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REGULARITY CRITERIA FOR WEAK SOLUTIONS OF THE MAGNETO-MICROPOLAR EQUATIONS

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Abstract. In this paper, we show that a weak solution $(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$ of the magneto-micropolar equations, defined in $[0, T)$, which satisfies $\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b} \in L^{\frac{32}{7}}(0, T; L^2(\mathbb{R}^3))$ or $\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b} \in L^\infty(0, T; L^2(\mathbb{R}^3))$, is regular in $\mathbb{R}^3 \times (0, T)$ and can be extended as a C^∞ solution beyond T.

1. **Introduction.** In this paper we present some regularity criteria for weak solutions of the following *magneto-micropolar fluid system* in three space dimensions:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla (p + \frac{1}{2} |\,\mathbf{b}\,|^2) = (\mu + \chi) \Delta \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{b} + \chi \nabla \times \mathbf{w}, \\ \mathbf{w}_t + \mathbf{u} \cdot \nabla \mathbf{w} = \gamma \Delta \mathbf{w} + \kappa \nabla (\nabla \cdot \mathbf{w}) + \chi \nabla \times \mathbf{u} - 2\chi \mathbf{w}, \\ \mathbf{b}_t + \mathbf{u} \cdot \nabla \mathbf{b} = \nu \Delta \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot), \;\; \mathbf{w}(\cdot, 0) = \mathbf{w}_0(\cdot), \;\; \mathbf{b}(\cdot, 0) = \mathbf{b}_0(\cdot), \end{cases} \tag{1}$$

where $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, $\mathbf{w}(x, t) = (w_1(x, t), w_2(x, t), w_3(x, t)) \in \mathbb{R}^3$ denotes the micro-rotational velocity, $\mathbf{b}(x, t) = (b_1(x, t), b_2(x, t), b_3(x, t)) \in \mathbb{R}^3$ the divergence free magnetic field and $p(x, t) \in \mathbb{R}$ the hydrostatic pressure. As usual, $x \in \mathbb{R}^3$ denotes the space variable and $0 \le t < T$ the time variable. The positive constants μ, χ, ν, κ, and γ are associated with specific properties of the fluid; precisely, μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities and, lastly, ν^{-1} is the magnetic Reynolds number. The initial data for the velocity and magnetic fields, given by $\mathbf{u}_0$ and $\mathbf{b}_0$ in (1), are divergence free, i.e., $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$.

Let us list some recent papers which discuss regularity of weak solutions of the magneto-micropolar equations (1) and systems that are particular cases of these equations as, for example, the classical Navier-Stokes equations.

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In 2010, Y. Baoquan [18] established the following regularity criteria: a solution $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)$ of (1) can be extended smoothly beyond $t = T$ if

$$\int_0^T \|\mathbf{u}(t)\|_\alpha^\beta \, dt < \infty, \quad \text{where } \frac{3}{\alpha} + \frac{2}{\beta} \le 1,\ 3 < \alpha \le \infty,$$

or

$$\int_0^T \|\nabla \mathbf{u}(t)\|_\alpha^\beta \, dt < \infty, \quad \text{where } \frac{3}{\alpha} + \frac{2}{\beta} \le 2,\ \frac{3}{2} < \alpha \le \infty,$$

provided that the initial data $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$ belong to $H^1(\mathbb{R}^3)$ and $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. Here $(\mathbf{u}, \mathbf{w}, \mathbf{b}) \in C([0, T); H^1(\mathbb{R}^3)) \cap C((0, T); H^2(\mathbb{R}^3))$ is assumed. Notice that Y. Baoquan [18] used only the velocity field or its gradient in the regularity criteria described above.

In 2013, Y. Wang [14] showed that a weak solution $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)$ of (1), defined on the interval $[0, T)$, can be extended smoothly beyond $t = T$ if one assumes

$$\int_0^T \|\partial_3 \mathbf{u}(t)\|_\alpha^\beta \, dt < \infty \quad \text{where } \frac{3}{\alpha} + \frac{2}{\beta} \le 1,\ \alpha \ge 3, \tag{2}$$

provided that $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in H^1(\mathbb{R}^3)$ and $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. Let us point out that Y. Wang [14] established the regularity criterion given above by considering only one component of the gradient of the velocity field (see (2)). The paper [14] deals with an extension of a regularity criterion obtained for weak solutions of the magneto-hydrodynamic equations (see [2]). This latter system is the special case of(1) obtained for $\mathbf{w} = 0$ and $\chi = 0$. Further results related to the current paper are in [2, 21].

The papers [14] and [18] (see also [3, 8, 12, 13, 15, 17, 19, 20, 22, 23, 24, 28, 29]), raised our interest to obtain regularity criteria for weak solutions of the magneto-micropolar system (1), which involve only one component of the velocity field $\mathbf{u}$.

Note that the magneto-micropolar system (1) reduces to the classical Navier-Stokes equations for the velocity field $\mathbf{u}$ and the pressure p if one assumes that $\mathbf{w} = \mathbf{b} = 0$ and $\chi = 0$. Regularity criteria for weak solutions of Navier-Stokes equations, involving only one component of the velocity field, have been published recently. We want to comment on two results.

First, Z. Zhang and X. Yang [22] present a regularity criterion for the Navier-Stokes equations involving the gradient of one component of the velocity field. Precisely, if $\mathbf{u}(t)$ is a weak solution of the Navier-Stokes equations in $(0, T)$ and

$$\int_0^T \|\nabla u_3(t)\|_2^{\frac{32}{7}} \, dt < \infty, \tag{3}$$

then $\mathbf{u}$ is C^∞ in $\mathbb{R}^3 \times (0, T)$ and the solution can be extended as a C^∞ function beyond T. Here $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \mathbf{u}_0 = 0$ is assumed. In the current paper we prove that the criterion (3) can be extended from the Navier-Stokes equations to the magneto-micropolar system (1) under appropriate conditions on the other fields $\mathbf{w}$ and $\mathbf{b}$. See Theorem 1.1 below.

The second paper we cite is [23]. Z. Zhang and X. Yang [23] deal with one component of the gradient of one component of the velocity field. More precisely, regularity of a weak solution $\mathbf{u}(t)$ of the Navier-Stokes equations is obtained under the assumption

$$\partial_3 u_3(t) \in L^\infty(0, T; L^2(\mathbb{R}^3)). \tag{4}$$

Theorem 1.2 below establishes an extension of criterion (4) from the Navier-Stokes equations to the magneto-micropolar system (1).

Further regularity results for weak solutions of the Navier-Stokes equations are established in [1, 4, 5, 6, 7, 10, 25, 26].

The main results of the current paper are:

Theorem 1.1. *Let* $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in L^2(\mathbb{R}^3)$ *with* $\nabla \cdot \mathbf{u}_0, \nabla \cdot \mathbf{b}_0 = 0$. *Let* $T > 0$ *and let*

$$(\mathbf{u}, \mathbf{w}, \mathbf{b}) \in C([0,T); H^1(\mathbb{R}^3)) \cap C((0,T); H^2(\mathbb{R}^3)) \tag{5}$$

denote a weak solution of the magneto-micropolar equations (1) *in* $(0,T)$ *satisfying the initial condition* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(0) = (\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$. *If*

$$\nabla u_3, \nabla_h \mathbf{w}, \nabla_h \mathbf{b} \in L^{\frac{32}{7}}(0,T; L^2(\mathbb{R}^3)), \tag{6}$$

then $(\mathbf{u}, \mathbf{w}, \mathbf{b})$ *is* C^∞ *in* $\mathbb{R}^3 \times (0,T)$ *and the solution can be extended as a* C^∞ *function beyond* T.

Theorem 1.2. *Let* $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in L^2(\mathbb{R}^3)$ *with* $\nabla \cdot \mathbf{u}_0, \nabla \cdot \mathbf{b}_0 = 0$. *Let* $T > 0$ *and let*

$$(\mathbf{u}, \mathbf{w}, \mathbf{b}) \in C([0,T); H^1(\mathbb{R}^3)) \cap C((0,T); H^2(\mathbb{R}^3)) \tag{7}$$

denote a weak solution of the magneto-micropolar equations (1) *in* $(0,T)$ *satisfying the initial condition* $(\mathbf{u}, \mathbf{w}, \mathbf{b})(0) = (\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$. *If*

$$\partial_3 u_3, \partial_3 \mathbf{w}, \partial_3 \mathbf{b} \in L^\infty(0,T; L^2(\mathbb{R}^3)), \tag{8}$$

then $(\mathbf{u}, \mathbf{w}, \mathbf{b})$ *is* C^∞ *in* $\mathbb{R}^3 \times (0,T)$ *and the solution can be extended as a* C^∞ *function beyond* T.

An outline of the paper follows: There are two sections after the Introduction. In Section 2, we list definitions and notations used throughout the paper and recall results that play an important role in our proofs of the main results. Section 3 presents the proofs of Theorems 1.1 and 1.2.

2. Preliminaries.

We introduce notations and definitions used in the paper.

- Boldface letters denote vector fields; for example,
$$\mathbf{a} = \mathbf{a}(x,t) = (a_1(x,t), a_2(x,t), a_3(x,t)), \quad x \in \mathbb{R}^3,\ t \geq 0.$$
- The Euclidean norm of any vector $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ is denoted and given by $|\mathbf{a}| = \sqrt{\sum_{i=1}^n a_i^2}$.
- The notation L^α is used for the standard Lebesgue space equipped with the norm $\|\cdot\|_\alpha$, where $1 \leq \alpha \leq \infty$; more specifically,
$$\|\mathbf{a}\|_\alpha := \left(\int_{\mathbb{R}^3} |\mathbf{a}(x)|^\alpha \, dx \right)^{\frac{1}{\alpha}}, \quad 1 \leq \alpha < \infty,$$
and
$$\|\mathbf{a}\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}^3} \{|\mathbf{a}(x)|\},$$
where $\mathbf{a} : \mathbb{R}^3 \to \mathbb{R}^n$ $(n \in \mathbb{N})$ is a mensurable function. We define the L^2-inner product of two vector functions by
$$(\mathbf{a}, \mathbf{b})_2 := \int_{\mathbb{R}^3} \mathbf{a}(x) \cdot \mathbf{b}(x) \, dx,$$
where $\mathbf{c} \cdot \mathbf{d} := \sum_{i=1}^n c_i d_i$ for $\mathbf{c} = (c_1, ..., c_n), \mathbf{d} = (d_1, ..., d_n) \in \mathbb{R}^n$; and $\mathbf{a}, \mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^n$ $(n \in \mathbb{N})$ are mensurable functions.

- Let $\nabla \mathbf{a} = (\nabla a_1, ..., \nabla a_n)$ denote the gradient of $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$, where $\nabla a_j = (\partial_1 a_j, \partial_2 a_j, \partial_3 a_j)$, with $\partial_i = \partial/\partial x_i$ for all $i = 1, 2, 3$ and $j = 1, ..., n$.
- The horizontal gradient is denoted by $\nabla_h \mathbf{a} = (\nabla_h a_1, ..., \nabla_h a_n)$, where $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ and $\nabla_h a_j = (\partial_1 a_j, \partial_2 a_j)$, with $j = 1, ..., n$.
- Here $\mathbf{a} \cdot \nabla := \sum_{i=1}^{3} a_i \partial_i$, where $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.
- $\nabla \times \mathbf{a}$ denotes the curl of the vector field $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.
- Denote $\nabla \cdot \mathbf{a} = \sum_{i=1}^{3} \partial_i a_i$, where $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$.
- Δ represents the standard Laplacian operator.
- The horizontal Laplacian is denoted by $\Delta_h \mathbf{a} = (\Delta_h a_1, ..., \Delta_h a_n)$, where $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$ and $\Delta_h a_j = \sum_{i=1}^{2} \partial_i^2 a_j$, with $j = 1, ..., n$.
- Let $(X, \|\cdot\|)$ be a Banach space and assume that $1 \leq \beta \leq \infty$, $c, d \in \mathbb{R}, c < d$. We denote by $L^\beta(c, d; X)$ the space of all measurable functions $f : (c, d) \to X$ with $\|f(\cdot)\| \in L^\beta(c, d)$ endowed with the norm $\|f\|_{L^\beta(c,d;X)} := \left(\int_c^d \|f(t)\|^\beta \, dt \right)^{\frac{1}{\beta}}$, where $\beta < \infty$, and also $\|f\|_{L^\infty(c,d;X)} = \operatorname{ess\,sup}_{c<t<d} \{\|f(t)\|\}$.
- We define a weak solution of (1) as follows: Let $T > 0$ and $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in L^2(\mathbb{R}^3)$, with $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. A mensurable function $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)$ is called a weak solution for (1) on $[0, T)$ if the following conditions hold
  1. $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;
  2. the system (1) is satisfied in the sense of distributions;
  3. the energy inequality holds, i.e.,

$$\begin{aligned} &\|(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)\|_2^2 + 2(\mu + \chi) \int_0^t \|\nabla \mathbf{u}(\tau)\|_2^2 \, d\tau + 2\gamma \int_0^t \|\nabla \mathbf{w}(\tau)\|_2^2 \, d\tau \\ &+ 2\nu \int_0^t \|\nabla \mathbf{b}(\tau)\|_2^2 \, d\tau + 2\kappa \int_0^t \|\nabla \cdot \mathbf{w}(\tau)\|_2^2 \, d\tau + 2\chi \int_0^t \|\mathbf{w}(\tau)\|_2^2 \, d\tau \\ &\leq \|(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)\|_2^2, \end{aligned} \tag{9}$$

  for all $0 \leq t < T$.
- For brevity, dependencies on the variables x and t are often suppressed in our notation. For example, the function $t \to \|\mathbf{u}(\cdot, t)\|_2$ may also be written as $\|\mathbf{u}(t)\|_2$ or $\|\mathbf{u}\|_2$. Furthermore, as usual, the value of constants may change line by line in the paper.

Now, we enunciate the lemmas that will be applied in the proofs of our main results. The first one is proved in [16].

Lemma 2.1 (see [16])**.** *Let i, j, k be a permutation of $1, 2, 3$. Assume that*

$$f, g, \partial_i g, \partial_j g, h, \partial_j h, \partial_k h \in L^2(\mathbb{R}^3).$$

Then,

$$\int_{\mathbb{R}^3} fgh \, dx \leq C \|f\|_2 \|g\|_2^{\frac{1}{4}} \|\partial_i g\|_2^{\frac{1}{2}} \|\partial_j g\|_2^{\frac{1}{4}} \|h\|_2^{\frac{1}{4}} \|\partial_k h\|_2^{\frac{1}{2}} \|\partial_j h\|_2^{\frac{1}{4}}.$$

The second one was established in [23].

Lemma 2.2 (see [23])**.** *Let $f \in L^6(\mathbb{R}^3)$ and $g \in L^2(\mathbb{R}^3)$ with $\partial_3 f \in L^2(\mathbb{R}^3)$ and $\nabla_h g \in L^2(\mathbb{R}^3)$. Then,*

$$\int_{\mathbb{R}^3} f^2 g^2 \, dx \leq 2\sqrt{2} \|f\|_6^{\frac{3}{2}} \|\partial_3 f\|_2^{\frac{1}{2}} \|g\|_2 \|\nabla_h g\|_2.$$

The third one was written in [2].

Lemma 2.3 (see [2]). *Assume that* $\theta, \lambda, \vartheta \in \mathbb{R}$ *satisfy*

$$1 \le \theta, \lambda, \vartheta < \infty,\ \frac{1}{\theta} + \frac{2}{\lambda} > 1,\ 1 + \frac{3}{\vartheta} = \frac{1}{\theta} + \frac{2}{\lambda}.$$

Consider that $f \in H^1(\mathbb{R}^3)$, $\partial_1 f, \partial_2 f \in L^\lambda(\mathbb{R}^3)$ *and* $\partial_3 f \in L^\theta(\mathbb{R}^3)$. *Then, there exists a positive constant* C *such that*

$$\|f\|_\vartheta \le C\|\partial_1 f\|_\lambda^{\frac{1}{3}}\|\partial_2 f\|_\lambda^{\frac{1}{3}}\|\partial_3 f\|_\theta^{\frac{1}{3}}.$$

In particular, if $\lambda = 2$ *and* $f \in H^1(\mathbb{R}^3)$, $\partial_3 f \in L^\theta(\mathbb{R}^3)$ (*with* $1 \le \theta < \infty$), *then there is a positive constant* C *such that*

$$\|f\|_{3\theta} \le C\|\partial_1 f\|_2^{\frac{1}{3}}\|\partial_2 f\|_2^{\frac{1}{3}}\|\partial_3 f\|_\theta^{\frac{1}{3}}.$$

3. **Proof of the main results.** In this section we prove Theorems 1.1 and 1.2. In both results, it is necessary to consider $\epsilon \in (0,T)$ arbitrary in order to obtain $\delta \in (0,\epsilon)$ such that $(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(\delta) \in L^2(\mathbb{R}^3)$. It is known that there is a unique strong solution $(\mathbf{u}', \mathbf{w}', \mathbf{b}') \in C([\delta, T^*); H^1(\mathbb{R}^3)) \cap L^2(\delta, T^*; H^2(\mathbb{R}^3))$ for the system (1) (see [9, 11] and references therein) that satisfies $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(\delta) = (\mathbf{u}, \mathbf{w}, \mathbf{b})(\delta)$, where $t = T^*$ is the maximum time of existence for this solution, and $(\mathbf{u}', \mathbf{w}', \mathbf{b}') \in C^\infty(\mathbb{R}^3 \times (0, T^*))$ (since $\epsilon > 0$ is arbitrary). Therefore, if $T < T^*$, one concludes that $(\mathbf{u}, \mathbf{w}, \mathbf{b})(t)$ is regular in $\mathbb{R}^3 \times (0,T)$. On the other hand, assuming $T^* \le T$, we will prove below that $\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(t)\|_2$ is uniformly bounded for $t \in [\delta, T^*)$ (in Theorem 1.2) and bounded as $t \nearrow T^*$ (in Theorem 1.1). However, this is not possible. In fact, this boundedness would imply that $(\mathbf{u}', \mathbf{w}', \mathbf{b}')(t)$ could be extended beyond $t = T^*$, which contradicts the definition of T^*.

3.1. **Proof of Theorem 1.1.** First, notice that by applying the product $(\cdot, \Delta_h\mathbf{u})_2$ to the first equation of the magneto-micropolar system (1) and using the fact that $\nabla\cdot\mathbf{u} = 0$, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{u}\|_2^2 + (\mu+\chi)\|\nabla\nabla_h\mathbf{u}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{u})_2\\&-\chi(\nabla\times\mathbf{w}, \Delta_h\mathbf{u})_2.\end{aligned} \tag{10}$$

Similarly, from the second and third equations in (1), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{w}\|_2^2 + \gamma\|\nabla\nabla_h\mathbf{w}\|_2^2 + \kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h\mathbf{w}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{w}, \Delta_h\mathbf{w})_2\\&-\chi(\nabla\times\mathbf{u}, \Delta_h\mathbf{w})_2\end{aligned} \tag{11}$$

and also

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h\mathbf{b}\|_2^2 + \nu\|\nabla\nabla_h\mathbf{b}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{b})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{b})_2. \tag{12}$$

By adding the results (10), (11) and (12) one infers

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u}, \nabla_h\mathbf{w}, \nabla_h\mathbf{b})\|_2^2 + (\mu+\chi)\|\nabla\nabla_h\mathbf{u}\|_2^2 + \gamma\|\nabla\nabla_h\mathbf{w}\|_2^2 + \nu\|\nabla\nabla_h\mathbf{b}\|_2^2\\&+\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla_h\mathbf{w}\|_2^2 = (\mathbf{u}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{u})_2\\&-\chi(\nabla\times\mathbf{w}, \Delta_h\mathbf{u})_2 + (\mathbf{u}\cdot\nabla\mathbf{w}, \Delta_h\mathbf{w})_2 - \chi(\nabla\times\mathbf{u}, \Delta_h\mathbf{w})_2 + (\mathbf{u}\cdot\nabla\mathbf{b}, \Delta_h\mathbf{b})_2\\&-(\mathbf{b}\cdot\nabla\mathbf{u}, \Delta_h\mathbf{b})_2.\end{aligned} \tag{13}$$

Let us examine the terms on the right hand side of the above equality. We have

$$-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2 = -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i b_j\partial_k^2 u_j\,dx$$
$$= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\,dx + \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_k\partial_i b_j\partial_k u_j\,dx.$$

Similarly, we get

$$-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta_h\mathbf{b})_2 = -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i u_j\partial_k^2 b_j\,dx$$
$$= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\,dx - \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_k\partial_i b_j\partial_k u_j\,dx,$$

where we have used that $\mathbf{b}$ is divergence free. Hence,

$$-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{u},\Delta_h\mathbf{b})_2 = \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\,dx$$
$$+\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\,dx \le C\Big(\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla_h\mathbf{u}|\,dx +$$
$$\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{u}||\nabla_h\mathbf{b}|\,dx\Big).$$

Furthermore,

$$(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2 = \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i w_j\partial_k^2 w_j\,dx$$
$$= -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\partial_i w_j\partial_k w_j\,dx - \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_k\partial_i w_j\partial_k w_j\,dx.$$

On the other hand, by analysing the last term above it is easy to prove that it is actually null. In fact, since $\nabla\cdot\mathbf{u}=0$, one has

$$-\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_k\partial_i w_j\partial_k w_j\,dx = \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_k w_j\partial_i\partial_k w_j\,dx.$$

Therefore,

$$(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2 \le C\int_{\mathbb{R}^3}|\nabla_h\mathbf{w}||\nabla\mathbf{w}||\nabla_h\mathbf{u}|\,dx.$$

Similarly, we obtain

$$(\mathbf{u}\cdot\nabla\mathbf{b},\Delta_h\mathbf{b})_2 \le C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla_h\mathbf{u}|\,dx.$$

Also notice that, by applying Cauchy-Schwarz's inequality, one has

$$-\chi(\nabla\times\mathbf{u},\Delta_h\mathbf{w})_2 - \chi(\nabla\times\mathbf{w},\Delta_h\mathbf{u})_2 \le \chi\|\nabla\nabla_h\mathbf{u}\|_2^2 + \chi\|\nabla_h\mathbf{w}\|_2^2. \tag{14}$$

By [22], the following estimate holds:

$$(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2 \leq C\int_{\mathbb{R}^3}|\nabla u_3||\nabla\mathbf{u}||\nabla_h\mathbf{u}|\,dx.$$

Consequently, from (13), we infer

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\mu\|\nabla\nabla_h\mathbf{u}\|_2^2+\gamma\|\nabla\nabla_h\mathbf{w}\|_2^2+\nu\|\nabla\nabla_h\mathbf{b}\|_2^2\\
&+\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2+\chi\|\nabla_h\mathbf{w}\|_2^2\\
&\leq C\int_{\mathbb{R}^3}|(\nabla u_3,\nabla_h\mathbf{w},\nabla_h\mathbf{b})||(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})||(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})|\,dx.
\end{aligned}$$

By applying Lemma 2.1, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\mu\|\nabla\nabla_h\mathbf{u}\|_2^2+\gamma\|\nabla\nabla_h\mathbf{w}\|_2^2+\nu\|\nabla\nabla_h\mathbf{b}\|_2^2\\
&+\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2+\chi\|\nabla_h\mathbf{w}\|_2^2\leq\\
&\leq C\|(\nabla u_3,\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^{\frac{1}{4}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^{\frac{1}{4}}\\
&\times\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^{\frac{3}{2}}.
\end{aligned}$$

By using Young's inequality, it follows that

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\alpha\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^2\\
&+2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2+2\chi\|\nabla_h\mathbf{w}\|_2^2\leq C\|(\nabla u_3,\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^4\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\\
&\times\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2,
\end{aligned}$$

where $\alpha=\min\{\mu,\nu,\gamma\}$. Now, by integrating over $[T^*-\tau,t]$ the inequality above (τ will be chosen a posteriori), we obtain

$$\begin{aligned}
&\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(t)\|_2^2+\alpha\int_{T^*-\tau}^t\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})(s)\|_2^2\,ds\\
&\leq C+C\int_{T^*-\tau}^t\|(\nabla u_3,\nabla_h\mathbf{w},\nabla_h\mathbf{b})(s)\|_2^4\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(s)\|_2\\
&\times\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(s)\|_2\,ds.
\end{aligned}\tag{15}$$

In order to estimate the term $\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(t)\|_2$ for all $t\in[T^*-\tau,T^*)$, we define

$$\begin{aligned}
I(t):=&\sup_{s\in[T^*-\tau,t]}\{\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(s)\|_2\}\\
&+\left(\int_{T^*-\tau}^t\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})(s)\|_2^2\,ds\right)^{\frac{1}{2}}
\end{aligned}\tag{16}$$

and

$$J(t):=\sup_{s\in[T^*-\tau,t]}\{\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(s)\|_2\}+\left(\int_{T^*-\tau}^t\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})(s)\|_2^2\,ds\right)^{\frac{1}{2}},\tag{17}$$

where $T^*-\tau\leq t<T^*$. First of all, let us establish a relationship between I and J. By (15), one has

$$I^2(t)\leq 2\sup_{s\in[T^*-\tau,t]}\{\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(s)\|_2^2\}$$

$$
\begin{aligned}
&+2\int_{T^*-\tau}^{t}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})(s)\|_2^2\,ds\\
&\leq C+CI(t)J(t)^{\frac{3}{4}}\int_{T^*-\tau}^{t}\|(\nabla u_3,\nabla_h\mathbf{w},\nabla_h\mathbf{b})(s)\|_2^{\frac{4}{2}}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(s)\|_2^{\frac{1}{4}}\,ds\\
&\leq C+CI(t)J(t)^{\frac{3}{4}}\int_0^T\|(\nabla u_3,\nabla_h\mathbf{w},\nabla_h\mathbf{b})(s)\|_2^{\frac{32}{7}}\,ds\\
&+CI(t)J(t)^{\frac{3}{4}}\int_0^T\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(s)\|_2^2\,ds\\
&\leq C+CI(t)J(t)^{\frac{3}{4}},
\end{aligned}
$$

where we have applied Young's inequality, (6) and (9). By using Young's inequality again, we get

$$I^2(t)\leq C+CJ^{\frac{3}{2}}(t)+\frac{1}{2}I^2(t),$$

or equivalently,

$$I(t)\leq C+CJ^{\frac{3}{4}}(t),\quad\forall\,t\in(T^*-\tau,T^*).\tag{18}$$

The inequality (18) is useful to prove that $\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(t)\|_2$ is bounded in the interval $[T^*-\tau,T^*)$. In order to establish this last statement, we start noting that the system (1) lets us conclude the following:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2+(\mu+\chi)\|\Delta\mathbf{u}\|_2^2=(\mathbf{u}\cdot\nabla\mathbf{u},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2-\chi(\nabla\times\mathbf{w},\Delta\mathbf{u})_2,$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2+\gamma\|\Delta\mathbf{w}\|_2^2+\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2+2\chi\|\nabla\mathbf{w}\|_2^2=(\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2\\
&-\chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2
\end{aligned}
$$

and also

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{b}\|_2^2+\nu\|\Delta\mathbf{b}\|_2^2=(\mathbf{u}\cdot\nabla\mathbf{b},\Delta\mathbf{b})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2,$$

where we used the fact that $\nabla\cdot\mathbf{u}=0$. Hence, by adding the three equalities above, one obtains

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2+(\mu+\chi)\|\Delta\mathbf{u}\|_2^2+\gamma\|\Delta\mathbf{w}\|_2^2+\nu\|\Delta\mathbf{b}\|_2^2+\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2\\
&+2\chi\|\nabla\mathbf{w}\|_2^2=(\mathbf{u}\cdot\nabla\mathbf{u},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2-\chi(\nabla\times\mathbf{w},\Delta\mathbf{u})_2\\
&+(\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2-\chi(\nabla\times\mathbf{u},\Delta\mathbf{w})_2+(\mathbf{u}\cdot\nabla\mathbf{b},\Delta\mathbf{b})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2.
\end{aligned}\tag{19}
$$

Let us examine all the terms on the right hand side of the equality above. We have

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2&=\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}u_i\partial_iw_j\partial_k^2w_j\,dx\\
&=\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_3\partial_3w_j\partial_k^2w_j\,dx+\sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}u_i\partial_iw_j\partial_k^2w_j\,dx\\
&\quad+\sum_{j=1}^{3}\int_{\mathbb{R}^3}u_3\partial_3w_j\partial_3^2w_j\,dx\\
&=:I_1(t)+I_2(t)+I_3(t).
\end{aligned}\tag{20}
$$

Here

$$
\begin{aligned}
I_1(t) &= \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_3 w_j \partial_k^2 w_j\,dx \\
&= -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3\partial_3 w_j \partial_k w_j\,dx - \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_k\partial_3 w_j \partial_k w_j\,dx \\
&= -\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_3\partial_3 w_j \partial_k w_j\,dx + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_3 u_3(\partial_k w_j)^2\,dx \\
&\le C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla \mathbf{w}||\nabla_h \mathbf{w}|\,dx + C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla_h \mathbf{w}|^2\,dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2(t) &= \sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i w_j \partial_k^2 w_j\,dx \\
&= -\sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3} \partial_k u_i\partial_i w_j \partial_k w_j\,dx - \sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3} u_i\partial_k\partial_i w_j \partial_k w_j\,dx \\
&= -\sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3} \partial_k u_i\partial_i w_j \partial_k w_j\,dx + \frac{1}{2}\sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3} \partial_i u_i(\partial_k w_j)^2\,dx \\
&\le C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla_h \mathbf{w}||\nabla \mathbf{w}|\,dx + C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{w}|^2\,dx.
\end{aligned}
$$

By using that $\mathbf{u}$ is divergence free, one also has

$$
\begin{aligned}
I_3(t) &= \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3\partial_3 w_j \partial_3^2 w_j\,dx \\
&= -\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 u_3(\partial_3 w_j)^2\,dx \\
&= \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} \partial_k u_k(\partial_3 w_j)^2\,dx \\
&\le C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{w}|^2\,dx.
\end{aligned}
$$

Therefore, using the above estimates, the equality (20) yields

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla\mathbf{w}, \Delta\mathbf{w})_2 \le C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla_h \mathbf{w}|^2\,dx + C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla_h \mathbf{w}||\nabla \mathbf{w}|\,dx \\
&+C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{w}|^2\,dx.
\end{aligned}
$$

Following the same process, we conclude that

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla\mathbf{b}, \Delta\mathbf{b})_2 \le C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla_h \mathbf{b}|^2\,dx + C\int_{\mathbb{R}^3} |\nabla \mathbf{u}||\nabla_h \mathbf{b}||\nabla \mathbf{b}|\,dx \\
&+C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{b}|^2\,dx.
\end{aligned}
$$

It is important to point out that the technique applied to $(\mathbf{u}\cdot\nabla\mathbf{w},\Delta\mathbf{w})_2$ may not be useful when we consider the terms $-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2$ and $-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2$ (such expressions were obtained in (19)). However, we can argue as follows.

$$\begin{aligned}
&-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2=-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i\partial_i b_j\partial_k^2 u_j\,dx\\
&-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i\partial_i u_j\partial_k^2 b_j\,dx=\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\,dx+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i\partial_k\partial_i b_j\partial_k u_j\,dx\\
&+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\,dx+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i\partial_k\partial_i u_j\partial_k b_j\,dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2=\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\,dx\\
&+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i\partial_k\partial_i b_j\partial_k u_j\,dx+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\,dx\\
&-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_i\partial_k\partial_i b_j\partial_k u_j\,dx.
\end{aligned}$$

By using $\nabla\cdot\mathbf{b}=0$, we have

$$\begin{aligned}
&-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta\mathbf{u})_2-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta\mathbf{b})_2=\sum_{i,j,k=1}^{3}\Big[\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\,dx\\
&+\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\,dx\Big] \qquad (21)\\
&=\sum_{j=1}^{3}\sum_{k=1}^{2}\Big[\int_{\mathbb{R}^3}\partial_k b_3\partial_3 b_j\partial_k u_j\,dx+\int_{\mathbb{R}^3}\partial_k b_3\partial_3 u_j\partial_k b_j\,dx\Big]\\
&\quad+\sum_{j,k=1}^{3}\sum_{i=1}^{2}\Big[\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\,dx+\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\,dx\Big]\\
&\quad+\sum_{j=1}^{3}\Big[\int_{\mathbb{R}^3}\partial_3 b_3\partial_3 b_j\partial_3 u_j\,dx+\int_{\mathbb{R}^3}\partial_3 b_3\partial_3 u_j\partial_3 b_j\,dx\Big]\\
&=: J_1(t)+J_2(t)+J_3(t). \qquad (22)
\end{aligned}$$

Let us estimate each term $J_i(t)$ for $i=1,2,3$. Hence,

$$\begin{aligned}
J_1(t)&=\sum_{j=1}^{3}\sum_{k=1}^{2}\Big[\int_{\mathbb{R}^3}\partial_k b_3\partial_3 b_j\partial_k u_j\,dx+\int_{\mathbb{R}^3}\partial_k b_3\partial_3 u_j\partial_k b_j\,dx\Big]\\
&\le C\int_{\mathbb{R}^3}|\nabla_h\mathbf{b}||\nabla\mathbf{b}||\nabla\mathbf{u}|\,dx.
\end{aligned}$$

Similarly, one obtains

$$J_2(t) = \sum_{j,k=1}^{3}\sum_{i=1}^{2}\left[\int_{\mathbb{R}^3} \partial_k b_i \partial_i b_j \partial_k u_j \, dx + \int_{\mathbb{R}^3} \partial_k b_i \partial_i u_j \partial_k b_j \, dx\right]$$
$$\leq C\int_{\mathbb{R}^3} |\nabla \mathbf{b}||\nabla_h \mathbf{b}||\nabla \mathbf{u}| \, dx + C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{b}|^2 \, dx$$

and, by applying $\nabla \cdot \mathbf{b} = 0$, we have

$$J_3(t) = \sum_{j=1}^{3}\left[\int_{\mathbb{R}^3} \partial_3 b_3 \partial_3 b_j \partial_3 u_j \, dx + \int_{\mathbb{R}^3} \partial_3 b_3 \partial_3 u_j \partial_3 b_j \, dx\right]$$
$$= -\sum_{j=1}^{3}\sum_{k=1}^{2}\left[\int_{\mathbb{R}^3} \partial_k b_k \partial_3 b_j \partial_3 u_j \, dx + \int_{\mathbb{R}^3} \partial_k b_k \partial_3 u_j \partial_3 b_j \, dx\right]$$
$$\leq C\int_{\mathbb{R}^3} |\nabla_h \mathbf{b}||\nabla \mathbf{b}||\nabla \mathbf{u}| \, dx.$$

Replacing, in (21), the estimates obtained for $J_i(t)$, $i = 1, 2, 3$, we get

$$-(\mathbf{b}\cdot\nabla\mathbf{b}, \Delta\mathbf{u})_2 - (\mathbf{b}\cdot\nabla\mathbf{u}, \Delta\mathbf{b})_2 \leq C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{b}|^2 \, dx + C\int_{\mathbb{R}^3} |\nabla_h \mathbf{b}||\nabla \mathbf{b}||\nabla \mathbf{u}| \, dx.$$

Furthermore, notice that

$$-\chi(\nabla\times\mathbf{w}, \Delta\mathbf{u})_2 - \chi(\nabla\times\mathbf{u}, \Delta\mathbf{w})_2 \leq \chi\|\nabla\mathbf{w}\|_2^2 + \chi\|\Delta\mathbf{u}\|_2^2,$$

where we have applied Cauchy-Schwarz's inequality. At last, Y. Zhou and M. Pokorný [29] proved that

$$(\mathbf{u}\cdot\nabla\mathbf{u}, \Delta\mathbf{u})_2 \leq C\int_{\mathbb{R}^3} |\nabla_h \mathbf{u}||\nabla \mathbf{u}|^2 \, dx.$$

Therefore, (19) reads

$$\frac{1}{2}\frac{d}{dt}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2 + \mu\|\Delta\mathbf{u}\|_2^2 + \gamma\|\Delta\mathbf{w}\|_2^2 + \nu\|\Delta\mathbf{b}\|_2^2 + \kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2$$
$$+\chi\|\nabla\mathbf{w}\|_2^2 \leq C\int_{\mathbb{R}^3} |(\nabla_h\mathbf{u}, \nabla_h\mathbf{w}, \nabla_h\mathbf{b})||(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})|^2 \, dx. \tag{23}$$

By using Lemma 2.1, one gets

$$\frac{d}{dt}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^2 + 2\alpha\|(\Delta\mathbf{u}, \Delta\mathbf{w}, \Delta\mathbf{b})\|_2^2 + 2\kappa\|\nabla(\nabla\cdot\mathbf{w})\|_2^2 + 2\chi\|\nabla\mathbf{w}\|_2^2$$
$$\leq C\|(\nabla_h\mathbf{u}, \nabla_h\mathbf{w}, \nabla_h\mathbf{b})\|_2\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})\|_2^{\frac{1}{2}}\|(\nabla\nabla_h\mathbf{u}, \nabla\nabla_h\mathbf{w}, \nabla\nabla_h\mathbf{b})\|_2$$
$$\times\|(\Delta\mathbf{u}, \Delta\mathbf{w}, \Delta\mathbf{b})\|_2^{\frac{1}{2}},$$

where $\alpha = \min\{\mu, \gamma, \nu\}$. Now, by integrating over $[T^*-\tau, s]$, $s \leq t$, the inequality above yields

$$\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(s)\|_2^2 + 2\alpha\int_{T^*-\tau}^{s}\|(\Delta\mathbf{u}, \Delta\mathbf{w}, \Delta\mathbf{b})(\tau)\|_2^2 \, d\tau \leq C$$
$$+CI(t)\int_{T^*-\tau}^{s}\|(\nabla\mathbf{u}, \nabla\mathbf{w}, \nabla\mathbf{b})(\tau)\|_2^{\frac{1}{2}}\|(\nabla\nabla_h\mathbf{u}, \nabla\nabla_h\mathbf{w}, \nabla\nabla_h\mathbf{b})(\tau)\|_2$$
$$\times\|(\Delta\mathbf{u}, \Delta\mathbf{w}, \Delta\mathbf{b})(\tau)\|_2^{\frac{1}{2}} \, d\tau,$$

where we applied the definition of $I(t)$ given in (16). By Hölder's inequality, we have

$$\begin{aligned}&\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(s)\|_2^2+2\alpha\int_{T^*-\tau}^{s}\|(\Delta\mathbf{u},\Delta\mathbf{w},\Delta\mathbf{b})(\tau)\|_2^2\,d\tau\\&\leq C+CI^2(t)J^{\frac{1}{2}}(t)\left(\int_{T^*-\tau}^{s}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\tau)\|_2^2\,d\tau\right)^{\frac{1}{4}},\end{aligned}$$

for all $s\leq t$. In the inequality above the definitions of $I(t)$ and $J(t)$ were applied (see (16) and (17)). It follows, by using (17), that

$$J^2(t)\leq C+CI^2(t)J^{\frac{1}{2}}(t)\left(\int_{T^*-\tau}^{t}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\tau)\|_2^2\,d\tau\right)^{\frac{1}{4}}.$$

By using Young's inequality, we infer

$$J^2(t)\leq C+CI^{\frac{8}{3}}(t)\left(\int_{T^*-\tau}^{t}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\tau)\|_2^2\,d\tau\right)^{\frac{1}{3}}+\frac{1}{2}J^2(t).$$

Consequently, by applying (18), we obtain

$$J(t)\leq C+[C+CJ(t)]\left(\int_{T^*-\tau}^{t}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\tau)\|_2^2\,d\tau\right)^{\frac{1}{6}}. \tag{24}$$

From the energy inequality (9), one concludes that there exists $0<\tau\ll 1$ such that

$$\int_{T^*-\tau}^{T}\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\tau)\|_2^2\,d\tau\leq\frac{1}{(2C)^6}.$$

Now, we can obtain the desired estimate for $\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(t)\|$ in $[T^*-\tau,T^*)$. In fact, by replacing the bound above in (24), we get

$$J(t)\leq C,\quad\forall\, t\in[T^*-\tau,T^*).$$

The definition (17) establishes the proof of Theorem 1.1.

3.2. **Proof of Theorem 1.2.** In order to prove Theorem 1.2 let us examine all the terms on the right hand side of (13) in an alternative way. We have

$$\begin{aligned}(\mathbf{u}\cdot\nabla\mathbf{w},\Delta_h\mathbf{w})_2&=\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}u_i\partial_iw_j\partial_k^2w_j\,dx\\&=-\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_ku_i\partial_iw_j\partial_kw_j\,dx\\&=\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}w_j\partial_ku_i\partial_k\partial_iw_j\,dx\\&\leq C\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{w}|\,dx,\end{aligned}$$

since $\nabla\cdot\mathbf{u}=0$. By arguing in the same way, one gets

$$(\mathbf{u}\cdot\nabla\mathbf{b},\Delta_h\mathbf{b})_2\leq C\int_{\mathbb{R}^3}|\mathbf{b}||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{b}|\,dx.$$

Notice that

$$-(\mathbf{b}\cdot\nabla\mathbf{b},\Delta_h\mathbf{u})_2 = -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i b_j\partial_k^2 u_j\,dx \leq C\int_{\mathbb{R}^3}|\mathbf{b}||\nabla\mathbf{b}||\nabla\nabla_h\mathbf{u}|\,dx$$

and also

$$-(\mathbf{b}\cdot\nabla\mathbf{u},\Delta_h\mathbf{b})_2 = -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i u_j\partial_k^2 b_j\,dx \leq C\int_{\mathbb{R}^3}|\mathbf{b}||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{b}|\,dx.$$

The reader might check that (14) assures the following estimate:

$$-\chi(\nabla\times\mathbf{u},\Delta_h\mathbf{w})_2-\chi(\nabla\times\mathbf{w},\Delta_h\mathbf{u})_2\leq\chi\|\nabla\nabla_h\mathbf{u}\|_2^2+\chi\|\nabla_h\mathbf{w}\|_2^2.$$

At last, Y. Zhou and M. Pokorný [29] proved that

$$(\mathbf{u}\cdot\nabla\mathbf{u},\Delta_h\mathbf{u})_2\leq C\int_{\mathbb{R}^3}|u_3||\nabla\mathbf{u}||\nabla\nabla_h\mathbf{u}|\,dx.$$

By replacing all these last results obtained above in (13) and by using Young's inequality, one has

$$\frac{1}{2}\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\alpha\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^2+\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 +\chi\|\nabla_h\mathbf{w}\|_2^2\leq C\int_{\mathbb{R}^3}|(u_3,\mathbf{w},\mathbf{b})|^2|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})|^2\,dx +\frac{\alpha}{2}\int_{\mathbb{R}^3}|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})|^2\,dx,$$

where $\alpha=\min\{\mu,\gamma,\nu\}$. Hence,

$$\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\alpha\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^2+2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 +2\chi\|\nabla_h\mathbf{w}\|_2^2\leq C\int_{\mathbb{R}^3}|(u_3,\mathbf{w},\mathbf{b})|^2|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})|^2\,dx.$$

By Lemmas 2.2 and 2.3, and also by (8), we obtain

$$\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\alpha\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^2+2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 +2\chi\|\nabla_h\mathbf{w}\|_2^2\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2 \times\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2.$$

By Young's inequality, one concludes

$$\frac{d}{dt}\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2+\frac{\alpha}{2}\|(\nabla\nabla_h\mathbf{u},\nabla\nabla_h\mathbf{w},\nabla\nabla_h\mathbf{b})\|_2^2+2\kappa\|\nabla_h(\nabla\cdot\mathbf{w})\|_2^2 +2\chi\|\nabla_h\mathbf{w}\|_2^2\leq C\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})\|_2^2\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})\|_2^2.$$

By applying Gronwall's inequality, we get

$$\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(t)\|_2\leq\|(\nabla_h\mathbf{u},\nabla_h\mathbf{w},\nabla_h\mathbf{b})(\delta)\|_2 \times\exp\left\{C\int_\delta^T\|(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(s)\|_2^2\,ds\right\},$$

for all $t \in [\delta, T^*)$. By energy inequality (9), we can guarantee the following estimate:

$$\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})(t)\|_2 \leq C, \quad \forall\, t \in [\delta, T^*). \tag{25}$$

In order to prove that the term $\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(t)\|_2$ is bounded in $[\delta, T^*)$, we recall that (23) and Hölder's inequality imply

$$\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + 2\alpha\|(\Delta \mathbf{u}, \Delta \mathbf{w}, \Delta \mathbf{b})\|_2^2 + 2\kappa\|\nabla(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla \mathbf{w}\|_2^2$$
$$\leq C\|(\nabla_h \mathbf{u}, \nabla_h \mathbf{w}, \nabla_h \mathbf{b})\|_2\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_4^2,$$

where $\alpha = \min\{\mu, \gamma, \nu\}$. By using (25) and Gagliardo-Nirenberg's inequality, one has

$$\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + 2\alpha\|(\Delta \mathbf{u}, \Delta \mathbf{w}, \Delta \mathbf{b})\|_2^2 + 2\kappa\|\nabla(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla \mathbf{w}\|_2^2$$
$$\leq C\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^{\frac{1}{2}}\|(\Delta \mathbf{u}, \Delta \mathbf{w}, \Delta \mathbf{b})\|_2^{\frac{3}{2}},$$

for all $t \in [\delta, T^*)$. By Young's inequality, we infer

$$\frac{d}{dt}\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2 + \alpha\|(\Delta \mathbf{u}, \Delta \mathbf{w}, \Delta \mathbf{b})\|_2^2 + 2\kappa\|\nabla(\nabla \cdot \mathbf{w})\|_2^2 + 2\chi\|\nabla \mathbf{w}\|_2^2$$
$$\leq C\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})\|_2^2.$$

By Gronwall's inequality,

$$\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(t)\|_2 \leq C\|(\nabla \mathbf{u}, \nabla \mathbf{w}, \nabla \mathbf{b})(\delta)\|_2, \quad \forall\, t \in [\delta, T^*).$$

This completes the proof of Theorem 1.2.

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