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GLOBAL WEAK SOLUTIONS FOR THE TWO-COMPONENT NOVIKOV EQUATION

Cheng He and Changzheng Qu

School of Mathematics and Statistics and Center for Nonlinear Studies
Ningbo University
Ningbo 315211, China

Abstract. The two-component Novikov equation is an integrable generalization of the Novikov equation, which has the peaked solitons in the sense of distribution as the Novikov and Camassa-Holm equations. In this paper, we prove the existence of the H^1-weak solution for the two-component Novikov equation by the regular approximation method due to the existence of three conserved densities. The key elements in our approach are some a priori estimates on the approximation solutions.

1. **Introduction.** This paper is devoted to the existence of weak solutions to the Cauchy problem for the two-component Novikov equation [18]

$$\begin{cases} m_t + uvm_x + (2vu_x + uv_x)m = 0, \quad m = u - u_{xx}, \quad t > 0, \\ n_t + uvn_x + (2uv_x + vu_x)n = 0, \quad n = v - v_{xx}. \end{cases} \tag{1}$$

Note that this system reduces respectively to the Novikov equation [23]

$$m_t + 3uu_x m + u^2 m_x = 0, \tag{2}$$

when $v = u$, and the celebrated Camassa-Holm (CH) equation [1]

$$m_t + 2u_x m + um_x = 0, \tag{3}$$

when $v = 1$.

The CH equation was proposed as a nonlinear model describing the unidirectional propagation of the shallow water waves over a flat bottom [1]. Based on the Hamiltonian theory of integrable systems, it was found earlier by using the method of recursion operator due to Fuchssteiner and Fokas [10]. It can also be obtained by using the tri-Hamiltonian duality approach related to the bi-Hamiltonian representation of the Korteweg-de Vries (KdV) equation [9, 25]. The CH equation exhibits several remarkable properties. One is the the existence of the multi-peaked solitons on the line $\mathbb{R}$ and unit circle $\mathbb{S}^1$ [1, 2], where the peaked solitons are the weak solution in the sense of distribution. Second, it can describes wave breaking phenomena [4], which is different from the classical integrable systems. The existence of H^1-conservation law to the CH equation enables ones to define the H^1-weak solution [28]. There have been a number of results concerning about integrability,

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well-posedness, blow up and wave breaking, orbital stability in the energy space and geometric formulations etc, see for instance [4, 5, 6, 8, 28] and references therein.

The Novikov equation (2) can be viewed as a cubic generalization of the CH equation, which was introduced by Novikov [23, 24] in the classification for a class of equations while they possesses higher-order generalized symmetries. Eq. (2) was proved to be integrable since it enjoys Lax-pair and bi-Hamiltonian structure [14], and is equivalent to the first equation in the negative flow of the Sawada-Kotera hierarchy via Liouville transformation [16]. The Novikov equation (2) also admits peaked solitons over the line $\mathbb{R}$ and unit circle $\mathbb{S}^1$ [14, 20], which can be derived by the inverse spectral method. Orbital stability of peaked solitons over the line $\mathbb{R}$ and unit circle $\mathbb{S}^1$ of (2) in the energy space were verified [20] based on the conservation laws and the structure of peaked solitons of the Novikov equation (2). The well-posedness and wave breaking of the Novikov equation have been discussed in a number of papers, and it reveals that the Cauchy problem of the Novikov equation (2) has global strong solutions when the initial data $u_0 \in H^s$, $s > 3/2$ [3, 15, 26, 27]. The existence of global weak solutions to the Cauchy problem of the Novikov equation (2) was also discussed in [17].

As the two-component generalization of Novikov equation (2), the so-called Geng-Xue system [11]

$$\begin{aligned} &m_t + 3vu_x m + uvm_x = 0, \\ &n_t + 3uv_x n + uvn_x = 0, \end{aligned} \tag{4}$$

has been studied extensively [11, 13]. The integrability [11, 19], dynamics and structure of the peaked solitons of (4) [21] were discussed. In [13], well-posedness and wave breaking phenomena of the Cauchy problem of (4) were discussed. The single peakons and multi-peakons of system (4) were constructed in [21] by using compatibility of Lax-pair, which are not the weak solutions in the sense of distribution. Furthermore, the Geng-Xue system does not have the H^1-conserved density, this is different from the CH and Novikov equations. The weak solution in H^1 is not well-defined since it does not obey the H^1-conservation law.

The main object in this work is to investigate the existence of weak solutions to system (1). It is of great interest to understand the effect from interactions among the two-components, nonlinear dispersion and various nonlinear terms. More specifically, we shall consider the Cauchy problem of (1) and aim to leverage ideas from previous works on CH and Novikov equations. The weak solution of the Cauchy problem associated with (1) is established in Theorem 3.1.

The remainder of this paper is organized as follows. In the next section 2, we review some basic results and lemmas as well as invariant properties of momentum densities m and n. In Section 3, we establish the existence of weak solutions, our approach is the regular approximation method together with some a priori estimates.

2. **Strong solutions and some a priori estimates.**

In this section, we recall the local well-posedness, some properties of strong and weak solutions to equation (1) and several approximation results.

First, we introduce some notations. Throughout the paper, we denote the convolution by $*$. Let $\|\cdot\|_X$ denote the norm of Banach space X and let $\langle \cdot, \cdot \rangle$ denote the duality paring between $H^1(\mathbb{R})$ and $H^{-1}(\mathbb{R})$. Let $\mathcal{M}(\mathbb{R})$ be the space of Radon measures on $\mathbb{R}$ with bounded total variation and $\mathcal{M}^+(\mathbb{R})$ be the subset of positive Radon measures. Moreover, we write $BV(\mathbb{R})$ for the space of functions with bounded variation, $\mathbb{V}(f)$ being the total variation of $f \in BV(\mathbb{R})$. Furthermore, for $0 < p < \infty$, $s \geq 0$, let $\|\cdot\|_{L^p}$ and $\|\cdot\|_s$ denote the norm of $L^p(\mathbb{R})$ space and $H^s(\mathbb{R})$ space, respectively.

With $m = u - u_{xx}$ and $n = v - v_{xx}$, the Cauchy problem of equation (1) takes the form:

$$\begin{cases} m_t + uvm_x + (2vu_x + uv_x)m = 0, \quad m = u - u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ n_t + uvn_x + (2uv_x + vu_x)n = 0, \quad n = v - v_{xx}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{5}$$

Note that if $P(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, we have $(1-\partial_x^2)^{-1}f = P * f$ for all the $f \in L^2(\mathbb{R})$ and $P * m = u$, $P * n = v$. Then we can rewrite the equation (5) as follows:

$$\begin{cases} u_t + uvu_x + P_x * (\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v) + \frac{1}{2}P * (u_x^2 v_x) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ v_t + uvv_x + P_x * (\frac{1}{2}v_x^2 u + vv_x u_x + v^2 u) + \frac{1}{2}P * (v_x^2 u_x) = 0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{6}$$

Next we recall the local well-posedness and the conservation laws.

Lemma 2.1. [12] *Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$. Assume that $T = T(u_0, v_0) > 0$ be the maximal existence time of the corresponding strong solution (u, v). Then the initial value problem of system (1) possesses a strong solution*

$$u, v \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $(u_0, v_0) \to (u(\cdot, u_0), v(\cdot, v_0)) : H^s(\mathbb{R}) \times H^s(\mathbb{R}) \to C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})) \times C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.

Lemma 2.2. [12] *Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $(u(t,x), v(t,x))$ be the corresponding solution to equation (1) with the initial data (u_0, v_0). Then we have*

$$\begin{aligned} &\int_{\mathbb{R}} \left(u^2(t,x) + u_x^2(t,x)\right) dx = \int_{\mathbb{R}} \left(u_0^2 + u_{0x}^2\right) dx, \\ &\int_{\mathbb{R}} \left(v^2(t,x) + v_x^2(t,x)\right) dx = \int_{\mathbb{R}} \left(v_0^2 + v_{0x}^2\right) dx, \\ &\int_{\mathbb{R}} \left(u(t,x)v(t,x) + u_x(t,x)v_x(t,x)\right) dx = \int_{\mathbb{R}} \left(u_0 v_0 + u_{0x} v_{0x}\right) dx. \end{aligned}$$

Moreover, we have

$$|u(t,x)| \leq \frac{\sqrt{2}}{2}\|u_0\|_1, \quad |v(t,x)| \leq \frac{\sqrt{2}}{2}\|v_0\|_1.$$

Note that equation (1) has the solitary waves with corner at their peaks. Obviously, such solitons are not strong solutions to equation (6). In order to provide a mathematical framework for the study of these solitons, we define the notion of weak solutions to equation (6). Let

$$\begin{aligned} F_u(u,v) &= uvu_x + P_x * (\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v) + \frac{1}{2}P * (u_x^2 v_x), \\ F_v(u,v) &= uvv_x + P_x * (\frac{1}{2}v_x^2 u + vv_x u_x + v^2 u) + \frac{1}{2}P * (v_x^2 u_x). \end{aligned}$$

Then equation (6) can be written as

$$\begin{cases} u_t + F_u(u,v) = 0, \\ v_t + F_v(u,v) = 0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \end{cases} \tag{7}$$

Lemma 2.3. [22] *Let $T > 0$. If*

$$f, g \in L^2((0,T); H^1(\mathbb{R})) \quad \text{and} \quad \frac{df}{dt}, \frac{dg}{dt} \in L^2((0,T); H^{-1}(\mathbb{R})),$$

then f, g are a.e. equal to functions continuous from $[0,T]$ into $L^2(\mathbb{R})$ and

$$\langle f(t), g(t)\rangle - \langle f(s), g(s)\rangle = \int_s^t \left\langle \frac{df(\tau)}{d\tau}, g(\tau)\right\rangle d\tau + \int_s^t \left\langle \frac{dg(\tau)}{d\tau}, f(\tau)\right\rangle d\tau$$

for all $s, t \in [0,T]$.

Throughout this paper, let $\{\rho_n\}_{n\geq 1}$ denote the mollifiers

$$\rho_n = \left(\int_{\mathbb{R}} \rho(\xi) d\xi\right)^{-1} n\rho(nx), \quad x \in \mathbb{R}, \ \ n \geq 1,$$

where $\rho \in C_c^\infty(\mathbb{R})$ is defined by

$$\rho(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & \text{for } \ |x| < 1, \\ 0, & \text{for } \ |x| \geq 1. \end{cases}$$

Next, we recall two crucial approximation results and two identities.

Lemma 2.4. [7] *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. If $\mu \in \mathcal{M}(\mathbb{R})$, then*

$$[\rho_n * (f\mu) - (\rho_n * f)(\rho_n * \mu)] \to 0, \quad \text{as } n \to \infty \quad \text{in } L^1(\mathbb{R}).$$

Lemma 2.5. [7] *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. If $g \in L^\infty(\mathbb{R})$, then*

$$\rho_n * (fg) - (\rho_n * f)(\rho_n * g) \to 0, \quad \text{as } n \to \infty \quad \text{in } L^\infty(\mathbb{R}).$$

Lemma 2.6. [7] *Assume that $u(t,\cdot) \in W^{1,1}(\mathbb{R})$ is uniformly bounded in $W^{1,1}(\mathbb{R})$ for all $t \in \mathbb{R}_+$. Then for a.e. $t \in \mathbb{R}_+$, there hold*

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * u| dx = \int_{\mathbb{R}} (\rho_n * u_t) \mathrm{sgn}(\rho_n * u) dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * u_x| dx = \int_{\mathbb{R}} (\rho_n * u_{xt}) \mathrm{sgn}(\rho_n * u_x) dx.$$

Consider the flow governed by $(uv)(t,x)$:

$$\begin{cases} \dfrac{dq(t,x)}{dt} = (uv)(t,q), \quad t > 0, \quad x \in \mathbb{R}, \\ q(0,x) = x, \quad x \in \mathbb{R}. \end{cases} \tag{8}$$

Applying classical results in the theory of ODEs, one can obtain the following useful result on the above initial value problem.

Lemma 2.7. [12] *Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$, and $T > 0$ be the life-span of the corresponding strong solution (u, v) to equation (5) with the initial data (u_0, v_0). Then equation (8) has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism over $\mathbb{R}$ with*

$$q_x = \exp\left(\int_0^t (uv)_x(s, q(s,x))ds\right), \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Furthermore, setting $m = u - u_{xx}$ and $n = v - v_{xx}$, we obtain

$$m(t,q) = \exp\left(-\int_0^t (2vu_x + uv_x)(s, q(s,x))ds\right) m_0,$$
$$n(t,q) = \exp\left(-\int_0^t (2uv_x + vu_x)(s, q(s,x))ds\right) n_0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Theorem 2.8. *Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$. Assume that $m_0 = u_0 - \partial_x^2 u_0$ and $n_0 = v_0 - \partial_x^2 v_0$ are nonnegative, and $T > 0$ be the maximal existence time of the corresponding strong solution (u, v). Then the initial value problem of system (1) possesses a pair of unique strong solution (u, v), where*

$$u, v \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Set $m(t,\cdot) = u(t,\cdot) - u_{xx}(t,\cdot)$ and $n(t,\cdot) = v(t,\cdot) - v_{xx}(t,\cdot)$. Then, $E_u(u) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx$, $E_v(v) = \int_{\mathbb{R}} \left(v^2 + v_x^2\right) dx$, $H(u,v) = \int_{\mathbb{R}} (uv + u_x v_x)\, dx$ and $E_0(u,v) = \int_{\mathbb{R}} (mn)^{\frac{1}{3}}\, dx$ are four conservation laws and we have for all $t \in \mathbb{R}_+$

(i). $m(t,\cdot) \geq 0, n(t,\cdot) \geq 0, u(t,\cdot) \geq 0, v(t,\cdot) \geq 0$ and $|u_x(t,\cdot)| \leq u(t,\cdot)$, $|v_x(t,\cdot)| \leq v(t,\cdot)$ on $\mathbb{R}$;

(ii). $\|u(t,\cdot)\|_{L^1} \leq \|m(t,\cdot)\|_{L^1}, \|u(t,\cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2}\|u(t,\cdot)\|_1 = \frac{\sqrt{2}}{2}\|u_0\|_1$,
and $\|v(t,\cdot)\|_{L^1} \leq \|n(t,\cdot)\|_{L^1}, \|v(t,\cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2}\|v(t,\cdot)\|_1 = \frac{\sqrt{2}}{2}\|v_0\|_1$;

(iii). $\|u_x(t,\cdot)\|_{L^1} \leq \|m(t,\cdot)\|_{L^1}$ and $\|v_x(t,\cdot)\|_{L^1} \leq \|n(t,\cdot)\|_{L^1}$.

Moreover, if $m_0, n_0 \in L^1(\mathbb{R})$, we obtain

$$\|m(t,\cdot)\|_{L^1} \leq e^{\|u_0\|_1\|v_0\|_1 t}\|m_0\|_{L^1} \quad \text{and} \quad \|n(t,\cdot)\|_{L^1} \leq e^{\|u_0\|_1\|v_0\|_1 t}\|n_0\|_{L^1}.$$

Proof. Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the solution (u, v) to equation (5) with the initial data (u_0, v_0). If $m_0 \geq 0$ and $n_0 \geq 0$, then Lemma 2.7 ensures that $m(t,\cdot) \geq 0$ and $n(t,\cdot) \geq 0$ for all $t \in [0,\infty)$. By $u = P * m$, $v = P * n$ and the positivity of P, we infer that $u(t,\cdot) \geq 0$ and $v(t,\cdot) \geq 0$ for all $t \geq 0$. Note that v is analogous as u and

$$u(t,x) = \frac{e^{-x}}{2}\int_{-\infty}^x e^y m(t,y)dy + \frac{e^x}{2}\int_x^\infty e^{-y} m(t,y)dy, \tag{9}$$

and

$$u_x(t,x) = -\frac{e^{-x}}{2}\int_{-\infty}^x e^y m(t,y)dy + \frac{e^x}{2}\int_x^\infty e^{-y} m(t,y)dy. \tag{10}$$

From the above two relations and $m \geq 0$, we deduce that

$$|u_x(t,x)| \leq u(t,x) \leq \frac{\sqrt{2}}{2}\|u(t,x)\|_1.$$

In view of Lemma 2.2, we obtain that $E_u(u)$ and $E_v(v)$ are conserved and

$$u(t,x) \le \frac{\sqrt{2}}{2}\|u_0\|_1, \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Since $m(t,x) = u - u_{xx}$, it follows that $u = P * m$ and $u_x = P_x * m$. Note that $\|P\|_{L^1} = \|P_x\|_{L^1} = 1$. Applying Young's inequality, one can easily obtain $(i)-(iii)$. Since equation (1) can be used to derive the following form

$$\left((mn)^{\frac{1}{3}}\right)_t + \left((mn)^{\frac{1}{3}}uv\right)_x = 0,$$

it immediately follows that $E_0(u,v)$ is a conserved density. On the other hand, by equation (5), we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} m(t,x)dx &= -\int_{-\infty}^{\infty} (uvm_x + (2vu_x + uv_x)m)\,dx \\ &= \int_{-\infty}^{\infty} (vu_x m - (uvm)_x)\,dx \le \|u\|_{L^\infty}\|v\|_{L^\infty}\int_{-\infty}^{\infty} m(t,x)dx \\ &\le \|u_0\|_1\|v_0\|_1 \int_{-\infty}^{\infty} m(t,x)dx.\end{aligned}$$

Since $m_0 \in L^1(\mathbb{R})$, in view of Gronwall's inequality, we can get

$$\|m(t,\cdot)\|_{L^1} \le e^{\|u_0\|_1\|v_0\|_1 t}\|m_0\|_{L^1}.$$

Similarly, we find

$$\|n(t,\cdot)\|_{L^1} \le e^{\|u_0\|_1\|v_0\|_1 t}\|n_0\|_{L^1}.$$

This completes the proof of Theorem 2.8. □

3. Global weak solutions.

In this section, we will prove that there exists a unique global weak solution to equation (6), provided the initial data (u_0, v_0) satisfy certain sign-invariant conditions.

Theorem 3.1. *Let u_0, $v_0 \in H^1(\mathbb{R})$. Assume $m_0 = u_0 - u_{0xx}$ and $n_0 = v_0 - v_{0xx} \in \mathcal{M}^+(\mathbb{R})$. Then equation (6) has a pair of unique weak solution (u,v), where*

$$u, v \in W^{1,\infty}(\mathbb{R}_x \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

with the initial data $u(0,x) = u_0$, $v(0,x) = v_0$ and such that $m = u - u_{xx}$, $n = v - v_{xx} \in \mathcal{M}^+(\mathbb{R})$ are bounded on $[0,T)$, for any fixed $T > 0$. Moreover, $E_u(u) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx$, $E_v(v) = \int_{\mathbb{R}} \left(v^2 + v_x^2\right) dx$ and $H(u,v) = \int_{\mathbb{R}} (uv + v_x u_x)\, dx$ are conserved densities.

Proof. First, we shall prove $u, v \in W^{1,\infty}(\mathbb{R}_x \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$. Let $u_0, v_0 \in H^1(\mathbb{R})$ and assume that $m_0 = u_0 - u_{0,xx}$, $n_0 = v_0 - v_{0,xx} \in \mathcal{M}^+(\mathbb{R})$. Note that $u_0 = P * m_0$ and $v_0 = P * n_0$. Thus, we have for any $f \in L^\infty(\mathbb{R})$,

$$\begin{aligned}\|u_0\|_{L^1} = \|P * m_0\|_{L^1} &= \sup_{\|f\|_{L^\infty}\le 1} \int_{\mathbb{R}} f(x)(P * m_0)(x)dx \\ &= \sup_{\|f\|_{L^\infty}\le 1} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} P(x-y)dm_0(y)dx \\ &= \sup_{\|f\|_{L^\infty}\le 1} \int_{\mathbb{R}} (P * f)(y)dm_0(y) \\ &\le \sup_{\|f\|_{L^\infty}\le 1} \|P\|_{L^1}\|f\|_{L^\infty}\|m_0\|_{M(\mathbb{R})} = \|m_0\|_{\mathcal{M}(\mathbb{R})}.\end{aligned} \tag{11}$$

Similarly, we have

$$\|v_0\|_{L^1} \leq \|n_0\|_{\mathcal{M}(\mathbb{R})}. \tag{12}$$

We first prove that there exists a corresponding (u, v) with the initial data (u_0, v_0), which belongs to $H^1_{loc}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \times H^1_{loc}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$, satisfying equation (6) in the sense of distributions.

Let us define $u_0^n = \rho_n * u_0 \in H^\infty(\mathbb{R})$ and $v_0^n = \rho_n * v_0 \in H^\infty(\mathbb{R})$ for $n \geq 1$. Obviously, we have

$$\begin{aligned} u_0^n &\longrightarrow u_0 \quad H^1(\mathbb{R}), \quad n \to \infty, \\ v_0^n &\longrightarrow v_0 \quad H^1(\mathbb{R}), \quad n \to \infty, \end{aligned} \tag{13}$$

and for all $n \geq 1$,

$$\begin{aligned} &\|u_0^n\|_1 = \|\rho_n * u_0\|_1 \leq \|u_0\|_1, \quad \|v_0^n\|_1 \leq \|v_0\|_1, \\ &\|u_0^n\|_{L^1} = \|\rho_n * u_0\|_{L^1} \leq \|u_0\|_{L^1}, \quad \|v_0^n\|_{L^1} \leq \|v_0\|_{L^1}, \end{aligned} \tag{14}$$

in view of Young's inequality. Note that for all $n \geq 1$,

$$m_0^n = u_0^n - u_{0,xx}^n = \rho_n * m_0 \geq 0, \quad \text{and} \quad n_0^n = v_0^n - v_{0,xx}^n = \rho_n * v_0 \geq 0.$$

Comparing with the proof of relation (11) and (12), we get

$$\|m_0^n\|_{L^1} \leq \|m_0\|_{\mathcal{M}(\mathbb{R})}, \quad \text{and} \quad \|n_0^n\|_{L^1} \leq \|n_0\|_{\mathcal{M}(\mathbb{R})}, \quad n \geq 1. \tag{15}$$

By Theorem 2.8, we obtain that there exists a global strong solution

$$u^n = u^n(\cdot, u_0^n), v^n = v^n(\cdot, v_0^n) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$$

for every $s \geq 3$, and we have $u^n(t,x) - u^n_{xx}(t,x) \geq 0$ and $v^n(t,x) - v^n_{xx}(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. In view of theorem 2.8 and (14),we obtain for $n \geq 1$ and $t \geq 0$,

$$\begin{aligned} \|u_x^n(t,\cdot)\|_{L^\infty} \leq \|u^n(t,\cdot)\|_{L^\infty} \leq \|u^n(t,\cdot)\|_1 = \|u_0^n\|_1 \leq \|u_0\|_1, \\ \|v_x^n(t,\cdot)\|_{L^\infty} \leq \|v^n(t,\cdot)\|_{L^\infty} \leq \|v^n(t,\cdot)\|_1 = \|v_0^n\|_1 \leq \|v_0\|_1. \end{aligned} \tag{16}$$

By the above inequality, we have

$$\begin{aligned} \|u^n(t,\cdot)v^n(t,\cdot)u_x^n(t,\cdot)\|_{L^2} &\leq \|u^n(t,\cdot)\|_{L^\infty}\|v^n(t,\cdot)\|_{L^\infty}\|u_x^n(t,\cdot)\|_{L^2} \\ &\leq \|u^n(t,\cdot)\|_1^2\|v^n(t,\cdot)\|_1 \leq \|u_0\|_1^2\|v_0\|_1. \end{aligned} \tag{17}$$

Similarly, we have

$$\|v^n(t,\cdot)u^n(t,\cdot)v_x^n(t,\cdot)\|_{L^2} \leq \|v_0\|_1^2\|u_0\|_1. \tag{18}$$

By Young's inequality and (16), for all $t \geq 0$ and $n \geq 1$, we obtain

$$\begin{aligned} &\left\| P_x * \left(\frac{1}{2}(u_x^n)^2 v^n + u^n u_x^n v_x^n + (u^n)^2 v^n \right) + \frac{1}{2} P * \left((u_x^n)^2 v_x^n \right) \right\|_{L^2} \\ &\leq \|P_x\|_{L^2} \left\| \frac{1}{2}(u_x^n)^2 v^n + u^n u_x^n v_x^n + (u^n)^2 v^n \right\|_{L^1} + \frac{1}{2}\|P\|_{L^2} \left\|(u_x^n)^2 v_x^n\right\|_{L^1} \\ &\leq \frac{1}{2}\|u_x^n\|_{L^2}^2\|v^n\|_{L^\infty} + \frac{1}{2}\|u^n\|_{L^\infty}\|u_x^n\|_{L^2}\|v_x^n\|_{L^2} + \|u^n\|_{L^2}^2\|v^n\|_{L^\infty} + \frac{1}{2}\|v_x^n\|_{L^\infty}\|u_x^n\|_{L^2}^2 \\ &\leq \frac{5}{2}\|u^n\|_1^2\|v^n\|_1 \leq \frac{5}{2}\|u_0\|_1^2\|v_0\|_1. \end{aligned} \tag{19}$$

Similarly, we get

$$\left\| P_x * \left(\frac{1}{2}(v_x^n)^2 u^n + v^n u_x^n v_x^n + (v^n)^2 u^n \right) + \frac{1}{2} P * \left((v_x^n)^2 u_x^n \right) \right\|_{L^2} \leq \frac{5}{2}\|v_0\|_1^2\|u_0\|_1. \tag{20}$$

Combining (17)-(20) with equation (6) for all $t \geq 0$ and $n \geq 1$, we find

$$\left\| \frac{d}{dt} u^n(t,\cdot) \right\|_{L^2} \leq \frac{7}{2} \|u_0\|_1^2 \|v_0\|_1, \quad \text{and} \quad \left\| \frac{d}{dt} v^n(t,\cdot) \right\|_{L^2} \leq \frac{7}{2} \|v_0\|_1^2 \|u_0\|_1. \tag{21}$$

For fixed $T > 0$, by Theorem 2.8 and (21), we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left([u^n(t,x)]^2 + [u_x^n(t,x)]^2 + [u_t^n(t,x)]^2 \right) dxdt \leq \left(\|u_0\|_1^2 + \frac{49}{4} \|u_0\|_1^4 \|v_0\|_1^2 \right) T, \\ &\int_0^T \int_{\mathbb{R}} \left([v^n(t,x)]^2 + [v_x^n(t,x)]^2 + [v_t^n(t,x)]^2 \right) dxdt \leq \left(\|v_0\|_1^2 + \frac{49}{4} \|v_0\|_1^4 \|u_0\|_1^2 \right) T. \end{aligned} \tag{22}$$

It follows that the sequence $\{u^n\}_{n \geq 1}$ is uniformly bounded in the space $H^1((0,T) \times \mathbb{R})$.Thus we can extract a subsequence such that

$$u^{n_k} \rightharpoonup u \quad \text{weakly in } H^1(0,T) \times \mathbb{R}) \text{ for } n_k \to \infty \tag{23}$$

and

$$u^{n_k} \longrightarrow u, \quad a.e. \text{ on } (0,T) \times \mathbb{R} \text{ for } n_k \to \infty, \tag{24}$$

for some $u \in H^1((0,T) \times \mathbb{R})$. By Theorem 2.8, (11) and (14), we have that for fixed $t \in (0,T)$, the sequence $u_x^{n_k}(t,\cdot) \in BV(\mathbb{R})$ satisfies

$$\begin{aligned} \mathbb{V}[u_x^{n_k}(t,\cdot)] =& \|u_{xx}^{n_k}(t,\cdot)\|_{L^1} \leq \|u^{n_k}(t,\cdot)\|_{L^1} + \|m^{n_k}(t,\cdot)\|_{L^1} \\ \leq & 2\|m^{n_k}(t,\cdot)\|_{L^1} \leq 2e^{\|u_0^{n_k}\|_1 \|v_0^{n_k}\|_1 t} \|m_0^{n_k}\|_{L^1} \\ \leq & 2e^{\|u_0\|_1 \|v_0\|_1 t} \|m_0\|_{\mathcal{M}(\mathbb{R})} \end{aligned}$$

and

$$\|u_x^{n_k}(t,\cdot)\|_{L^\infty} \leq \|u^{n_k}(t,\cdot)\|_1 = \|u_0^{n_k}(t,\cdot)\|_1 \leq \|u_0\|_1.$$

Applying Helly's theorem, we obtain that there exists a subsequence, denoted again by $\{u_x^{n_k}(t,\cdot)\}$, which converges at every point to some function $\hat{u}(t,\cdot)$ of finite variation with

$$\mathbb{V}[\hat{u}(t,\cdot)] \leq 2e^{\|u_0\|_1 \|v_0\|_1 t} \|m_0\|_{\mathcal{M}(\mathbb{R})}.$$

Since for almost all $t \in (0,T)$, $u_x^{n_k}(t,\cdot) \to u_x(t,\cdot)$ in $D'(\mathbb{R})$ in view of (24), it follows that $\hat{u}(t,\cdot) = u_x(t,\cdot)$ for a.e. $t \in (0,T)$. Therefore, we have

$$u_x^{n_k} \longrightarrow u_x \quad a.e. \text{ on } (0,T) \times \mathbb{R} \text{ for } n_k \to \infty, \tag{25}$$

and for a.e. $t \in (0,T)$,

$$\mathbb{V}[u_x(t,\cdot)] = \|u_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} \leq 2e^{\|u_0\|_1 \|v_0\|_1 t} \|m_0\|_{\mathcal{M}(\mathbb{R})}.$$

We can analogously extract a subsequence of $\{v^{n_k}\}$, denote again by $\{v^{n_k}\}$ such that

$$\begin{aligned} v^{n_k} &\longrightarrow v \quad a.e. \text{ on } (0,T) \times \mathbb{R} \text{ for } n_k \to \infty \\ \text{and} \quad v_x^{n_k} &\longrightarrow v_x \quad a.e. \text{ on } (0,T) \times \mathbb{R} \text{ for } n_k \to \infty. \end{aligned} \tag{26}$$

By Theorem 2.8 $(ii) - (iii)$ and (16), we have

$$\left\| \frac{1}{2}(u_x^n)^2 v^n + u^n u_x^n v_x^n + (u^n)^2 v^n + \frac{1}{2}(u_x^n)^2 v_x^n \right\|_{L^1} \leq 3\|u_0\|_1^2 \|v_0\|_1.$$

For fixed $t \in (0,T)$, it follows that the sequence $\{\frac{1}{2}(u_x^n)^2 v^n + u^n u_x^n v_x^n + (u^n)^2 v^n + \frac{1}{2}(u_x^n)^2 v_x^n\}$ is uniformly bounded in $L^1(\mathbb{R})$. Therefore, it has a subsequence converging weakly in $L^1(\mathbb{R})$, denoted again by $\{\frac{1}{2}(u_x^n)^2 v^n + u^n u_x^n v_x^n + (u^n)^2 v^n + \frac{1}{2}(u_x^n)^2 v_x^n\}$.

By (24) and (25), we deduce that the weak $L^1(\mathbb{R})$-limit is $\frac{1}{2}(u_x)^2v^n + uu_xv_x + u^2v + \frac{1}{2}(u_x)^2v_x$. Note that P, $P_x \in L^\infty(\mathbb{R})$. It follows that

$$\begin{aligned} &P_x * \left[\frac{1}{2}(u_x^n)^2v^n + u^nu_x^nv_x^n + (u^n)^2v^n\right] + P * \left(\frac{1}{2}(u_x^n)^2v_x^n\right) \\ &\quad \longrightarrow P_x * \left[\frac{1}{2}u_x^2v^n + uu_xv_x + u^2v\right] + P * \left(\frac{1}{2}u_x^2v_x\right), \quad \text{as } n \to \infty. \end{aligned} \tag{27}$$

We can analogously obtain that

$$\begin{aligned} &P_x * [\frac{1}{2}(v_x^n)^2u^n + v^nv_x^nu_x^n + (v^n)^2u^n] + P * \left(\frac{1}{2}(v_x^n)^2u_x^n\right) \\ &\quad \longrightarrow P_x * [\frac{1}{2}v_x^2u^n + vv_xu_x + v^2u] + P * \left(\frac{1}{2}v_x^2u_x\right), \quad \text{as } n \to \infty. \end{aligned} \tag{28}$$

Combining (24)-(26) with (27) and (28), we deduce that (u, v) satisfies equation (6) in $D'((0,T) \times \mathbb{R})$.

Since $u_t^{n_k}(t,\cdot)$ is uniformly bounded in $L^2(\mathbb{R})$ for all $t \in \mathbb{R}_+$ and $\|u^{n_k}(t,\cdot)\|_1$ has a uniform bound as $t \in \mathbb{R}_+$ and all $n \geq 1$. Hence the family $t \mapsto u^{n_k}(t,\cdot) \in H^1(\mathbb{R})$ is weakly equicontinuous on $[0,T]$ for any $T > 0$. An application of the Arzela-Ascoli theorem yields that $\{u^{n_k}\}$ contains a subsequence, denoted again by $\{u^{n_k}\}$, which converges weakly in $H^1(\mathbb{R})$, uniformly in $t \in [0,T]$. The limit function is u. Because T is arbitrary, we have that u is locally and weakly continuous from $[0,\infty)$ into $H^1(\mathbb{R})$, i.e.

$$u \in C_{w,loc}(\mathbb{R}_+; H^1(\mathbb{R})).$$

For a.e. $t \in \mathbb{R}_+$, since $u^{n_k}(t,\cdot) \rightharpoonup u(t,\cdot)$ weakly in $H^1(\mathbb{R})$, in view of (15) and (16), we obtain

$$\begin{aligned} \|u(t,\cdot)\|_{L^\infty} \leq &\|u(t,\cdot)\|_1 \leq \liminf_{n_k \to \infty} \|u^n(t,\cdot)\|_1 \\ = &\liminf_{n_k \to \infty} \|u_0^{n_k}(t,\cdot)\|_1 \leq \liminf_{n_k \to \infty} \|P\|_1 \|m_0^{n_k}(t,\cdot)\|_{L^1} \\ \leq &\|m_0\|_{\mathcal{M}(\mathbb{R})}, \end{aligned} \tag{29}$$

for a.e. $t \in \mathbb{R}_+$. The previous relation implies that

$$u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})).$$

Note that by Theorem 2.8 and (15), we have

$$\begin{aligned} \|u_x^n(t,\cdot)\|_{L^\infty} \leq &\|u^n(t,\cdot)\|_{L^\infty} \leq \|u^n(t,\cdot)\|_1 \\ \leq &\|P\|_1 \|m_0^n(t,\cdot)\|_{L^1} \leq \|m_0(t,\cdot)\|_{\mathcal{M}(\mathbb{R})}. \end{aligned} \tag{30}$$

Combining this with (25), we deduce that

$$u_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R}).$$

This shows that

$$u \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})).$$

Taking the same way as u, we get

$$v \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})).$$

Please note that we use the subsequence of $\{v^{n_k}\}$ which is determined after using the Arzela-Ascoli theorem.

Now, by a regularization technique, we prove that $E_u(u)$, $E_v(v)$ and $H(u,v)$ are conserved densities. As (u,v) solves equation (6) in the sense of distributions, we see that for a.e. $t \in \mathbb{R}_+$, $n \geq 1$,

$$\begin{cases} \rho_n * u_t + \rho_n * (uvu_x) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) \\ \qquad\qquad + \frac{1}{2}\rho_n * P * \left(u_x^2 v_x\right) = 0, \\ \rho_n * v_t + \rho_n * (uvv_x) + \rho_n * P_x * \left(\frac{1}{2}v_x^2 v + vu_x v_x + v^2 u\right) \\ \qquad\qquad + \frac{1}{2}\rho_n * P * \left(v_x^2 u_x\right) = 0. \end{cases} \tag{31}$$

By differentiation of the first equation of (31), we obtain

$$\begin{aligned} &\rho_n * u_{xt} + \rho_n * (uvu_x)_x + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v_x\right) \\ &\qquad + \rho_n * P_{xx} * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) = 0. \end{aligned} \tag{32}$$

Note that $\partial^2(P * f) = P * f - f$, $f \in L^2(\mathbb{R})$. We can rewrite (32) as

$$\begin{aligned} &\rho_n * u_{xt} + \rho_{nx} * (uvu_x) + \rho_n * P * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) \\ &\qquad - \rho_n * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v_x\right) = 0. \end{aligned} \tag{33}$$

Take these two equation (32) and (33) into the integration below, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (\rho_n * u)^2 + (\rho_n * u_x)^2 dx \\ =& \int_{\mathbb{R}} (\rho_n * u)(\rho_n * u_t) + (\rho_n * u_x)(\rho_n * u_{xt}) dx \\ =& -\int_{\mathbb{R}} (\rho_n * u)\left(\rho_n * (uvu_x) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right)\right. \\ &\qquad \left. + \rho_n * P * \left(\frac{1}{2}u_x^2 v_x\right)\right) dx \\ &- \int_{\mathbb{R}} (\rho_n * u_x)\left(\rho_{nx} * (uvu_x) + \rho_n * P * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right)\right. \\ &\qquad \left. -\rho_n * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v_x\right)\right) dx. \end{aligned} \tag{34}$$

Note that

$$\lim_{n\to\infty} \|\rho_n * u - u\|_{L^2} = \lim_{n\to\infty} \|\rho_n * (uvu_x) - uvu_x\|_{L^2} = 0.$$

Therefore, by using Hölder inequality, we have for a.e. $t \in \mathbb{R}^+$

$$\int_{\mathbb{R}} (\rho_n * u)(\rho_n * (uvu_x)) dx \longrightarrow \int_{\mathbb{R}} u^2 v u_x dx, \quad \text{as} \quad n \to \infty.$$

Similarly, for a.e. $t \in \mathbb{R}$

$$\int_{\mathbb{R}} (\rho_n * u) \left(\rho_n * P_x * \left(\frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) \right) dx \longrightarrow \int_{\mathbb{R}} u P_x * \left(\frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) dx, \qquad \text{as} \quad n \to \infty,$$

$$\int_{\mathbb{R}} (\rho_n * u) \left(\rho_n * P * \left(\frac{1}{2} u_x^2 v_x \right) \right) dx \longrightarrow \int_{\mathbb{R}} u P * \left(\frac{1}{2} u_x^2 v_x \right) dx, \qquad \text{as} \quad n \to \infty,$$

$$\int_{\mathbb{R}} (\rho_n * u_x) \left(\rho_n * P * \left(\frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) \right) dx \longrightarrow \int_{\mathbb{R}} u_x P * \left(\frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) dx, \qquad \text{as} \quad n \to \infty,$$

$$\int_{\mathbb{R}} (\rho_n * u_x) \left(\rho_n * \left(\frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) \right) dx \longrightarrow \int_{\mathbb{R}} u_x \left(\frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) dx, \qquad \text{as} \quad n \to \infty,$$

$$\int_{\mathbb{R}} (\rho_n * u_x) \left(\rho_n * P_x * \left(\frac{1}{2} u_x^2 v_x \right) \right) dx \longrightarrow \int_{\mathbb{R}} u_x P_x * \left(\frac{1}{2} u_x^2 v_x \right) dx, \qquad \text{as} \quad n \to \infty,$$

as $u(t,\cdot), v(t,\cdot) \in H^1(\mathbb{R})$ and $u_x, v_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. Furthermore, note that

$$\begin{aligned} \int_{\mathbb{R}} (\rho_n * u_x)(\rho_{nx} * (uvu_x)) dx = &- \int_{\mathbb{R}} (\rho_{n,xx} * u)(\rho * (uvu_x)) dx \\ &+ \int_{\mathbb{R}} (\rho_{n,xx} * u)(\rho_n * uv)(\rho_n * u_x) dx + \frac{1}{2} \int_{\mathbb{R}} (\rho_n * u_x)^2 (\rho_n * (uv)_x) dx. \end{aligned} \tag{35}$$

Observe that

$$\int_{\mathbb{R}} (\rho_n * u_x)^2 \left(\rho_n * (uv)_x \right) dx \longrightarrow \int_{\mathbb{R}} u_x^2 (uv)_x dx, \quad \text{as} \quad n \to \infty.$$

On the other hand

$$\|\rho_{nxx} * u\|_{L^1} \le \|u_{xx}\|_{\mathcal{M}(\mathbb{R})} \le 2e^{\|u_0\|_1 \|v_0\|_1 t} \|m_0\|_{\mathcal{M}(\mathbb{R})}, \ \forall t \in [0,T).$$

As $u(t,\cdot), v(t,\cdot) \in H^1(\mathbb{R})$ and $u_x, v_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, by Lemma 2.5, it follows that

$$\|(\rho_n * uv)(\rho_n * u_x) - (\rho_n * (uvu_x))\|_{L^\infty} \to 0, \qquad n \to \infty.$$

Therefore,

$$\int_{\mathbb{R}} (\rho_{n,xx} * u) \left((\rho_n * uv)(\rho_n * u_x) - \rho_n * (uvu_x) \right) dx \to 0, \qquad n \to \infty.$$

In view of the above relations and (35), we obtain

$$\int_{\mathbb{R}} (\rho_n * u_x)(\rho_{nx} * (uvu_x)) dx \to \frac{1}{2} \int_{\mathbb{R}} u_x^2 (uv)_x dx, \qquad n \to \infty. \tag{36}$$

Let us define

$$E_n^u(t) = \int_{\mathbb{R}} (\rho_n * u)^2 + (\rho_n * u_x)^2 dx, \tag{37}$$

and

$$\begin{aligned} G_n^u(t) = & -2\int_{\mathbb{R}}(\rho_n * u)\left(\rho_n * (uvu_x) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right)\right. \\ & \left. + \rho_n * P * \left(\frac{1}{2}u_x^2 v_x\right)\right) dx \\ & - 2\int_{\mathbb{R}}(\rho_n * u_x)\left(\rho_{nx} * (uvu_x) + \rho_n * P * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right)\right. \\ & \left. -\rho_n * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v_x\right)\right) dx. \end{aligned}$$

We have proved that for fixed $T > 0$, for a.e. $t \in [0, T)$,

$$\begin{cases} \dfrac{d}{dt}E_n^u(t) = G_n^u(t), & n \geq 1, \\ G_n^u(t) \to 0, & n \to \infty. \end{cases} \tag{38}$$

Therefore, we get

$$E_n^u(t) - E_n^u(0) = \int_0^t G_n^u(s)ds, \qquad t \in [0, T), \quad n \geq 1. \tag{39}$$

By Young's inequality and Hölder's inequality, it follows that there is a $K^u(T) > 0$ such that

$$|G_n^u(t)| \leq K^u(T), \quad n \geq 1.$$

In view of (38) and (39), an application of Lebesgue's dominated convergence theorem yields that for fixed a.e. $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty}\left(E_n^u(t) - E_n^u(0)\right) = 0.$$

By (24) and the above relation, for fixed $t \in \mathbb{R}_+$, we can get

$$E_u(u) = \lim_{n\to\infty} E_n^u(t) = \lim_{n\to\infty} E_n^u(0) = E_u(u_0).$$

By Theorem 2.8, we infer that for all fixed $t \in \mathbb{R}_+$, $E_u(u)$ is conserved. Similarly, we can show that $E_v(v)$ is also conserved.

Next, we prove that $H(u, v)$ is a conserved density.

By differentiation of the second equation of (31), we obtain this relation:

$$\begin{aligned} & \rho_n * v_{xt} + \rho_{nx} * (uvv_x) + \rho_n * P * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) \\ & \quad - \rho_n * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) + \rho_n * P_x * \left(\frac{1}{2}v_x^2 u_x\right) = 0. \end{aligned} \tag{40}$$

In view of (31), (33) and (40), we obtain

$$\begin{aligned} & \frac{d}{dt}\int_{\mathbb{R}}(\rho_n * u)(\rho_n * v) + (\rho_n * u_x)(\rho_n * v_x)dx \\ = & \int_{\mathbb{R}}(\rho_n * u)(\rho_n * v_t) + (\rho_n * u_x)(\rho_n * v_{xt}) + (\rho_n * u_t)(\rho_n * v) + (\rho_n * u_{xt})(\rho_n * v_x)dx \\ = & -\int_{\mathbb{R}}(\rho_n * u)\left(\rho_n * (uvv_x) + \rho_n * P_x * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+\rho_n * P * \left(\frac{1}{2}v_x^2 u_x\right)\Big) dx \\
&-\int_{\mathbb{R}} (\rho_n * u_x) \Big(\rho_{nx} * (uvv_x) + \rho_n * P * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) \\
&\quad -\rho_n * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) + \rho_n * P_x * \left(\frac{1}{2}v_x^2 u_x\right)\Big) dx \\
&-\int_{\mathbb{R}} (\rho_n * v) \Big(\rho_n * (vuu_x) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) \qquad (41)\\
&\quad +\rho_n * P * \left(\frac{1}{2}u_x^2 v_x\right)\Big) dx \\
&-\int_{\mathbb{R}} (\rho_n * v_x) \Big(\rho_{nx} * (uvu_x) + \rho_n * P * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) \\
&\quad -\rho_n * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v_x\right)\Big) dx.
\end{aligned}
$$

We can analogously get the similar convergence like the case $\frac{d}{dt}\int_{\mathbb{R}}(\rho_n * u)^2 + (\rho_n * u_x)^2 dx$ by using Lemma 2.5, $u(t,\cdot), v(t,\cdot) \in H^1(\mathbb{R})$ and $u_x, v_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$. It is nature to define

$$
H_n(t) = \int_{\mathbb{R}} (\rho_n * u)(\rho_n * v) + (\rho_n * u_x)(\rho_n * v_x) dx, \qquad (42)
$$

and

$$
\begin{aligned}
&G_n^{u,v}(t) \\
=&-\int_{\mathbb{R}} (\rho_n * u) \Big(\rho_n * (uvv_x) + \rho_n * P_x * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) \\
&+\rho_n * P * \left(\frac{1}{2}v_x^2 u_x\right)\Big) dx \\
&-\int_{\mathbb{R}} (\rho_n * u_x) \Big(\rho_{nx} * (uvv_x) + \rho_n * P * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) \\
&-\rho_n * \left(\frac{1}{2}v_x^2 u + vu_x v_x + v^2 u\right) + \rho_n * P_x * \left(\frac{1}{2}v_x^2 u_x\right)\Big) dx \qquad (43)\\
&-\int_{\mathbb{R}} (\rho_n * v) \Big(\rho_n * (vuu_x) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) \\
&+\rho_n * P * \left(\frac{1}{2}u_x^2 v_x\right)\Big) dx \\
&-\int_{\mathbb{R}} (\rho_n * v_x) \Big(\rho_{nx} * (uvu_x) + \rho_n * P * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) \\
&-\rho_n * \left(\frac{1}{2}u_x^2 v + uu_x v_x + u^2 v\right) + \rho_n * P_x * \left(\frac{1}{2}u_x^2 v_x\right)\Big) dx.
\end{aligned}
$$

And it is easy to get

$$
H_n(t) - H_n(0) = \int_0^t G_n^{u,v}(s) ds, \quad t \in [0, T), \ \ n \geq 1. \qquad (44)
$$

Similarly, we get this estimate by using Young's inequality and Holder's inequality:

$$
|G_n^{u,v}(t)| \leq K^{u,v}(T), \quad n \geq 1.
$$

An application of Lebesgue's dominated convergence theorem yields that for fixed a.e. $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty}[H_n(t) - H_n(0)] = 0.$$

By these convergence above, for fixed $t \in \mathbb{R}_+$, we can get

$$H(u,v) = \lim_{n\to\infty} H_n(t) = \lim_{n\to\infty} H_n(0) = H(u_0, v_0),$$

which indicates that $H(u,v)$ is a conserved density.

Since $L^1(\mathbb{R}) \subset (L^\infty(\mathbb{R}))^* \subset (C_0(\mathbb{R}))^* = \mathcal{M}(\mathbb{R})$. It is not too hard to show that for a.e. $t \in [0,T)$,

$$\|m(t,\cdot)\| \le 3e^{\|u_0\|_1 \|v_0\|_1 t}\|m_0\|_{\mathcal{M}(\mathbb{R})}.$$

For any fixed T, $\forall t \in [0,T)$, we have proved

$$(u(t,\cdot) - u_{xx}(t,\cdot)) \in \mathcal{M}(\mathbb{R}).$$

Therefore, in view of (24) and (25), we obtain that for all $t \in [0,T)$, as $n \to \infty$,

$$u^{n_k}(t,\cdot) - u^{n_k}_{xx}(t,\cdot) \to u(t,\cdot) - u_{xx}(t,\cdot) \quad \text{in} \quad D'(\mathbb{R}).$$

Since $u^{n_k}(t,\cdot) - u^{n_k}_{xx}(t,\cdot) \ge 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, we deduce that for a.e. $t \in [0,T)$

$$u(t,\cdot) - u_{xx}(t,\cdot) \in \mathcal{M}^+(\mathbb{R}).$$

Similarly, we arrive at the conclusion:

$$v(t,\cdot) - v_{xx}(t,\cdot) \in \mathcal{M}^+(\mathbb{R}).$$

Finally, we show the uniqueness of the weak solutions of equation (6). Let (u,v) and $(\bar{u},\bar{v})$ be two weak solutions of equation (6) in the class

$$(f,g) \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \times W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

Note that

$$\begin{aligned}
&\|u(t,\cdot) - u_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} \le 3e^{\|u_0\|_1 \|v_0\|_1 t}\|m_0\|_{\mathcal{M}(\mathbb{R})},\\
&\|v(t,\cdot) - v_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} \le 3e^{\|u_0\|_1 \|v_0\|_1 t}\|n_0\|_{\mathcal{M}(\mathbb{R})} \quad \text{for} \quad \text{a.e.} \quad t \in [0,T).
\end{aligned}$$

Define

$$\begin{aligned}
M(T) = \sup_{t\in[0,T)} \big\{ &\|u(t,\cdot) - u_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} + \|v(t,\cdot) - v_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} \\
&+ \|\bar{u}(t,\cdot) - \bar{u}_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} + \|\bar{v}(t,\cdot) - \bar{v}_{xx}(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} \big\}.
\end{aligned}$$

Then for fixed T, we obtain $M(T) < \infty$. For all $(t,x) \in [0,T) \times \mathbb{R}$, in view of (11), we find that

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^1} \le \|P\|_{L^1} M(T) = M(T),\\
&\|u_x(t,\cdot)\|_{L^1} \le \|P_x\|_{L^1} M(T) = M(T),\\
&\|v(t,\cdot)\|_{L^1}, \|v_x(t,\cdot)\|_{L^1}, \|\bar{u}(t,\cdot)\|_{L^1}, \|\bar{u}_x(t,\cdot)\|_{L^1}, \|\bar{v}(t,\cdot)\|_{L^1} \text{ and } \|\bar{v}_x(t,\cdot)\|_{L^1} \le M(T).
\end{aligned} \tag{45}$$

On the other hand, from (29) and (30), we have

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^\infty} \le \|m_0\|_{\mathcal{M}(\mathbb{R})} \le N, \qquad \|u_x(t,\cdot)\|_{L^\infty} \le \|m_0\|_{\mathcal{M}(\mathbb{R})} \le N,\\
&\|v(t,\cdot)\|_{L^\infty} \le \|n_0\|_{\mathcal{M}(\mathbb{R})} \le N, \qquad \|v_x(t,\cdot)\|_{L^\infty} \le \|n_0\|_{\mathcal{M}(\mathbb{R})} \le N,\\
&\|\bar{u}(t,\cdot)\|_{L^\infty}, \|\bar{u}_x(t,\cdot)\|_{L^\infty}, \|\bar{v}(t,\cdot)\|_{L^\infty} \text{ and } \|\bar{v}_x(t,\cdot)\|_{L^\infty} \le N.
\end{aligned} \tag{46}$$

Let us define

$$\hat{u}(t,x) = u(t,x) - \bar{u}(t,x) \quad \text{and} \quad \hat{v}(t,x) = v(t,x) - \bar{v}(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}.$$

Convoluting equation (6) for (u,v) and $(\bar{u},\bar{v})$ with ρ_n, we have that for a.e. $t\in[0,T)$ and all $n\geq 1$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{u}|dx=&\int_{\mathbb{R}}\rho_n*\hat{u}_t\text{sgn}(\rho_n*\hat{u})dx\\ =&-\int_{\mathbb{R}}\rho_n*(\hat{u}vu_x+\bar{u}u_x\hat{v}+\bar{u}\bar{v}\hat{u}_x)\,\text{sgn}(\rho_n*\hat{u})dx\\ &-\int_{\mathbb{R}}\rho_n*P_x*(\frac{1}{2}\hat{u}(u_x+\bar{u}_x)v+\frac{1}{2}\bar{u}_x^2\hat{v}+\hat{u}u_xv_x+\bar{u}v_x\hat{u}_x\\ &+\bar{u}\bar{u}_x\hat{v}_x+\hat{u}(u+\bar{u})v+u^2\hat{v})\text{sgn}(\rho_n*\hat{u})dx\\ &-\int_{\mathbb{R}}\rho_n*\frac{1}{2}P*\left(\hat{u}_x(u_x+\bar{u}_x)v_x+\bar{u}_x^2\hat{v}_x\right)\text{sgn}(\rho_n*\hat{u})dx.\end{aligned}\tag{47}$$

Using (46) and Young's inequality, we infer that for a.e. $t\in[0,T)$ and all $n\geq 1$

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{u}|dx\\ &\leq C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right).\end{aligned}\tag{48}$$

where C is a constant depending on N. Similarly, convoluting equation (6) for (u,v) and $(\bar{u},\bar{v})$ with $\rho_{n,x}$, it follows that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx=&\int_{\mathbb{R}}\rho_n*\hat{u}_{xt}\text{sgn}(\rho_{nx}*\hat{u})dx\\ =&-\int_{\mathbb{R}}\rho_n*(\hat{u}vu_x+\bar{u}u_x\hat{v}+\bar{u}\bar{v}\hat{u}_x)_x\,\text{sgn}(\rho_{nx}*\hat{u})dx\\ &-\int_{\mathbb{R}}\rho_n*P_{xx}*(\frac{1}{2}\hat{u}(u_x+\bar{u}_x)v+\frac{1}{2}\bar{u}_x^2\hat{v}+\hat{u}u_xv_x+\bar{u}v_x\hat{u}_x\\ &\qquad+\bar{u}\bar{u}_x\hat{v}_x+\hat{u}(u+\bar{u})v+u^2\hat{v})\text{sgn}(\rho_{nx}*\hat{u})dx\\ &-\int_{\mathbb{R}}\rho_n*\frac{1}{2}P_x*\left(\hat{u}_x(u_x+\bar{u}_x)v_x+\bar{u}_x^2\hat{v}_x\right)\text{sgn}(\rho_{nx}*\hat{u})dx\\ =&I_1+I_2+I_3.\end{aligned}\tag{49}$$

For the term I_1, we have

$$\begin{aligned}&I_1\\ =&-\int_{\mathbb{R}}\rho_n*(\hat{u}_xvu_x+\hat{u}u_xv_x+\hat{u}vu_{xx}+\bar{u}_xu_x\hat{v}+\bar{u}u_{xx}\hat{v}+\bar{u}u_x\hat{v}_x\\ &\qquad+\bar{u}_x\bar{v}\hat{u}_x+\bar{u}\bar{v}_x\hat{u}_x+\bar{u}\bar{v}\hat{u}_{xx})\text{sgn}(\rho_{nx}*\hat{u})dx\\ \leq& C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)\\ &\qquad-\int_{\mathbb{R}}\rho_n*(\hat{u}vu_{xx}+\bar{u}u_{xx}\hat{v}+\bar{u}\bar{v}\hat{u}_{xx})\text{sgn}(\rho_{nx}*\hat{u})dx\\ \leq& C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)\\ &-\int_{\mathbb{R}}(\rho_n*\hat{u}v)(\rho_n*u_{xx})\text{sgn}(\rho_{nx}*\hat{u})dx-\int_{\mathbb{R}}(\rho_n*\bar{u}\hat{v})(\rho_n*u_{xx})\text{sgn}(\rho_{nx}*\hat{u})dx\\ &\qquad-\int_{\mathbb{R}}(\rho_n*\bar{u}\bar{v})(\rho_n*\hat{u}_{xx})\text{sgn}(\rho_{nx}*\hat{u})dx+R_n(t)\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)\\
&+\int_{\mathbb{R}}(\rho_n*(\hat{u}v)_x)(\rho_n*u_x)\mathrm{sgn}(\rho_{nx}*\hat{u})dx+\int_{\mathbb{R}}(\rho_n*(\bar{u}\hat{v})_x)(\rho_n*u_x)\mathrm{sgn}(\rho_{nx}*\hat{u})dx\\
&\qquad+\int_{\mathbb{R}}(\rho_n*(\bar{u}\bar{v})_x)(\rho_n*\hat{u}_x)\mathrm{sgn}(\rho_{nx}*\hat{u})dx+R_n(t)\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)+R_n(t),
\end{aligned}
\tag{50}
$$

where C is a constant depending on $M(T)$, N, $\|u_0\|_1$ and $\|v_0\|_1$ and $R_n(t)$ satisfies

$$
\begin{cases} R_n(t)\longrightarrow 0, & n\to\infty,\\ |R_n(t)|\le\kappa(T), & n\ge 1,\quad t\in[0,T). \end{cases}
\tag{51}
$$

For the second term I_2, we find

$$
\begin{aligned}
&I_2\\
&=-\int_{\mathbb{R}}\rho_n*P_{xx}*\left(\frac{1}{2}\hat{u}(u_x+\bar{u}_x)v+\frac{1}{2}\bar{u}_x^2\hat{v}+\hat{u}u_xv_x+\bar{u}v_x\hat{u}_x\right.\\
&\qquad\left.+\,\bar{u}\bar{u}_x\hat{v}_x+\hat{u}(u+\bar{u})v+u^2\hat{v}\right)\mathrm{sgn}(\rho_{nx}*\hat{u})dx\\
&\le 2\int_{\mathbb{R}}\rho_n*\left|\frac{1}{2}\hat{u}(u_x+\bar{u}_x)v+\frac{1}{2}\bar{u}_x^2\hat{v}+\hat{u}u_xv_x+\bar{u}v_x\hat{u}_x+\bar{u}\bar{u}_x\hat{v}_x+\hat{u}(u+\bar{u})v+u^2\hat{v}\right|dx\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right).
\end{aligned}
\tag{52}
$$

For the final term I_3, we have

$$
\begin{aligned}
I_3&=-\int_{\mathbb{R}}\rho_n*\frac{1}{2}P_x*\left(\hat{u}_x(u_x+\bar{u}_x)v_x+\bar{u}_x^2\hat{v}_x\right)\mathrm{sgn}(\rho_{nx}*\hat{u})dx\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right).
\end{aligned}
\tag{53}
$$

Adding these three terms, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)+R_n(t).
\end{aligned}
\tag{54}
$$

For these terms $\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{v}|dx$ and $\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx$, we have similar results:

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{v}|dx\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right),\\
&\frac{d}{dt}\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)+R_n(t).
\end{aligned}
\tag{55}
$$

From (48), (54) and (55), we infer that

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)\\
&\le C\left(\int_{\mathbb{R}}|\rho_n*\hat{u}|dx+\int_{\mathbb{R}}|\rho_n*\hat{u}_x|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}|dx+\int_{\mathbb{R}}|\rho_n*\hat{v}_x|dx\right)+R_n(t).
\end{aligned}
\tag{56}
$$

If $\int_{\mathbb{R}} |\rho_n * \hat{u}|dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x|dx \neq 0$, then by Gronwall's inequality, we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}} |\rho_n * \hat{u}|dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x|dx\right)\\ &\leq e^{\int_0^t C + \tilde{R}_n(\tau)d\tau} \left(|\rho_n * \hat{u}| + |\rho_n * \hat{u}_x| + |\rho_n * \hat{v}| + |\rho_n * \hat{v}_x|\right)(0,x),\end{aligned} \tag{57}$$

where $\tilde{R}_n(t) = R_n(t)\left(\int_{\mathbb{R}} |\rho_n * \hat{u}|dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x|dx\right)^{-1}$. From Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}&\left(\int_{\mathbb{R}} |\rho_n * \hat{u}|dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}|dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x|dx\right)\\ &\leq e^{Ct} \left(|\rho_n * \hat{u}| + |\rho_n * \hat{u}_x| + |\rho_n * \hat{v}| + |\rho_n * \hat{v}_x|\right)(0,x),\end{aligned} \tag{58}$$

As T is arbitrary, $\hat{u}_0 = \hat{u}_{0x} = \hat{v}_0 = \hat{v}_{0,x} = 0$, we obtain $(u,v) = (\bar{u},\bar{v})$. This completes the proof of theorem 3.1. □

REFERENCES

[1] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664.

[2] R. Camassa, D. D. Holm and J. M. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.*, **31** (1994), 1–33.

[3] R. M. Chen, F. Guo, Y. Lu and C. Qu, Analysis on the blow-up of solutions to a class of interable peakon equations, *J. Funct. Anal.*, **270** (2016), 2343–2374.

[4] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229–243.

[5] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **26** (1998), 303–328.

[6] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. Math.*, **173** (2011), 559–568.

[7] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Commun. Math. Phys.*, **211** (2000), 45–61.

[8] A. Constantin and W. Strauss, Stability of peakons, *Comm. Pure Appl. Math.*, **53** (2000), 603–610.

[9] A. S. Fokas, P. J. Olver and P. Rosenau, A plethora of integrable bi-Hamiltonian equations, in *Algebraic Aspects of Integrable Systems*, Progr. Nonlinear Differential Equations Appl., 26, Birkhäuser, Boston, MA, 1997, 93–101.

[10] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D*, **4** (1981/82), 47–66.

[11] X. Geng and B. Xue, An extension of integrable peakon equations with cubic nonlinearity, *Nonlinearity*, **22** (2009), 1847–1856.

[12] C. He, X. C. Liu and C. Qu, Orbital stability of peakons and the trains of peakons for an integrable two-component Novikov system, work in progress.

[13] A. Himonas and D. Mantzavinos, The initial value problem for a Novikov system, *J. Math. Phys.* **57** (2016), 21pp.

[14] A. N. W. Hone and J. P. Wang, Integrable peakon equations with cubic nonlinearity, *J. Phys. A*, **41** (2008), 10pp.

[15] Z. Jiang and L. Ni, Blow-up phenomenon for the integrable Novikov equation, *J. Math. Anal. Appl.*, **385** (2012), 551–558.

[16] J. Kang, X. Liu, P. J. Olver and C. Qu, Liouville correspondences between integrable hierarchies, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **13** (2017), 26pp.

[17] S. Lai, Global weak solutions to the Novikov equation, *J. Funct. Anal.*, **265** (2013), 520–544.

[18] H. Li, Two-component generalizations of the Novikov equation, *J. Nonlinear Math. Phys.*, **26** (2019), 390–403.

[19] N. Li and Q. P. Liu, On bi-Hamiltonian structure of two-component Novikov equation, *Phys. Lett. A*, **377** (2013), 257–261.

[20] X. Liu, Y. Liu and C. Qu, Stability of peakons for the Novikov equation, *J. Math. Pures Appl.*, **101** (2014), 172–187.

[21] H. Lundmark and J. Szmigielski, An inverse spectral problem related to the Geng-Xue two-component peakon equation, *Mem. Amer. Math. Soc.*, **244** (2016), 87pp.

[22] J. Málek, J. Nečas, M. Rokyta and M. Růžička, *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Applied Mathematics and Mathematical Computation, 13, Chapman & Hall, London, 1996.

[23] A. V. Mikhailov and V. S. Novikov, Perturbative symmetry approach, *J. Phys. A*, **35** (2002), 4775–4790.

[24] V. Novikov, Generalizations of the Camassa-Holm equation, *J. Phys. A*, **42** (2009), 14pp.

[25] P. J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E*, **53** (1996), 1900–1906.

[26] F. Tiğlay, The periodic Cauchy problem for Novikov's equation, *Int. Math. Res. Not.*, **2011** (2011), 4633–4648.

[27] X. Wu and Z. Yin, Global weak solutions for the Novikov equation, *J. Phys. A.*, **44** (2011), 17pp.

[28] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.*, **53** (2000), 1411–1433.

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E-mail address: 1811071003@nbu.edu.cn
E-mail address: quchangzheng@nbu.edu.cn