

ODD RANDOM ATTRACTORS FOR STOCHASTIC NON-AUTONOMOUS KURAMOTO-SIVASHINSKY EQUATIONS WITHOUT DISSIPATION

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ABSTRACT. We study the random dynamics for the stochastic non-autonomous Kuramoto-Sivashinsky equation in the possibly non-dissipative case. We first prove the existence of a pullback attractor in the Lebesgue space of odd functions, then show that the fiber of the odd pullback attractor semi-converges to a nonempty compact set as the time-parameter goes to minus infinity and finally prove the measurability of the attractor. In a word, we obtain a longtime stable random attractor formed from odd functions. A key tool is the existence of a bridge function between Lebesgue and Sobolev spaces of odd functions.

1. Introduction. The deterministic Kuramoto-Sivashinsky (KS) equation just describes pattern formation phenomena with the phase turbulence, which was first introduced by Kuramoto[16], with developments even in the recent papers [15, 20] (deterministic) or [7, 8, 10, 27] (stochastic).

In this paper, we are concerned with the existence and longtime stability of a pullback random attractor for the stochastic KS equation with additive white noise, time-dependent forces and space-periodic conditions:

$$\begin{cases} du + (\nu D^4 u + D^2 u + uDu)dt = f(t, x)dt + g(x)dW, & t \geq \tau, \\ D^i u(t, -l/2) = D^i u(t, l/2), & i = 0, 1, 2, 3, \\ u(\tau, x) = u_\tau(x), & x \in (-l/2, l/2) =: \mathcal{I}, \end{cases} \quad (1)$$

where $\nu, l > 0$, $D = \frac{\partial}{\partial x}$ and W is a two-sided scalar Wiener process on a probability (Ω, \mathcal{F}, P) .

The deterministic form ($f = g = 0$) is deduced from the the original KS equation (Temam [22]) with an unknown variable \tilde{u} and $u = D\tilde{u}$. Then the periodicity of \tilde{u} and the integration by parts yield

$$\int_{\mathcal{I}} u(t, x)dx = 0, \quad \forall t \in \mathbb{R}. \quad (2)$$

By a similar method as in [7, 8, 10, 27], under the local integrability of the force f and the higher regularity of g , the problem (1)-(2) is well-posed and thus generates a non-autonomous random dynamical system (cocycle) (see [4]).

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However, it is not easy to obtain an attractor since the equation is possibly non-dissipative. Indeed, the first eigenvalue of the differential operator $\nu D^4 + D^2$ is given by

$$\lambda_1 := \left(\frac{2\pi}{l}\right)^2 \left(\nu \left(\frac{2\pi}{l}\right)^2 - 1\right),$$

with two eigenvectors given by $\cos \frac{2\pi x}{l}$ and $\sin \frac{2\pi x}{l}$. If the viscosity is small, e.g. $\nu < l^2/(4\pi^2)$, then $\lambda_1 < 0$ and thus the equation is not dissipative (which is different from dissipative equations in [11]).

In this article, we will show that the cocycle has a pullback random attractor $\mathcal{A}(t, \omega)$ even in the non-dissipative case. Such a bi-parametric attractor (depending on time and sample) seems to have first been introduced by Wang [23] with developments in [9, 14, 26, 28] and the references therein.

In order to overcome the difficulty of non-dissipation, we fix attention on the Lebesgue space H_o of odd functions, where

$$H := \dot{L}^2(\mathcal{I}) = \{w \in L^2(\mathcal{I}) : \int_{\mathcal{I}} w(x) dx = 0\} \text{ and } H_o = \{w \in H : w \text{ is odd}\}.$$

By proving the existence of a bridge function between H_o and V_o (the Sobolev space of odd functions), see Lemma 2.2, we can establish the pullback absorption and the pullback asymptotic compactness of the cocycle in H_o if the force is tempered, and thus obtain a pullback attractor. Of course, this attractor consists of odd functions and thus we call it an odd attractor.

A further topic is to study the longtime stability of the pullback attractor. More precisely, \mathcal{A} is called backward stable if there is a nonempty compact set $E(\omega)$ in H_o such that

$$\lim_{t \rightarrow -\infty} \text{dist}_{H_o}(\mathcal{A}(t, \omega), E(\omega)) = 0. \quad (3)$$

Such a backward stability indicates that the attractor is not explosive and the system has more strong attraction ability in the past. The criteria for the longtime stability are given in terms of backward uniform asymptotic compactness of the cocycle, see [25] in the stochastic case and see [12, 13] in the deterministic case.

If we further assume that the force f is backward tempered, then we can prove the backward uniform asymptotic compactness, which leads to the longtime stability as in (3). We conveniently obtain the backward compactness of the pullback attractor.

Since the backward absorbing set is an uncountable union of random sets, the measurability of the absorbing set (and thus the attractor) seems to be unknown. In order to overcome this difficulty, we consider two universes, one is the usual tempered universe and another is backward tempered. We prove an important result that the pullback attractors are the same set with respect to different universes and thus the measurability of the tempered attractor implies the measurability of the backward tempered attractor.

In a word, the stochastic equation (1) has a longtime stable random attractor formed from odd functions.

2. Random dynamical systems in the space of odd functions. In this section, we prove the existence of a bridge function and define a cocycle in the Lebesgue or Sobolev spaces of odd functions.

2.1. Sobolev spaces of odd functions. Let $H = \dot{L}^2(\mathcal{I})$ with the L^2 -norm $\|\cdot\|$ and equip $V = \dot{H}_{\text{per}}^2(\mathcal{I}) = H_{\text{per}}^2(\mathcal{I}) \cap H$ with the following scalar product and norm:

$$((u, v)) = \int_{\mathcal{I}} D^2u D^2v dx, \quad \|u\|_V^2 = ((u, u)) = \|D^2u\|^2, \quad \forall u, v \in V.$$

Let H_o be the subset of H formed from all odd functions and $V_o = V \cap H_o$.

Lemma 2.1. (i) H_o and V_o are closed linear subspaces of H and V respectively, and thus they are Hilbert spaces too.

(ii) Any bounded set in V_o is pre-compact in H_o .

Proof. (i) The linearity follows from the fact that the linear combination of two odd functions is still odd.

We then prove the closedness. Let $u_n \in H_o$ and $u \in H$ such that $\|u_n - u\| \rightarrow 0$. Then there are an index subsequence n_k and a set $\mathcal{I}_1 \subset \mathcal{I}$ with Lebesgue measure zero such that

$$u_{n_k}(x) \rightarrow u(x), \quad \text{for all } x \in \mathcal{I} \setminus \mathcal{I}_1.$$

Let $\mathcal{I}_2 = \{x \in \mathcal{I} : -x \in \mathcal{I}_1\}$. Then, for all $x \in \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2)$, we know $-x \in \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2)$ and

$$u(-x) = \lim_{k \rightarrow \infty} u_{n_k}(-x) = - \lim_{k \rightarrow \infty} u_{n_k}(x) = -u(x),$$

which means u is odd on $\mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2)$. Since the Lebesgue measure of $\mathcal{I}_1 \cup \mathcal{I}_2$ is zero, u almost everywhere equals to an odd function and thus $u \in H_o$.

(ii) Let $\{u_n\}$ be a bounded sequence in V_o . By the compactness of the Sobolev embedding $V \hookrightarrow H$, there is a subsequence such that $u_{n_k} \rightarrow u$ in H . By the closedness of H_o in H as proved above, we know $u \in H_o$. Hence $\{u_n\}$ is pre-compact in H_o as desired. \square

The following result means the existence of a bridge function between H_o and V_o , which improves [22, lemma III 4.1] (see also [21]) and will be very useful.

Lemma 2.2. For any $a, b > 0$, there is an odd function $\xi \in \dot{C}^\infty[-l/2, l/2]$, given by

$$\xi(x) = -\frac{al}{\pi} \sum_{k=1}^M \frac{1}{k} \sin \frac{2k\pi x}{l},$$

where $M := M(a, b, l)$ is large enough, such that

$$a\|u\|^2 \leq b\|D^2u\|^2 + (uD\xi, u), \quad \forall u \in V_o. \tag{4}$$

Proof. From the definition of ξ it is easy to show that

$$-D\xi(x) = 2a \sum_{k=1}^M \cos \frac{2k\pi x}{l} = a \sum_{0 < |k| \leq M} e^{2k\pi ix/l},$$

which further implies that, for every $u \in V_o$,

$$\begin{aligned} a\|u\|^2 - (uD\xi, u) &= \int_{-l/2}^{l/2} (a - D\xi(x))u^2(x)dx \\ &= a \sum_{|k| \leq M} \int_{-l/2}^{l/2} u^2(x)e^{2k\pi ix/l}dx = a \sum_{|k| \leq M} f_k, \end{aligned}$$

where f_k is the k th Fourier coefficient of the function $u^2(\cdot)$. Since $u \in V$, the Sobolev embedding $H^2(\mathcal{I}) \hookrightarrow C^{1,1/2}(\mathcal{I})$ yields the continuity of $u(\cdot)$ and so is $u^2(\cdot)$. Hence the Fourier series converges:

$$u^2(x) = \sum_{k \in \mathbb{Z}} f_k e^{2k\pi i x/l} \text{ and particularly } u^2(0) = \sum_{k \in \mathbb{Z}} f_k.$$

Since u is odd, we have $u(0) = 0$ and thus $\sum_{k \in \mathbb{Z}} f_k = 0$, which further implies

$$\begin{aligned} \left| \sum_{|k| \leq M} f_k \right| &= \left| \sum_{|k| > M} f_k \right| \\ &\leq \left(\sum_{|k| > M} (2k\pi/l)^4 |f_k|^2 \right)^{1/2} \left(\sum_{|k| > M} (2k\pi/l)^{-4} \right)^{1/2}. \end{aligned}$$

By the Parserval identity $\|D^2 u^2\|^2 = l \sum_{k \in \mathbb{Z}} (2k\pi/l)^4 f_k^2$, we obtain

$$\left| \sum_{|k| \leq M} f_k \right| \leq c_1 \|D^2 u^2\| \left(\sum_{|k| > M} k^{-4} \right)^{1/2} \leq c_2 \|D^2 u^2\| M^{-\frac{3}{2}}.$$

Since $H^2(\mathcal{I})$ is a Banach algebra in dimension one, it follows that

$$\|D^2 u^2\| \leq c_3 \|u^2\|_{H^2} \leq c_4 \|u\|_{H^2}^2 \leq c_5 \|D^2 u\|^2.$$

All above inequalities further imply

$$a \|u\|^2 - (uD\xi, u) \leq c_6 M^{-\frac{3}{2}} \|D^2 u\|^2.$$

We can choose $M \geq (c_6/b)^{2/3}$ to obtain (4) as desired. □

2.2. Random dynamical systems on odd spaces. Consider the quadruple $(\Omega, \mathcal{F}, P, \theta_s)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with the compact-open topology, \mathcal{F} is the Borel algebra, P is the two-sided Wiener measure and $\theta_s \omega(\cdot) = \omega(s + \cdot) - \omega(s)$.

As usual [1, 2, 3, 17], we identify $W(s, \omega) = \omega(s)$ and then the solution of $dz + z = dW$ can be denoted by $z(\theta_s \omega)$. The mapping $s \rightarrow z(\theta_s \omega)$ is continuous and the random variable $|z(\omega)|$ is tempered:

$$\lim_{t \rightarrow \pm\infty} \frac{|z(\theta_t \omega)|}{t} = 0, \int_{-\infty}^0 e^{\varepsilon s} |z(\theta_s \omega)|^M ds < +\infty, \forall \varepsilon, M > 0. \tag{5}$$

By the ergodic theorem, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_{-t}^0 |z(\theta_s \omega)| ds = \mathbb{E}|z| > 0. \tag{6}$$

Both (5) and (6) hold true in a θ -invariant full-measure subset of Ω , but we do not distinguish the full-measure subset and Ω . Put

$$v(t, \tau, \omega) = u(t, \tau, \omega) - \xi - gz(\theta_t \omega), \tag{7}$$

where $\xi \in \dot{C}^\infty[-l/2, l/2]$ is given in (4) and we assume $g \in \dot{H}_{\text{per}}^4(\mathcal{I}) \cap H_o$.

By the change (7) of variables, we can rewrite the equation (1) as a random equation (without the stochastic derivative):

$$\begin{cases} \frac{dv}{dt} + \nu D^4 v + D^2 v + v Dv + D(\xi v) + z(\theta_t \omega) D(gv) = \\ f(t) - \nu D^4 \xi - D^2 \xi - \xi D\xi + z(\theta_t \omega)(g - \nu D^4 g - D^2 g - zg Dg - g D\xi), \\ D^i v(t, -l/2) = D^i v(t, l/2), \quad i = 0, 1, 2, 3, \quad t > \tau, \\ v(\tau, x) = v_\tau(x), \quad x \in \mathcal{I}, \quad \tau \in \mathbb{R}. \end{cases} \quad (8)$$

Lemma 2.3. *Suppose $f \in L^2_{loc}(\mathbb{R}, H)$ and $g \in \dot{H}^4_{per}(\mathcal{I})$. Then, for $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in H$, the equation (8) has a unique weak solution $v(t, \tau, \omega; v_\tau)$ such that $v(\tau, \tau, \omega; v_\tau) = v_\tau$,*

$$v \in C([\tau, +\infty); H) \cap L^2_{loc}(\tau, +\infty; V)$$

and the solution $v(s, \tau, \omega; v_\tau)$ is (joint) continuous in $s \geq \tau$ and $v_\tau \in H$. If we further assume that $f(s), g$ and v_τ are odd functions, then we have

$$v \in C([\tau, +\infty); H_o) \cap L^2_{loc}(\tau, +\infty; V_o).$$

Proof. By the similar method as in [7, 8, 10, 27], the equation (1) is well-posed in H and so is the equation (8).

We prove the assertion about odd functions. Let u be the solution of the equation (1) and define $\hat{u}(x) = -u(-x)$ for all $x \in \mathcal{I}$. Since $f(s)$ and g are odd, it follows that \hat{u} fulfills (1) too. Since v_τ, ξ, g are odd, it follows that $u_\tau = v_\tau + \xi + gz(\theta_\tau \omega)$ is odd. Then $\hat{u}(\tau, \tau, \omega)(x) = -u_\tau(-x) = u_\tau(x)$, which means the initial conditions are the same. By the uniqueness of solutions, we have $\hat{u}(x) = u(x)$, i.e. $-u(-x) = u(x)$. Hence, the solution u of the equation (1) is odd and so is v (by(7)). \square

By Lemma 2.3, under the assumptions of $f \in L^2_{loc}(\mathbb{R}, H_o)$ and $g \in \dot{H}^4_{per} \cap H_o$, we obtain a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H_o \rightarrow H_o$ defined by

$$\begin{aligned} \Phi(t, \tau, \omega) u_\tau &= u(\tau + t, \tau, \theta_{-\tau} \omega; u_\tau) \\ &= v(\tau + t, \tau, \theta_{-\tau} \omega; v_\tau) + \xi + gz(\theta_t \omega), \quad \forall t \geq 0, \tau \in \mathbb{R}, u_\tau \in H_o, \end{aligned} \quad (9)$$

where the \mathfrak{F} -measurability of Φ can be proved by the same method as given in [6] and the uniqueness of solutions implies the cocycle property:

$$\Phi(t_1 + t_2, \tau, \omega) = \Phi(t_1, \tau + t_2, \theta_{t_2} \omega) \Phi(t_2, \tau, \omega), \quad \forall t_1, t_2 \geq 0. \quad (10)$$

We then take the universe \mathfrak{B} by the collection of all backward tempered bi-parametric sets $B = \{B(\tau, \omega) \subset H_o : \tau \in \mathbb{R}, \omega \in \Omega\}$, where B is called backward tempered if

$$\lim_{t \rightarrow +\infty} e^{-\alpha t} \sup_{s \leq \tau} \|B(s - t, \theta_{-t} \omega)\|^2 = 0, \quad \forall \alpha > 0, \tau \in \mathbb{R}, \omega \in \Omega. \quad (11)$$

In order to prove the existence of a \mathfrak{B} -pullback attractor, we further assume $f \in L^2_{loc}(\mathbb{R}, H_o)$ such that there is a $\alpha_0 > 0$ such that

$$\sup_{s \leq \tau} \int_{-\infty}^0 e^{\alpha_0 r} \|f(r + s)\|^2_{H_o} dr < \infty, \quad \forall \tau \in \mathbb{R}. \quad (12)$$

By [25], the assumption (12) implies that it holds true for all positive rates:

$$\sup_{s \leq \tau} \int_{-\infty}^0 e^{\alpha r} \|f(r + s)\|^2_{H_o} dr < \infty, \quad \forall \alpha > 0, \tau \in \mathbb{R}. \quad (13)$$

3. **Backward-uniform estimates.** We will frequently use the trilinear form

$$b(u, v, w) = \int_{\mathcal{I}} u(Dv)w dx, \quad \forall u, v, w \in V$$

and the following relationships

$$b(u, u, u) = 0, \quad b(u, v, u) = -2b(v, u, u), \tag{14}$$

$$|b(u, u, v)| \leq c\|u\|\|D^2u\|\|v\|. \tag{15}$$

By the Sobolev embedding $H^4(\mathcal{I}) \hookrightarrow C^2(\bar{\mathcal{I}})$, the assumption $g \in \dot{H}_{\text{per}}^4(\mathcal{I})$ implies $g \in C^2(\bar{\mathcal{I}}) \subset W^{2,\infty}(\mathcal{I})$ and thus

$$\beta := (\mathbb{E}|z| + 1)\|Dg\|_{\infty} + 1 < +\infty.$$

Lemma 3.1. *Assume $f \in L^2_{\text{loc}}(\mathbb{R}, H_0)$ fulfilling (12) and $g \in \dot{H}_{\text{per}}^4(\mathcal{I}) \cap H_0$. Then, for $B \in \mathfrak{B}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is $T = T(B, \tau, \omega) \geq 1$ such that for all $t \geq T$ and $u_{s-t} \in B(s-t, \theta_{-t}\omega)$ with $s \leq \tau$,*

$$\sup_{s \leq \tau} \|u(s, s-t, \theta_{-s}\omega; u_{s-t})\|^2 \leq cR(\tau, \omega), \tag{16}$$

$$\sup_{s \leq \tau} \int_{s-t}^s e^{\beta(r-s) + \|Dg\|_{\infty} \int_r^s |z(\theta_{\hat{r}-s}\omega)| d\hat{r}} \|D^2v(r)\|^2 dr \leq cR(\tau, \omega), \tag{17}$$

$$\sup_{s \leq \tau} \sup_{\sigma \in [s-1, s]} \|v(\sigma, s-t, \theta_{-s}\omega; v_{s-t})\|^2 \leq C(\omega)(1 + F(\tau)), \tag{18}$$

where $R(\tau, \omega) = 1 + \rho(\omega) + R_f(\tau, \omega)$,

$$\rho(\omega) = |z(\omega)| + \int_{-\infty}^0 e^{\beta r + \|Dg\|_{\infty} \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} (|z(\theta_r\omega)|^4 + 1) dr, \tag{19}$$

$$R_f(\tau, \omega) = \sup_{s \leq \tau} \int_{-\infty}^0 e^{\beta r + \|Dg\|_{\infty} \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} \|f(r+s)\|^2 dr, \tag{20}$$

$C(\omega)$ is an intrinsic positive random variable and

$$F(\tau) = \sup_{s \leq \tau} \int_{-\infty}^0 e^r \|f(r+s)\|^2 dr < +\infty.$$

Proof. Multiplying (8) by $v(r, s-t, \theta_{-s}\omega; u_{s-t})$ and integrating over \mathcal{I} yield

$$\begin{aligned} & \frac{d}{dr} \|v\|^2 + 2\nu\|D^2v\|^2 - 2\|Dv\|^2 + 2(vDv, v) + 2(D(\xi v), v) \\ &= -2z(\theta_{r-s}\omega)(D(gv), v) + 2(f(r), v) + 2(h(\xi), v) + 2z(\theta_{r-s}\omega)(\hat{h}(g), v), \end{aligned}$$

where

$$\begin{aligned} h(\xi) &= -\nu D^4\xi - D^2\xi - \xi D\xi, \\ \hat{h}(g) &= g - \nu D^4g - D^2g - z(\theta_{r-s}\omega)gDg - gD\xi. \end{aligned} \tag{21}$$

By the integration by parts, we have $\|Dv\|^2 \leq \|v\|\|D^2v\|$ and thus

$$-2\|Dv\|^2 \geq -2\|v\|\|D^2v\| \geq -\frac{\nu}{2}\|D^2v\|^2 - \frac{2}{\nu}\|v\|^2.$$

By (14), $(vDv, v) = 0$ and

$$2(D(\xi v), v) = 2(vD\xi, v) + 2(\xi Dv, v) = (vD\xi, v).$$

By (14) again,

$$-2z(\theta_{r-s}\omega)(D(gv), v) = -z(\theta_{r-s}\omega)(vDg, v) \leq \|Dg\|_\infty |z(\theta_{r-s}\omega)| \|v\|^2.$$

By the Young inequality,

$$2(f(r), v) \leq \|f(r)\|^2 + \|v\|^2.$$

Since $\xi \in \dot{C}^\infty[-l/2, l/2]$ as in Lemma 2.2, it follows that

$$2(h(\xi), v) = -2(\nu D^4 \xi + D^2 \xi + \xi D \xi, v) \leq \|v\|^2 + c.$$

Since $g \in H^4(\mathcal{I})$, it follows that

$$\begin{aligned} 2z(\theta_{r-s}\omega)(h(g), v) &= 2z(\theta_s\omega)(g - \nu D^4 g - D^2 g - z(\theta_{r-s}\omega)gDg - gD\xi, v) \\ &\leq \|v\|^2 + c(|z(\theta_{r-s}\omega)|^2 + |z(\theta_{r-s}\omega)|^4) \leq \|v\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4). \end{aligned}$$

From all above estimates, we obtain

$$\begin{aligned} \frac{d}{dr} \|v\|^2 + \frac{3}{2} \nu \|D^2 v\|^2 + (vD\xi, v) - \left(\frac{2}{\nu} + 3 + |z(\theta_{r-s}\omega)| \|Dg\|_\infty \right) \|v\|^2 \\ \leq \|f(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4). \end{aligned} \tag{22}$$

Given $\beta = (\mathbb{E}|z| + 1) \|Dg\|_\infty + 1$. Applying Lemma 2.2 with $a = \beta + \frac{2}{\nu} + 3$ and $b = \frac{\nu}{2}$, we can choose ξ such that

$$\frac{\nu}{2} \|D^2 v\|^2 + (vD\xi, v) \geq (\beta + \frac{2}{\nu} + 3) \|v\|^2.$$

Then we can rewrite (22) as

$$\begin{aligned} \frac{d}{dr} \|v\|^2 + (\beta - |z(\theta_{r-s}\omega)| \|Dg\|_\infty) \|v\|^2 + \nu \|D^2 v\|^2 \\ \leq \|f(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4). \end{aligned} \tag{23}$$

Multiplying (23) by

$$e^{\int_{s-t}^r (\beta - \|Dg\|_\infty |z(\theta_{\hat{r}-s}\omega)|) d\hat{r}} = e^{\beta(r-s+t) - \|Dg\|_\infty \int_{s-t}^r |z(\theta_{\hat{r}-s}\omega)| d\hat{r}},$$

and then integrating the product over $r \in [s-t, \sigma]$ with $\sigma \geq s-t$, we obtain

$$\begin{aligned} &\|v(\sigma, s-t, \theta_{-s}\omega; v_{s-t})\|^2 \\ &+ \nu \int_{s-t}^\sigma e^{\beta(r-\sigma) + \|Dg\|_\infty \int_r^\sigma |z(\theta_{\hat{r}-s}\omega)| d\hat{r}} \|D^2 v(r)\|^2 dr \\ &\leq \|v_{s-t}\|^2 e^{-\beta(\sigma - (s-t)) + \|Dg\|_\infty \int_{s-t}^\sigma |z(\theta_{r-s}\omega)| dr} \\ &+ \int_{s-t}^\sigma e^{\beta(r-\sigma) + \|Dg\|_\infty \int_r^\sigma |z(\theta_{\hat{r}-s}\omega)| d\hat{r}} (\|f(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4)) dr. \end{aligned} \tag{24}$$

For all $s \leq \tau$, we take $\sigma = s$ in (24) to obtain

$$\begin{aligned}
 & \|v(s, s - t, \theta_{-s}\omega; v_{s-t})\|^2 \\
 & + \nu \int_{s-t}^s e^{\beta(r-s) + \|Dg\|_\infty \int_r^s |z(\theta_{\hat{r}-s}\omega)| d\hat{r}} \|D^2v(r)\|^2 dr \\
 & \leq \|v_{s-t}\|^2 e^{-\beta t + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} \\
 & + \int_{s-t}^s e^{\beta(r-s) + \|Dg\|_\infty \int_{r-s}^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} (\|f(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4)) dr \\
 & = \|v_{s-t}\|^2 e^{-\beta t + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} \\
 & + \int_{-t}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} (\|f(r+s)\|^2 + c(1 + |z(\theta_r\omega)|^4)) dr. \tag{25}
 \end{aligned}$$

By the change (7) of variables, we have

$$\begin{aligned}
 v(s, s - t, \theta_{-s}\omega; v_{s-t}) &= u(s, s - t, \theta_{-s}\omega; u_{s-t}) - \xi - gz(\omega), \\
 v_{s-t} &= u_{s-t} - \xi - gz(\theta_{-t}\omega),
 \end{aligned} \tag{26}$$

which together with (25) implies

$$\begin{aligned}
 & \|u(s, s - t, \theta_{-s}\omega; u_{s-t})\|^2 \leq c\|v(s, s - t, \theta_{-s}\omega; v_{s-t})\|^2 + c(1 + |z(\omega)|) \\
 & \leq c(\|u_{s-t}\|^2 + 1 + |z(\theta_{-t}\omega)|) e^{-\beta t + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} + c(1 + |z(\omega)|) \\
 & + c \int_{-t}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} (\|f(r+s)\|^2 + |z(\theta_r\omega)|^4 + 1) dr. \tag{27}
 \end{aligned}$$

By the ergodic limit (6), there is $T_1 > 0$ such that

$$\int_{-t}^0 |z(\theta_r\omega)| dr \leq (\mathbb{E}|z| + 1)t, \quad \forall t \geq T_1.$$

Since $\beta = (\mathbb{E}|z| + 1)\|Dg\|_\infty + 1$, we have for all $t \geq T_1$,

$$e^{-\beta t + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} \leq e^{-\beta t + \|Dg\|_\infty (\mathbb{E}|z| + 1)t} = e^{-t} \leq 1, \tag{28}$$

which together with (5) implies that there is $T_2 \geq T_1$ such that for all $t \geq T_2$,

$$|z(\theta_{-t}\omega)| e^{-\beta t + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} \leq |z(\theta_{-t}\omega)| e^{-t} \leq 1.$$

Since $u_{s-t} \in B(s - t, \theta_{-t}\omega)$, we see from (11) and (28) that there is a $T_3 \geq T_2$ such that for all $t \geq T_3$,

$$\sup_{s \leq \tau} \|u_{s-t}\|^2 e^{-\beta t + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} \leq e^{-t} \sup_{s \leq \tau} \|B(s - t, \theta_{-t}\omega)\|^2 \leq 1.$$

Hence, by taking the supremum of (27) over $s \in (-\infty, \tau]$, we have for all $t \geq T_3$,

$$\begin{aligned}
 & \sup_{s \leq \tau} \|u(s, s - t, \theta_{-s}\omega; u_{s-t})\|^2 \leq c(1 + |z(\omega)|) \\
 & + c \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} (|z(\theta_r\omega)|^4 + 1) dr \\
 & + c \sup_{s \leq \tau} \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} \|f(r+s)\|^2 dr \\
 & = c(1 + \rho(\omega) + R_f(\tau, \omega)),
 \end{aligned}$$

which proves (16). Consider the second term in (25) and take the supremum over $(-\infty, \tau]$, we obtain (17) as follows:

$$\begin{aligned} & \sup_{s \leq \tau} \int_{s-t}^s e^{\beta(r-s) + \|Dg\|_\infty \int_r^s |z(\theta_{\hat{r}-s}\omega)| d\hat{r}} \|D^2v(r)\|^2 dr \\ & \leq c(1 + \rho(\omega) + R_f(\tau, \omega)), \quad \forall t \geq T_3. \end{aligned}$$

Finally, by the ergodic limit (6), one can prove that there is an intrinsic random variable $C(\omega) > 0$ such that

$$\int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r} \leq -r(\mathbb{E}|z| + 1) + C(\omega), \quad \forall r \leq 0,$$

which implies that for another intrinsic random variable (still denoted by $C(\omega)$),

$$e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} \leq C(\omega)e^r, \quad \forall r \leq 0. \tag{29}$$

Using (28)-(29), we see from (24) that for all $\sigma \in [s - 1, s]$, $s \leq \tau$ and $t \geq T_3$,

$$\begin{aligned} & \|v(\sigma, s - t, \theta_{-s}\omega; v_{s-t})\|^2 \\ & \leq \|v_{s-t}\|^2 e^{-\beta(\sigma-(s-t)) + \|Dg\|_\infty \int_{-t}^0 |z(\theta_r\omega)| dr} \\ & + c \int_{-t}^{\sigma-s} e^{\beta(r+s-\sigma) + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} (\|f(r+s)\|^2 + |z(\theta_r\omega)|^4 + 1) dr \\ & \leq \|v_{s-t}\|^2 e^{\beta(s-\sigma)} e^{-t} + C(\omega) \int_{-t}^0 e^{\beta(s-\sigma)} e^r (\|f(r+s)\|^2 + |z(\theta_r\omega)|^4 + 1) dr \\ & \leq c \|v_{s-t}\|^2 e^{-t} + C(\omega) \int_{-t}^0 e^r (\|f(r+s)\|^2 + |z(\theta_r\omega)|^4 + 1) dr. \end{aligned} \tag{30}$$

By taking the supremum in $s \in (-\infty, \tau]$ and $\sigma \in [s - 1, s]$, we have for all $t \geq T_3$,

$$\begin{aligned} & \sup_{s \leq \tau} \sup_{\sigma \in [s-1, s]} \|v(\sigma, s - t, \theta_{-s}\omega; v_{s-t})\|^2 \\ & \leq C(\omega) \left(1 + \sup_{s \leq \tau} \int_{-\infty}^0 e^r (\|f(r+s)\|^2) dr \right), \end{aligned}$$

which is finite in view of (13). Hence, (18) holds true. In addition, by (29), $\rho(\omega)$ and $R_f(\tau, \omega)$ are finite. The proof is complete. \square

We need the non-autonomous version of the uniform Gronwall lemma (see [19]).

Lemma 3.2. *If y, h_1, h_2 is non-negative and locally integrable on \mathbb{R} such that*

$$y'(r) \leq h_1(r)y(r) + h_2(r), \quad \forall r \geq s - 1,$$

where $s \in \mathbb{R}$. Then

$$y(s) \leq \left(\int_{s-1}^s y(r) dr + \int_{s-1}^s h_2(r) dr \right) e^{\int_{s-1}^s h_1(r) dr}. \tag{31}$$

Lemma 3.3. *For $B \in \mathfrak{B}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is $T = T(B, \tau, \omega) \geq 1$ such that for all $t \geq T$,*

$$\sup_{s \leq \tau} \|D^2u(s, s - t, \theta_{-s}\omega; u_{s-t})\|^2 \leq R_V(\tau, \omega) < +\infty, \tag{32}$$

uniformly in $u_{s-t} \in B(s - t, \theta_{-t}\omega)$ for all $s \leq \tau$.

Proof. Multiplying (8) by $D^4v(r, s - t, \theta_{-s}\omega; u_{s-t})$ and integrating over \mathcal{I} yield

$$\begin{aligned} & \frac{d}{dr} \|D^2v\|^2 + 2\nu \|D^4v\|^2 - 2\|D^3v\|^2 + 2(vDv, D^4v) + 2(D(\xi v), D^4v) \\ &= -2z(\theta_{r-s}\omega)(D(gv), D^4v) + 2(f(r), D^4v) \\ &+ 2(h(\xi), D^4v) + 2z(\theta_{r-s}\omega)(\hat{h}(g), D^4v), \end{aligned}$$

where $h(\xi)$ and $\hat{h}(g)$ are defined by (21). By the interpolation inequality,

$$-2\|D^3v\|^2 \geq -2\|D^2v\| \|D^4v\| \geq -\frac{\nu}{8} \|D^4v\|^2 - c\|D^2v\|^2.$$

By (15),

$$2(vDv, D^4v) \geq -c\|v\| \|D^2v\| \|D^4v\| \geq -\frac{\nu}{8} \|D^4v\|^2 - c\|v\|^2 \|D^2v\|^2.$$

Note that $H^2(\mathcal{I})$ is a Banach algebra, by the Poincare inequality,

$$\begin{aligned} 2(D(\xi v), D^4v) &\geq -c\|D(\xi v)\| \|D^4v\| \geq -c\|D^2(\xi v)\| \|D^4v\| \\ &\geq -c\|D^2\xi\| \|D^2v\| \|D^4v\| \geq -\frac{\nu}{8} \|D^4v\|^2 - c\|D^2v\|^2. \end{aligned}$$

By the similar method,

$$\begin{aligned} -2z(\theta_{r-s}\omega)(D(gv), D^4v) &\leq c|z(\theta_{r-s}\omega)| \|D^2g\| \|D^2v\| \|D^4v\| \\ &\leq \frac{\nu}{8} \|D^4v\|^2 + c|z(\theta_{r-s}\omega)|^2 \|D^2v\|^2. \end{aligned}$$

By the Young inequality,

$$2(f(r), D^4v) \leq \frac{\nu}{8} \|D^4v\|^2 + c\|f(r)\|^2.$$

Since $\xi \in \dot{C}^\infty[-l/2, l/2]$ as in Lemma 2.2, it follows that

$$2(h(\xi), D^4v) = -2(\nu D^4\xi + D^2\xi + \xi D\xi, D^4v) \leq \frac{\nu}{8} \|D^4v\|^2 + c.$$

Since $g \in H^4(\mathcal{I})$, it follows that

$$\begin{aligned} & 2z(\theta_{r-s}\omega)(h(g), D^4v) \\ &= 2z(\theta_s\omega)(g - \nu D^4g - D^2g - z(\theta_{r-s}\omega)gDg - gD\xi, D^4v) \\ &\leq \frac{\nu}{8} \|D^4v\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4). \end{aligned}$$

From all above estimates, we obtain

$$\begin{aligned} & \frac{d}{dr} \|D^2v\|^2 \leq c(\|v\|^2 + |z(\theta_{r-s}\omega)|^2 + 1) \|D^2v\|^2 \\ &+ c(\|f(r)\|^2 + |z(\theta_{r-s}\omega)|^4 + 1). \end{aligned} \tag{33}$$

By using the uniform Gronwall lemma on (33) with

$$\begin{aligned} y(r) &= \|D^2v(r, s - t, \theta_{-s}\omega; v_{s-t})\|^2, \\ h_1(r) &= c(\|v(r, s - t, \theta_{-s}\omega; v_{s-t})\|^2 + |z(\theta_{r-s}\omega)|^2 + 1), \\ h_2(r) &= c(\|f(r)\|^2 + |z(\theta_{r-s}\omega)|^4 + 1), \end{aligned}$$

we see from (31) that for all $s \leq \tau$,

$$\|D^2v(s, s - t, \theta_{-s}\omega; v_{s-t})\|^2 \leq \left(\int_{s-1}^s y(r)dr + \int_{s-1}^s h_2(r)dr \right) e^{\int_{s-1}^s h_1(r)dr}. \tag{34}$$

We then estimate the supremum of each term in (34) with respect to $s \leq \tau$. By (17) and the continuity of $|z(\theta, \omega)|$, for all $t \geq T \geq 1$,

$$\begin{aligned} \sup_{s \leq \tau} \int_{s-1}^s y(r) dr &= \sup_{s \leq \tau} \int_{s-1}^s \|D^2 v(r, s-t, \theta_{-s} \omega; v_{s-t})\|^2 dr \\ &\leq C(\omega) \sup_{s \leq \tau} \int_{s-1}^s e^{\beta(r-s) + \|Dg\|_\infty \int_r^s |z(\theta_{\hat{r}-s} \omega)| d\hat{r}} \|D^2 v(r)\|^2 dr \\ &\leq C(\omega) R(\tau, \omega) < +\infty. \end{aligned}$$

By (13) and the continuity of $|z(\theta, \omega)|$,

$$\begin{aligned} \sup_{s \leq \tau} \int_{s-1}^s h_2(r) dr &= c \sup_{s \leq \tau} \int_{s-1}^s (\|f(r)\|^2 + |z(\theta_{r-s} \omega)|^4 + 1) dr \\ &\leq c \sup_{s \leq \tau} \int_{s-1}^s e^{r-s} \|f(r)\|^2 dr + c \sup_{\sigma \in [-1, 0]} |z(\theta_\sigma \omega)|^4 + c \\ &\leq c \sup_{s \leq \tau} \int_{-\infty}^0 e^r \|f(r+s)\|^2 dr + C(\omega) \leq cF(\tau) + C(\omega) < +\infty. \end{aligned}$$

By (18),

$$\begin{aligned} \sup_{s \leq \tau} \int_{s-1}^s h_1(r) dr &= c \sup_{s \leq \tau} \int_{s-1}^s (\|v(r, s-t, \theta_{-s} \omega; v_{s-t})\|^2 + |z(\theta_{r-s} \omega)|^2 + 1) dr \\ &\leq c \sup_{s \leq \tau} \sup_{r \in [s-1, s]} \|v(r, s-t, \theta_{-s} \omega; v_{s-t})\|^2 + C(\omega) \\ &\leq C(\omega)(1 + F(\tau)) < +\infty. \end{aligned}$$

Hence,

$$\sup_{s \leq \tau} \|D^2 v(s, s-t, \theta_{-s} \omega; v_{s-t})\|^2 \leq C(\omega)(F(\tau) + R(\tau, \omega))e^{C(\omega)(1+F(\tau))}.$$

Finally, by the change (26) of variables and (34),

$$\begin{aligned} \sup_{s \leq \tau} \|D^2 u(s, s-t, \theta_{-s} \omega; v_{s-t})\|^2 &\leq 8(\sup_{s \leq \tau} \|D^2 v(s, s-t, \theta_{-s} \omega; v_{s-t})\|^2 + \|D^2 \xi\|^2 + \|D^2 g\|^2 |z(\omega)|) \\ &\leq C(\omega)(F(\tau) + R(\tau, \omega))e^{C(\omega)(1+F(\tau))} + c(1 + |z(\omega)|) =: R_V(\tau, \omega) < +\infty. \end{aligned}$$

The proof is complete. □

4. Existence and backward stability of odd random attractors. We need the concept of a pullback random attractor as introduced by Wang[23].

Definition 4.1. Let Φ be a cocycle (fulfilling (10)) on a Polish space X over the quadruple $(\Omega, \mathcal{F}, P, \theta)$ and \mathfrak{D} a universe of some bi-parametric sets on X . Then $\mathcal{A} \in \mathfrak{D}$ is called a pullback attractor for Φ if it is compact, invariant under Φ , and \mathfrak{D} -pullback attracting. The pullback attractor is called a pullback random attractor if it is further random, i.e. each $\mathcal{A}(\tau, \cdot)$ is a random set.

For a pullback attractor, we can consider its longtime stability, i.e. the limiting behavior of its fiber when the time-parameter goes to infinity, see [5, 25] in the stochastic case and see [12, 13] in the deterministic case.

Definition 4.2. A pullback attractor \mathcal{A} for a cocycle is called **backward stable** if, for each $\omega \in \Omega$, there is a nonempty compact set $K(\omega)$ such that

$$\lim_{\tau \rightarrow -\infty} \text{dist}_X(\mathcal{A}(\tau, \omega), K(\omega)) = 0. \tag{35}$$

While, the minimal compact set fulfilling (35) (if exists) is called a **backward controller**.

The backward controller means the minimal compact set controlled the attractor from the past.

Theorem 4.3. Suppose $f \in L^2_{loc}(\mathbb{R}, H_o)$ with the backward tempered condition and $g \in \dot{H}^4_{per}(\mathcal{I}) \cap H_o$. Then the cocycle Φ generated from the stochastic KS equation has the following properties in the space $H_o = \dot{L}^2_o(\mathcal{I})$.

- (i) Φ has a \mathfrak{B} -pullback attractor $\mathcal{A} \in \mathfrak{B}$.
- (ii) \mathcal{A} is backward stable with a backward controller, given by

$$\mathcal{A}(-\infty, \omega) := \bigcap_{\tau \in \mathbb{R}} \overline{\bigcup_{s \leq \tau} \mathcal{A}(s, \omega)}. \tag{36}$$

- (iii) \mathcal{A} is a \mathfrak{B} -pullback **random** attractor.

Proof. Step 1. Prove the \mathfrak{B} -pullback absorption. By Lemma 3.1, Φ has a \mathfrak{B} -pullback absorbing set given by

$$\mathcal{K}(\tau, \omega) = \{w \in H_o : \|w\|^2 \leq cR(\tau, \omega) = c(1 + \rho(\omega) + R_f(\tau, \omega))\} \tag{37}$$

where ρ and R_f are defined by (19) and (20) respectively. In fact, by (16), \mathcal{K} is backward absorbing in the following sense

$$\bigcup_{s \leq \tau} \Phi(t, s - t, \theta_{-t}\omega)B(s - t, \theta_{-t}\omega) \subset \mathcal{K}(\tau, \omega), \quad \forall t \geq T(B), \quad B \in \mathfrak{B}.$$

To prove $\mathcal{K} \in \mathfrak{B}$, we first claim that

$$\begin{aligned} \rho(\omega) &= |z(\omega)| + \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} dr \\ &\quad + \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} |z(\theta_r\omega)|^4 dr \end{aligned}$$

is tempered with any rate $\alpha > 0$. Indeed, by (5), $|z(\omega)|$ is tempered, i.e. $e^{-\alpha t} |z(\theta_{-t}\omega)| \rightarrow 0$ as $t \rightarrow +\infty$. If $\|Dg\|_\infty = 0$, then $\rho(\omega)$ is obviously tempered.

If $\|Dg\|_\infty > 0$, then we assume without loss of the generality that $\alpha < \|Dg\|_\infty \min(1, \mathbb{E}|z|)$. Then, by the ergodic limit (6), there is $T > 0$ such that for all $t \geq T$ and $r \leq -t$,

$$\int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r} \leq \left(\mathbb{E}|z| + \frac{\alpha}{4\|Dg\|_\infty}\right)|r|, \quad \int_{-t}^0 |z(\theta_{\hat{r}}\omega)| d\hat{r} \geq \left(\mathbb{E}|z| - \frac{\alpha}{4\|Dg\|_\infty}\right)t.$$

Since $\beta = \|Dg\|_\infty(\mathbb{E}|z| + 1) + 1 > \|Dg\|_\infty(\mathbb{E}|z| + \frac{\alpha}{2\|Dg\|_\infty})$, it follows that for all $t \geq T$ and $r \leq -t$,

$$\begin{aligned} &e^{\beta(r+t) + \|Dg\|_\infty \int_r^{-t} |z(\theta_{\hat{r}}\omega)| d\hat{r}} \\ &\leq e^{(\|Dg\|_\infty \mathbb{E}|z| + \frac{\alpha}{2})(r+t) + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r} - \|Dg\|_\infty \int_{-t}^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} \\ &\leq e^{(\|Dg\|_\infty \mathbb{E}|z| + \frac{\alpha}{2})(r+t) - (\|Dg\|_\infty \mathbb{E}|z| + \frac{\alpha}{4})r - (\|Dg\|_\infty \mathbb{E}|z| - \frac{\alpha}{4})t} \leq e^{\frac{\alpha}{4}r} e^{\frac{3\alpha}{4}t}. \end{aligned} \tag{38}$$

By (38), we have

$$\begin{aligned} & e^{-\alpha t} \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}-t}\omega)| d\hat{r}} dr \\ &= e^{-\alpha t} \int_{-\infty}^{-t} e^{\beta(r+t) + \|Dg\|_\infty \int_r^{-t} |z(\theta_{\hat{r}}\omega)| d\hat{r}} dr \\ &\leq e^{-\alpha t} \int_{-\infty}^{-t} e^{\frac{\alpha}{4}r} e^{\frac{3\alpha}{4}t} dr \leq \frac{4}{\alpha} e^{-\frac{\alpha}{4}t} \rightarrow 0 \text{ as } t \rightarrow +\infty, \end{aligned}$$

which means the second term of $\rho(\omega)$ is tempered. By (38) and (5),

$$\begin{aligned} & e^{-\alpha t} \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}-t}\omega)| d\hat{r}} |z(\theta_{r-t}\omega)|^4 dr \\ &= e^{-\alpha t} \int_{-\infty}^{-t} e^{\beta(r+t) + \|Dg\|_\infty \int_r^{-t} |z(\theta_{\hat{r}}\omega)| d\hat{r}} |z(\theta_r\omega)|^4 dr \\ &\leq e^{-\frac{\alpha}{4}t} \int_{-\infty}^0 e^{\frac{\alpha}{4}r} |z(\theta_r\omega)|^4 dr \rightarrow 0, \text{ as } t \rightarrow +\infty, \end{aligned}$$

which means that the third term of $\rho(\omega)$ is tempered. Hence $\rho(\omega)$ is tempered and thus backward tempered (since it is independent of τ). We then claim

$$R_f(\tau, \omega) = \sup_{s \leq \tau} \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} \|f(r+s)\|^2 dr$$

is backward tempered. Let α be the rate mentioned above and $\tau \in \mathbb{R}$. Since $R_f(\cdot, \omega)$ is increasing, it follows from (38) and (13) that

$$\begin{aligned} & e^{-\alpha t} \sup_{s \leq \tau} R_f(s-t, \theta_{-t}\omega) = e^{-\alpha t} R_f(\tau-t, \theta_{-t}\omega) \\ &= e^{-\alpha t} \sup_{s \leq \tau-t} \int_{-\infty}^0 e^{\beta r + \|Dg\|_\infty \int_r^0 |z(\theta_{\hat{r}-t}\omega)| d\hat{r}} \|f(r+s)\|^2 dr \\ &= e^{-\alpha t} \sup_{s \leq \tau-t} \int_{-\infty}^{-t} e^{\beta(r+t) + \|Dg\|_\infty \int_r^{-t} |z(\theta_{\hat{r}}\omega)| d\hat{r}} \|f(r+t+s)\|^2 dr \\ &\leq e^{-\frac{\alpha}{4}t} \sup_{s \leq \tau-t} \int_{-\infty}^{-t} e^{\frac{\alpha}{4}r} \|f(r+t+s)\|^2 dr \leq e^{-\frac{\alpha}{4}t} \sup_{s \leq \tau} \int_{-\infty}^{-t} e^{\frac{\alpha}{4}r} \|f(r+s)\|^2 dr \\ &\leq e^{-\frac{\alpha}{4}t} \sup_{s \leq \tau} \int_{-\infty}^0 e^{\frac{\alpha}{4}r} \|f(r+s)\|^2 dr \rightarrow 0, \text{ as } t \rightarrow +\infty, \end{aligned}$$

which means $R_f(\cdot, \cdot)$ is backward tempered and thus $\mathcal{K} \in \mathfrak{B}$. Note that $R_f(\tau, \omega)$ is the supremum of uncountable random variables, its measurability is unknown.

Step 2. Prove backward \mathfrak{B} -pullback asymptotic compactness. We need to prove that, for any $s_n \leq \tau$, $t_n \rightarrow +\infty$, $u_{0,n} \in B(s_n - t_n, \theta_{-t_n}\omega)$, where $\tau \in \mathbb{R}$, $B \in \mathfrak{B}$ and $\omega \in \Omega$ are fixed, the solution sequence $\{u(s_n, s_n - t_n, \theta_{-s_n}\omega, u_{0,n})\}$ has a convergent subsequence in H_σ .

Indeed, by (32) in Lemma 3.3, we have

$$\|D^2 u(s_n, s_n - t_n, \theta_{-s_n}\omega, u_{0,n})\|^2 \leq R_V(\tau, \omega) < +\infty$$

provided n is large enough. Then the sequence $\{u(s_n, s_n - t_n, \theta_{-s_n}\omega, u_{0,n})\}$ is bounded in V_σ . By the compactness of the Sobolev embedding $V \hookrightarrow H$, the sequence

is pre-compact in H . By Lemma 2.1, $\{u(s_n, s_n - t_n, \theta_{-s_n}\omega, u_{0,n})\}$ is pre-compact in H_o .

Step 3. Prove the existence of a \mathfrak{B} -pullback attractor $\mathcal{A} \in \mathfrak{B}$. By the abstract result as in [23], the existence of a \mathfrak{B} -pullback attractor follows from the \mathfrak{B} -pullback absorption (by taking $s = \tau$ in Step 1) and the \mathfrak{B} -pullback asymptotic compactness (by taking $s_n \equiv \tau$ in Step 2). But the measurability of \mathcal{A} is temporarily unknown since we cannot prove the measurability of the absorbing set \mathcal{K} (it is an uncountable union of random sets).

Step 4. Prove the backward stability of \mathcal{A} and the existence of a backward controller. By the backward asymptotic compactness in Step 2, we can prove that \mathcal{A} is backward compact, i.e. $\cup_{s \leq \tau} \mathcal{A}(s, \tau)$ is pre-compact (cf. [18, 24]). Then the theorem of nested compact sets implies that $\mathcal{A}(-\infty, \omega)$ (as defined by (36)) is nonempty compact. By the same method as in [25], we can prove

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{H_o}(\mathcal{A}(\tau, \omega), \mathcal{A}(-\infty, \omega)) = 0$$

and thus \mathcal{A} is backward stable. Suppose $E(\omega)$ is another nonempty compact set such that

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{H_o}(\mathcal{A}(\tau, \omega), E(\omega)) = 0.$$

For any $w \in \mathcal{A}(-\infty, \omega)$, we can take a sequence $w_n \in \mathcal{A}(\tau_n, \omega)$, where $\tau_n \rightarrow -\infty$, such that $w_n \rightarrow w$. While,

$$\text{dist}_{H_o}(w_n, E(\omega)) \leq \text{dist}_{H_o}(\mathcal{A}(\tau_n, \omega), E(\omega)) \rightarrow 0$$

and thus $w \in E(\omega)$, which proves the minimality. Therefore, $\mathcal{A}(-\infty, \omega)$ is the backward controller.

Step 5. Prove the measurability of \mathcal{A} . Let \mathfrak{D} be the usual universe forming from all tempered set in H_o , i.e. $D = \{D(\tau, \omega)\} \in \mathfrak{D}$ if and only if

$$\lim_{t \rightarrow +\infty} e^{-\varepsilon t} \|D(\tau - t, \theta_{-t}\omega)\|^2 = 0, \quad \forall \varepsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega, \tag{39}$$

where we have omitted the supremum in the definition (11) of \mathfrak{B} . Then, by the same method as in Lemma 3.1, one can prove that Φ has a \mathfrak{D} -pullback absorbing set given by

$$\mathcal{K}_{\mathfrak{D}}(\tau, \omega) = \{w \in H_o : \|w\|^2 \leq c(1 + \rho(\omega) + R_{\mathfrak{D}}(\tau, \omega))\} \tag{40}$$

where

$$R_{\mathfrak{D}}(\tau, \omega) = \int_{-\infty}^0 e^{\beta r + \|Dg\|_{\infty} \int_r^0 |z(\theta_{\hat{r}}\omega)| d\hat{r}} \|f(r + \tau)\|^2 dr$$

such that $\sup_{s \leq \tau} R_{\mathfrak{D}}(s, \omega) = R_f(\tau, \omega)$ in view of the definition of R_f in (20). As an integral of random variables, $R_{\mathfrak{D}}(\tau, \cdot)$ is measurable (although we do not know the measurability of R_f). By the same method as in Step 1, we know $\mathcal{K}_{\mathfrak{D}} \in \mathfrak{D}$ (it may not belong to \mathfrak{B}).

By the same method as in Lemma 3.3 and Step 2, one can prove that Φ is \mathfrak{D} -pullback asymptotically compact in H_o . Then the abstract result in [23] can be applied to obtain a \mathfrak{D} -pullback **random** attractor $\mathcal{A}_{\mathfrak{D}}$ such that $\mathcal{A}_{\mathfrak{D}}$ is just constructed by the omega-limit set of $\mathcal{K}_{\mathfrak{D}}$.

Since $R_{\mathfrak{D}}(\tau, \omega) \leq R_f(\tau, \omega)$, we have $\mathcal{K}_{\mathfrak{D}} \subset \mathcal{K}$ and thus their omega-limit sets fulfill $\mathcal{A}_{\mathfrak{D}} \subset \mathcal{A}$. On the other hand, since $\mathcal{A} \in \mathfrak{B} \subset \mathfrak{D}$, it follows that \mathcal{A} can be

attracted by $\mathcal{A}_{\mathfrak{D}}$, which, together with the invariance of \mathcal{A} , implies $\mathcal{A} \subset \mathcal{A}_{\mathfrak{D}}$. So, $\mathcal{A} = \mathcal{A}_{\mathfrak{D}}$ and thus \mathcal{A} is random too. \square

Remark 1. The method for proving $\mathcal{K} \in \mathfrak{B}$ (where the absolute value $|z(\theta_s\omega)|$ is involved) differs from those in the literature. The method for proving the measurability of \mathcal{A} also differs from the usual.

On the other hand, every function of the (nonempty) attractor $\mathcal{A}(\tau, \omega)$ is odd and smooth, where the smoothness follows from the Sobolev embedding $V \hookrightarrow C^1(\mathcal{I})$ and the invariance of the attractor.

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