

STABILITY AND HOPF BIFURCATION OF THE COEXISTENCE EQUILIBRIUM FOR A DIFFERENTIAL-ALGEBRAIC BIOLOGICAL ECONOMIC SYSTEM WITH PREDATOR HARVESTING

KERIOUI NADJAH AND ABDELOUAHAB MOHAMMED SALAH*

Laboratory of mathematics and their interactions
University Abdelhafid Boussouf, Mila, 43000, Algeria

(Communicated by Moxun Tang)

ABSTRACT. The objective of the current paper is to investigate the dynamics of a new bioeconomic predator prey system with only predator's harvesting and Holling type III response function. The system is equipped with an algebraic equation because of the economic revenue. We offer a detailed mathematical analysis of the proposed model to illustrate some of the significant results. The boundedness and positivity of solutions for the model are examined. Coexistence equilibria of the bioeconomic system have been thoroughly investigated and the behaviours of the model around them are described by means of qualitative theory of dynamical systems (such as local stability and Hopf bifurcation). The obtained results provide a useful platform to understand the role of the economic revenue v . We show that a positive equilibrium point is locally asymptotically stable when the profit v is less than a certain critical value v_1^* , while a loss of stability by Hopf bifurcation can occur as the profit increases. It is evident from our study that the economic revenue has the capability of making the system stable (survival of all species). Finally, some numerical simulations have been carried out to substantiate the analytical findings.

1. Introduction. Human beings face the twin problems of food scarcity and environmental destruction. There is a great interest in studying and designing bioeconomic models with regard to the biodiversity for humanity's long-term gains. Researchers strive to produce certain potentially advantageous outcomes in order to ensure the sustainable growth of the ecosystem and to preserve the enduring prosperity.

More recently, the study of population dynamics with harvesting has become an interesting topic in mathematical bio-economics due to its importance related to the optimal management of renewable resources [5]. In 1954, Gordon introduced a common property resource economic theory [6], studying the impact of harvest effort on the ecosystem from an ecological viewpoint and suggests the following economic principle:

$$\text{Net Economic Revenue (NER)} = \text{Total Revenue (TR)} - \text{Total Cost (TC)}. \quad (1)$$

2020 *Mathematics Subject Classification.* Primary: 34D20, 34C23; Secondary: 37N25.

Key words and phrases. Harvesting, differential-algebraic equation, stability, Hopf bifurcation.

* Corresponding author: Abdelouahab Mohammed Salah.

Many research efforts have been focused on the investigation of this sort of dynamics. In [13–16], the authors have studied the dynamical behaviour of a class of predator-prey ecosystems formulated from several differential equations and an algebraic equation. They have obtained interesting results, such as stability of interior equilibrium, Hopf bifurcation, limit cycle, singularity induced bifurcation, and its control, and so on. But in all of these studied models, only the prey population is subjected to harvesting. The interaction between predator and their prey was investigated by using different functional response such as Holling I and II with the assumption that there is a natural mortality of the isolated predator species. In [17], the authors have studied the dynamics of the Beddington-DeAngelis predator-prey system with predator harvesting.

As far as we are aware, the dynamical analysis of a predator-prey model where both prey and predator grow logistically with Holling III functional response, subject to predator harvesting, has not been previously investigated. Thus, in the present research, we investigate this type of models and discuss its dynamical behaviours, such as stability and Hopf bifurcation [1,2]. Moreover, we aim to find some principles which are theoretically beneficial for the management and the control of the renewable resources.

To achieve the ahead set goals, we organized the present article as follows: we begin our study by describing the concept behind model building and specifying its biological significance. Sequentially, we establish the positivity and boundedness of solutions for the model. Next, we examine the existence of the positive equilibrium, then we provide a detailed description of the stability and the Hopf bifurcation analysis of the system. Finally, we give a numerical simulation experiments to confirm the derived theoretical results.

2. Model formulation. In this section, we aim to develop a model that combines both economic and biological aspects in resource management. The model is structured as follows: starting by the predation rate, it is known that the physiological prey absorption capabilities by a predator are limited even if a large number of prey is available. Such a response function presenting a plateau for large prey densities is called Holling II functional response [10–12] in which the rate of capture increases with increasing prey density and approaches saturation gradually. Type III of Holling response function is similar to Type II except at low prey density, where the rate of prey capture accelerates. In our proposed model, we assume that there exists an upper limit for the maximum predation rate. To achieve this aim, we have considered the predation term as $\frac{ax^2y}{d+x^2}$. Point out that

$$\lim_{x \rightarrow \infty} \frac{ax^2}{d+x^2} = a.$$

Moreover, one way to add realism to the model is to consider the effects of crowding. Space and resources are limited even if there are more density of the populations. Therefore, the growth rates of both preys and predators are supposed to be logistic. Taking into account the above hypothesis, we propose a model which consists of prey having density x and predator having density y of the form

$$\begin{cases} \dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{ax^2y}{d+x^2}, \\ \dot{y} &= sy \left(1 - \frac{y}{N}\right) + \frac{bx^2y}{d+x^2}. \end{cases} \quad (2)$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$. Here r and s are positive constants stand for the intrinsic growth rates of prey and predator population respectively. K and N are positive constants representing the carrying capacity of the two species. d and a are positive constants stand for half capturing saturation constant and the maximal efficiency of predation respectively. b is a positive constant that represents a conversion coefficient.

It is known that the harvest effort is an important factor to construct a useful bioeconomic mathematical model, for this reason, and taking unto-account (1), we extend the system (2) by considering the following algebraic equation which describes the economic profit v of the harvest effort on predator:

$$E(t)(py(t) - c) = v, \tag{3}$$

where $0 \leq E(t) \leq E_{max}$ and $y(t) \geq 0$ represent the harvest effort and the density of predator respectively. p represents the unit price of the harvested population and c is the cost of the harvest effort, the total revenue is $TR = pE(t)y(t)$ and the total cost is $TC = cE(t)$.

Based on (2) and (3), a singular differential-algebraic model that consists of two differential equations and an algebraic equation can be established as follows:

$$\begin{cases} \dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{ax^2y}{d+x^2}, \\ \dot{y} &= sy \left(1 - \frac{y}{N}\right) + \frac{bx^2y}{d+x^2} - Ey, \\ 0 &= E(py - c) - v, \end{cases} \tag{4}$$

which is a semiexplicit DAE of the form

$$\begin{cases} \dot{z} &= f(v, X), \\ 0 &= g(v, X), \end{cases} \tag{5}$$

where we denote $X = (x, y, E)^T$, with $(x, y)^T$ the differential variable, E the algebraic variable and v is the bifurcation parameter, f and g are smooth functions given by

$$f(v, X) = \begin{pmatrix} f_1(v, X) \\ f_2(v, X) \end{pmatrix} = \begin{pmatrix} x \left(r \left(1 - \frac{x}{K} \right) - \frac{axy}{d+x^2} \right) \\ y \left(s \left(1 - \frac{y}{N} \right) + \frac{bx^2}{d+x^2} - E \right) \end{pmatrix},$$

$$g(v, X) = E(py - c) - v.$$

3. Mathematical analysis and main results. For biological considerations, we are only interested in the dynamics of this model in the positive octant \mathbb{R}_+^3 . Thus, we consider the biologically meaningful initial condition

$$x(0) = x_0 \geq 0, y(0) = y_0 \geq 0, E(0) = E_0 = \frac{v}{py_0 - c}, py_0 - c > 0. \tag{6}$$

3.1. Existence and uniqueness.

Proposition 1. *The system (4) equipped by the initial conditions (6) have a unique maximal solution $(x(t), y(t), E(t))$ in an open subset U of $\Omega = \{(x, y, E)^T \in \mathbb{R}_+^3 / py - c > 0\}$ defined on some maximal interval $[0, T[$.*

Proof. Let $(x, y, E)^T \in U$ then, from the algebraic equation $g(x, y, E, v) = 0$ we get $E = \frac{v}{py - c}$, substituting in the second differential equation of (4), the DAE is transformed to the following ODE that have the same solution with respect to the differential variables $z = (x, y)^T$:

$$\begin{cases} \dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{ax^2y}{d + x^2}, \\ \dot{y} &= sy \left(1 - \frac{y}{N}\right) + \frac{bx^2y}{d + x^2} - \frac{vy}{py - c}, \end{cases} \quad (7)$$

its vectorial form is $\dot{z} = F(z)$, where

$$F(z) = \begin{pmatrix} x \left(r \left(1 - \frac{x}{K} \right) - \frac{axy}{d + x^2} \right) \\ y \left(s \left(1 - \frac{y}{N} \right) + \frac{bx^2}{d + x^2} - \frac{v}{py - c} \right) \end{pmatrix}.$$

Clearly $F \in C^1(U')$, where U' is an open subset of $\Omega' = \{(x, y)^T \in \mathbb{R}_+^2 / py - c > 0\}$. Thus, by applying the Cauchy-Lipschitz's theorem for ODE [8], we deduce the local existence and uniqueness of a maximal solution $(x, y)^T$ to (7) for any $(x_0, y_0) \in U'$, then, the local existence and uniqueness of solution for (4) is straightforward. \square

3.2. Positivity and boundedness. Regarding the positivity of solution for the system (4), we introduce the following proposition:

Proposition 2. *Any smooth solution of (4), defined on the maximal interval $[0, T[$, with positive initial conditions (6), remain positive for all $t \in [0, T[$.*

Proof. From the system (7), it follows that $x = 0 \Rightarrow \frac{dx}{dt} = 0$ and $y = 0 \Rightarrow \frac{dy}{dt} = 0$ thus, $x = 0$ and $y = 0$ are invariant sets showing that $x(t) \geq 0$ and $y(t) \geq 0$ whenever $x(0) > 0$ and $y(0) > 0$.

From the second equation of (7), we deduce that for all $t \in [0, T[$:

$$py(t) - c \neq 0. \quad (8)$$

Suppose that there exist $t^* \in [0, T[$ such that $E(t^*) < 0$, it follows that $py(t^*) - c < 0$ then, by applying the intermediate value theorem to the continuous function $py(t) - c$ on the interval $[0, t^*]$, we deduce the existence of $\tilde{t} \in]0, t^*[$ such that $py(\tilde{t}) - c = 0$ which contradict with (8), thus, $E(t) \geq 0$ for all $t \in [0, T[$. \square

In the predator-prey ecosystem, the impulse process of the ecosystem is typically related to the accelerated development of the species population. If this trend persists for a period of time, the biomass of population will be outside of the environment carrying capacity and the predator-prey ecosystem will be out of control, which is catastrophic for the ecosystem.

Clearly, when the predator biomass y approaches to the critical value $y_c = \frac{c}{p}$, the fishing effort E will be unbounded which is not realistic.

To answer the boundedness of the solution for the system (4), we impose a realistic ecological constraint in the context that the economic policy requires a minimum level $y_{min} > 0$ for the resource given by:

$$y(t) \geq y_{min} > \frac{c}{p}, \forall t \geq 0. \quad (9)$$

This constraint will affect the fishing effort E that will be constrained by a fixed production capacity (capital and labor involved in the production process remain constant). We denote this limit capacity E_{max} , then

$$0 < E(t) \leq E_{max} = \frac{v}{py_{min} - c}, \forall t \geq 0. \tag{10}$$

Proposition 3. *All solutions of the system (4) subject to the initial conditions (6) and constraint (10) are bounded in \mathbb{R}_+^3 , with ultimate bound.*

Proof. Suppose that $E(t)$ is subject to the constraint (10). Defining a function $\psi(t) = bx(t) + ay(t)$, then its time derivative along the solutions of the system (4) is given by:

$$\begin{aligned} \frac{d\psi}{dt} &= b \frac{dx}{dt} + a \frac{dy}{dt}, \\ &= rbx \left(1 - \frac{x}{K}\right) + asy \left(1 - \frac{y}{N}\right) - aEy. \end{aligned}$$

Hence for each $\mu > 0$, we have

$$\begin{aligned} \frac{d\psi}{dt} + \mu\psi &= rbx \left(1 - \frac{x}{K}\right) + asy \left(1 - \frac{y}{N}\right) - aEy + \mu bx + \mu ay, \\ &= \left(rbx - \frac{rb}{K}x^2 + \mu bx \right) + \left(asy - \frac{as}{N}y^2 - aEy + \mu ay \right), \\ &= bx \left[(r + \mu) - \frac{r}{K}x \right] + ay \left[(s + \mu - E) - \frac{s}{N}y \right], \\ &\leq bx \left[(r + \mu) - \frac{r}{K}x \right] + ay \left[(s + \mu) - \frac{s}{N}y \right], \\ &\leq \frac{bK(r + \mu)^2}{4r} + \frac{aN(s + \mu)^2}{4s} := \eta. \end{aligned}$$

By using the theory of differential inequality [3], we obtain

$$0 \leq \psi(t) \leq \frac{\eta}{\mu}(1 - e^{-\mu t}) + \psi(0)e^{-\mu t} \leq \max \left(\psi(0), \frac{\eta}{\mu} \right).$$

Taking limit $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} \psi(t) \leq \frac{\eta}{\mu}.$$

Hence all the solutions of the system (4) subject to initial conditions (6) and constraint (10) are confined in the region

$$H = \left\{ (x, y, E)^T \in \mathbb{R}_+^3 : 0 < E \leq E_{max}, 0 \leq \psi = bx + ay \leq \frac{\eta}{\mu} + \epsilon, \text{ for } \epsilon > 0 \right\}.$$

□

4. Existence and number of positive equilibrium. Our objective in this section is to inspect the existence of the positive equilibrium points and to study their stabilities.

An equilibrium point of the system (4) is a solution of the following equations:

$$\begin{cases} f_1(v, X) = 0, \\ f_2(v, X) = 0, \\ g(v, X) = 0. \end{cases} \tag{11}$$

By the analysis of the roots for (11), it follows that

- (i) If $v = 0$, then there exist at least three boundary equilibrium points $X_{e1} = (0, 0, 0)$, $X_{e2} = (K, 0, 0)$, $X_{e3} = (0, N, 0)$, and at most five other boundary equilibrium $X_{ei} = (x_i^*, y_i^*, 0)$, $i = 1, 2, \dots, 5$, where x_i^* are the roots of the equation

$$r \left(1 - \frac{x}{K} \right) \frac{d + x^2}{ax} - N \left(1 + \frac{bx^2}{s(d + x^2)} \right) = 0,$$

or equivalently the fifth degree equation

$$x^5 - Kx^4 + (2d + NabK/(rs) + NaK/r)x^3 - 2dKx^2 + (d^2 + NadK/r)x - d^2K = 0,$$

satisfying $0 < x_i^* < K$, and

$$y_i^* = \frac{r}{Kax_i^*} (K - x_i^*) (d + (x_i^*)^2), \quad i = 1, 2, \dots, 5.$$

- (ii) If $v > 0$, then there exist at most two boundary equilibrium points $X_{ei} = (0, y_i^*, \frac{v}{py_i^* - c})$, $i = 1, 2$, where y_i^* are the roots of the quadratic equation

$$psy^2 - s(Np + c)y + N(cs + v) = 0,$$

satisfying $y_i^* > \frac{c}{p}$, and at most eight interior equilibrium points $X_{ei} = (\bar{x}_i, \bar{y}_i, \frac{v}{p\bar{y}_i - c})$, $i = 1, 2, \dots, 8$, where $\bar{y}_i = \frac{r}{Ka\bar{x}_i} (K - \bar{x}_i) (d + \bar{x}_i^2) > \frac{c}{p}$ and \bar{x}_i is a solution of the equation

$$s \left(1 - r \left(1 - \frac{x}{K} \right) \frac{d + x^2}{Nax} \right) + \frac{bx^2}{d + x^2} - \frac{v}{p(r(1 - \frac{x}{K}) \frac{d+x^2}{ax}) - c} = 0,$$

or equivalently

$$\begin{cases} P(x) = \sum_{i=0}^8 p_i x^i = 0, \\ Q(x) = \sum_{i=0}^3 q_i x^i > 0, \end{cases} \tag{12}$$

where p_i , $i = 0, 1, 2, \dots, 8$ are given by

$$\begin{aligned} p_0 &= d^3 K^2 pr^2 s, \\ p_1 &= -(acd^2 K^2 rs + ad^2 K^2 Nprs + 2d^3 Kpr^2 s), \\ p_2 &= a^2 cdK^2 Ns + acd^2 Krs + ad^2 KNprs + d^3 pr^2 s + 3d^2 K^2 pr^2 s + a^2 dK^2 Nv, \\ p_3 &= -(abdK^2 Npr + 2acdK^2 rs + 2adK^2 Nprs + 6d^2 Kpr^2 s), \\ p_4 &= a^2 bcK^2 N + abdKNpr + a^2 cK^2 Ns + 2acdKrs + 2adKNprs + 3d^2 pr^2 s \\ &\quad + 3dK^2 pr^2 s + a^2 K^2 Nv, \\ p_5 &= -(abK^2 Npr + acK^2 rs + aK^2 Nprs + 6dKpr^2 s), \\ p_6 &= abKNpr + acKrs + aKNprs + 3dpr^2 s + K^2 pr^2 s, \\ p_7 &= -2Kpr^2 s, \\ p_8 &= pr^2 s, \end{aligned}$$

and q_i , $i = 0, 1, 2, 3$ are given by

$$\begin{aligned} q_0 &= dKpr, \\ q_1 &= -(acK + dpr), \\ q_2 &= Kpr, \\ q_3 &= -pr. \end{aligned}$$

Since there are three sign changes in the sequence of coefficients $q_i, i = 0, 1, 2, 3$, then, by Descartes' rule of signs [9], there are either one positive root \bar{x}_1 or three positive roots $\bar{x}_1 \leq \bar{x}_2 \leq \bar{x}_3$ of $Q(x)$.

Let P_0, P_1, \dots, P_l be the sequence of polynomials generated by the Euclidean algorithm [9] started with $P_0 = P, P_1 = P'$. The exact number of the interior equilibrium of (4) is given in the following proposition:

Proposition 4. *The number of the interior equilibrium of (4) is exactly m , where*

$$m = \mu(0) - \mu(\bar{x}_1), \text{ if } Q(x) \text{ has one root,} \tag{13}$$

or

$$m = \mu(0) - \mu(\bar{x}_1) + \mu(\bar{x}_2) - \mu(\bar{x}_3), \text{ if } Q(x) \text{ has three roots,} \tag{14}$$

where $\mu(x)$ denotes the number of changes of sign in the sequence $\{P_i(x)\}$.

Proof. The number of the interior equilibrium of (4) is equal to the number of the positive solutions of (12). Thus we have

- If $Q(x)$ has one root, then for all $x > 0$, we have $Q(x) > 0 \Leftrightarrow x \in]0, \bar{x}_1[$, by theorem 6.3d in [9] $P(x) = 0$ has exactly $\mu(0) - \mu(\bar{x}_1)$ solutions in the interval $]0, \bar{x}_1[$. Thus $P(x) = 0$ has exactly $m = \mu(0) - \mu(\bar{x}_1)$ positive solutions satisfying $Q(x) > 0$.
- If $Q(x)$ has three roots, then for all $x > 0$, we have $Q(x) > 0 \Leftrightarrow x \in]0, \bar{x}_1[\cup]\bar{x}_2, \bar{x}_3[$, it follows that $P(x) = 0$ has exactly $\mu(0) - \mu(\bar{x}_1)$ solutions in the interval $]0, \bar{x}_1[$, and exactly $\mu(\bar{x}_2) - \mu(\bar{x}_3)$ solutions in the interval $]\bar{x}_2, \bar{x}_3[$. Thus $P(x) = 0$ has exactly

$$m = \mu(0) - \mu(\bar{x}_1) + \mu(\bar{x}_2) - \mu(\bar{x}_3),$$

positive solutions satisfying $Q(x) > 0$.

□

5. Dynamic analysis near the coexistence equilibria. In this section we study the stability of an interior equilibrium X_e and analyse the bifurcation through it using the bifurcation theory and the normal form theory.

5.1. Local stability analysis. For the analysis of the local stability of X_e , we let $X = Q\bar{X}$, here

$$\bar{X} = (x, y, \bar{E})^T, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{E_e p}{py_e - c} & 1 \end{pmatrix},$$

then we get $D_X g(X_e)Q = (0, 0, py_e - c)$, and

$$x = x, \quad y = y, \quad \bar{E} = E + \frac{E_e py}{py_e - c}.$$

Then, the system can be expressed as follows:

$$\begin{cases} \dot{x} &= x \left(r \left(1 - \frac{x}{K} \right) - \frac{axy}{d + x^2} \right), \\ \dot{y} &= y \left(s \left(1 - \frac{y}{N} \right) + \frac{bx^2}{d + x^2} - \bar{E} + \frac{E_e py}{py_e - c} \right), \\ 0 &= \left(\bar{E} - \frac{E_e py}{py_e - c} \right) (py - c) - v. \end{cases} \tag{15}$$

We denote also by

$$f(v, \bar{X}) = \begin{pmatrix} f_1(v, \bar{X}) \\ f_2(v, \bar{X}) \end{pmatrix} = \begin{pmatrix} x \left(r \left(1 - \frac{x}{K} \right) - \frac{axy}{d+x^2} \right) \\ y \left(s \left(1 - \frac{y}{N} \right) + \frac{bx^2}{d+x^2} - \bar{E} + \frac{E_e py}{py_e - c} \right) \end{pmatrix},$$

$$g(v, \bar{X}) = \left(\bar{E} - \frac{E_e py}{py_e - c} \right) (py - c) - v, \quad \bar{X} = (x, y, \bar{E})^T,$$

and we can conclude that the system (15) has a positive equilibrium point

$$\bar{X}_e = (x_e, y_e, \bar{E}_e)^T = \left(x_e, y_e, E_e + \frac{E_e py}{py_e - c} \right)^T,$$

and $D_X g(\bar{X}_e)Q = (0, 0, py_e - c)$.

For the system (15), we consider the following local parametrization:

$$\bar{X} = \varphi(v, Y) = \bar{X}_e + U_0 Y + V_0 h(v, Y), \quad g(v, \varphi(v, Y)) = 0.$$

Here $Y = (y_1, y_2)$, $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth mapping. More information about the local parametrization can be found in [4, 7]. Then, we can deduce that the parametric system of (15) takes the form

$$\begin{cases} \dot{y}_1 &= f_1(v, \varphi(v, Y)), \\ \dot{y}_2 &= f_2(v, \varphi(v, Y)). \end{cases} \tag{16}$$

Consequently, the Jacobian matrix $A(v)$ of the parametric system (16) at $Y = 0$ takes the form

$$\begin{aligned} A(v) &= \begin{pmatrix} D_{y_1} f_1(v, \varphi(v, Y)) & D_{y_2} f_1(v, \varphi(v, Y)) \\ D_{y_1} f_2(v, \varphi(v, Y)) & D_{y_2} f_2(v, \varphi(v, Y)) \end{pmatrix}, \\ &= \begin{pmatrix} D_{\bar{X}} f_1(v, \bar{X}_e) \\ D_{\bar{X}} f_2(v, \bar{X}_e) \end{pmatrix} \begin{pmatrix} D_{\bar{X}} g(v, \bar{X}_e) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix}, \\ &= \begin{pmatrix} D_x f_1(v, \bar{X}_e(v)) & D_y f_1(v, \bar{X}_e(v)) \\ D_x f_2(v, \bar{X}_e(v)) & D_y f_2(v, \bar{X}_e(v)) \end{pmatrix}, \\ &= \begin{pmatrix} x_e \left(-\frac{r}{K} + \frac{ay_e(x_e^2 - d)}{(x_e^2 + d)^2} \right) & -\frac{ax_e^2}{x_e^2 + d} \\ \frac{2bdx_e y_e}{(x_e^2 + d)^2} & y_e \left(-\frac{s}{N} + \frac{pE_e}{py_e - c} \right) \end{pmatrix}. \end{aligned}$$

Therefore, the characteristic equation of the matrix $A(v)$ can be expressed as

$$\lambda^2 + a_1(v)\lambda + a_2(v) = 0, \tag{17}$$

where

$$a_1(v) = x_e \left(\frac{r}{K} - \frac{ay_e(x_e^2 - d)}{(x_e^2 + d)^2} \right) + y_e \left(\frac{s}{N} - \frac{pE_e}{py_e - c} \right),$$

$$a_2(v) = x_e y_e \left(-\frac{r}{K} + \frac{ay_e(x_e^2 - d)}{(x_e^2 + d)^2} \right) \left(-\frac{s}{N} + \frac{pE_e}{py_e - c} \right) + \frac{2abdx_e^3 y_e}{(x_e^2 + d)^3}.$$

Result 1. For the positive equilibrium point \bar{X}_e of the system (15), we have

- (i) If $a_1^2(v) \geq 4a_2(v)$ and $a_2(v) > 0$, then, when $a_1(v) > 0$, \bar{X}_e is locally asymptotically stable node. When $a_1(v) < 0$, \bar{X}_e is unstable node.
- (ii) If $a_2(v) < 0$, then, \bar{X}_e is an unstable saddle point.
- (iii) If $a_1^2(v) < 4a_2(v)$, then, when $a_1(v) > 0$, \bar{X}_e is locally asymptotically stable focus. When $a_1(v) < 0$, \bar{X}_e is unstable focus.

Remark 1. The positive equilibrium point \bar{X}_e of the system (15) corresponds to the equilibrium point $Y = 0$ of the system (16).

5.2. Hopf bifurcation analysis. The Hopf bifurcation is a very interesting type of bifurcations of systems. It refers to the local birth or death of a periodic solution from an equilibrium point as a parameter crosses a critical value named bifurcation value.

In this fragment, we discuss the Hopf bifurcation in the system (15) from the equilibrium point X_e by considering the economic profit v as a bifurcation value. If we let $a_1^2(v) \leq 4a_2(v)$, then the equation (17) has a pair of conjugate complex roots:

$$\begin{aligned} \lambda_{1,2} &= -\frac{1}{2}a_1(v) \pm i\sqrt{a_2(v) - \frac{a_1^2(v)}{4}}, \\ &:= \alpha(v) \pm i\omega(v). \end{aligned}$$

Let $a_1(v) = 0$, we get the bifurcation value v^* that satisfies

$$v^* = \frac{(py_e(v^*) - c)^2}{py_e(v^*)} \left(\frac{s}{N}y_e(v^*) + x_e(v^*) \left(\frac{r}{K} - \frac{ay_e(v^*)((x_e(v^*))^2 - d)}{((x_e(v^*))^2 + d)^2} \right) \right),$$

if

$$\frac{r}{K} = \frac{ay_e(v^*)((x_e(v^*))^2 - d)}{((x_e(v^*))^2 + d)^2}, \tag{18}$$

then

$$v^* = \frac{s(py_e(v^*) - c)^2}{pN}. \tag{19}$$

Moreover

$$\alpha(v^*) = 0, \quad \omega(v^*) = \sqrt{\frac{2abdy_e(v^*)(x_e(v^*))^3}{((x_e(v^*))^2 + d)^3}} > 0,$$

which implies that if $\alpha'(v^*) = \frac{d}{dv} \left(\frac{Npvy_e(v) - sy_e(v)(py_e(v) - c)^2}{N(py_e(v) - c)^2} \right)_{v=v^*} \neq 0$, then, Hopf bifurcation occurs at the value v^* . The signal of the number σ given by

$$\sigma = \frac{1}{8} \left[\frac{3a^3x_e^5(d - x_e^2)}{(d + x_e^2)^6} (-y_e(3d - x_e^2) + 4d) + \frac{sp\omega^{*2}}{N(py_e - c)} + \frac{3p^2cE_e\omega^{*2}}{(py_e - c)^3} \right], \tag{20}$$

which determines the direction of the Hopf bifurcation through the interior equilibrium $X_e(v)$ of the system (4) as stated in the following theorem.

Theorem 5.1. For the system (4), there exist a positive constant ε and two small neighborhoods of the positive equilibrium point $X_e(v)$: Z_1 and Z_2 , where $0 < \varepsilon \ll 1$ and $Z_1 \subset Z_2$.

Case 1.: If $\sigma > 0$, then

- (i) When $v^* < v < v^* + \varepsilon$, $X_e(v)$ rejects all the points in Z_2 , so it is unstable.

- (ii) When $v^* - \varepsilon < v < v^*$, the system (4) has at least a periodic solution located in \bar{Z}_1 (the closure of Z_1), one of them rejects all the points in $\bar{Z}_1 \setminus X_e(v)$, at the same time another periodic solution (may be the same one) rejects all the point in $Z_2 \setminus \bar{Z}_1$, and $X_e(v)$ is locally asymptotic stable.

Case 2.: If $\sigma < 0$, then

- (i) When $v^* - \varepsilon < v < v^*$, $X_e(v)$ attracts all the points in Z_2 , and $X_e(v)$ is locally asymptotic stable.
- (ii) When $v^* < v < v^* + \varepsilon$, the system (4) has at least a periodic solution located in \bar{Z}_1 , one of them attracts all the points in $\bar{Z}_1 \setminus X_e(v)$, at the same time another periodic solution (may be the same one) attracts all the point in $Z_2 \setminus \bar{Z}_1$, then $X_e(v)$ is unstable.

Proof. The proof of Theorem 5.1 is detailed in Appendix A. \square

6. Numerical simulations. Now the computer simulation modelling using MATLAB software will be carried out to illustrate the analytical results that we have established in the previous sections. The next numerical example shows the different dynamical behaviours when the economic profit increases through a certain value v^* . Let consider the following parameter values:

$$r = 0.728025, a = 1, b = 0.72, c = 0.28, d = 0.3, p = 3, s = 0.75, K = 4, N = 0.8. \quad (21)$$

6.1. Number of the interior equilibria. For the set of parameter values (21), we calculate the coefficients p_i , $i = 0, \dots, 8$ and q_i , $i = 0, 1, 2, 3$ of $P(x)$ and $Q(x)$ respectively, defined in (12) as shown in table 1.

The polynomial $Q(x)$ has a unique positive root $\bar{x}_1 \approx 3.8701$. Thus, by proposition .4, the exact number of the interior equilibria of (4) is $m = \mu(0) - \mu(\bar{x}_1)$. A Matlab code based on the Euclidean algorithm is developed to calculate this number, for $0 < v \leq 5$, and the results are depicted in figure (1). We observe that for $0 < v < v_c \approx 1.436$, there are two interior equilibria, and for $v_c < v \leq 5$, there are no interior equilibria. Figure (2) illustrates the biological coordinates of the two interior equilibria X_{e1} and X_{e2} versus the economic profit v , showing that the two equilibria coincides when $v = v_c \approx 1.436$, while they collapse for $v > v_c$.

6.2. Local stability of the interior equilibria. We analyse the local stability of the two equilibria X_{e1} and X_{e2} in the existence interval $I_v =]0, v_c[$. We calculate the trace Tr and the determinant Det of the Jacobian matrix A at the two interior equilibria, as shown in figure (3).

- For the second equilibrium X_{e2} , we observe that $Tr(A(X_{e2})) \geq 0$ and $Det(A(X_{e2})) \leq 0$ for all $v \in I_v$, indicating that X_{e2} is always an unstable saddle point.
- For the first equilibrium X_{e1} , we observe that $Tr(A(X_{e1}))$ change its sign and $Det(A(X_{e1})) > 0$ and $\Delta_{X_{e1}} < 0$ for all $v \in I_v$ indicating that X_{e1} is always a focus point and changes its stability property. Figure (4) illustrates $Tr(A(X_{e1}))$ versus the economic profit v showing that X_{e1} is a locally stable focus for $0 < v < v_1^* \approx 0.9596$ or $1.4147 \approx v_2^* < v < v_c$ and unstable focus for $v_1^* < v < v_2^*$.

TABLE 1. Evaluation of the coefficients p_i and q_i of $P(x)$ and $Q(x)$ respectively.

Coefficients p_i		Coefficients q_i	
p_0	0.51518	q_0	2.62089
p_1	-2.36479	q_1	-1.77522
p_2	$6.5172 + 3.84v$	q_2	8.7363
p_3	-22.6624	q_3	-2.18408
p_4	$27.7848 + 12.8v$		
p_5	-52.1281		
p_6	31.0395		
p_7	-9.54038		
p_8	1.19255		

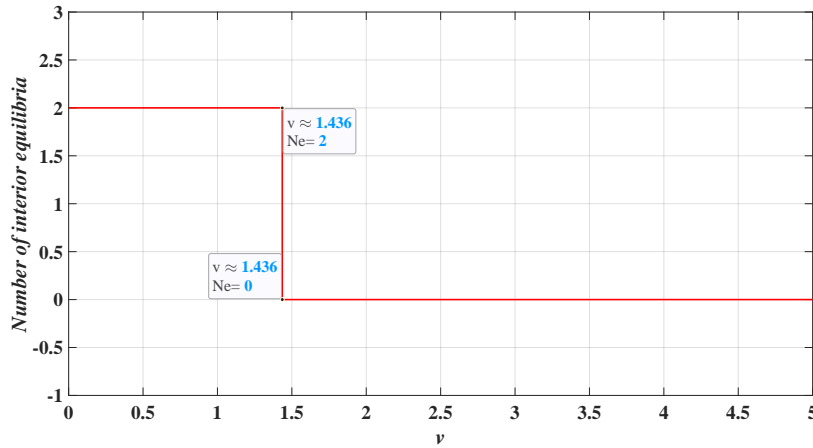


FIGURE 1. Number of the interior equilibria of system (4) versus the economic profit v , for $0 < v \leq 5$.

6.3. Hopf bifurcation through the interior equilibria. Since $Det(A(X_{e2})) \leq 0$ for all $v \in I_v$, the Hopf bifurcation is not expected through the second interior equilibrium, thus we investigate the Hopf bifurcation only through the first interior equilibrium X_{e1} . From the local stability study, there are two possible Hopf bifurcation at $v = v_1^*$ and $v = v_2^*$. We focused on the Hopf bifurcation at $v = v_1^*$.

In order to determine a high precision Hopf bifurcation values v_1^* through X_{e1} , we solve numerically the equation (19). First, we define the function

$$h(v) = \frac{s(py_e(v) - c)^2}{pN} - v,$$

then (19) can be written as

$$h(v) = 0, \tag{22}$$

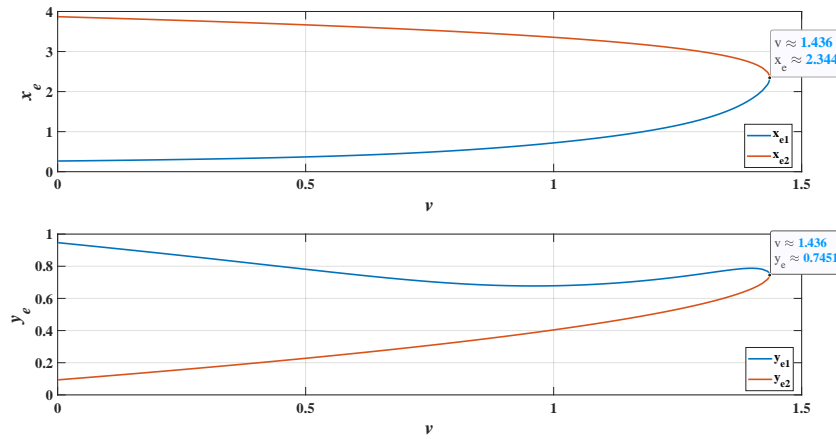


FIGURE 2. The biological coordinates x_e , y_e of the two interior equilibria X_{e1} and X_{e2} versus the economic profit v .

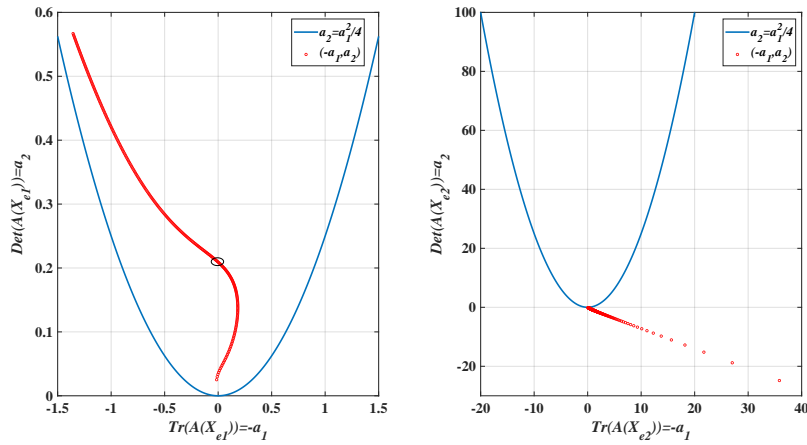


FIGURE 3. Representation of the trace Tr and the determinant Det of the Jacobian matrix A at the two interior equilibria X_{e1} and X_{e2} for $v \in I_v$.

to approximate its solution we develop a Matlab code based on the bisection method applied to the interval $I_0 = [0.9, 1]$.

We have $h(0.9).h(1) \approx -6.6 \times 10^{-4} < 0$, it follow that (22) has at least one solution in I_0 . We choose a maximum error $\epsilon = 10^{-13}$, then, we obtain $v_1^* \approx 0.959607613852853$. Substituting in (36) we get $\sigma \approx 0.0232833778979292 > 0$ which satisfies Case 1 of theorem 5.1. Then, the system (4) undergoes a sub-critical Hopf bifurcation through X_{e1} at $v = v_1^*$, where X_{e1} is locally asymptotic stable for v close to v_1^* with $v < v_1^*$ and it is surrounded by a bifurcating unstable limit cycle as illustrated in figure 5. X_{e1} becomes a center for $v = v_1^*$ as shown in figure 6.

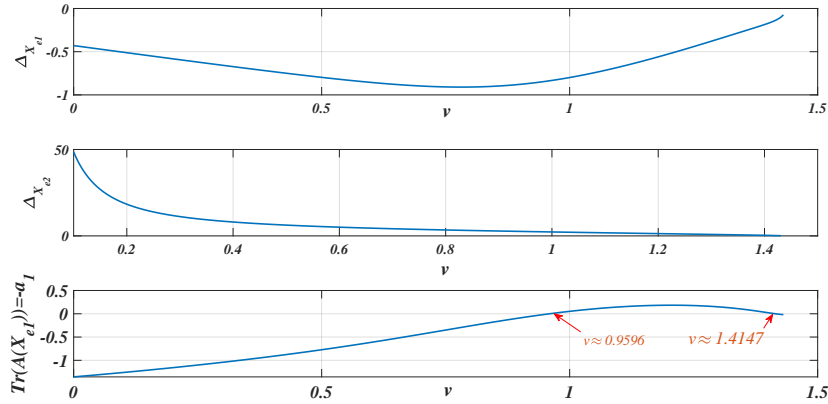


FIGURE 4. Representation of the discriminants of X_{e1} and X_{e2} and the trace $Tr(A(X_{e1}))$ versus the economic profit v .

Finally X_{e1} is an unstable focus for v close to v_1^* with $v > v_1^*$ as depicted in figure 7.

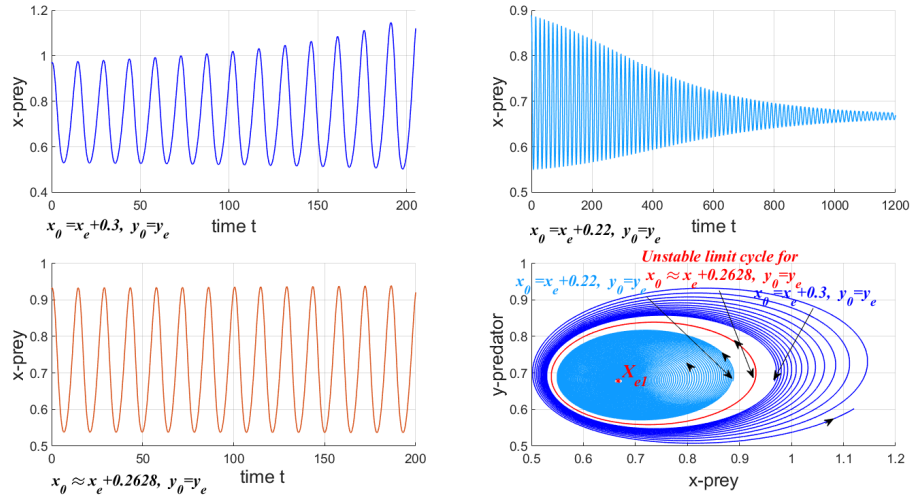


FIGURE 5. Time evolution of prey x , and the phase trajectory of the system (4) for $v = 0.955 < v_1^*$, showing stable behaviour of the first positive equilibrium point $X_{e1}(v)$ with the initial conditions $x_0 = x_e + 0.22$, $y_0 = y_e$, $E_0 = E_e$, surrounded by the bifurcating unstable limit cycle γ and an unstable behaviour in the exterior of γ .

Remark 2. Compared with the systems proposed in [14, 15, 17] in which logistic growth for prey or predator species and Holling type II or Beddington-DeAngelis

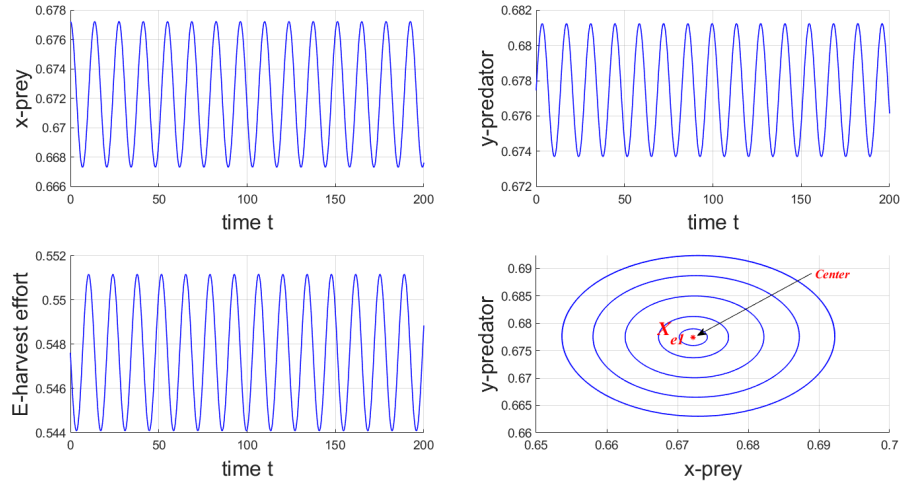


FIGURE 6. Time evolution of species x , y , the harvest effort E and phase portrait of the system (4), for $v \approx v_1^*$, indicating that $X_{e1}(v_1^*)$ is a center surrounded by a band of continues cycles.

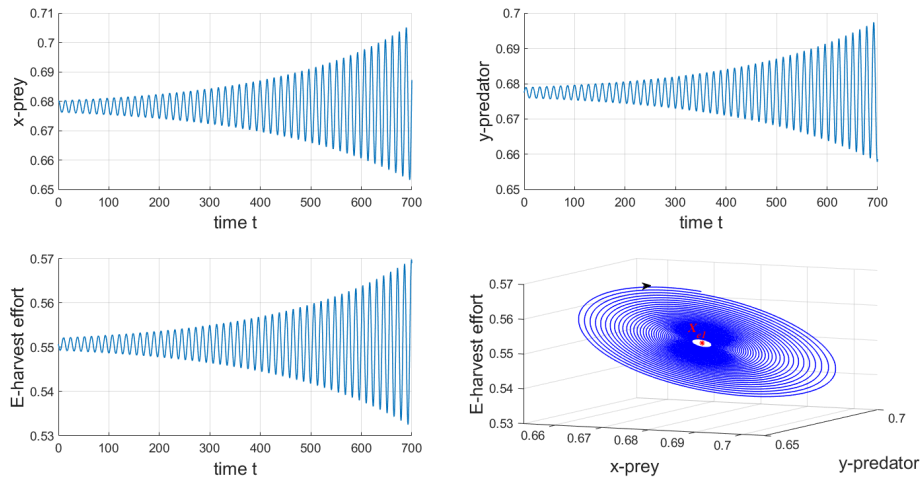


FIGURE 7. Time evolution of species x , y , the harvest effort E and the phase trajectory of the system (4) depicting unstable behaviour of the positive equilibrium point $X_{e1}(v)$ for $v = 0.961 > v_1^*$ with initial conditions $x_0 = x_e + 0.02$, $y_0 = y_e$, $E_0 = E_e$.

functional response are considered, our model consider logistic growth for both prey and predator species and Holing type III functional response, which make our model more realistic, moreover it focuses on economic interest of commercial harvest effort on predator. Another advantage is that the proposed model has multiple interior equilibria which gives more opportunities for fishermen in control theory to stabilize the ecosystem at the interior equilibrium point that represents its ideal performance.

7. Conclusion. This paper has dealt with a differential-algebraic biological economic system. We have taken predator functional response to prey in a form that approaches to a constant even when the prey population increases. We consider the dynamical behaviour of the system when only the predator is subjected to harvesting. From the biological perspective, we are only interested on the positive equilibrium points. The number of positive equilibria is investigated by means of Descartes' rule of signs and is calculated numerically using a Matlab code which has been developed based on the Euclidean algorithm. The obtained results have shown that the proposed system has an even number of positive equilibria between 0 and 8. This gives special importance to the proposed system because the diversity of positive equilibria gives more opportunities in control theory to choose the point that represents the ideal performance of the ecosystem. The local stability of the interior equilibria is carried out by analysing their corresponding characteristic equation and the proposed numerical example has shown that the system has two interior equilibria one of them is unstable saddle point and the other one is a focus point that changes its stability property when varying the economic revenue v . Moreover, one parameter bifurcation analysis is done with respect to the economic revenue. It has been assumed that the positive economic revenue is responsible for the stability of the proposed model. The stability analysis has revealed that when the economic profit v is less than a bifurcation value v_1^* both species converge to their steady states and they will coexist over the time. Moreover, it is shown that when the economic profit is larger than the bifurcation value, then the state of prey population, predator population and, the harvest effort will be unstable which can result in serious imbalance of the ecosystem. The proposed study allows us to point out that it is important for the government to adjust revenue and draw up beneficial strategies to support, encourage, and improve fishery or mitigate emissions so that the community can be driven to steady states that will lead to the survival and sustainable growth of the predator-prey ecosystem. We'll improve our model in the forthcoming papers by introducing several aspects such as time delays that would make the model more realistic. We should incorporate the stage structure of the model where the predator population can be separated into adolescents and adults and only the adults can be captured by fishermen which is economically feasible. Furthermore, some other types of bifurcations such as transcritical bifurcation and singularity induced bifurcation will be investigated in future work.

Acknowledgments. This research was supported by the Algerian General Directorate for Scientific Research and Technological Development (DGRSDT).

Appendix A. Proof of Theorem 5.1. In order to explore the direction of the Hopf bifurcation in the system (15) according to [4, 7] when $v = v^*$, $\bar{X} = \bar{X}_e$ we need to lead the normal form of this system as follows:

$$\begin{cases} \dot{y}_1 &= \omega^* y_2 + \frac{1}{2} a_{11}^1 y_1^2 + a_{12}^1 y_1 y_2 + \frac{1}{2} a_{22}^1 y_2^2 + \frac{1}{6} a_{111}^1 y_1^3 \\ &+ \frac{1}{2} a_{112}^1 y_1^2 y_2 + \frac{1}{2} a_{122}^1 y_1 y_2^2 + \frac{1}{6} a_{222}^1 y_2^3 + O(|Y|^4), \\ \dot{y}_2 &= -\omega^* y_1 + \frac{1}{2} a_{11}^2 y_1^2 + a_{12}^2 y_1 y_2 + \frac{1}{2} a_{22}^2 y_2^2 + \frac{1}{6} a_{111}^2 y_1^3 \\ &+ \frac{1}{2} a_{112}^2 y_1^2 y_2 + \frac{1}{2} a_{122}^2 y_1 y_2^2 + \frac{1}{6} a_{222}^2 y_2^3 + O(|Y|^4). \end{cases} \tag{23}$$

where $\omega^* := \omega(v^*) = \sqrt{\frac{2abdy_e x_e^3}{(x_e^2 + d)^3}}$.

It can be proved that the system (16) with $v = v^*, \bar{X} = \bar{X}_e$ takes the form

$$\left\{ \begin{aligned} \dot{y}_1 &= f_{1y_1}(v^*, \bar{X}_e)y_1 + f_{1y_2}(v^*, \bar{X}_e)y_2 + \frac{1}{2}f_{1y_1y_1}(v^*, \bar{X}_e)y_1^2 \\ &\quad + f_{1y_1y_2}(v^*, \bar{X}_e)y_1y_2 + \frac{1}{2}f_{1y_2y_2}(v^*, \bar{X}_e)y_2^2 + \frac{1}{6}f_{1y_1y_1y_1}(v^*, \bar{X}_e)y_1^3 \\ &\quad + \frac{1}{2}f_{1y_1y_1y_2}(v^*, \bar{X}_e)y_1^2y_2 + \frac{1}{2}f_{1y_1y_2y_2}(v^*, \bar{X}_e)y_1y_2^2 \\ &\quad + \frac{1}{6}f_{1y_2y_2y_2}(v^*, \bar{X}_e)y_2^3 + O(|Y|^4), \\ \dot{y}_2 &= f_{2y_1}(v^*, \bar{X}_e)y_1 + f_{2y_2}(v^*, \bar{X}_e)y_2 + \frac{1}{2}f_{2y_1y_1}(v^*, \bar{X}_e)y_1^2 \\ &\quad + f_{2y_1y_2}(v^*, \bar{X}_e)y_1y_2 + \frac{1}{2}f_{2y_2y_2}(v^*, \bar{X}_e)y_2^2 + \frac{1}{6}f_{2y_1y_1y_1}(v^*, \bar{X}_e)y_1^3 \\ &\quad + \frac{1}{2}f_{2y_1y_1y_2}(v^*, \bar{X}_e)y_1^2y_2 + \frac{1}{2}f_{2y_1y_2y_2}(v^*, \bar{X}_e)y_1y_2^2 \\ &\quad + \frac{1}{6}f_{2y_2y_2y_2}(v^*, \bar{X}_e)y_2^3 + O(|Y|^4). \end{aligned} \right. \tag{24}$$

In the following, we shall calculate the coefficients of the parametric system (24). We derive

$$\begin{aligned} D_{\bar{X}}f_1(v, \bar{X}) &= \left(r \left(1 - \frac{x}{K} \right) - \frac{axy}{x^2 + d} + x \left(-\frac{r}{K} + \frac{ay(x^2 - d)}{(x^2 + d)^2} \right), -\frac{ax^2}{x^2 + d}, 0 \right), \\ D_{\bar{X}}f_2(v, \bar{X}) &= \left(\frac{2bdxy}{(x^2 + d)^2}, s \left(1 - \frac{y}{N} \right) + \frac{bx^2}{x^2 + d} + \frac{pE_e y}{py_e - c} - \bar{E} \right. \\ &\quad \left. + y \left(-\frac{s}{N} + \frac{pE_e}{py_e - c} \right), -y \right), \\ D_{\bar{X}}g(v, \bar{X}) &= \left(0, \bar{E}p - \frac{2p^2E_e y}{py_e - c} + \frac{pE_e c}{py_e - c}, py - c \right), \\ D\varphi(v, Y) &= (D_{y_1}\varphi(v, Y), D_{y_2}\varphi(v, Y)), \\ &= \begin{pmatrix} D_{\bar{X}}g(v, \bar{X}) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{py - c} \left(-\bar{E}p + \frac{2p^2E_e y}{py_e - c} - \frac{pE_e c}{py_e - c} \right) \end{pmatrix}. \end{aligned} \tag{25}$$

Therefore

$$\begin{aligned} f_{1y_1}(v, \bar{X}) &= D_{\bar{X}}f_1(v, \bar{X})D_{y_1}\varphi(v, Y) = r \left(1 - \frac{x}{K} \right) - \frac{axy}{x^2 + d} \\ &\quad + x \left(-\frac{r}{K} + \frac{ay(x^2 - d)}{(x^2 + d)^2} \right), \\ f_{1y_2}(v, \bar{X}) &= D_{\bar{X}}f_1(v, \bar{X})D_{y_2}\varphi(v, Y) = -\frac{ax^2}{x^2 + d}, \\ f_{2y_1}(v, \bar{X}) &= D_{\bar{X}}f_2(v, \bar{X})D_{y_1}\varphi(v, Y) = \frac{2bdxy}{(x^2 + d)^2}, \end{aligned}$$

$$\begin{aligned}
 f_{2y_2}(v, \bar{X}) &= D_{\bar{X}}f_2(v, \bar{X})D_{y_2}\varphi(v, Y) = s \left(1 - \frac{y}{N}\right) + \frac{bx^2}{x^2 + d} + \frac{pE_e y}{py_e - c} - \bar{E} \\
 &+ y \left(-\frac{s}{N} + \frac{pE_e}{py_e - c}\right) - \frac{y}{py - c} \left(-\bar{E}p + \frac{2p^2E_e y}{py_e - c} - \frac{pE_e c}{py_e - c}\right). \quad (26)
 \end{aligned}$$

Substituting v^* and \bar{X}_e into equations (26), we get

$$\begin{aligned}
 f_{1y_1}(v^*, \bar{X}_e) &= 0, \quad f_{2y_2}(v^*, \bar{X}_e) = 0, \quad f_{1y_2}(v^*, \bar{X}_e) = -\frac{ax_e^2}{x_e^2 + d}, \\
 f_{2y_1}(v^*, \bar{X}_e) &= \frac{2bdx_e y_e}{(x_e^2 + d)^2}. \quad (27)
 \end{aligned}$$

In view of equations (26), we can deduce that

$$\begin{aligned}
 D_{\bar{X}}f_{1y_1}(v, \bar{X}) &= \left(2 \left(-\frac{r}{K} + \frac{ay(x^2 - d)}{(x^2 + d)^2}\right) + \frac{2ayx^2(3d - x^2)}{(x^2 + d)^3}, -\frac{2axd}{(x^2 + d)^2}, 0\right), \\
 D_{\bar{X}}f_{1y_2}(v, \bar{X}) &= \left(-\frac{2axd}{(x^2 + d)^2}, 0, 0\right), \\
 D_{\bar{X}}f_{2y_1}(v, \bar{X}) &= \left(\frac{2bdy(d - 3x^2)}{(x^2 + d)^3}, \frac{2bdx}{(x^2 + d)^2}, 0\right), \\
 D_{\bar{X}}f_{2y_2}(v, \bar{X}) &= \left(\frac{2bdx}{(x^2 + d)^2}, -\frac{2s}{N} - \frac{\bar{E}pc}{(py - c)^2} + \frac{pE_e c^2}{(py - c)^2(py_e - c)}, \frac{py}{py - c} - 1\right). \quad (28)
 \end{aligned}$$

From equations (25) and (28), we get

$$\begin{aligned}
 f_{1y_1 y_1}(v, \bar{X}) &= D_{\bar{X}}f_{1y_1}(v, \bar{X})D_{y_1}\varphi(v, Y) = 2 \left(-\frac{r}{K} + \frac{ay(x^2 - d)}{(x^2 + d)^2}\right) + \frac{2ayx^2(3d - x^2)}{(x^2 + d)^3}, \\
 f_{1y_1 y_2}(v, \bar{X}) &= D_{\bar{X}}f_{1y_1}(v, \bar{X})D_{y_2}\varphi(v, Y) = -\frac{2axd}{(x^2 + d)^2}, \\
 f_{1y_2 y_1}(v, \bar{X}) &= D_{\bar{X}}f_{1y_2}(v, \bar{X})D_{y_1}\varphi(v, Y) = -\frac{2axd}{(x^2 + d)^2}, \\
 f_{1y_2 y_2}(v, \bar{X}) &= D_{\bar{X}}f_{1y_2}(v, \bar{X})D_{y_2}\varphi(v, Y) = 0, \\
 f_{2y_1 y_1}(v, \bar{X}) &= D_{\bar{X}}f_{2y_1}(v, \bar{X})D_{y_1}\varphi(v, Y) = \frac{2bdy(d - 3x^2)}{(x^2 + d)^3}, \\
 f_{2y_1 y_2}(v, \bar{X}) &= D_{\bar{X}}f_{2y_1}(v, \bar{X})D_{y_2}\varphi(v, Y) = \frac{2bdx}{(x^2 + d)^2}, \\
 f_{2y_2 y_1}(v, \bar{X}) &= D_{\bar{X}}f_{2y_2}(v, \bar{X})D_{y_1}\varphi(v, Y) = \frac{2bdx}{(x^2 + d)^2}, \\
 f_{2y_2 y_2}(v, \bar{X}) &= D_{\bar{X}}f_{2y_2}(v, \bar{X})D_{y_2}\varphi(v, Y) = -\frac{2s}{N} - \frac{2pc\bar{E}}{(py - c)^2} + \frac{2p^2E_e cy}{(py - c)^2(py_e - c)}. \quad (29)
 \end{aligned}$$

substituting v^* and \bar{X}_e into equations (29), yield

$$\begin{aligned}
 f_{1y_1 y_1}(v^*, \bar{X}_e) &= \frac{2ay_e x_e^2(3d - x_e^2)}{(x_e^2 + d)^3}, \\
 f_{1y_1 y_2}(v^*, \bar{X}_e) &= f_{1y_2 y_1}(v^*, \bar{X}_e) = -\frac{2adx_e}{(x_e^2 + d)^2}, \\
 f_{2y_1 y_2}(v^*, \bar{X}_e) &= f_{2y_2 y_1}(v^*, \bar{X}_e) = \frac{2bdx_e}{(x_e^2 + d)^2},
 \end{aligned}$$

$$\begin{aligned}
f_{2y_1y_1}(v^*, \bar{X}_e) &= \frac{2bdy_e(d - 3x_e^2)}{(x_e^2 + d)^3}, \\
f_{2y_2y_2}(v^*, \bar{X}_e) &= -\frac{2spy_e}{N(py_e - c)}, \\
f_{1y_2y_2}(v^*, \bar{X}_e) &= 0.
\end{aligned} \tag{30}$$

By equations (29), we get

$$\begin{aligned}
D_{\bar{X}}f_{1y_1y_1}(v, \bar{X}) &= \left(\frac{24ady_x(d - x^2)}{(x^2 + d)^4}, \frac{2ad(3x^2 - d)}{(x^2 + d)^3}, 0 \right), \\
D_{\bar{X}}f_{1y_1y_2}(v, \bar{X}) &= D_{\bar{X}}f_{1y_2y_1}(v, \bar{X}) = \left(\frac{2ad(3x^2 - d)}{(x^2 + d)^3}, 0, 0 \right), \\
D_{\bar{X}}f_{1y_2y_2}(v, \bar{X}) &= (0, 0, 0), \\
D_{\bar{X}}f_{2y_1y_1}(v, \bar{X}) &= \left(\frac{24bdy_x(x^2 - d)}{(x^2 + d)^4}, \frac{2bd(d - 3x^2)}{(x^2 + d)^3}, 0 \right), \\
D_{\bar{X}}f_{2y_1y_2}(v, \bar{X}) &= D_{\bar{X}}f_{2y_2y_1}(v, \bar{X}) = \left(\frac{2bd(d - 3x^2)}{(x^2 + d)^3}, 0, 0 \right), \\
D_{\bar{X}}f_{2y_2y_2}(v, \bar{X}) &= \left(0, \frac{4p^2cE}{(py - c)^3} + \frac{2p^2cE_e}{(py - c)^2(py_e - c)}, -\frac{2pc}{(py - c)^2} \right).
\end{aligned} \tag{31}$$

Substituting v^*, \bar{X}_e into equations (25) and (31), we obtain

$$\begin{aligned}
D_{\bar{X}}f_{1y_1y_1}(v^*, \bar{X}_e) &= \left(\frac{24ady_ex_e(d - x_e^2)}{(x_e^2 + d)^4}, \frac{2ad(3x_e^2 - d)}{(x_e^2 + d)^3}, 0 \right), \\
D_{\bar{X}}f_{1y_1y_2}(v^*, \bar{X}_e) &= D_{\bar{X}}f_{1y_2y_1}(v^*, \bar{X}_e) = \left(\frac{2ad(3x_e^2 - d)}{(x_e^2 + d)^3}, 0, 0 \right), \\
D_{\bar{X}}f_{1y_2y_2}(v^*, \bar{X}_e) &= (0, 0, 0), \\
D_{\bar{X}}f_{2y_1y_1}(v^*, \bar{X}_e) &= \left(\frac{24bdy_ex_e(x_e^2 - d)}{(x_e^2 + d)^4}, \frac{2bd(d - 3x_e^2)}{(x_e^2 + d)^3}, 0 \right), \\
D_{\bar{X}}f_{2y_1y_2}(v^*, \bar{X}_e) &= D_{\bar{X}}f_{2y_2y_1}(v^*, \bar{X}_e) = \left(\frac{2bd(d - 3x_e^2)}{(x_e^2 + d)^3}, 0, 0 \right), \\
D_{\bar{X}}f_{2y_2y_2}(v^*, \bar{X}_e) &= \left(0, \frac{6p^2cE_e}{(py_e - c)^3}, -\frac{2pc}{(py_e - c)^2} \right), \\
D\varphi(v^*, 0) &= (D_{y_1}\varphi(v^*, 0), D_{y_2}\varphi(v^*, 0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{32}$$

From equations (32), we have

$$\begin{aligned}
f_{1y_1y_1y_1}(v^*, \bar{X}_e) &= \frac{24ady_ex_e(d - x_e^2)}{(x_e^2 + d)^4}, \\
f_{2y_1y_1y_1}(v^*, \bar{X}_e) &= \frac{24bdy_ex_e(x_e^2 - d)}{(x_e^2 + d)^4}, \\
f_{2y_2y_2y_2}(v^*, \bar{X}_e) &= \frac{6p^2cE_e}{(py_e - c)^3}, \\
f_{1y_1y_2y_1}(v^*, \bar{X}_e) &= f_{1y_2y_1y_1}(v^*, \bar{X}_e) = f_{1y_1y_1y_2}(v^*, \bar{X}_e) = \frac{2ad(3x_e^2 - d)}{(x_e^2 + d)^3},
\end{aligned}$$

$$\begin{aligned}
 f_{2y_1y_1y_2}(v^*, \bar{X}_e) &= f_{2y_1y_2y_1}(v^*, \bar{X}_e) = f_{2y_2y_1y_1}(v^*, \bar{X}_e) = \frac{2ad(d - 3x_e^2)}{(x_e^2 + d)^3}, \\
 f_{1y_2y_2y_1}(v^*, \bar{X}_e) &= f_{1y_2y_2y_2}(v^*, \bar{X}_e) = f_{1y_1y_2y_2}(v^*, \bar{X}_e) = f_{1y_2y_1y_2}(v^*, \bar{X}_e) = 0, \\
 f_{2y_1y_2y_2}(v^*, \bar{X}_e) &= f_{2y_2y_1y_2}(v^*, \bar{X}_e) = f_{2y_2y_2y_1}(v^*, \bar{X}_e) = 0.
 \end{aligned}
 \tag{33}$$

According to equations (24), (27), (30) and (33), the parametric system of the system (16) with $v = v^*$, $X = \bar{X}_e$ can be written as

$$\left\{ \begin{aligned}
 \dot{y}_1 &= -\frac{ax_e^2}{d + x_e^2}y_2 + \frac{ay_ex_e^2(3d - x_e^2)}{(d + x_e^2)^3}y_1^2 - \frac{2ax_ed}{(d + x_e^2)^2}y_1y_2 + \frac{4adx_ey_e(d - x_e^2)}{(d + x_e^2)^4}y_1^3 \\
 &\quad + \frac{ad(3x_e^2 - d)}{(d + x_e^2)^3}y_1^2y_2 + O(|Y|^4), \\
 \dot{y}_2 &= \frac{2bdx_ey_e}{(d + x_e^2)^2}y_1 + \frac{bdy_e(d - 3x_e^2)}{(d + x_e^2)^3}y_1^2 + \frac{2bdx_e}{(d + x_e^2)^2}y_1y_2 - \frac{spy_e}{N(py_e - c)}y_2^2 \\
 &\quad + \frac{4bdx_ey_e(x_e^2 - d)}{(d + x_e^2)^4}y_1^3 + \frac{bd(d - 3x_e^2)}{(d + x_e^2)^3}y_1^2y_2 + \frac{p^2cE_e}{(py_e - c)^3}y_2^3 + O(|Y|^4).
 \end{aligned} \right.
 \tag{34}$$

Compared with the normal form (24), we should normalize the parametric system (34) with the following nonsingular linear transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $P = \begin{pmatrix} \frac{ax_e^2}{x_e^2 + d} & 0 \\ 0 & -\omega^* \end{pmatrix}$, $U = (u_1, u_2)^T$. For convenience, we use Y instead of U . Thus, the normal form of the system (15) with $v = v^*$ and $X = \bar{X}_e$ takes the form

$$\left\{ \begin{aligned}
 \dot{y}_1 &= \omega^*y_2 + \frac{a^2x_e^4y_e(3d - x_e^2)}{(d + x_e^2)^4}y_1^2 + \frac{2adx_e\omega^*}{(d + x_e^2)^2}y_1y_2 \\
 &\quad + \frac{4da^3x_e^5(d - x_e^2)}{(d + x_e^2)^6}y_1^3 - \frac{a^2d\omega^*x_e^2(3x_e^2 - d)}{(d + x_e^2)^4}y_1^2y_2 + O(|Y|^4), \\
 \dot{y}_2 &= -\omega^*y_1 - \frac{ax_e\omega^*(d - 3x_e^2)}{2(d + x_e^2)^2}y_1^2 + \frac{\omega^{*2}}{y_e}y_1y_2 + \frac{spy_e\omega^*}{N(py_e - c)}y_2^2 \\
 &\quad - \frac{2a^2x_e^4\omega^*(x_e^2 - d)}{(d + x_e^2)^4}y_1^3 + \frac{ax_e\omega^{*2}(d - 3x_e^2)}{2(d + x_e^2)^2}y_1^2y_2 + \frac{p^2cE_e\omega^{*2}}{(py_e - c)^3}y_2^3 \\
 &\quad + O(|Y|^4).
 \end{aligned} \right.
 \tag{35}$$

According to the Hopf bifurcation theory [7], the direction of the Hopf bifurcation is determined by the signal of σ given by

$$16\sigma = \frac{1}{\omega^*} \{ a_{11}^1 (a_{11}^2 - a_{12}^1) + a_{22}^2 (a_{12}^2 - a_{22}^1) + (a_{11}^2 a_{12}^2 - a_{12}^1 a_{22}^1) \}$$

$$\begin{aligned}
& + (a_{111}^1 + a_{122}^1 + a_{112}^2 + a_{222}^2), \\
& = \frac{6a^3 x_e^5 (d - x_e^2)}{(d + x_e^2)^6} (-y_e(3d - x_e^2) + 4d) + \frac{2sp\omega^{*2}}{N(py_e - c)} + \frac{6p^2 c E_e \omega^{*2}}{(py_e - c)^3}. \quad (36)
\end{aligned}$$

REFERENCES

- [1] M.-S. Abdelouahab, N.-E. Hamri and J. Wang, [Hopf bifurcation and chaos in fractional-order modified hybrid optical system](#), *Nonlinear Dynamics*, **69** (2012), 275–284.
- [2] M.-S. Abdelouahab and R. Lozi, [Hopf-like bifurcation and mixed mode oscillation in a fractional-order FitzHugh-Nagumo model](#), *AIP Conference Proceedings* **2183**, 100003, (2019).
- [3] G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, 4th edition, John Wiley and Sons, Inc., New York, 1989.
- [4] B. S. Chen, X. X. Liao and Y. Q. Liu, Normal forms and bifurcations for the differential-algebraic systems, *Acta Math. Appl. Sinica*, **23** (2000), 429–443.
- [5] C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, 2nd edition, John Wiley and Sons, New York, 1990.
- [6] H. S. Gordon, [The economic theory of a common property resource: The fishery](#), *Journal of Political Economy*, **62** (1954), 124–142.
- [7] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
- [8] J. Hale, *Theory of Functional Differential Equations*, 2nd edition, Applied Mathematical Sciences, Vol. 3. Springer-Verlag, New York-Heidelberg, 1977.
- [9] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1: Power Series-Integration-Conformal Mapping-Location of Zeros, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
- [10] C. S. Holling, [The components of predation as revealed by a study of small mammal predation of the European pine sawfly](#), *Canadian Entomology*, **91** (1959a), 293–320.
- [11] C. S. Holling, [Some characteristics of simple types of predation and parasitism](#), *Canadian Entomology*, **91** (1959), 385–398.
- [12] C. S. Holling, [The functional response of predators to prey density and its role in mimicry and population regulation](#), *Memoirs of the Entomological Society of Canada*, **97** (1965), 5–60.
- [13] T. K. Kar and K. Chakraborty, Effort dynamics in a prey-predator model with harvesting, *Int. J. Inf. Syst. Sci.*, **6** (2010), 318–332.
- [14] W. Liu, L. Biwen, C. Fu and B. Chen, [Dynamics of a predator-prey ecological system with nonlinear harvesting rate](#), *Wuhan Univ. J. Nat. Sci.*, **20** (2015), 25–33.
- [15] W. Liu, C. J. Fu and B. Chen, [Hopf bifurcation for a predator-prey biological economic system with Holling type II functional response](#), *J. Franklin Inst.*, **348** (2011), 1114–1127.
- [16] C. Liu, Q. Zhang, Y. Zhang and X. D. Duan, [Bifurcation and control in a differential-algebraic harvested prey-predator model with stage structure for predator](#), *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **18** (2008), 3159–3168.
- [17] W. Zhu, J. Huang and W. Liu, [The stability and Hopf bifurcation of the differential-algebraic biological economic system with single harvesting](#), *2015 Sixth International Conference on Intelligent Control and Information Processing (ICICIP)*, Wuhan, (2015), 92–97.

Received February 2020; 1st revision July 2020; 2nd revision July 2020.

E-mail address: n.kerious@centre-univ-mila.dz

E-mail address: m.abdelouahab@centre-univ-mila.dz