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# GORENSTEIN GLOBAL DIMENSIONS RELATIVE TO BALANCED PAIRS

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Abstract. Let $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$ be Gorenstein subcategories induced by an admissible balanced pair $(\mathcal{X},\mathcal{Y})$ in an abelian category $\mathcal{A}$. In this paper, we establish Gorenstein homological dimensions in terms of these two subcategories and investigate the Gorenstein global dimensions of $\mathcal{A}$ induced by the balanced pair $(\mathcal{X},\mathcal{Y})$. As a consequence, we give some new characterizations of pure global dimensions and Gorenstein global dimensions of a ring $R$.

1. **Introduction.** Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a contravariantly finite subcateogry of $\mathcal{A}$. This means that for each object $M \in \mathcal{A}$ there exists a (not necessarily exact) complex $\cdots \to X_1 \to X_0 \to M \to 0$, where $X_i \in \mathcal{X}$ for every $i \geq 0$, which is exact after applying the functor $\mathrm{Hom}_{\mathcal{A}}(X,-)$ for each $X \in \mathcal{X}$. Such a complex is called an $\mathcal{X}$-*resolution* of $M$ and denoted by $X^\bullet \rightarrow M$, where $X^\bullet$ is the corresponding deleted complex. Since $X^\bullet$ is unique up to homotopy, so we can compute right derived functors of Hom, denoted by $\mathrm{Ext}^n_{\mathcal{X}}$. In many cases there is "balance" in the computation of such functors, meaning that there exists a convariantly finite subcateogry $\mathcal{Y}$ of $\mathcal{A}$ such that $\mathrm{Ext}^n_{\mathcal{X}}(M,N)$ can be also obtained from the right derived functors $\mathrm{Ext}^n_{\mathcal{Y}}$ computed from a $\mathcal{Y}$-*coresolution* of $N$, $0 \to N \to Y_0 \to Y_1 \to \cdots$, where $Y_i \in \mathcal{Y}$ for every $i \geq 0$. This phenomenon can be summarized by saying that the pair $(\mathcal{X},\mathcal{Y})$ is a balanced pair (in the sense of Chen [3]) or equivalently that the functor Hom is right balanced by $\mathcal{X} \times \mathcal{Y}$ (see Enochs and Jenda [5, Section 8.2]). In this case, we donote by $\mathrm{Ext}^i_*(M,N)$ the abelian groups $\mathrm{Ext}^i_{\mathcal{X}}(M,N) \cong \mathrm{Ext}^i_{\mathcal{Y}}(M,N)$ (see [5]). Let $\mathcal{P}$ and $\mathcal{I}$ be classes of projective and injective left $R$-modules, respectively. Then the pair $(\mathcal{P},\mathcal{I})$ is a classical example of balanced pairs. Balanced pairs have gained attention in the last years and it is very useful in relative homological algebra because balanced pairs are very closely related to resolutions, triangle-equivalences, cotorsion pairs and recollements (see for instance [3, 5, 6, 12, 18]).

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Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Sather-Wagstaff, Sharif and White introduced in [13] the Gorenstein subcategory $\mathcal{G}(\mathcal{X})$, which unifies the following notions: modules of G-dimension zero, Gorenstein projective modules, Gorenstein injective modules and so on. The Gorenstein subcategory $\mathcal{G}(\mathcal{X})$ of $\mathcal{A}$ is defined as $\mathcal{G}(\mathcal{X}) = \{M \in \mathcal{A} \mid$ there exists an exact sequence $X^\bullet = \cdots \to X_1 \to X_0 \to X_{-1} \to \cdots$ in $\mathcal{X}$, which is both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact, such that $M \cong \ker(X_0 \to X_{-1})\}$. Such a complex $X^\bullet$ is called a *complete $\mathcal{X}$-resolution* of $M$. Recently, Gorenstein subcategories have been extensively studied by many authors, see [2, 11, 13, 16] for instance.

Let $(\mathcal{X}, \mathcal{Y})$ be an admissible balanced pair of abelian category $\mathcal{A}$. Then we have two Gorenstein subcategories: $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$. We can establish Gorenstein homological dimensions in terms of these categories: the $\mathcal{G}(\mathcal{X})$ *dimension* of an object $A \in \mathcal{A}$, denoted by $\mathcal{G}(\mathcal{X})$-dim$A$, is the minimal integer $n \geqslant 0$ such that there is an $\mathcal{X}$-resolution $0 \twoheadrightarrow G_n \twoheadrightarrow \cdots \twoheadrightarrow G_0 \twoheadrightarrow A \twoheadrightarrow 0$ with each $G_i \in \mathcal{G}(\mathcal{X})$. If there is no such an integer, set $\mathcal{G}(\mathcal{X})$-dim$A = \infty$. Dually, we can define the notion of $\mathcal{G}(\mathcal{Y})$ *codimension* of an object $A \in \mathcal{A}$, denoted by $\mathcal{G}(\mathcal{Y})$-codim$A$. If $\mathcal{A}$ is the category of left $R$-modules and $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}, \mathcal{I})$, then $\mathcal{G}(\mathcal{X})$ dimension and $\mathcal{G}(\mathcal{Y})$ codimension are exactly Gorenstein projective dimension and Gorenstein injective dimension defined by Holm in [10], respectively. Denote by silp $R$ (respectively, spli $R$) the supremum of injective (respectively, projective) dimensions of the projective (respectively, injective) modules (see [7]). Let Mod $R$ the category of left $R$-modules. The following is well known.

**Theorem 1.1.** *(see* [4, Theorem 4.1]*) Let $R$ be a ring. Then the following are equivalent for any nonnegative integer $n$:*

*(1) For any $R$-module $M$, $\mathcal{G}(\mathcal{P})$-dim$M \leqslant n$;*

*(2) For any $R$-module $N$, $\mathcal{G}(\mathcal{I})$-codim$N \leqslant n$;*

*(3)* silp $R =$ spli $R \leqslant n$.

*Moreover,*

$$\sup\{\mathcal{G}(\mathcal{P})\text{-dim}M \mid M \in \mathrm{Mod}\ R\} = \sup\{\mathcal{G}(\mathcal{I})\text{-codim}M \mid M \in \mathrm{Mod}\ R\}.$$

*In this case, we say that $R$ has finite left Gorenstein global dimension and define the common value of these two numbers to be its left Gorenstein global dimension.*

The main goal of this paper is to generalize Theorem 1.1 to Gorenstein subcategories $\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(\mathcal{Y})$ induced by any admissible balanced pair $(\mathcal{X}, \mathcal{Y})$ over an abelian category $\mathcal{A}$, without assuming that $\mathcal{A}$ has projective or injective objects. The following is the main result of this paper:

**Theorem 1.2.** *Let $(\mathcal{X}, \mathcal{Y})$ be an admissible balanced pair over an abelian category $\mathcal{A}$. Then the following are equivalent for any nonnegative integer $n$:*

*(1) For any object $A$ in $\mathcal{A}$, $\mathcal{G}(\mathcal{X})$-dim$A \leqslant n$.*

*(2) For any object $A$ in $\mathcal{A}$, $\mathcal{G}(\mathcal{Y})$-codim$A \leqslant n$.*

*(3)* $\sup\{\mathcal{Y}\text{-cores.dim}P \mid P \in \mathcal{X}\} = \sup\{\mathcal{X}\text{-res.dim}I \mid I \in \mathcal{Y}\} \leqslant n$.

*Moreover,*

$$\sup\{\mathcal{G}(\mathcal{X})\text{-dim}M \mid M \in \mathcal{A}\} = \sup\{\mathcal{G}(\mathcal{Y})\text{-codim}M \mid M \in \mathcal{A}\}.$$

The common value of the last equality is called the Gorenstein global dimension of the abelian category $\mathcal{A}$ relative to $(\mathcal{X}, \mathcal{Y})$. What interests us is that if we consider the balanced pair induced by pure projective and pure injective modules over a ring $R$, then the induced Gorenstein global dimension is exactly the pure global dimension (see Corollary 2.8). If we consider the balanced pair induced by Gorenstein projective and Gorenstein injective modules over a Ding-Chen ring, then the induced Gorenstein global dimension is exactly the Gorenstein global dimension (see Corollary 2.9).

The proof of the above results will be carried out in the next section.

## 2. The proof of the main theorem.

Throughout this section, we always assume that $\mathcal{A}$ is an abelian category (not necessarily with projective objects or injective objects). Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ which is closed under taking direct summands. Let $M \in \mathcal{A}$. A morphism $f : X \to M$ is called a *right $\mathcal{X}$-approximation* of $M$, if $X \in \mathcal{X}$ and any morphism $g : X' \to M$ with $X' \in \mathcal{X}$ factors through $f$. The subcategory $\mathcal{X}$ is called *contravariantly finite* if each object in $\mathcal{A}$ has a right $\mathcal{X}$-approximation. Dually one has the notion of *left $\mathcal{Y}$-approximation* and then the notion of *covariantly finite subcategories*. We start by recalling the definition of balanced pairs.

**Definition 2.1.** (see [3, Definition 1.1]) A pair $(\mathcal{X}, \mathcal{Y})$ of additive subcategories in $\mathcal{A}$ is called a *balanced pair* if the following conditions are satisfied:

(BP0) the subcategory $\mathcal{X}$ is contravariantly finite and $\mathcal{Y}$ is covariantly finite;

(BP1) for each object $M$, there is an $\mathcal{X}$-resolution $X^\bullet \to M$ such that it is acyclic by applying the functors $\mathrm{Hom}_\mathcal{A}(-, Y)$ for all $Y \in \mathcal{Y}$;

(BP2) for each object $N$, there is a $\mathcal{Y}$-coresolution $N \to Y^\bullet$ such that it is acyclic by applying the functors $\mathrm{Hom}_\mathcal{A}(X, -)$ for all $X \in \mathcal{X}$.

We say that a contravariantly finite subcategory $\mathcal{X} \subseteq \mathcal{A}$ is *admissible* if each right $\mathcal{X}$-appoximation is epic. Dually one has the notion of *coadmissible* covariantly finite subcategory. It turns out that $\mathcal{X}$ is admissible if and only if $\mathcal{Y}$ is coadmissible for a balanced pair $(\mathcal{X}, \mathcal{Y})$ (see [3, Corollary 2.3]). In this case, we say the balanced pair is *admissible*. **In what follows, we always assume that $(\mathcal{X}, \mathcal{Y})$ is an admissible balanced pair in $\mathcal{A}$.**

**Lemma 2.2.** *If the short exact sequence $0 \to X \to Y \to Z \to 0$ is $\mathrm{Hom}_\mathcal{A}(\mathcal{X}, -)$-exact, then the following hold:*

*(1) If $Z \in \mathcal{G}(\mathcal{X})$, then $X \in \mathcal{G}(\mathcal{X})$ if and only if $Y \in \mathcal{G}(\mathcal{X})$. Moreover, $\mathcal{G}(\mathcal{X})$ is closed under direct summands.*

*(2) If $X, Y \in \mathcal{G}(\mathcal{X})$, then $Z \in \mathcal{G}(\mathcal{X})$ if and only if $\mathrm{Ext}^1_*(Z, Q) = 0$ for any $Q \in \mathcal{X}$.*

*(3) If $X \in \mathcal{G}(\mathcal{Y})$, then $Y \in \mathcal{G}(\mathcal{Y})$ if and only if $Z \in \mathcal{G}(\mathcal{Y})$. Moreover, $\mathcal{G}(\mathcal{Y})$ is closed under direct summands.*

*(4) If $Y, Z \in \mathcal{G}(\mathcal{Y})$, then $X \in \mathcal{G}(\mathcal{Y})$ if and only if $\mathrm{Ext}^1_*(I, Z) = 0$ for any $I \in \mathcal{Y}$.*

*Proof.* We just prove (1) and (2) since (3) and (4) follow by duality.

(1) The first statement follows from [1, Proposition 2.13(1)]. One can prove that $\mathcal{G}(\mathcal{X})$ is closed under direct summands by the proof similar to that of [17, Theorem 2.5].

(2) The "only if" part is clear. For the "if" part, since $X \in \mathcal{G}(\mathcal{X})$, there exists a $\mathrm{Hom}_\mathcal{A}(\mathcal{X}, -)$-exact short exact sequence $0 \to X \to P \to K \to 0$ with $P \in \mathcal{X}$ and $K \in \mathcal{G}(\mathcal{X})$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & G & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

By [3, Propsotion 2.2], all rows and columns are $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact. It follows from (1) that $G \in \mathcal{X}$ since $Y, K \in \mathcal{X}$. By assumption $\mathrm{Ext}^1_*(Z,P)=0$, we know that the middle row in the above diagram splits. So $Z \in \mathcal{G}(\mathcal{X})$ by (1). $\square$

Recall that the $\mathcal{X}$*-resolution dimension* $\mathcal{X}$-res.dim$A$ of an object $A$ is defined to be the minimal integer $n \geqslant 0$ such that there is an $\mathcal{X}$-resolution

$$0 \to X_n \to \cdots \to X_0 \to A \to 0.$$

If there is no such an integer, set $\mathcal{X}$-res.dim$A = \infty$. Let $\mathcal{Y} \subseteq \mathcal{A}$ be another full subcategory which is closed under taking direct summands. Dually one has the notion of $\mathcal{Y}$*-coresolution dimension* $\mathcal{Y}$-cores.dim$B$ of an object $B$. For details, see [5, 8.4]. We let $\widetilde{\mathcal{X}}$ (respectively, $\widetilde{\mathcal{Y}}$) to denote the full subcategory of $\mathcal{A}$ whose objects are of finite $\mathcal{X}$-resolution (respectively, $\mathcal{Y}$-coresolution) dimension.

**Lemma 2.3.** *The following are true for any object $A \in \mathcal{A}$:*
*(1) If $A \in \widetilde{\mathcal{X}}$ or $A \in \widetilde{\mathcal{Y}}$, then $\mathrm{Ext}^i_*(G,A)=0$ for any $i \geqslant 1$ and $G \in \mathcal{G}(\mathcal{X})$.*
*(2) If $A \in \widetilde{\mathcal{X}}$ or $A \in \widetilde{\mathcal{Y}}$, then $\mathrm{Ext}^i_*(A,H)=0$ for any $i \geqslant 1$ and $H \in \mathcal{G}(\mathcal{Y})$.*

*Proof.* We just prove (1) since (2) follows by duality. If $G \in \mathcal{G}(\mathcal{X})$, then it is easy to see that $\mathrm{Ext}^i_*(G,P)=0$ for any $P \in \mathcal{X}$ and all $i \geqslant 1$. If $A \in \widetilde{\mathcal{X}}$, then there exists an $\mathcal{X}$-resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

with all $P_i \in \mathcal{X}$. Hence $\mathrm{Ext}^i_*(G,A) \cong \mathrm{Ext}^{i+n}_*(G,P_n)=0$ for any integer $i \geqslant 1$. Since $G \in \mathcal{G}(\mathcal{X})$, there exist a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact sequence

$$0 \to K_{-j} \to Q_{-j-1} \to K_{-j-1} \to 0$$

where $Q_{-j-1} \in \mathcal{X}$ and $K_{-j} \in \mathcal{G}(\mathcal{X})$ for any integer $j \geqslant 0$, here $K_0 = G$. Therefore $\mathrm{Ext}^i_*(G,A) \cong \mathrm{Ext}^{i+1}_*(K_{-1},A) \cong \cdots \cong \mathrm{Ext}^{i+n}_*(K_{-n},A) \cong \cdots$. If $A \in \widetilde{\mathcal{Y}}$, it is easy to see that $\mathrm{Ext}^i_*(G,A)=0$ for any integer $i \geqslant 1$, as desired. $\square$

We let $\widetilde{\mathcal{G}}(\mathcal{X})$ (respectively, $\widetilde{\mathcal{G}}(\mathcal{Y})$) to denote the full subcategory of $\mathcal{A}$ whose objects are of finite $\mathcal{G}(\mathcal{X})$ (respectively, $\mathcal{G}(\mathcal{Y})$) (co)dimension.

**Proposition 2.4.** *The following are true for any $0 \neq A \in \mathcal{A}$:*
*(1) If $A \in \widetilde{\mathcal{G}}(\mathcal{X})$, then the following are equivalent:*
*(i) $\mathcal{G}(\mathcal{X})$-dim$A \leqslant n$;*

(*ii*) *For any* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact complex* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to A \to 0$ *with all* $G_i \in \mathcal{G}(\mathcal{X})$*, then so is* $K_n$*;*

(*iii*) $\mathrm{Ext}^i_*(A,Q)=0$ *for all* $i>n$ *and all* $Q \in \widetilde{\mathcal{X}}$*;*

(*iv*) $\mathrm{Ext}^i_*(A,Q)=0$ *for all* $i>n$ *and all* $Q \in \mathcal{X}$*.*

*In this case,* $\mathcal{G}(\mathcal{X})\text{-dim}A = \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_*(A,Q) \neq 0 \text{ for some } Q \in \widetilde{\mathcal{X}}\}$.

*(2) If* $A \in \widetilde{\mathcal{G}}(\mathcal{Y})$*, then the following are equivalent:*

(*i*) $\mathcal{G}(\mathcal{Y})\text{-codim}A \leqslant n$*;*

(*ii*) *For any* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact complex* $0 \to A \to G^0 \to \cdots \to G^{n-1} \to K^n \to 0$ *with all* $G^i \in \mathcal{G}(\mathcal{Y})$*, then so is* $K^n$*;*

(*iii*) $\mathrm{Ext}^i_*(I,A)=0$ *for all* $i>n$ *and all* $I \in \widetilde{\mathcal{Y}}$*;*

(*iv*) $\mathrm{Ext}^i_*(I,A)=0$ *for all* $i>n$ *and all* $I \in \mathcal{Y}$*;*

*In this case,* $\mathcal{G}(\mathcal{Y})\text{-codim}A = \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_*(I,A) \neq 0 \text{ for some } I \in \widetilde{\mathcal{Y}}\}$.

*Proof.* We just prove (1) since (2) follows by duality.

(*i*) $\Rightarrow$ (*ii*) It is obvious by a argument similar to that of [10, Proposition 2.7].

(*ii*) $\Rightarrow$ (*iii*) Note that $(\mathcal{X},\mathcal{Y})$ is a balanced pair. Then there exists a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact complex $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ with all $P_i \in \mathcal{X}$. Hence $K_n \in \mathcal{G}(\mathcal{X})$ by (ii) and $\mathrm{Ext}^i_*(A,Q) \cong \mathrm{Ext}^{i-n}_*(K_n,Q)=0$ for all $i>n$ and all $Q \in \widetilde{\mathcal{X}}$ by Lemma 2.3(1).

(*iii*) $\Rightarrow$ (*iv*) This is obvious.

(*iv*) $\Rightarrow$ (*i*) Assume that $\mathcal{G}(\mathcal{X})\text{-dim}A = m > n$. Since $(\mathcal{X},\mathcal{Y})$ is a balanced pair, there exists a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact sequence $0 \to K_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$ with all $P_i \in \mathcal{X}$ and $K_m \in \mathcal{G}(\mathcal{X})$ by hypothesis. Then there is a short exact sequence $0 \to K_j \to P_{j-1} \to K_{j-1} \to 0$ which is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact for each $1 \leqslant j \leqslant m$ with $K_0 = A$. Hence $\mathrm{Ext}^1_*(K_{m-1},Q) \cong \mathrm{Ext}^m_*(A,Q)=0$ for any $Q \in \mathcal{X}$ by (iv). Consider the $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact sequence $0 \to K_m \to P_{m-1} \to K_{m-1} \to 0$ where $K_m, P_{m-1} \in \mathcal{G}(\mathcal{X})$, it follows from Lemma 2.2(2) that $K_{m-1} \in \mathcal{G}(\mathcal{X})$. Therefore $\mathcal{G}(\mathcal{X})\text{-dim}A \leqslant m-1$, a contradiction. So $\mathcal{G}(\mathcal{X})\text{-dim}A \leqslant n$. □

**Proposition 2.5.** *The following are true for any* $0 \neq A \in \mathcal{A}$*:*

*(1) If* $\mathcal{G}(\mathcal{X})\text{-dim}A = n < \infty$*, then there exist* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact sequences*

$$0 \to K \to G \to A \to 0 \quad \text{and} \quad 0 \to A \to L \to G' \to 0$$

*such that* $G, G' \in \mathcal{G}(\mathcal{X})$ *and* $\mathcal{X}\text{-res.dim}K \leqslant n-1$ *(if* $n=0$*, this should be interpreted as* $K=0$*) and* $\mathcal{X}\text{-res.dim}L \leqslant n$*.*

*(2) If* $\mathcal{G}(\mathcal{Y})\text{-codim}A = n < \infty$*, then there exist* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$*-exact sequences*

$$0 \to A \to G \to K \to 0 \quad \text{and} \quad 0 \to G' \to L \to A \to 0$$

*such that* $G, G' \in \mathcal{G}(\mathcal{Y})$ *and* $\mathcal{Y}\text{-cores.dim}K \leqslant n-1$ *(if* $n=0$*, this should be interpreted as* $K=0$*) and* $\mathcal{Y}\text{-cores.dim}L \leqslant n$*.*

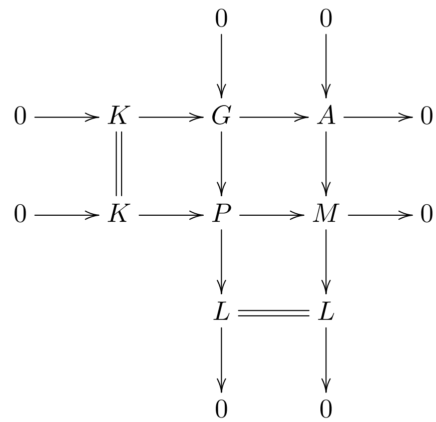
*Proof.* According to Lemma 2.2(1), the results follows by an argument similar to that of Proposition 3.3 in [17]. □

**Corollary 2.6.** *The following are true for any object* $A \in \mathcal{A}$*:*

*(1) If* $A \in \widetilde{\mathcal{Y}}$*, then* $\mathcal{X}\text{-res.dim}A = \mathcal{G}(\mathcal{X})\text{-dim}A$*.*

*(2) If* $A \in \widetilde{\mathcal{X}}$*, then* $\mathcal{Y}\text{-cores.dim}A = \mathcal{G}(\mathcal{Y})\text{-codim}A$*.*

*Proof.* We just prove (1) since (2) follows by duality. It is clear $\mathcal{G}(\mathcal{X})\text{-dim}A \leqslant \mathcal{X}\text{-res.dim}A$, so it is enough to show that $\mathcal{X}\text{-res.dim}A \leqslant \mathcal{G}(\mathcal{X})\text{-dim}A$, which is trivial when $\mathcal{G}(\mathcal{X})$-dim$A = \infty$. Now assume $\mathcal{G}(\mathcal{X})$-dim$A = n < \infty$, then there exists a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact sequence $0 \to K \to G \to A \to 0$ such that $G \in \mathcal{G}(\mathcal{X})$ and $\mathcal{X}$-res.dim$K = n - 1$ by Proposition 2.5(1). Since $G \in \mathcal{G}(\mathcal{X})$, there exists a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact sequence $0 \to G \to P \to L \to 0$ such that $P \in \mathcal{X}$ and $L \in \mathcal{G}(\mathcal{X})$. Hence there exists the following commutative diagram:



where all rows and columns are $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact. Hence $\mathcal{X}$-res.dim$M \leqslant \mathcal{X}$-res.dim$K + 1 = n$ by the middle row in above diagram. Note that $A \in \widetilde{\mathcal{Y}}$ by hypothesis, $\mathrm{Ext}^1_*(L, A) = 0$ by Lemma 2.3(1). So the third column in this diagram splits, and $M \cong L \oplus A$. This implies that $\mathcal{X}$-res.dim$A \leqslant \mathcal{X}$-res.dim$M \leqslant n = \mathcal{G}(\mathcal{X})$-dim$A$, as desired. □

**Proposition 2.7.** *If $\mathcal{G}(\mathcal{X})$-dim$A \leqslant n$ or $\mathcal{G}(\mathcal{Y})$-codim$A \leqslant n$ for any object $A \in \mathcal{A}$ and any positive integer $n$, then*

$$\sup\{\mathcal{Y}\text{-cores.dim}P \mid P \in \mathcal{X}\} = \sup\{\mathcal{X}\text{-res.dim}I \mid I \in \mathcal{Y}\} \leqslant n.$$

*Proof.* Suppose $\mathcal{G}(\mathcal{X})$-dim$A \leqslant n$ for any object $A \in \mathcal{A}$. It is easy to see that $\sup\{\mathcal{X}$-res.dim$I \mid I \in \mathcal{Y}\} < \infty$ by Corollary 2.6. Now assume that $P \in \mathcal{X}$. For any object $A$, we have $\mathcal{G}(\mathcal{X})$-dim$A \leqslant n$ by hypothesis. It follows from Proposition 2.4(1) that $\mathrm{Ext}^i_*(A, P) = 0$ for all $i > n$. Hence $\mathcal{Y}$-cores.dim$P \leqslant n$, so $\sup\{\mathcal{Y}$-cores.dim$P \mid P \in \mathcal{X}\} \leqslant n$. In the following, we claim that if both $\sup\{\mathcal{Y}$-cores.dim$P \mid P \in \mathcal{X}\}$ and $\sup\{\mathcal{X}$-res.dim$I \mid I \in \mathcal{Y}\}$ are finite, then they are equal. Indeed, assume that $\sup\{\mathcal{Y}$-cores.dim$P \mid P \in \mathcal{X}\} = t$ and $\sup\{\mathcal{X}$-res.dim$I \mid I \in \mathcal{Y}\} = s$. So there exists $I \in \mathcal{Y}$ such that $\mathcal{X}$-res.dim$I = s$. It is easy to check that there exists $P \in \mathcal{X}$ such that $\mathrm{Ext}^s_*(I, P) \neq 0$, this implies that $\mathcal{Y}$-cores.dim$P \geqslant s$. Hence $\sup\{\mathcal{Y}$-cores.dim$P \mid P \in \mathcal{X}\} = t \geqslant s$. Similarly, one can get $s \geqslant t$. Then the result follows. □

Let $0 \to K \to P \to A \to 0$ be a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact sequence with $P \in \mathcal{X}$. Then object $K$ is called a *first syzygy* of $A$, denoted by $\Omega_1(A)$. An *$n$-th syzygy* of $A$, denoted by $\Omega_n(A)$, is defined as usual by induction. Dually, one can define *$n$-th cosyzygy* of $A$, denoted by $\Omega^n(A)$. We are now in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* (1) ⇒ (3) and (2) ⇒ (3) follow from Proposition 2.7.

(3) ⇒ (1) First, we claim that any $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact complex of objects in $\mathcal{X}$ is a complete $\mathcal{X}$-resolution. Indeed, let $T$ be a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact complex

with each term in $\mathcal{X}$. We show that for each integer $n$, the relevant exact sequence $0 \to K_{d+1} \to T_d \to K_d \to 0$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact. If $P \in \mathcal{X}$, then $\mathcal{Y}$-cores.dim$P = t \leqslant n$ by hypothesis. Hence $\mathrm{Ext}^1_*(K_d, P) \cong \mathrm{Ext}^{t+1}_*(K_{d-t}, P) = 0$ because each $T_i \in \mathcal{X}$ for any integer $i$, which implies that these $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact sequences are $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{X})$-exact. So $T$ is a complete $\mathcal{X}$ resolution.

Let $A$ be an object in $\mathcal{A}$. Assume that

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is a $\mathcal{Y}$-coresolution of $A$ and consider the $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact sequences

$$0 \to K^i \to I^i \to K^{i+1} \to 0, \;\; i \geqslant 0.$$

Here $K^0 = A$. The following commutative diagram follows from the Horseshoe Lemma

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega_n(K^i) & \to & P^i_{n-1} & \to & \cdots & \to & P^i_1 & \to & P^i_0 & \to & K^i \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega_n(I^i) & \to & P^i_{n-1}\oplus P^{i+1}_{n-1} & \to & \cdots & \to & P^i_1\oplus P^{i+1}_1 & \to & P^i_0\oplus P^{i+1}_0 & \to & I^i \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega_n(K^{i+1}) & \to & P^{i+1}_{n-1} & \to & \cdots & \to & P^{i+1}_1 & \to & P^{i+1}_0 & \to & K^{i+1} \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & & & 0 & & 0 & & 0
\end{array}$$

with $\Omega_n(K^i) \to \Omega_n(I^i) \to \Omega_n(K^{i+1})$. Since $\sup\{\mathcal{X}\text{-res.dim}I \mid I \in \mathcal{Y}\} \leqslant n$, we have $\Omega_n(I^i) \in \mathcal{X}$ for any $i \geqslant 0$. Paste these $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact sequences

$$\Omega_n(K^i) \to \Omega_n(I^i) \to \Omega_n(K^{i+1}),\; i \geqslant 0,$$

together, there is a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact complex

$$0 \to \Omega_n(A) \to \Omega_n(I^0) \to \Omega_n(I^1) \to \cdots$$

with $\Omega_n(I^i) \in \mathcal{X}$ for any $i \geqslant 0$. Let $\cdots \to P_1 \to P_0 \to A \to 0$ be an $\mathcal{X}$-resolution of $A$. Similarly, one can get a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact complex

$$\cdots \to \Omega_n(P_1) \to \Omega_n(P_0) \to \Omega_n(A) \to 0$$

with $\Omega_n(P_i) \in \mathcal{X}$ for any $i \geqslant 0$. Hence there is a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},-)$-exact complex

$$\cdots \to \Omega_n(P_1) \to \Omega_n(P_0) \to \Omega_n(I^0) \to \Omega_n(I^1) \to \cdots$$

with each term in $\mathcal{X}$. By the claim before, we obtain that the complex above is a complete $\mathcal{X}$-resolution of $\Omega_n(A)$. So $\Omega_n(A) \in \mathcal{G}(\mathcal{X})$, and hence $\mathcal{G}(\mathcal{X})$-dim$A \leqslant n$.

Dually, one can prove (3) $\Rightarrow$ (2).

The last equality is immediate from above equivalences and Proposition 2.7. □

Let $R$ be a ring and Mod $R$ the category of left $R$-modules, and let $\mathcal{PP}(R)$ and $\mathcal{PI}(R)$ be the subcategories of Mod $R$ consisting of pure projective modules and pure injective modules, respectively. Then $(\mathcal{PP}(R), \mathcal{PI}(R))$ is an admissible balanced pair in Mod $R$ (see [5, Example 8.3.2]). In this case, $\mathcal{G}(\mathcal{PP}(R))$ coincides with pure projective modules. Indeed, any pure acyclic complex of pure projective modules is necessarily contractible. Dually, $\mathcal{G}(\mathcal{PI}(R))$ coincides with the pure injective modules. Denote by $\mathrm{ppd}_R M$ (respectively, $\mathrm{pid}_R M$) the pure projective (respectively, pure injective) dimension of $M$. Recall that the pure global dimension of $R$ is the supremum of pure projective dimensions of left $R$-modules, which is also equal to the supremum of pure injective dimensions of left $R$-modules (see [14, Section 2]). As a consequence of Theorem 1.2, we have the following result.

**Corollary 2.8.** *Let $R$ be a ring. Then $R$ has finite pure global dimension if and only if there exists a nonnegative integer $n$, such that*

$$\sup\{\mathrm{pid}_R P \mid P \in \mathcal{PP}(R)\} = \sup\{\mathrm{ppd}_R I \mid I \in \mathcal{PI}(R)\} \leqslant n.$$

Let $R$ be a Ding-Chen ring (that is, a left and right coherent ring $R$ such that both ${}_RR$ and $R_R$ have finite absolutely pure dimension). Then the pair $(\mathcal{G}(\mathcal{P}), \mathcal{G}(\mathcal{I}))$ is an admissible balanced pair in Mod $R$ (see [9, Theorem 1.1] and [8, Theorem 4.7]). In this case, by the stability of Gorenstein projective modules (see [13, Theorem A]), we see that $\mathcal{G}(\mathcal{G}(\mathcal{P}))$ coincides with Gorenstein projective modules. Dually, $\mathcal{G}(\mathcal{G}(\mathcal{I}))$ coincides with Gorenstein injective modules. Note that $R$ has finite left Gorenstein global dimension if and only if $\sup\{\mathcal{G}(\mathcal{I})\text{-codim}P \mid P \in \mathcal{G}(\mathcal{P})\} = \sup\{\mathcal{G}(\mathcal{P})\text{-dim}I \mid I \in \mathcal{G}(\mathcal{I})\} < \infty$ by Theorem 1.1. Thus we have the following result which is a consequence of Theorem 1.2.

**Corollary 2.9.** *Let $R$ be a Ding-Chen ring. Then $R$ has finite left Gorenstein global dimension if and only if there exists a nonnegative integer $n$, such that*

$$\sup\{\mathcal{G}(\mathcal{I})\text{-codim}P \mid P \in \mathcal{G}(\mathcal{P})\} = \sup\{\mathcal{G}(\mathcal{P})\text{-dim}I \mid I \in \mathcal{G}(\mathcal{I})\} \leqslant n.$$

Let $\Lambda$ be an artin algebra over a commutative artinian ring $k$ and mod $\Lambda$ the category of finitely generated left $\Lambda$-modules. Suppose $F$ is an additive subbifunctor of the additive bifunctor $\mathrm{Ext}^1_\Lambda(\ ,\ ) : (\mathrm{mod}\ \Lambda)^{op} \times \mathrm{mod}\ \Lambda \to \mathbf{Ab}$. A short exact sequence $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in mod $\Lambda$ is said to be $F$-*exact* if $\eta$ is in $\mathrm{F}(Z, X)$. The full subcategory of mod $\Lambda$ consisting of all $F$-projective (respectively, $F$-injective) modules is denoted by $\mathcal{P}(F)$ (respectively, $\mathcal{I}(F)$). $F$ is said to have enough projectives (respectively, injectives) if for any $A \in \mathrm{mod}\ \Lambda$ there is an $F$-exact sequence $0 \to B \to P \to A \to 0$ (respectively, $0 \to A \to I \to C \to 0$) with $P$ in $\mathcal{P}(F)$ (respectively, $I$ in $\mathcal{I}(F)$). Assume that $F$ has enough projectives and injectives. Then $(\mathcal{P}(F), \mathcal{I}(F))$ is an admissible balanced pair in mod $\Lambda$. The notions of $F$-Gorenstein projective modules and $F$-Gorenstein injective modules were introduced in [15]. It is not hard to see that in this case, $\mathcal{G}(\mathcal{P}(F))$ coincides with $F$-Gorenstein projective modules. Dually, $\mathcal{G}(\mathcal{I}(F))$ coincides with the $F$-Gorenstein injective modules. Denote by $\mathrm{pd}_F M$ (respectively, $\mathrm{id}_F M$) the $F$-projective (respectively, $F$-injective) dimension of $M$. As a consequence of Theorem 1.2, we have the following result which contains [15, Theorem 3.4].

**Corollary 2.10.** *Let $\Lambda$ be an artin algebra and $F$ an additive subbifunctor of the additive bifunctor $\mathrm{Ext}^1_\Lambda(\ ,\ )$. Then the following are equivalent for any nonnegative integer $n$:*

*(1) For any finitely generated left $\Lambda$-module $A$, $\mathcal{G}(\mathcal{P}(F))\text{-dim}A \leqslant n$;*
*(2) For any finitely generated left $\Lambda$-module $A$, $\mathcal{G}(\mathcal{I}(F))\text{-codim}A \leqslant n$;*
*(3) $\sup\{\mathrm{id}_F P \mid P \in \mathcal{P}(F)\} = \sup\{\mathrm{pd}_F I \mid I \in \mathcal{I}(F)\} \leqslant n$.*

*Moreover,*

$$\sup\{\mathcal{G}(\mathcal{P}(F))\text{-dim}M \mid M \in \text{mod } \Lambda\} = \sup\{\mathcal{G}(\mathcal{I}(F))\text{-codim}M \mid M \in \text{mod } \Lambda\}.$$

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