infty} \times \mathbb{R}.$$

Let $X = (\eta, U, V, M, N, \sigma, h) \in C([0, T], W)$ be the solution of system (18) followed Theorem 3.2. Then

$$X \in C^1([0, T], [W_{per}^{1,\infty}]^3 \times [L_{per}^\infty]^2 \times W_{per}^{1,\infty} \times \mathbb{R}),$$

and the functions $\alpha(t, \theta), \beta(t, \theta), \gamma(t, \theta), \delta(t, \theta)$, which solve system (19) for any fixed θ with the specified initial data above, coincide for almost every θ and for all time with $(\eta_\theta, U_\theta, V_\theta, \sigma_\theta)$, that is, for all $t \in [0, T]$, for almost every θ ,

$$(\alpha(t, \theta), \beta(t, \theta), \gamma(t, \theta), \delta(t, \theta)) = (\eta_\theta(t, \theta), U_\theta(t, \theta), V_\theta(t, \theta), \sigma_\theta(t, \theta)). \tag{20}$$

Proof. Firstly, let's introduce a key space B_{per}^∞ in which elements are bounded periodic functions with the norm $\|f\|_{B_{per}^\infty} = \sup_{\theta \in [0,1]} |f(\theta)|$, clearly, this space is complete. According to the initial data conditions in the lemma, the solution for system (19) in space $[B_{per}^\infty]^4$ can be defined as $(\alpha(t, \theta), \beta(t, \theta), \gamma(t, \theta), \delta(t, \theta))$. It is not very difficult to check that the solutions exists on the interval $[0, T]$ on which $(\eta, U, V, M, N, \sigma, h)$ is defined, since the system (19) is linear in $(\alpha, \beta, \gamma, \delta)$. According system (18), we get that

$$\begin{cases} \eta_\theta(t, \theta) = \bar{\eta}_\theta + \int_0^t (U_\theta(\tau, \theta) + V_\theta(\tau, \theta)) d\tau, \\ U_\theta(t, \theta) = \bar{U}_\theta + \int_0^t \left(\frac{1}{2}(\bar{h} + \delta(\tau, \theta)) + \left(\frac{U^2}{2} + MN - N^2 - P_{1,x} - P_2 \right) (1 + \alpha(\tau, \theta)) \right) d\tau, \\ V_\theta(t, \theta) = \bar{V}_\theta + \int_0^t \left(\frac{1}{2}(\bar{h} + \delta(\tau, \theta)) + \left(\frac{V^2}{2} + MN - M^2 - Q_{1,x} - Q_2 \right) (1 + \alpha(\tau, \theta)) \right) d\tau, \\ \sigma_\theta(t, \theta) = \bar{\sigma}_\theta + \int_0^t \left((3U^2 - 2P_2 - 2P_{1,x} + 2V^2)\beta(\tau, \theta) + (3V^2 - 2Q_2 - 2Q_{1,x} + 2U^2)\gamma(\tau, \theta) \right. \\ \left. - 2(UP_{2,x} + UP_1 + VQ_{2,x} + VQ_1)(1 + \delta(\tau, \theta)) \right) d\tau. \end{cases} \tag{21}$$

Since B_{per}^∞ imbeds in L_{per}^2 , the integral form of system (19) holds in L_{per}^2 sense. Now, we know that $(\alpha, \beta, \gamma, \delta)$ and $(\eta_\theta, U_\theta, V_\theta, \sigma_\theta)$ satisfy the linear system (19) with the same initial data in L^2 sense on the interval $[0, T]$. By uniqueness, we get that

$$(\alpha(t), \beta(t), \gamma(t), \delta(t)) = (\eta_\theta(t), U_\theta(t), V_\theta(t), \sigma_\theta(t))$$

in L_{per}^2 on $[0, T]$. □

Now, we give the initial data as follow

$$\begin{cases} \bar{U}(\theta) = \bar{u} \circ \bar{y}(\theta), & \bar{V}(\theta) = \bar{v} \circ \bar{y}(\theta), \\ \bar{M}(\theta) = \bar{u}_x \circ \bar{y}(\theta), & \bar{N}(\theta) = \bar{v}_x \circ \bar{y}(\theta), \\ \bar{H}(\theta) = \int_0^{\bar{y}(\theta)} (\bar{u}^2 + \bar{u}_x^2 + \bar{v}^2 + \bar{v}_x^2) dx, \\ \int_0^{\bar{y}(\theta)} (\bar{u}^2 + \bar{u}_x^2 + \bar{v}^2 + \bar{v}_x^2) dx + \bar{y}(\theta) = (1 + \bar{h})\theta, \end{cases} \tag{22}$$

where $\bar{h} = \int_0^1 (\bar{u}^2 + \bar{u}_x^2 + \bar{v}^2 + \bar{v}_x^2) dx$. By (22), we know that $\bar{y}(\theta)$ is continuous, strictly increasing. In the following work, we will testify that $(\bar{y} - Id, \bar{U}, \bar{V}, \bar{m}, \bar{N}, \bar{H} - \bar{h}Id, \bar{h})$ belongs to G which is defined as follows.

Definition 3.4. The set G is made up of all $(\eta, U, V, M, N, \sigma, h) \in W$ such that

$$\begin{cases} (\eta, U, V, M, N, \sigma, h) \in [W_{per}^{1,\infty}]^3 \times [L_{per}^\infty]^2 \times W_{per}^{1,\infty} \times \mathbb{R}, \\ y_\theta \geq 0, H_\theta \geq 0, y_\theta + H_\theta \geq 0 \quad \text{almost everywhere,} \\ y_\theta H_\theta = U^2 y_\theta^2 + U_\theta^2 + V^2 y_\theta^2 + V_\theta^2 \quad \text{almost everywhere,} \end{cases} \quad (23)$$

where $y = \eta(\theta) + \theta$, $H(\theta) = \sigma(\theta) + h\theta$.

We know that the initial data $(\bar{\eta}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{\sigma}, \bar{h})$ belongs to G . We will certify that the solution of system (18) exists globally in time with any initial data in G .

Lemma 3.5. Given the initial data $\bar{X} = (\bar{\eta}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{\sigma}, \bar{h})$ in G . Let

$$X(t) = (\eta(t), U(t), V(t), M(t), N(t), \sigma(t), h(t))$$

be the local solution of system (18) in $C([0, T], W)$ for some $T > 0$, with the above initial data. Then

- (i) $X(t)$ belongs to G for all $t \in [0, T]$;
- (ii) almost every $t \in [0, T], y_\theta(t, \theta) > 0$ for almost every θ .

Proof. (i) We continue to use S defined above. Therefore, from Lemma 3.3, the first one of (23) holds for all $t \in [0, T]$ and $X \in C^1([0, T], W)$. We will prove that the second and third inequality of (23) hold for all $\theta \in S$. Fixed $\theta \in S$ and dropped it in the notation without ambiguity. By system (17), we get

$$\begin{aligned} (y_\theta H_\theta)_t &= y_{\theta t} H_\theta + y_\theta H_{\theta t} \\ &= (U_\theta + V_\theta) H_\theta + y_\theta [(3U^2 - 2P_2 - 2P_{1,x} + 2V^2)U_\theta \\ &\quad + (3V^2 - 2Q_2 - 2Q_{1,x} + 2U^2)V_\theta \\ &\quad - 2(UP_{2,x} + UP_1 + VQ_{2,x} + VQ_1)y_\theta], \end{aligned}$$

and on the other hand,

$$\begin{aligned} (U^2 y_\theta^2 + U_\theta^2 + V^2 y_\theta^2 + V_\theta^2)_t &= 2UU_t y_\theta + 2U^2 y_\theta y_{\theta t} + 2U_\theta U_{\theta t} + 2VV_t y_\theta + 2V^2 y_\theta y_{\theta t} + 2V_\theta V_{\theta t} \\ &= (U_\theta + V_\theta) H_\theta + y_\theta [(3U^2 - 2P_2 - 2P_{1,x} + 2V^2)U_\theta + (3V^2 - 2Q_2 - 2Q_{1,x} + 2U^2)V_\theta \\ &\quad - 2(UP_{2,x} + UP_1 + VQ_{2,x} + VQ_1)y_\theta]. \end{aligned}$$

It is not very difficult to check that $(My_\theta)_t = (U_\theta)_t, \bar{M}\bar{y}_\theta = \bar{U}_\theta, (Ny_\theta)_t = (V_\theta)_t$ and $\bar{N}\bar{y}_\theta = \bar{V}_\theta$. We can get that $M(t, \theta)y_\theta(t, \theta) = V_\theta(t, \theta)$ for $t \in [0, T]$, since the uniqueness of ordinary differential equation. Thus, (23) is defined. Now, we define t^* as follow

$$t^* = \sup\{t \in [0, T] | y_\theta(t') \geq 0 \text{ for all } t' \in [0, t]\}.$$

If $t^* \leq T$, we obtain

$$y_\theta(t^*) = 0$$

for the continuity of $y_\theta(t)$ on time. From (23), we get that $U_\theta(t^*) = V_\theta(t^*) = 0$. Then, utilizing (17), we have

$$y_{\theta t}(t^*) = U_\theta(t^*) + V_\theta(t^*) = 0.$$

From system (17) and the fact $y_\theta(t^*) = U_\theta(t^*) + V_\theta(t^*) = 0$, we can infer

$$y_{\theta tt}(t^*) = U_{\theta t}(t^*) + V_{\theta t}(t^*) = H_\theta(t^*).$$

If $H_\theta(t^*) = 0$, combining with (23), we can deduce that

$$y_\theta(t^*) = U_\theta(t^*) = V_\theta(t^*) = H_\theta(t^*) = 0.$$

This is a contradiction to the uniqueness of system (17). If $H_\theta(t^*) < 0$, then $y_{\theta tt} < 0$. Thus, $y_\theta(t^*)$ is the strict maximum. This contradicts the definition of t^* . Hence $H_\theta(t^*) > 0$ implies that $y_\theta(t^*)$ is the strict maximum. This contradicts the fact $t^* < T$. Therefore, we get $y_\theta(t^*) \geq 0$ for all $t \in [0, T]$. Let us certify that $H_\theta(t) \geq 0$. This follows from (23) when $y_\theta(t) > 0$. If $y_\theta(t) = 0$ then as above, $H_\theta(t) < 0$ implies that $y_\theta(t) = 0$ is the strict maximum. This contradicts the fact $y_\theta(t) \geq 0$ on $[0, T]$. Hence $H_\theta(t) \geq 0$ on $[0, T]$. Now we have that $y_\theta(t) + H_\theta(t) \geq 0$. If the equality holds, it then follows that $y_\theta(t) = U_\theta(t) = V_\theta(t) = H_\theta(t) = 0$. This contradicts the uniqueness of system (17) for $\bar{y}_\theta > 0$.

(ii) Let

$$A = \{(t, \theta) \in [0, T] \times \mathbb{R} | y_\theta(t, \theta) = 0\}.$$

Fubini's theorem infers that

$$meas(A) = \int_{\mathbb{R}} meas(A_\theta) d\theta = \int_0^T meas(A_t) dt, \tag{24}$$

where

$$A_\theta = \{t \in [0, T] | y_\theta(t, \theta) = 0\}, \quad A_t = \{\theta \in \mathbb{R} | y_\theta(t, \theta) = 0\}.$$

From the above proof, we know that for $\theta \in A$, the time points t satisfying $y_\theta(t, \theta) = 0$ are isolated. Thus, we have that $meas(A_\theta) = 0$. It follows from (24) and $meas(A^c) = 0$ that

$$meas(A_t) = 0 \text{ for almost every } t \in [0, T].$$

We denote by K the set of times such that $meas(A_t) > 0$, i.e.

$$K = \{t \in \mathbb{R}^+ | meas((A)_t) > 0\}.$$

Then, $meas(K) = 0$. For all $t \in K^c, y_\theta > 0$ almost everywhere. Therefore, $y(t, \theta)$ is strictly increasing and invertible. \square

Theorem 3.6. *Given any $\bar{X} = (\bar{y}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{H}) \in G$. Then the system (16) admits a global solution $X(t) = (y(t), U(t), V(t), M(t), N(t), h(t))$ in $C^1(\mathbb{R}^+, W)$ with the initial data $\bar{X} = (\bar{y}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{H})$ and $X(t) \in G$ for all $t \in \mathbb{R}^+$. Moreover, by equipping G with the topology induced by the E -norm then the map $D: G \times \mathbb{R}^+ \mapsto G$ defined as*

$$D_t(\bar{X}) = X(t)$$

is a continuous semigroup.

Proof. Let us write (y, U, V, M, N, H) to denote $\eta, U, V, M, N, \sigma, h$ with $y = \eta + Id$ and $H = \sigma + hId$. Assuming $(\eta, U, V, M, N, \sigma, h)$ be a solution of system (18) in $C^1(\mathbb{R}^+, W)$ with the initial data $(\bar{\eta}, \bar{U}, \bar{V}, \bar{M}, \bar{N}, \bar{\sigma}, \bar{h})$, we have

$$\sup_{t \in [0, T]} \|(\eta(t, \cdot), U(t, \cdot), V(t, \cdot), M(t, \cdot), N(t, \cdot), \sigma(t, \cdot), h(t, \cdot)) \|_W < \infty. \tag{25}$$

It is clear that $h(t) = \bar{h}$ for all $t \in \mathbb{R}^+$. According to system (16) we get that $H(t, 0) = 0$. Because $H_\theta \geq 0$, we obtain $\|H\|_{L^\infty} \leq H(t, 1) = \bar{h}$. Hence, $\|\sigma\|_{L^\infty} \leq$

$2\bar{h}$, and $\sup_{t \in [0, T]} \|\sigma\|_{L^\infty} \leq 2\bar{h}$. For θ and θ' in $[0, 1]$, we have that $|y(\theta) - y(\theta')| \leq 1$ for $y_\theta \geq 0$ and $y(1) - y(0) = 1$. Thus, we claim that

$$U^2 y_\theta \leq H_\theta, U_\theta M \leq H_\theta, V^2 y_\theta \leq H_\theta, V_\theta N \leq H_\theta. \tag{26}$$

Indeed, using (23), we obtain that $y_\theta = 0$ which deduces $U_\theta = 0$. Thus, (26) holds if $y_\theta = 0$. Othetwise, if $y_\theta \geq 0$, from (23), we have

$$U^2 + \frac{U_\theta^2}{y_\theta} + V^2 + \frac{V_\theta^2}{y_\theta} = H_\theta \tag{27}$$

and

$$\begin{cases} U_\theta M \leq \frac{U_\theta^2}{y_\theta} + M^2 y_\theta, & U_\theta N \leq \frac{V_\theta^2}{y_\theta} + N^2 y_\theta, \\ V_\theta N \leq \frac{V_\theta^2}{y_\theta} + N^2 y_\theta, & V_\theta M \leq \frac{U_\theta^2}{y_\theta} + M^2 y_\theta. \end{cases} \tag{28}$$

Utilizing (27)-(28), we get (26). From (13)-(22), we have that

$$\sup_{t \in [0, T]} \|\rho_i\|_{L^\infty} \leq C\bar{h}, \quad \sup_{t \in [0, T]} \|\rho_{i,x}\|_{L^\infty} \leq C\bar{h},$$

where the constant C is independent of t and the initial data. From system (18), we have following estimates

$$\begin{aligned} \sup_{t \in [0, T]} \|U(t)\|_{L^\infty} &\leq \infty, & \sup_{t \in [0, T]} \|V(t)\|_{L^\infty} &\leq \infty, \\ \sup_{t \in [0, T]} \|M(t)\|_{L^\infty} &\leq \infty, & \sup_{t \in [0, T]} \|N(t)\|_{L^\infty} &\leq \infty. \end{aligned}$$

For $\eta_t = U + V$, therefore

$$\sup_{t \in [0, T]} \|\eta(t)\|_{L^\infty} \leq \infty.$$

Now, we have certified that

$$\begin{aligned} C_1 = \sup_{t \in [0, T]} \{ &\|U(t)\|_{L^\infty} + \|V(t)\|_{L^\infty} + \|M(t)\|_{L^\infty} + \|N(t)\|_{L^\infty} \\ &+ \|\rho_1\|_{L^\infty} + \|\rho_{1,x}\|_{L^\infty} + \|\rho_2\|_{L^\infty} + \|\rho_{2,x}\|_{L^\infty} \\ &+ \|\rho_3\|_{L^\infty} + \|\rho_{3,x}\|_{L^\infty} + \|\rho_4\|_{L^\infty} + \|\rho_{4,x}\|_{L^\infty} \} \end{aligned}$$

is finite. Let

$$Z(t) = \|y_\theta(t)\|_{L^2} + \|U_\theta(t)\|_{L^2} + \|V_\theta(t)\|_{L^2} + \|H_\theta(t)\|_{L^2}.$$

Since the system (17) is semilinear, we have that

$$Z(t) = Z(0) + C \int_0^t Z(\tau) d\tau,$$

where C is only depend on C_1 . Applying Gronwall's inequality, we get (25). By the standard theorem of ordinary differential equation, we obtain that D_t is a continuous semigroup. □

4. The original system. In this section, we will investigate how to obtain a global conservative solution of system (4) from the global solution of system (17) with the initial variables (t, x) . Let (y, U, V, M, N, H) be the global solution of system (17). Note that

$$u(t, x) \doteq u(t, \theta), \quad v(t, \theta) \doteq v(t, \theta), \quad \text{if } y(t, \theta) = x. \tag{29}$$

Theorem 4.1. *Let (y, U, V, M, N, H) be a global solution of system (17). Then the pair function $(u(t, x), v(t, x))$ defined by (29) is the global solution to the system (4). Moreover, this solution (u, v) satisfies the following property:*

$$\|u(t, \cdot)\|_{H^1}^2 + \|v(t, \cdot)\|_{H^1}^2 = \|u(0, \cdot)\|_{H^1}^2 + \|v(0, \cdot)\|_{H^1}^2, \quad \text{a.e. } t \geq 0. \tag{30}$$

Furthermore, let (\bar{u}_n, \bar{v}_n) be a sequence of the initial data such that

$$\|\bar{u}_n - \bar{u}\|_{H^1} \rightarrow 0, \quad \|\bar{v}_n - \bar{v}\|_{H^1} \rightarrow 0.$$

Then the corresponding solutions (\bar{u}_n, \bar{v}_n) converge to $(u(t, x), v(t, x))$ uniformly for all (t, x) in any bounded set.

Proof. Firstly, what we have to do is to show that the definition of u and v make sense. Given $x \in [0, 1)$, if $\theta_1 \leq \theta_2$, $x = y(t, \theta_1) = y(t, \theta_2)$, then

$$y_\theta(\theta) = 0 \text{ in } [\theta_1, \theta_2].$$

We can obtain that $U_\theta = V_\theta = 0$ in $[\theta_1, \theta_2]$. Therefore, $U(\theta_1) = U(\theta_2)$, $V(\theta_1) = V(\theta_2)$, and we can get that $(u(t, x), v(t, x))$ from the above definition is meaningful. It is clear that $u(x + 1) = u(x)$ and $v(x + 1) = v(x)$. Now we will certify $u \in H^1$. Obviously, $u \in L^\infty$, which yields $u \in L^2$. So does v . Next, we will show that $u_x, v_x \in L^2$. From (23), we have

$$\begin{aligned} \int_0^1 u_x^2 dx &= \int_{y^{-1}(0)}^{y^{-1}(1)} u_x^2(t, y(\theta)) y_\theta d\theta = \int_0^1 u_x^2(t, y(\theta)) y_\theta d\theta \\ &= \int_{\theta \in [0, 1] | y_\theta} \frac{U_\theta}{y_\theta} d\theta \leq (H(1) - H(0)) = \bar{h}. \end{aligned}$$

It is similar to v_x . Now, we testify the pair function (u, v) satisfied (4). Given $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$ with compact support. Let (y, U, V, M, N, H) be the solution of system (16), then

$$\begin{aligned} &\int_{\mathbb{R}^+ \times \mathbb{R}} [-(u + v)\phi_t + (u + v)(u_x + v_x)\phi(t, x)] dx dt \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}} [-(U + V)y_\theta\phi_t + (U + V)(U_\theta + V_\theta)\phi(t, Y)] d\theta dt. \end{aligned} \tag{31}$$

Utilizing $y_t = U + V$ and $y_{\theta t} = U_\theta = V_\theta$, we get

$$\begin{aligned} &[(U + V)y_\theta\phi \circ y]_t - [(U + V)^2]_\theta \\ &= (U_t + V_t)y_\theta\phi \circ y + (U + V)y_{\theta t}\phi - (U + V)(U_\theta + V_\theta)\phi. \end{aligned} \tag{32}$$

Integrating (32) over $\mathbb{R}^+ \times \mathbb{R}$, using (23) and taking $x = y(t, \theta)$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \mathbb{R}} [-(U + V)y_\theta \phi_t + (U + V)(U_\theta + V_\theta)\phi(t, Y)]d\theta dt \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}} (U_t + V_t)y_\theta \circ y_\theta \phi d\theta dt \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}} (-P_1 - P_{2,x} - Q_1 - Q_{2,x})\phi(t, x)dx dt \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}} (-P_1 - P_{2,x} - Q_1 - Q_{2,x})y_\theta(t, \theta)\phi(t, y(t, \theta))d\theta dt. \end{aligned} \tag{33}$$

By (31)-(33), we obtain the first two equation of system (4). When $t \in K^c$, we have $y_\theta(t, \theta) > 0$ a.e. Utilizing (23), we obtain

$$H_\theta = U^2 y_\theta + \frac{U_\theta}{y_\theta} + V^2 y_\theta + \frac{V_\theta}{y_\theta}$$

holds almost everywhere. By taking $x = y(t, \theta)$, we have

$$\begin{aligned} & \int_0^1 (u^2(t, x) + u_x^2(t, x) + v^2(t, x) + v_x^2(t, x))dx \\ &= \int_0^1 (u^2(0, x) + u_x^2(0, x) + v^2(0, x) + v_x^2(0, x))dx. \end{aligned}$$

Therefore, we get (30).

Finally, let (\bar{u}_n, \bar{v}_n) converge to (\bar{u}, \bar{v}) in $H^1 \times H^1$. From (22), it is not very hard to check that

$$\begin{aligned} & \|\bar{y}_n - \bar{y}\|_{L^\infty} \rightarrow 0, \quad \|\bar{U}_n - \bar{U}\|_{L^\infty} \rightarrow 0, \quad \|\bar{V}_n - \bar{V}\|_{L^\infty} \rightarrow 0, \\ & \|\bar{H}_n - \bar{H}\|_{L^\infty} \rightarrow 0, \quad \|\bar{h}_n - \bar{h}\|_{L^\infty} \rightarrow 0. \end{aligned}$$

Now, we certify that

$$\begin{aligned} & \|\bar{y}_{n\theta} - \bar{y}_\theta\|_{L^2} \rightarrow 0, \quad \|\bar{U}_{n\theta} - \bar{U}_\theta\|_{L^2} \rightarrow 0, \quad \|\bar{V}_{n\theta} - \bar{V}_\theta\|_{L^2} \rightarrow 0, \\ & \|\bar{M}_n - \bar{M}\|_{L^2} \rightarrow 0, \quad \|\bar{N}_n - \bar{N}\|_{L^2} \rightarrow 0. \end{aligned}$$

Let $g_n = u_n^2 + u_{nx}^2 + v_n^2 + v_{nx}^2$ and $g = u^2 + u_x^2 + v^2 + v_x^2$. From (22), we have that

$$(1 + h)(\bar{y}_\theta - \bar{y}_{n\theta}) = (g_n \circ y_n - g \circ y)y_{n\theta}y_\theta + (h - h_n)y_\theta. \tag{34}$$

The first item on the right side of (34) can be written as follow

$$(g_n \circ y_n - g \circ y)y_{n\theta}y_\theta = (g_n \circ y_n - g \circ y_n)y_{n\theta}y_\theta + (g \circ y_n - g \circ y)y_{n\theta}y_\theta.$$

Because $0 \leq y_\theta \leq 1 + h$, it follows that

$$\int_0^1 |(g_n \circ y_n - g \circ y)y_{n\theta}y_\theta|d\theta \leq (1 + h)\|g_n - g\|_{L^1}.$$

Note that $g \in L^1$. For any $\varepsilon \geq 0$, there exists a continuous function r such that $\|g - v\|_{L^1} \leq \varepsilon$. Then

$$\begin{aligned} \int_0^1 |g \circ y - g \circ y_n|y_{n\theta}y_\theta d\theta &\leq \int_0^1 (|g \circ y - r \circ y| + |r \circ y - r \circ y_n|)y_{n\theta}y_\theta d\theta \\ &\quad + \int_0^1 |r \circ y_n - g \circ y_n|y_{n\theta}y_\theta d\theta. \end{aligned}$$

The first and third item tend to zero for the boundedness of $y_{n\theta}, y_\theta$ and the arbitrariness of ε . The second item tends to zero by utilizing the dominated convergence

theorem. Hence $\bar{y}_{n\theta} \rightarrow \bar{y}_\theta$ in L^1 . We finally obtain that $\bar{y}_{n\theta} \rightarrow \bar{y}_\theta$ in L^2 , because $y_{n\theta}$ is bounded in L^∞ . From (22), we get $\bar{H}_{n\theta} \rightarrow \bar{H}_\theta$ in L^2 ,

$$\bar{M}_n - \bar{M} = \bar{u}_{nx} \circ y_n - \bar{u}_x \circ y = \bar{u}_{nx} \circ y_n - \bar{u}_x \circ y_n + \bar{u}_x \circ y_n - \bar{u}_x \circ y.$$

For $\bar{u}_{nx} \rightarrow \bar{u}_x$ in L^∞ , we know that

$$\int_0^1 |\bar{u}_{nx} \circ y_n - \bar{u}_x \circ y_n|^2 d\theta \leq \|\bar{u}_{nx} - \bar{u}_x\|_{L^\infty}^2 \rightarrow 0.$$

Given any $\varepsilon > 0$, there exists a continuous function $r(\theta)$ such that

$$\int_0^1 |\bar{u}_x \circ y - r|^2 d\theta \leq \varepsilon. \tag{35}$$

Then

$$\begin{aligned} \bar{u}_{nx} \circ y_n - \bar{u}_x \circ y &= \bar{u}_{nx} \circ y \circ y^{-1} \circ y_n - \bar{u}_x \circ y \\ &= \bar{u}_{nx} \circ y \circ y^{-1} \circ y_n - r \circ y^{-1} \circ y_n + r \circ y^{-1} \circ y_n - r + r - \bar{u}_x \circ y. \end{aligned}$$

Utilizing (35), we have

$$\int_0^1 |\bar{u}_{nx} \circ y \circ y^{-1} \circ y_n - r \circ y^{-1} \circ y_n|^2 d\theta \leq \varepsilon,$$

and

$$\int_0^1 |r - \bar{u}_x \circ y|^2 d\theta \leq \varepsilon.$$

According to dominated convergence theorem, we get

$$\int_0^1 |r \circ y^{-1} \circ y_n - r|^2 d\theta \rightarrow 0.$$

Thus, we have $\bar{M}_n \rightarrow \bar{M}$ in L^2 . Similarly, $\bar{N}_n \rightarrow \bar{N}$ in L^2 .

According to (23), $\|\bar{U}_n - \bar{U}\|_{L^\infty} \rightarrow 0$, $\|\bar{V}_n - \bar{V}\|_{L^\infty} \rightarrow 0$, with $M_n \rightarrow M$, $y_{n\theta} \rightarrow y_\theta$ in L^2 , we get $\|U_{n\theta}\|_{L^2} \rightarrow \|U_\theta\|_{L^2}$. So does $\|V_{n\theta}\|_{L^2} \rightarrow \|V_\theta\|_{L^2}$. What we only need to do to prove $U_{n\theta} \rightarrow U_\theta$ and $V_{n\theta} \rightarrow V_\theta$ in L^2 is to show that for any continuous function Ψ with compact support, we have

$$\int_{\mathbb{R}} U_{n\theta} \Psi d\theta = \int_{\mathbb{R}} u_{nx} \circ y_n y_{n\theta} \Psi d\theta = \int_{\mathbb{R}} u_{nx} \Psi \circ y_n^{-1} dx.$$

Thus,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} U_{n\theta} \Psi d\theta = \int_{\mathbb{R}} u_x \Psi \circ y_n^{-1} dx = \int_{\mathbb{R}} U_\theta \Psi d\theta.$$

Then, we obtain $U_{n\theta} \rightarrow U_\theta$ in L^2 . With the same calculation we have $V_{n\theta} \rightarrow V_\theta$ in L^2 . Now we have

$$\begin{cases} y_n \rightarrow y \text{ in } H^1, U_n \rightarrow U \text{ in } H^1, V_n \rightarrow V \text{ in } H^1, \\ H_n \rightarrow H \text{ in } H^1, M_n \rightarrow M \text{ in } L^2, N_n \rightarrow N \text{ in } L^2. \end{cases} \tag{36}$$

Combining (16)-(17) and (36), we get

$$\begin{aligned} & \frac{d}{dt} (\|U_n(t) - U(t)\|_{L^\infty} + \|V_n(t) - V(t)\|_{L^\infty} + \|y_n(t) - y(t)\|_{L^\infty} \\ & + \|M_n(t) - M(t)\|_{L^2} + \|N_n(t) - N(t)\|_{L^2} + \|U_{n\theta}(t) - U_\theta(t)\|_{L^2} \\ & + \|V_{n\theta}(t) - V_\theta(t)\|_{L^2} + \|H_{n\theta}(t) - H_\theta(t)\|_{L^2} + \|y_{n\theta}(t) - y_\theta(t)\|_{L^2}) \\ & \leq C (\|U_n(t) - U(t)\|_{L^\infty} + \|V_n(t) - V(t)\|_{L^\infty} + \|y_n(t) - y(t)\|_{L^\infty} \\ & + \|M_n(t) - M(t)\|_{L^2} + \|N_n(t) - N(t)\|_{L^2} + \|U_{n\theta}(t) - U_\theta(t)\|_{L^2} \\ & + \|V_{n\theta}(t) - V_\theta(t)\|_{L^2} + \|H_{n\theta}(t) - H_\theta(t)\|_{L^2} + \|y_{n\theta}(t) - y_\theta(t)\|_{L^2}). \end{aligned}$$

According to Gronwall's inequality, we conclude that $y_n \rightarrow y, U_n \rightarrow U$ and $V_n \rightarrow V$ in L^∞ on any bounded time interval. This yields that

$$u_n(t, x) \rightarrow u(t, x), \quad v_n(t, x) \rightarrow v(t, x),$$

are uniformly Hölder continuous on any bounded time interval. □

Lastly, we shall certify that the solutions obtained in *Theorem 4.1* construct a semigroup.

Theorem 4.2. *Given initial data $(\bar{u}, \bar{v}) \in H^1_{per} \times H^1_{per}$. Let $(u(t), v(t)) = F_t(\bar{u}, \bar{v})$ be the corresponding global solution of system (4) constructed in *Theorem 4.1*. Then the map $F : H^1_{per} \times H^1_{per} \times [0, \infty] \mapsto H^1_{per} \times H^1_{per}$ is semigroup.*

Proof. Fix $(\bar{u}, \bar{v}) \in H^1_{per} \times H^1_{per}$ and $\tau > 0$. For every $t > 0$, what we need to do is prove

$$F_t(F_\tau(\bar{u}, \bar{v})) = F_{\tau+t}(\bar{u}, \bar{v}).$$

Let $(y(\tau, \theta), U(\tau, \theta), V(\tau, \theta), M(\tau, \theta), N(\tau, \theta), H(\tau, \theta))$ be the solution of system (16) with the initial data given by (22). For any time τ , we have the new initial data as follows

$$\int_0^{\hat{y}(\theta)} (u^2 + u_x^2 + v^2 + v_x^2) dx + \hat{y}(\theta) = (1 + \bar{h})\theta, \tag{37}$$

and

$$\begin{cases} \hat{H}(\theta) = \int_0^{\hat{y}(\theta)} (u^2 + u_x^2 + v^2 + v_x^2) dx, \\ \hat{U}(\theta) = u(t, \hat{y}(t, \theta)), \quad \hat{V}(\theta) = v(t, \hat{y}(t, \theta)), \\ \hat{M}(\theta) = u_x(t, \hat{y}(t, \theta)), \quad \hat{N}(\theta) = v_x(t, \hat{y}(t, \theta)). \end{cases} \tag{38}$$

Let $(\hat{y}(t + \tau, \theta), \hat{U}(t + \tau, \theta), \hat{V}(t + \tau, \theta), \hat{M}(t + \tau, \theta), \hat{N}(t + \tau, \theta), \hat{H}(t + \tau, \theta))$ be a solution of the system (16) with the initial data (37)-(38). We claim that

$$\begin{aligned} & (y(t + \tau, \theta), U(t + \tau, \theta), V(t + \tau, \theta), M(t + \tau, \theta), N(t + \tau, \theta), H(t + \tau, \theta)) \\ & = (\hat{y}(t + \tau, \tilde{\theta}), \hat{U}(t + \tau, \tilde{\theta}), \hat{V}(t + \tau, \tilde{\theta}), \hat{M}(t + \tau, \tilde{\theta}), \hat{N}(t + \tau, \tilde{\theta}), \hat{H}(t + \tau, \tilde{\theta})), \end{aligned}$$

where $\tilde{\theta}$ is defined as $\hat{y}(\tau + t, \tilde{\theta}) = y(\tau + t, \theta)$.

Actually, the equality $\hat{y}(\tau + t, \tilde{\theta}) = y(\tau + t, \theta)$ yields that

$$\hat{y}_\theta(\tau + t, \tilde{\theta}) d\theta = y_\theta(\tau + t, \theta) d\theta. \tag{39}$$

Utilizing (39), we claim that

$$Q_i(\tau + t, \tilde{\theta}) = Q_i(\tau + t, \theta(\tilde{\theta})), \quad Q_{i,x}(\tau + t, \tilde{\theta}) = Q_{i,x}(\tau + t, \theta(\tilde{\theta})), \quad i = 1, 2.$$

For $P_i, P_{i,x}$ ($i = 1, 2$), we can get the similar results. At the time τ , $y(\tau, \theta) = \hat{y}(\tau, \hat{\theta})$ implies that

$$\begin{aligned} & (y(\tau, \theta(\hat{\theta})), U(\tau, \theta(\hat{\theta})), V(\tau, \theta(\hat{\theta})), M(\tau, \theta(\hat{\theta})), N(\tau, \theta(\hat{\theta})), H(t + \tau, \theta)) \\ & = (\hat{y}(\tau, \tilde{\theta}), \hat{U}(\tau, \tilde{\theta}), \hat{V}(\tau, \tilde{\theta}), \hat{M}(\tau, \tilde{\theta}), \hat{N}(\tau, \tilde{\theta}), \hat{H}(\tau, \tilde{\theta})). \end{aligned}$$

Therefore, we get that $(y(\tau, \theta(\hat{\theta})), U(\tau, \theta(\hat{\theta})), V(\tau, \theta(\hat{\theta})), M(\tau, \theta(\hat{\theta})), N(\tau, \theta(\hat{\theta})), H(t + \tau, \theta))$ is a solution of system (16). Utilizing (29) and $\hat{y}(\tau + t, \tilde{\theta}) = y(\tau + t, \theta)$, we obtain that the solution of the system (4) constructs a semigroup. \square

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