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SOME PROPERTIES FOR ALMOST CELLULAR ALGEBRAS

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Abstract. In this paper, we will investigate some properties for almost cellular algebras. We compare the almost cellular algebras with quasi-hereditary algebras, which are known to carry any homological and categorical structures. We prove that any almost cellular algebra is the iterated inflation and obtain some sufficient and necessary conditions for an almost cellular algebra A to be quasi-hereditary.

1. **Introduction.** Cellular algebras were introduced by J. Graham and G. Lehrer in [6]. Their definition is based on the existence of a certain basis with some special properties motivated by KazhdanšCLusztig bases of Hecke algebras and is applicable to other families of algebras like Brauer algebras [6], cyclotomic Temperley-Lieb algebras [6, 8], cyclotomic Birman-Murakami-Wenzl algebras [5, 14], and so on. An equivalent definition of the cellular algebra was given by S. König and C.C. Xi in terms of cell ideals and a filtration by two-sided ideals [10] . Two different equivalent definitions have different advantages. The first one can be used to check concrete examples. The second one, however, is often more handy for theoretical and structural purposes (see [10], [11] and their references). One of the advantages of the concept of cellularity is that it provides a way to parametrize irreducible modules. The problem of determining a parameter set for, or even constructing bases of irreducible modules, is in this way reduced (but of course not solved in general) to questions of linear algebra. The relation between the cellular algebras and quasi-hereditary algebras was investigated by S. König and C.C. Xi. Precisely they obtained some sufficient and necessary conditions for a cellular algebra to be quasi-hereditary (More details could be found in [11, Theorem 3.1]).

There are several generalizations of cellular algebras. For example, affine cellular algebras if we extend the framework of cellular algebras to algebras that need not be finite dimensional over a field [13], relative cellular algebras if we allow different

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partial orderings relative to fixed idempotents [4], standardly based algebra by constructing a nice bases satisfy some conditions [3] and almost cellular algebras if we remove the compatible anti-involution from the definition of cellularity [7]. In this paper, we focus on the third generalization. Motivated by Schur-Weyl duality, the authors introduce the quantum walled Brauer-Clifford superalgebras $\mathrm{BC}_{r,s}(q)$, the quantum deformation of the walled Brauer-Clifford superalgebra $\mathrm{BC}_{r,s}$, which is the centralizer superalgebra of the action of $\mathrm{U}_q(\mathfrak{q}(n))$ on the mixed tensor space under some mild condition [1]. The Howe duality for quantum queer superalgebras was given [2]. Because of their similarity with Hecke-Clifford (super)algebras, the quantum walled Brauer-Clifford (super)algebras are not the cellular algebras since the absence of an anti-involution with the property that it fixes isomorphism classes of irreducible modules. However, these algebras also have many of the properties of cellular algebras, which belong to a large class of algebras, removing the anti-involution from the definition of cellularity, called almost cellular algebras[1] [7].

The aim of this paper is to study the structure of almost cellular algebras and determine some sufficient and necessary conditions for an almost cellular algebra to be quasi-hereditary, which is inspired by S. König and C. C. Xi's papers [10, 11]. In section two we review the definition of almost cellular algebras, and some examples. We end this section by determining the possibilities for a factor $\mathrm{J} \subseteq \mathrm{A}$ of an almost cellular algebra A. In last section, we give a list of homological properties of a factor and show the difference between cellular algebras and almost cellular algebras see Proposition 3.4, we prove that the determinant of the Cartan matrix C of an almost algebra A is a positive integer, and obtain some sufficient and necessary conditions for an almost cellular algebra to be quasi-hereditary.

2. **Definitions and basic properties.** Throughout the paper the symbols R and $\Bbbk$ stand for an arbitrary Noetherian commutative integral domain and a field, respectively. Denote the abelian group of two elements by $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$.

Definition 2.1. Suppose $\mathrm{A}, \mathrm{A}_1, \dots, \mathrm{A}_k$ are unital associative rings. We say A has a sandwich filtration over $\mathrm{A}_1, \dots, \mathrm{A}_k$ if it has a filtration by two-sided ideals

$$0 = \mathrm{J}_0 \subsetneq \mathrm{J}_1 \subsetneq \dots \subsetneq \mathrm{J}_k = \mathrm{A}$$

such that $\mathrm{J}_i/\mathrm{J}_{i-1} \cong \mathrm{V}_i \otimes_{\mathrm{A}_i} \mathrm{W}_i$ as an (A, A)-bimodule for some nonzero $(\mathrm{A}, \mathrm{A}_i)$-bimodule V_i and $(\mathrm{A}_i, \mathrm{A})$-bimodule W_i, both free of finite rank over A_i. We call $\mathrm{V}_i \otimes_{\mathrm{A}_i} \mathrm{W}_i$ the factors of A. In particular, a factor $\mathrm{J} \subseteq \mathrm{A}$ means the first layer in a sandwich filtration, that is, J is a two-sided ideal of A and $\mathrm{J} \cong \mathrm{V}_1 \otimes_{\mathrm{A}_1} \mathrm{W}_1$ as (A, A)-bimodules. If the rings $\mathrm{A}_1, \dots, \mathrm{A}_k$ all coincide, we simply say that A has a sandwich filtration over A_1. If A admits a sandwich filtration, we call A an almost cellular algebra.

Remark 2.2. The relationship between almost cellular algebras and cellular algebras is as follows: a cellular algebra admits a sandwich filtration over the base field and has a compatible anti-involution. The definition above is analogous to the iterated inflations in [12], since it allows each A_i to be an arbitrary ring. It is interesting to find an equivalent definition based on the bases for an almost cellular algebra, which can help us to determine whether an algebra is an almost cellular algebra or not?

[1]When equipped with a compatible anti-involution, the almost cellular algebras are cellular algebras, so called them almost cellular algebras.

Typical examples of almost cellular algebras can be found in [7].

Example 2.3. Cellular algebras and affine cellular algebras are almost cellular algebras.

Example 2.4. The finite Hecke-Clifford algebra $\mathrm{HC}_n(q)$ is the unital associative algebra over $\mathbb{C}(q)$ generated by elements t_i for $1 \leqslant i \leqslant n-1$ and elements c_i for $1 \leqslant i \leqslant n$ which satisfy the relations:

$$(t_i - q)(t_i + q^{-1}) = 0, i = 1, \cdots, n-1, \qquad t_i t_j = t_j t_i, |i-j| \geqslant 2,$$
$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, i = 1, \cdots, n-2,$$
$$c_i^2 = -1, \qquad c_i c_j = -c_j c_i, 1 \leqslant i \neq j \leqslant n,$$
$$t_i c_i = c_{i+1} t_i, \qquad t_i t_j = t_j t_i, j \neq i, i+1.$$

Example 2.5. The q-walled Brauer-Clifford algebra $\mathrm{WBC}_{r,s}(q)$ is the unital associative algebra over R generated by elements

$$t_1, \ldots, t_{r-1}, c_1, \ldots c_r, \bar{t}_1, \ldots, \bar{t}_{s-1}, \bar{c}_1, \ldots, \bar{c}_s, \text{and } e.$$

The elements $t_1, \ldots, t_{r-1}, c_1, \ldots c_r$ satisfy the relations of finite Hecke-Clifford algebra $\mathrm{HC}_r(q)$, and the elements $\bar{t}_1, \ldots, \bar{t}_{s-1}, \bar{c}_1, \ldots, \bar{c}_s$ satisfy the relations of finite Hecke-Clifford algebra $\mathrm{HC}_s(q)$ except that $\bar{c}_i^2 = 1$. Moreover $t_1, \ldots, t_{r-1}, c_1, \ldots c_r$ supercommute with $\bar{t}_1, \ldots, \bar{t}_{s-1}, \bar{c}_1, \ldots, \bar{c}_s$. The generator e commutes with

$$t_1, \ldots, t_{r-2}, c_1, \ldots c_{r-1}, \bar{t}_2, \ldots, \bar{t}_{s-1}, \bar{c}_2, \ldots, \bar{c}_s,$$

and satisfies

$$e^2 = 0, \quad e t_{r-1} e = e = e \bar{t}_1 e, \quad c_r e = \bar{c}_1 e, \quad e c_r = e \bar{c}_1, \quad e c_r e = 0,$$
$$e t_{r-1}^{-1} \bar{t}_1 e t_{r-1} = e t_{r-1}^{-1} \bar{t}_1 e \bar{t}_1, \quad t_{r-1} e \bar{t}_{r-1}^{-1} \bar{t}_1 e = \bar{t}_1 e t_{r-1}^{-1} \bar{t}_1 e.$$

Remark 2.6.
1. The Hecke-Clifford algebra $\mathrm{HC}_n(q)$ and the q-walled Brauer-Clifford algebra $\mathrm{WBC}_{r,s}(q)$ are almost cellular algebras. Moreover, roughly speaking, the q-walled Brauer-Clifford algebra $\mathrm{WBC}_{r,s}(q)$ has a sandwich filtration over the finite Hecke-Clifford algebras $\mathrm{HC}_{r-l}(q) \otimes \mathrm{HC}_{s-l}(q)$ for $0 \leqslant l \leqslant \min(r, s)$.
2. The finite Hecke-Clifford algebra $\mathrm{HC}_n(q)$ and the q-walled Brauer-Clifford algebra $\mathrm{WBC}_{r,s}(q)$ become $\mathbb{Z}_2$-graded algebras if we define $|t_i| = |\bar{t}_j| = |e| = \bar{0}$ and $|c_k| = |\bar{c}_l| = \bar{1}$ for all possible i, j, k, l.

Lemma 2.7. *([7, Lemma 2]) Let R be a Noetherian commutative integral domain and A be an R-algebra. Suppose we have an (A, A)- bimodule injection $\mathrm{V} \otimes_R \mathrm{W} \to \mathrm{A}$, where V is an (A, R)-bimodule, W is an (R, A)-bimodule, and V and W are both free over R. Then the multiplication map*

$$(\mathrm{V} \otimes_R \mathrm{W}) \otimes_{\mathrm{A}} (\mathrm{V} \otimes_R \mathrm{W}) \to \mathrm{V} \otimes_R \mathrm{W}$$

induced by this injection is given by

$$(v' \otimes_R w)(v \otimes_R w') = v' \varphi(w \otimes v) \otimes_R w',$$

where $\varphi : \mathrm{W} \otimes_{\mathrm{A}} \mathrm{V} \to R$ is an (R, R)-bimodule homomorphism uniquely determined by this formula.

Proposition 2.8. *Let A be an almost cellular algebra over a field $\Bbbk$ and $\mathrm{J} \subseteq \mathrm{A}$ a factor of A. Then J satisfies one of the following conditions*

1. *J has square zero.*

2. *There exists a primitive idempotent* $e \in \mathrm{A}$ *such that* $\mathrm{J} \cong \mathrm{A}e\mathrm{A} \cong \mathrm{A}e \otimes_{e\mathrm{A}e} e\mathrm{A}$ *as* (A, A)*-bimodules, and* $e\mathrm{A}e \cong \Bbbk$*. In particular,* $\mathrm{J} = \mathrm{J}^2$ *is a heredity ideal.*

Proof. By assumption, we may write $\mathrm{J} \cong \mathrm{V} \otimes_{\Bbbk} \mathrm{W}$ as (A, A)-bimodules, where V is a left A-module, W is a right A-module, V and W are finite-dimensional $\Bbbk$-vector spaces. By Lemma 2.7, if $\varphi(w \otimes v) = 0$ for all $w \in \mathrm{W}$ and $v \in \mathrm{V}$, then we have the situation (i).

Thus we may assume that there exists one $\varphi(w \otimes v) \neq 0$ for some $w \in \mathrm{W},\ v \in \mathrm{V}$. Then there exists a non-zero $k_0 \in \Bbbk$ such that $(v \otimes w)(v \otimes w) = k_0(v \otimes w)$. Hence J contains a primitive idempotent e, and $\mathrm{A}e$ is a left ideal which is contained in J. Then $\mathrm{J} \cong \mathrm{V}^{\oplus(\dim_{\Bbbk}\mathrm{W})}$ as a left A-module. However, $\mathrm{A}e$ is a submodule of J, and so $\mathrm{J} = \mathrm{A}e \oplus \mathrm{J}(1-e)$. It follows that $\mathrm{V} = \mathrm{A}e \oplus M$ for some left A-module M, and we can decompose $\mathrm{J} = (\mathrm{A}e)^m \oplus M^m$, where $m = \dim_{\Bbbk}\mathrm{W}$. Since $(\mathrm{A}e)^m$ is contained in the trace X of $\mathrm{A}e$ inside J, it follows that it is contained in the trace $\mathrm{A}e\mathrm{A}$ of $\mathrm{A}e$ in A. But the dimension of $\mathrm{A}e\mathrm{A}$ is less than or equal to the product of the dimension of $\mathrm{A}e$ with the dimension of $e\mathrm{A}$. This implies $\dim_{\Bbbk}\mathrm{W} \leqslant \dim_{\Bbbk}(e\mathrm{A})$.

On the other hand, $e\mathrm{A}$ is a right ideal which is contained in J. Then $\mathrm{J} \cong \mathrm{W}^{\oplus(\dim_{\Bbbk}\mathrm{V})}$ as a right A-module. Thus $\mathrm{J} = (1-e)\mathrm{J} \oplus e\mathrm{A}$. It follows that $\mathrm{W} = N \oplus e\mathrm{A}$ for some right A-module N and we can decompose $\mathrm{J} = N^n \oplus (e\mathrm{A})^n$, where $n = \dim_{\Bbbk}\mathrm{V}$. Since $(e\mathrm{A})^n$ is contained in the trace $\mathrm{A}e\mathrm{A}$ of $e\mathrm{A}$ in A, we get that $\dim_{\Bbbk}\mathrm{V} \leqslant \dim_{\Bbbk}(\mathrm{A}e)$. By above arguments, we have the following inequalities

$$\dim_{\Bbbk}\mathrm{A}e \leqslant \dim_{\Bbbk}\mathrm{V} \leqslant \dim_{\Bbbk}\mathrm{A}e, \quad \dim_{\Bbbk}e\mathrm{A} \leqslant \dim_{\Bbbk}\mathrm{W} \leqslant \dim_{\Bbbk}e\mathrm{A}.$$

Hence, $\dim_{\Bbbk}\mathrm{V} = \dim_{\Bbbk}\mathrm{A}e$ and $\dim_{\Bbbk}\mathrm{W} = \dim_{\Bbbk}e\mathrm{A}$. This means that $\mathrm{V} = \mathrm{A}e$ and $\mathrm{W} = e\mathrm{A}$. Since the multiplication $\mathrm{A}e \otimes_{\Bbbk} e\mathrm{A} \longrightarrow \mathrm{A}e\mathrm{A}$ is always surjective and $\dim(\mathrm{A}e)^m \leq \dim\mathrm{A}e\mathrm{A}$, it must be an isomorphism. Hence $\mathrm{J} \cong \mathrm{A}e\mathrm{A}$ and $e\mathrm{A}e \cong \Bbbk$. □

Remark 2.9. The proof of this proposition is a slight difference with the corresponding one of cellular algebra [10, Proposition 4.1].

The following corollary is immediately given.

Corollary 2.10. *Let* A *be an almost cellular* $\Bbbk$*-algebra with a sandwich filtration*

$$0 = \mathrm{J}_0 \subsetneq \mathrm{J}_1 \subsetneq \ldots \subsetneq \mathrm{J}_n = \mathrm{A}, \quad \textit{and} \quad \mathrm{J}_i/\mathrm{J}_{i-1} \cong \mathrm{V}_i \otimes_{\Bbbk} \mathrm{W}_i.$$

If all the square of $\mathrm{J}_i/\mathrm{J}_{i-1}$ *are nonzero in* $\mathrm{A}/\mathrm{J}_{i-1}$*, then* A *is a quasi-hereditary algebra and above sandwich filtration yields a heredity chain of* A*.*

Proof. By definition $\mathrm{J}_i/\mathrm{J}_{i-1} \subseteq \mathrm{A}/\mathrm{J}_{i-1}$ is a factor of $\mathrm{A}/\mathrm{J}_{i-1}$ and $\mathrm{J}_i/\mathrm{J}_{i-1}$ is an $(\mathrm{A}/\mathrm{J}_{i-1}, \mathrm{A}/\mathrm{J}_{i-1})$ -bimodule. By induction on the length of the sandwich filtration and Proposition 2.8 we get that A is a quasi-hereditary algebra and above sandwich filtration yields a heredity chain of A. □

3. **Some homological properties.**

In section two, we have seen that for a factor $\mathrm{J} \subseteq \mathrm{A}$ of an almost cellular algebra A there is exactly two possibilities, i.e., $\mathrm{J}^2 = 0$ or $\mathrm{J}^2 = \mathrm{J}$. In this section, we investigate some homological properties for an almost cellular algebra A.

Lemma 3.1. *(*[10, Proposition 6.1]*) For any ideal* J *in a* $\Bbbk$*-algebra* A*, the following two assertions are equivalent:*

1. $\mathrm{J}^2 = 0$,
2. $\mathrm{Tor}_2^{\mathrm{A}}(\mathrm{A}/\mathrm{J}, \mathrm{A}/\mathrm{J}) \cong \mathrm{J} \otimes_{\mathrm{A}} \mathrm{J}$.

Proposition 3.2. *Let* A *be an almost cellular* $\Bbbk$*-algebra and* $\mathrm{J} \subseteq \mathrm{A}$ *a nilpotent factor of* A *such that* $\mathrm{J} \cong \mathrm{V} \otimes_{\Bbbk} \mathrm{W}$ *as* (A, A)*-bimodules. Let* $\mathbf{D} := \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$*. Then the space* $\mathrm{Tor}_2^{\mathrm{A}}(\mathrm{A}/\mathrm{J}, \mathrm{A}/\mathrm{J})$ *is not zero if and only if* $\mathrm{Hom}_{\mathrm{A}}(\mathrm{V}, \mathbf{D}\mathrm{W}) \neq 0$*. In particular, if* $\mathrm{W} = \mathbf{D}\mathrm{V}$*, then* $\mathrm{Tor}_2^{\mathrm{A}}(\mathrm{A}/\mathrm{J}, \mathrm{A}/\mathrm{J}) \neq 0$*.*

Proof. Note from Lemma 3.1 that $\mathrm{J}^2 = 0$ if and only if $\mathrm{Tor}_2^{\mathrm{A}}(\mathrm{A}/\mathrm{J}, \mathrm{A}/\mathrm{J}) \cong \mathrm{J} \otimes_{\mathrm{A}} \mathrm{J}$. Since J is isomorphic to $\mathrm{V} \otimes_{\Bbbk} \mathrm{W}$, we get an isomorphism of $\Bbbk$-vector spaces $\mathrm{J} \otimes_{\mathrm{A}} \mathrm{J} \cong \mathrm{V} \otimes_{\Bbbk} (\mathrm{W} \otimes_{\mathrm{A}} \mathrm{V}) \otimes_{\Bbbk} \mathrm{W}$. Thus the Tor space $\mathrm{Tor}_2^{\mathrm{A}}(\mathrm{A}/\mathrm{J}, \mathrm{A}/\mathrm{J}) \neq 0$, provided that $\mathrm{W} \otimes_{\mathrm{A}} \mathrm{V}$ is not zero. Since the latter space is the $\Bbbk$-dual of $\mathrm{Hom}_{\mathrm{A}}(\mathrm{V}, \mathbf{D}\mathrm{W})$, it shows the assertion.

In particular, if $\mathrm{W} = \mathbf{D}\mathrm{V}$, then $0 \neq \mathrm{id}_{\mathrm{V}} \in \mathrm{Hom}_{\mathrm{A}}(\mathrm{V}, \mathbf{D}\mathrm{W})$. Thus $\mathrm{Tor}_2^{\mathrm{A}}(\mathrm{A}/\mathrm{J}, \mathrm{A}/\mathrm{J}) \neq 0$. □

Lemma 3.3. *Let* A *be a* $\Bbbk$*-algebra with an anti-involution* i *and* $\mathrm{J} \cong \Delta \otimes_{\Bbbk} i(\Delta)$ *be a cell ideal of* A*, where* $\Delta \subseteq \mathrm{J}$ *is a left ideal of* A*. Let* e *be an idempotent of* A *such that* $\Delta \subset \mathrm{J}e$*. Then* $e\mathrm{J}e$ *is a cell ideal of* $e\mathrm{A}e$ *with* $e\mathrm{J}e \cong e\Delta \otimes_{\Bbbk} i(e\Delta)$*.*

Proof. Since $i^2 = \mathrm{id}_{\mathrm{A}}$, we have that i can be regarded as an anti-involution of $e\mathrm{A}e$, and $i(e\mathrm{J}e) = e\mathrm{J}e$. By assumptions Δ is finite-dimensional and $\Delta \subset \mathrm{J}e$, we obtain that $e\Delta$ is finite-dimensional and $e\Delta \subset e\mathrm{J}e$. Moreover, $e\mathrm{J}e \cong e\mathrm{A} \otimes_{\mathrm{A}} \Delta \otimes_{\Bbbk} i(\Delta) \otimes_{\mathrm{A}} \mathrm{A}e \cong e\Delta \otimes_{\Bbbk} i(e\Delta)$ making the following diagram commutative, where α is the isomorphism $\mathrm{J} \cong \Delta \otimes_{\Bbbk} i(\Delta)$,

$$\begin{array}{ccc} e\mathrm{J}e & \xrightarrow{e\alpha e} & e\Delta \otimes_{\Bbbk} i(e\Delta) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle ex\otimes ye \longmapsto i(ye)\otimes i(ex)} \\ e\mathrm{J}e & \xrightarrow{e\alpha e} & e\Delta \otimes_{\Bbbk} i(e\Delta) \end{array}$$

□

Remark 3.4. It is known from [10, Proposition 4.3] that given a cellular algebra (A, i) and an idempotent e with $i(e) = e$, then $(e\mathrm{A}e, i)$ is a cellular algebra. It does mean that $e\mathrm{A}e$ is not necessarily cellular, which we also refer to [12, Section 7]. For example: let $\Bbbk$ be a commutative ring and A be the $\Bbbk$-algebra of two-by-two matrices. Let i be an involution of A given by

$$i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Then the $\Bbbk$-algebra A together with the involution i is cellular. Note that a two-by-two matrix is an idempotent matrix if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2$ if and only if the following equations are satisfied

$$a = a^2 + bc, \quad b = b(a+d), \quad c = c(a+d), \quad d = d^2 + bc.$$

Therefore, there is no idempotent matrix fixed by the above involution i when the character of $\Bbbk$ is 2, in this case, the $\Bbbk$-algebra A together with the involution i is not a cellular algebra. In the case of the character of $\Bbbk$ is not 2, then the matrix $e = \begin{pmatrix} 2^{-1} & 2^{-1} \\ 2^{-1} & 2^{-1} \end{pmatrix}$ is a primitive idempotent matrix which also fixed by the involution i, thus the $\Bbbk$-algebra A together with the involution i is a cellular algebra.

The next proposition shows the difference with those of almost cellular algebras.

Proposition 3.5. *Let* $\mathrm{A}, \mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n$ *be* $\Bbbk$*-algebras. Let* A *be an almost cellular algebra with a sandwich filtration*

$$0 = \mathrm{J}_0 \subsetneq \mathrm{J}_1 \subsetneq \ldots \subsetneq \mathrm{J}_n = \mathrm{A}, \quad \textit{and} \quad \mathrm{J}_i/\mathrm{J}_{i-1} \cong \mathrm{V}_i \otimes_{\mathrm{A}_i} \mathrm{W}_i,$$

for all $1 \leq i \leq n$*. Let* e *be an idemponent of* A*. Then* $e\mathrm{A}e$ *is an almost cellular algebra.*

Proof. Let s be the biggest integer such that $e\mathrm{J}_s e = 0$ and let t be the smallest integer such that $e\mathrm{J}_t e = e\mathrm{J}_{t+1}e$. Then

$$e\mathrm{J}_l e/e\mathrm{J}_{l-1}e \cong e(\mathrm{J}_l/\mathrm{J}_{l-1})e \cong e\mathrm{V}_l \otimes_{\mathrm{A}_l} \mathrm{W}_l e, \quad \text{for all } s+1 \leqslant l \leqslant t.$$

Note that eV_l is an $(e\mathrm{A}e, \mathrm{A}_l)$-bimodule and $\mathrm{W}_l e$ is an $(\mathrm{A}_l, e\mathrm{A}e)$-bimodule, both free of finite rank over A_l. Thus by above arguments $e\mathrm{A}e$ is an almost cellular algebra with a sandwich filtration

$$0 = e\mathrm{J}_s e \subsetneq e\mathrm{J}_{s+1}e \subsetneq \ldots \subsetneq e\mathrm{J}_t e = e\mathrm{A}e, \text{ and } e\mathrm{J}_i e/e\mathrm{J}_{i-1}e \cong e\mathrm{V}_i \otimes_{\mathrm{A}_i} \mathrm{W}_i e,$$

for all $s+1 \leqslant l \leqslant t$. □

From now on, we assume that R is a Noetherian commutative integral domain. Given an associative R-algebra B, two finitely generated free R-modules V and W, and a bilinear form $\varphi : \mathrm{V} \otimes_R \mathrm{W} \longrightarrow \mathrm{B}$ with values in B, we define an associative algebra A as follows: as an R-module, $\mathrm{A} = \mathrm{V} \otimes_R \mathrm{W} \otimes_R \mathrm{B}$. The multiplication is defined by

$$(a \otimes b \otimes x)(c \otimes d \otimes y) := a \otimes d \otimes x\varphi(c, b)y. \tag{1}$$

Proposition 3.6. *This definition makes* A *into an associative* R*-algebra.*

Proof. Since B is an associative algebra, we have

$$\begin{aligned}\big((a \otimes b \otimes x)(c \otimes d \otimes y)\big)(e \otimes f \otimes z) =& (a \otimes d \otimes x\varphi(c, b)y)(e \otimes f \otimes z)\\ =& a \otimes f \otimes x\varphi(c, b)y\varphi(e, d)z\end{aligned}$$

equals to

$$\begin{aligned}(a \otimes b \otimes x)\big((c \otimes d \otimes y)(e \otimes f \otimes z)\big) =& (a \otimes b \otimes x)(c \otimes f \otimes y\varphi(e, d)z)\\ =& a \otimes f \otimes x\varphi(c, b)y\varphi(e, d)z.\end{aligned}$$

□

Lemma 3.7. *Let* A *be an* R*-algebra with a factor* $\mathrm{J} = \mathrm{A}$*. Then* A *is isomorphic to a full matrix ring over the ground ring* R*.*

Proof. By the assumption $\mathrm{A} = \mathrm{V} \otimes_R \mathrm{W}$ for some free R-modules V and W. So there is an R-bimodule isomorphism

$$\mathrm{A} \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{A}) \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{V} \otimes_R \mathrm{W}, \mathrm{A}) \cong \mathrm{Hom}_R(\mathrm{W}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{V}, \mathrm{A}))$$

Denote the R-ranks of the free R-module V and W by n and m respectively. Then A has R-rank $n \times m$, and as a left module, A is isomorphic to m copies of V. Hence, $\mathrm{Hom}_{\mathrm{A}}(\mathrm{V}, \mathrm{A})$ has R-rank at least n. But by the above isomorphism it can not have larger rank. This means that the A-endomorphism ring E of V has rank one and is exactly R. We complete the proof. □

Definition 3.8. Let C be any algebra and let B be an algebra of the form $\mathrm{V} \otimes_R \mathrm{W} \otimes_R R$ with the multiplication defined in (3.1). Let $\mathrm{A} = \mathrm{C} \oplus \mathrm{B}$ such that B is a factor of A and A/B is isomorphic to C. Then we call A an inflation of C along B.

Theorem 3.9. *Any almost cellular algebra* A *over* R *with a sandwich filtration is the iterated inflation of finitely many copies of* R.

Proof. First we regard a factor $\mathrm{J} \subseteq \mathrm{A}$ as an algebra, which is always an inflation of the ground ring R. In fact, by Lemma 2.7 there exists an (R, R)-bimodule morphism $\varphi : \mathrm{W} \otimes_{\mathrm{A}} \mathrm{V} \longrightarrow R$ and we can identify J with $L \otimes_R L' \otimes_R R$ for two free R-modules L and L' having the same R-rank as V and W, respectively. Thus we can write J as an inflation.

Now we prove the theorem by induction on the length of the sandwich filtration. An almost cellular R-algebra A which is a factor in itself is just a full matrix ring over R of size $n \times m$ by Lemma 3.7. Choose L and L' to be free R-modules of rank n and m respectively, which we identify with $\mathrm{V} = \mathrm{A}e$ and $\mathrm{W} = e\mathrm{A}$, where e is a primitive idempotent. In this case, we identify $e\mathrm{A}e$ with R.

Using the above observation, we can rewrite matrix multiplication $\mathrm{A} \otimes_R \mathrm{A} \longrightarrow \mathrm{A}$ as

$$\mathrm{A} \otimes_R \mathrm{A} \cong (\mathrm{A}e \otimes_R e\mathrm{A}) \otimes_R (\mathrm{A}e \otimes_R e\mathrm{A}) \longrightarrow \mathrm{A}e \otimes_R e\mathrm{A}e \otimes_R e\mathrm{A} \cong \mathrm{A}e \otimes_R e\mathrm{A} \cong \mathrm{A}, \tag{2}$$

where all maps are (A, A)-bimodule homomorphisms. Thus, it provides us a bilinear form $\varphi : L' \otimes_R L \longrightarrow R$ and also shows how to write A as inflation of R along L and L'.

Now we assume that A is an almost cellular algebra with a sandwich chain of length greater than 1. We fix a factor $\mathrm{J} \subseteq \mathrm{A}$. By induction, the quotient algebra $\mathrm{B} = \mathrm{A}/\mathrm{J}$ is an iterated inflation of copies of R. Now we claim that A is an inflation of B along J. Indeed, we use the facts that J is an inflation of R by the first paragraph and $\mathrm{J} \subseteq \mathrm{A}$ is a factor of A. □

In the following subsection, we show sufficient and necessary conditions for an almost cellular algebra A to be quasi-hereditary.

Denote the simple A-modules $\mathrm{L}(1), \ldots, \mathrm{L}(m)$ and their projective covers by $\mathrm{P}(1)$, $\ldots$, $\mathrm{P}(m)$. Let $C = (c_{ij})$ be the Cartan matrix of an algebra A, where the entry c_{ij} is the composition multiplicity $[\mathrm{P}(i) : \mathrm{L}(j)]$. The determinant of C is called the Cartan determinant. In general this can be any integer. But the Cartan determinant of a cellular algebra is a positive integer. For our situation, we also have

Proposition 3.10. *Let* $\mathrm{A}, \mathrm{A}_1, \cdots, \mathrm{A}_n$ *be* R*-algebras over* R. *Let* A *be an almost cellular algebra with a sandwich filtration*

$$0 = \mathrm{J}_0 \subsetneq \mathrm{J}_1 \subsetneq \ldots \subsetneq \mathrm{J}_n = \mathrm{A}, \quad \textit{and} \quad \mathrm{J}_i/\mathrm{J}_{i-1} \cong \mathrm{V}_i \otimes_{\mathrm{A}_i} \mathrm{W}_i = \Delta(i).$$

Then the determinant $det(C)$ *of the Cartan matrix* C *of* A *is a positive integer.*

Proof. Denote the number of isomorphism classes of simple A_i-modules by m_i. Then the number of isomorphism classes of simple A-modules is $m \leq m_1 + \cdots m_n$ by [7, Theorem 3] and V_i is of finite A_i-rank. Let $d_{a,b} = [\Delta(a) : \mathrm{L}(b)]$, the composition multiplicity of the simple module $\mathrm{L}(b)$ in the standard module $\Delta(a)$, and $D = (d_{a,b})$ the corresponding matrix. Then D is an $n \times m$-matrix with integer entries. By the characterization of simple A-modules, we may assume that $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$, where D_1 and D_2 are integer matrices and D_2 (whose rows correspond to those indices i such that $\mathrm{J}_i^2 \not\subset \mathrm{J}_{i-1}$) is a square matrix by again by [7, Theorem 3]. Note that D_2 is a lower triangular matrix with all diagonal entries equal to one. This implies that $\det(D_2) = 1$.

The Cartan matrix C of A satisfies $C = D^T D$, where D^T is the transpose matrix of D. Indeed, the composition multiplicity $[\Delta(a) : \mathrm{L}(b)]$ equals to $\dim_{\Bbbk} \mathrm{W}_a e(b)$, where $e(b)$ is the primitive idempotent corresponding to $\mathrm{L}(b)$. Now,

$$C^T = \begin{pmatrix} D_1^T & D_2^T \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = (D_1^T D_1 + D_2^T D_2),$$

so hence $\det(C) = \det(I + (D_2^T D_2)^{-1} D_1^T D_1)$ by $\det(D_2^T D_2) = 1$. Note that $D_2^T D_2$ is positive definitive and $D_1^T D_1$ is positive semi-definitive. Then we can decompose $D_2^T D_2$ with $D_2^T D_2 = Z^2$ for some symmetric matrix Z, and furthermore, $B = Z^{-1} D_1^T D_1 Z^{-1}$ and $(D_2^T D_2)^{-1} D_1^T D_1$ have the same eigenvalues. Since B is symmetric and its eigenvalues λ are non-negative real numbers, it follows that $C = I + (D_2^T D_2)^{-1} D_1^T D_1$ has the eigenvalues of the form $1+\lambda$, and therefore $\det(C)$ is a positive integer. □

König and Xi obtained some equivalent conditions for a cellular algebra to be quasi-hereditary (see [11, Theorem 3.1]). Inspired by their work and combined with the above proposition, we have the following theorem.

Theorem 3.11. *Let* $\mathrm{A}, \mathrm{A}_1, \cdots, \mathrm{A}_n$ *be* R*-algebras over* R*. Let* A *be an almost cellular algebra. Then the following are equivalent:*

1. *There is a sandwich filtration of* A *over* $\mathrm{A}_1, \cdots \mathrm{A}_n$ *whose length equals the sum of the numbers of isomorphism classes of simple* A_i*-modules.*
2. *The determinant* $\det(C)$ *of Cartan matrix* C *of* A *is equal to one.*

In particular, if R *is a field, then* $\det(C) = 1$ *if and only if each* $\mathrm{A}_i = R$ *and* A *is quasi-hereditary algebra with the sandwich filtration to be exactly the heredity chain.*

Proof. From Proposition 3.10 we get that $\det(C)$ is equal to one if $\lambda = 0$ if and only if for the given sandwich filtration we have $n = m$. This means that $(i) \Longleftrightarrow (ii)$.

If R is a field, then from [7, Theorem 3] we get that the statement (i) holds if and only if each $\mathrm{A}_i = R$. Thus we complete the proof. □

Future Directions There are a number of interesting questions yet to be considered. For example, is there an equivalent definition based on the bases for an almost cellular algebra, which can help us to determine whether an algebra is an almost cellular algebra or not? An affine quasi-hereditary algebra with a balanced split involution is an affine cellular algebra [9, Proposition 9.8], and given an affine cellular algebra with an affine cell chain of ideals, one can ask how to decide whether it is affine quasi-hereditary? Furthermore, when is an almost cellular algebra with a sandwich filtration to be affine quasi-hereditary? So far as the authors are aware, this theory has yet to be developed.

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