ELECTRONIC RESEARCH ARCHIVE doi:10.3934/era.2020077
Volume **28**, Number **4**, December **2020** pp. **1459–1486**

CONVERGENCE AND QUASI-OPTIMALITY OF L^2−NORMS BASED AN ADAPTIVE FINITE ELEMENT METHOD FOR NONLINEAR OPTIMAL CONTROL PROBLEMS

Zuliang Lu*

Key Laboratory for Nonlinear Science and System Structure
Chongqing Three Gorges University
Chongqing 404000, China
Research Center for Mathematics and Economics
Tianjin University of Finance and Economics
Tianjin 300222, China

Fei Huang and Xiankui Wu

Key Laboratory for Nonlinear Science and System Structure
Chongqing Three Gorges University
Chongqing 404000, China

Lin Li

College of Computer Science and Technology
Chongqing University of Posts and Telecommunications
Chongqing 400065, China

Shang Liu

School of Mathematics and Statistics
Changsha University of Science and Technology
Changsha 410114, Hunan, China

Abstract. This paper aims at investigating the convergence and quasi-optimality of an adaptive finite element method for control constrained nonlinear elliptic optimal control problems. We derive a posteriori error estimation for both the control, the state and adjoint state variables under controlling by L^2-norms where bubble function is a wonderful tool to deal with the global lower error bound. Then a contraction is proved before the convergence is proposed. Furthermore, we find that if keeping the grids sufficiently mildly graded, we can prove the optimal convergence and the quasi-optimality for the adaptive finite element method. In addition, some numerical results are presented to verify our theoretical analysis.

2020 *Mathematics Subject Classification.* Primary: 49J20, 65N30; Secondary: 65N12.

Key words and phrases. Nonlinear optimal control problem, adaptive finite element method, convergence, quasi-optimality.

The first author is supported by NSFC grant 11201510, NSSFC grant 19BGL190, Postdoctoral Science Foundation grant 2017T100155 and 2015M580197, Innovation Team Building at Institutions of Higher Education in Chongqing grant CXTDX201601035, Scientific and Technological Research Program of Chongqing Municipal Education Commission and Chongqing Research Program of Basic Research and Frontier Technology grant cstc2019jcyj-msxmX0280.

* Corresponding author: Zuliang Lu.

1. **Introduction.** Since the pioneer work in adaptive finite element methods was proposed by Babuška and Rheinboldt [2], adaptive finite element methods have been applied successfully in engineering and scientific computations. The adaptive finite element method is based on the error information obtained by the computer to determine whether the solution is accurate enough. Hence the soul of adaptive finite elements is the a posteriori error estimation.

When Dörfler [15] presented a marking strategy aiming at electing the set of elements for refinement, based on the error indicators which was controlled by the control, the state and adjoint state, adaptive finite element algorithm was put on the stage of academic research. He provided a fineness assumption on the initial grid $\mathcal{T}_{h_0}$ which was used to prove the reduction of energy errors while in later investigations, Morin, Nochetto and Siebert [30] removed the assumption. Moreover, they proposed the interior node property in order to obtain the proof for convergence of adaptive finite element methods [31]. More pioneering works on adaptive finite element methods see literatures [5, 3, 9, 13, 16, 17, 22], in which the linear elliptic optimal control problem was mainly been investigated.

Convergence and quasi-optimality are the two key factors of adaptive finite element methods. It was noteworthy that Mekchay and Nochetto [29] extended the convergence result of Morin, Nochetto and Siebert [31] for general second order linear elliptic partial differential equations by introducing a novel concept that was the total error which was the sum of the energy errors adding the oscillations. This provided a valuable empirical basis for future scholars' work on convergence analysis. Meanwhile, Binev, Dahmen and Devore [4] firstly presented the property of optimality. Later, a large number of scholars participate in the study of the property. For example, Carstensen and Hoppe [6] proposed convergence and quasi-optimality which were established for the Raviart Thomas finite element method. Gong and Yan [20] considered the convergence analysis of adaptive finite element method for elliptic optimal control problems with pointwise control constraints.

As to our best knowledge, nonlinear optimal control problems have gradually penetrated into many fields of scientific research and engineering technology. Chen and Lu [11] investigated adaptive fully-discrete finite element methods for semilinear parabolic quadratic boundary optimal control problems. Gaevskaya, Hoppe, Iliash and Kieweg [17] developed an adaptive finite element method for a class of distributed optimal control problems with control constraints, and found requirement $h_0 \ll 1$ on the initial grid $\mathcal{T}_{h_0}$ is not restrictive for the convergence analysis of adaptive finite element for nonlinear problems. Chen, Gong, He and Zhou [10] studied an adaptive finite element method for a class of a nonlinear eigenvalue problems that may be of nonconvex energy functional and prove the convergence of adaptive finite element approximations.

Leng and Chen [23] proved the convergence and the quasi-optimality of an adaptive element method with integral control constraints while we extend the result of [23] to a nonlinear optimal control problem with integral control constraint on L^2-norms in this paper. We follow the idea of [27] to derive reliable and efficient posteriori error estimations and the idea of [19] to prove the posteriori upper and the global lower bound of the errors, in which the bubble function matters. Moreover, a contraction for an adaptive finite element method is obtained based on a mild assumption on initial mesh $\mathcal{T}_{h_0}$ which can be seen in [10, 14, 20, 21, 23, 24, 25, 29, 31]. Furthermore, we propose the optimal convergence rate. However, the quasi-optimality is the best obstacle mission for us to prove. Therefore, we continue to use the idea of [14, 21, 23]. Finally, we provide some numerical experiments to verify our theoretical analysis.

Here are some notations will be used in this paper. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and $\partial\Omega$ denote the boundary of Ω. We use the standard notation $W^{m,q}(\omega)$ with norm $\|\cdot\|_{m,q,\omega}$ and seminorm $|\cdot|_{m,q,\omega}$ to express the standard Sobolev space for $\omega \subset \Omega$. Moreover, we will omit the subscription if $\omega = \Omega$. For $q = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$. Also for $m = 0$ and $q = 2$, we denote $W^{0,2}(\omega) = L^2(\omega)$ and $\|\cdot\|_{0,2,\omega} = \|\cdot\|_{0,\omega}$. For $m = 0$ and $q = \infty$, we denote $W^{0,\infty}(\omega) = L^\infty(\omega)$ and $||\cdot||_{0,\infty,\omega} = \max_{\omega}|\cdot| = ||\cdot||_{\infty,\omega}$. Additionally, we observe that $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$. Beyond that, let c and C are the constants which independent of grids size, then we use $A \approx B$ to represent $cA \leq B \leq CA$. In addition, $(\cdot,\cdot)$ denotes the L^2 inner product.

The rest of our paper is arranged as follows. In Section 2, we give what the optimal control problems we want to investigate and some basic notations must be used. Then the a posteriori error estimation is obtained and an adaptive algorithm is proposed in Section 3. In Section 4, we use quasi-orthogonality and discrete local upper bound to prove the convergence of the adaptive finite element method and so is the quasi-optimality for details in Section 5. In the end, some numerical simulations is given to verify our theoretical analysis.

2. Nonlinear optimal control problem.

In this section we first introduce some basic notations, and then we show what the nonlinear optimal control problem we discussed about.

$\mathcal{T}_h$ is a regular triangulation of Ω such that $\bar{\Omega} = \cup_{T\in\mathcal{T}_h}\bar{T}$. T is an element of $\mathcal{T}_h$. Let $\mathcal{T}_{h_0}$ be the initial partition of $\bar{\Omega}$ into disjoint triangles. By newest-vertex bisections for $\mathcal{T}_{h_0}$, we can obtain a class $\mathbb{T}$ of conforming partitions. For $\mathcal{T}_h, \tilde{\mathcal{T}}_h \in \mathbb{T}$, we use $\mathcal{T}_h \subset \tilde{\mathcal{T}}_h$ to indicate that $\tilde{\mathcal{T}}_h$ is a refinement of $\mathcal{T}_h$ and $h_T = |T|^{1/2}$. According to [14], the continuous piecewise linear mesh function is defined by $h_{\mathcal{T}_h}$. Moreover, $h_{\mathcal{T}_h}(z)$ is the average of the $h_{T'}$ over all $T' \in \mathcal{T}_h$ for any vertex z of $\mathcal{T}_h$ with $z \in T'$. Then we have the following properties via keeping the meshs level low enough [21].

Lemma 2.1. [21] *For some constants c and C and fixed constant μ, there holds*

$$ch_T \leq h_{\mathcal{T}_h}|_T \leq Ch_T, \tag{1}$$

$$||\nabla h_{\mathcal{T}_h}||_\infty \leq \mu, \tag{2}$$

where all grids satisfied above are denoted by $\mathbb{T}_\mu$.

In this paper we mainly enter into meaningful discussions with the following nonlinear optimal control problem:

$$\min_{u\in U_{ad}} \left\{\frac{1}{2}\|y - y_d\|_0^2 + \frac{\alpha}{2}\|u\|_0^2\right\},$$

$$-\Delta y + \phi(y) = f + u, \quad \text{in } \Omega, \tag{3}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{4}$$

where $y_d \in L^2(\Omega)$, $U_{ad} = \{v : v \in L^2(\Omega),\ \int_\Omega v \geq 0\}$ is a closed convex subset of $U = L^2(\Omega)$ and $\phi(\cdot) \in W^{2,\infty}(-R,R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, $\phi' \geq 0$. Let $V = H_0^1(\Omega)$, we give the weak formulation to deal with state equation, namely, find $y \in V$ such that

$$a(y,v) + (\phi(y),v) = (f+u,v), \quad \forall\, v \in V,$$

where

$$a(y,v)=\int_\Omega \nabla y\cdot\nabla v dx \quad \text{and} \quad |||v|||=\sqrt{a(v,v)}.$$

Then the nonlinear optimal control problem can be restated as follows

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|y-y_d\|_0^2+\frac{\alpha}{2}\|u\|_0^2\right\}, \tag{5}$$

$$a(y,v)+(\phi(y),v)=(f+u,v), \quad \forall\, v\in V. \tag{6}$$

It is well known [26, 27] that the nonlinear optimal control problem has at least one solution (y,u), and that if a pair (y,u) is the solution of the optimal control problem, then there is a co-state $p\in V$ such that the triplet (y,p,u) satisfies the following optimality conditions:

$$a(y,v)+(\phi(y),v)=(f+u,v), \quad \forall\, v\in V, \tag{7}$$

$$a(q,p)+(\phi'(y)p,q)=(y-y_d,q), \quad \forall\, q\in V, \tag{8}$$

$$(\alpha u+p,v-u)\geq 0, \quad \forall\, v\in U_{ad}. \tag{9}$$

Since the coercivity of $a(\cdot,\cdot)$, we define a solution operator $S:L^2(\Omega)\to H_0^1(\Omega)$ of (4) such that $S(f+u)=y$ and let S^* be the adjoint of S such that $S^*(y-y_d)=p$. Suppose V_h is the continuous piecewise linear finite element space with respect to the partition $\mathcal{T}_h\in\mathbb{T}$. For $\mathcal{T}_h\in\mathbb{T}$, we define U^h as the piecewise constant finite element space with respect to $\mathcal{T}_h$. Let $U_{ad}^h=\{v_h\in U^h:\int_\Omega v_h\geq 0\}\subset L^2(\Omega)$. Then we derive the standard finite element discretization for the nonlinear optimal control problem as follows:

$$\min_{u_h\in U_{ad}^h}\left\{\frac{1}{2}\|y_h-y_d\|_0^2+\frac{\alpha}{2}\|u_h\|_0^2\right\}, \tag{10}$$

$$a(y_h,v)+(\phi(y_h),v)=(f+u_h,v), \quad \forall\, v\in V_h. \tag{11}$$

Similarly the nonlinear optimal control problem (10)-(11) has at least one solution (y_h,u_h), and that if a pair (y_h,u_h) is the solution of (10)-(11), then there is a co-state $p_h\in V_h$ such that the triplet (y_h,p_h,u_h) satisfies the following optimality conditions:

$$a(y_h,v)+(\phi(y_h),v)=(f+u_h,v), \quad \forall\, v\in V_h, \tag{12}$$

$$a(q,p_h)+(\phi'(y_h)p_h,q)=(y_h-y_d,q), \quad \forall\, q\in V_h, \tag{13}$$

$$(au_h+p_h,v_h-u_h)\geq 0, \quad \forall\, v_h\in U_{ad}^h. \tag{14}$$

Based on [12, 14, 21, 27], we have the following Lemmas in order to derive a L^2-norms posteriori error estimation for both the control, the state and adjoint state variables.

Lemma 2.2. [21] *Suppose that Ω is convex such that for any $f\in L^2(\Omega)$, (3)-(4) have at least one solution $y=S(f+u)\in H^2(\Omega)\cap H_0^1(\Omega)$ and*

$$||y||_2\leq C||f+u||_0,$$

and apparently the assumption is valid for S^*.

Lemma 2.3. [12] *Assume that Ω is convex, here holds*

$$\phi(y)-\phi(p)=-\tilde{\phi}'(y)(p-y)=-\phi'(p-y)+\tilde{\phi}''(y)(p-y),$$

for any $(y,p)\in V$ *and where*

$$\tilde{\phi}'(y)=\int_0^1 \phi'\big(y+s(p-y)\big)ds,\quad \tilde{\phi}''(y)=\int_0^1 (1-s)\phi''\big(p+s(p-y)\big)ds,$$

are bounded functions in $\bar{\Omega}$.

Lemma 2.4. [14, 27] *For all* $v\in H^1(\Omega)$, $\mathcal{T}_h\in\mathbb{T}$ *and* $T\in\mathcal{T}$, *we have*

$$||\nabla v||_{0,\partial T\backslash\partial\Omega}\le Ch_T^{-1/2}||\nabla v||_0+Ch_T^{1/2}|v|_2. \tag{15}$$

3. **A posteriori error estimation.** In this section, we will recall a residual-based a posteriori error estimation for nonlinear elliptic equations. For the model problem that we studied in Section 2, a reliable and efficient a posteriori estimation will be obtained. In the end of this section, an adaptive finite element algorithm will be introduced.

Here we define some error indicators. $\eta(\cdot)$ are error indicators and $osc(\cdot)$ represent the data oscillations. For $\mathcal{T}_h\in\mathbb{T},\ T\in\mathcal{T}_h$, we define

$$\begin{aligned}
&\eta^2_{1,\mathcal{T}_h}(p_h,T)=h_T^2\|\nabla p_h\|^2_{0,T},\\
&\eta^2_{2,\mathcal{T}_h}(u_h,y_h,T)=h_T^4\|f+u_h-\phi(y_h)\|^2_{0,T}+h_T^3\|(\nabla y_h)\cdot\mathbf{n}\|^2_{0,\partial T\backslash\partial\Omega},\\
&\eta^2_{3,\mathcal{T}_h}(y_h,p_h,T)=h_T^4\|y_h-y_d-\phi'(y_h)p_h\|^2_{0,T}+h_T^3\|(\nabla p_h)\cdot\mathbf{n}\|^2_{0,\partial T\backslash\partial\Omega},\\
&osc^2_{\mathcal{T}_h}(f,T)=h_T^4\|f-f_T\|^2_{0,T},\\
&osc^2_{\mathcal{T}_h}(y_h-y_d,T)=h_T^4\|(y_h-y_d)-(y_h-y_d)_T\|^2_{0,T},
\end{aligned}$$

where $u_h\in U^h_{ad}$, $y_h,p_h\in V_h$, $(\nabla y_h)\cdot\mathbf{n}$ denotes the jump of ∇y_h, and $\mathbf{n}$ denotes the outward normal oriented to $\partial T\backslash\partial\Omega$, and where f_T is L^2-projection of f onto piecewise constant space on T and $f_T=\frac{\int_T f}{|T|}$. For $\omega\subset\mathcal{T}_h$, we have

$$\eta^2_{1,\mathcal{T}_h}(p_h,\omega)=\sum_{T\in\omega}\eta^2_{1,\mathcal{T}_h}(p_h,T),$$
$$osc^2_{\mathcal{T}_h}(f,\omega)=\sum_{T\in\omega}osc^2_{\mathcal{T}_h}(f,T).$$

Similarly, we have $\eta^2_{2,\mathcal{T}_h}(u_h,y_h,\omega)$, $\eta^2_{3,\mathcal{T}_h}(y_h,p_h,\omega)$ and $osc^2_{\mathcal{T}_h}(y_h-y_d,\omega)$.

Lemma 3.1. *Let* $\mathcal{T}_h\in\mathbb{T}_\mu$ *under the conditions of Lemma 2.2, we have*

$$||y-y_h||_0\le C(||u-u_h||_0+|||h_{\mathcal{T}_h}(y-y_h)|||),$$
$$||p-p_h||_0\le C(||u-u_h||_0+|||h_{\mathcal{T}_h}(y-y_h)|||+|||h_{\mathcal{T}_h}(p-p_h)|||),$$

for sufficiently small μ.

Proof. Suppose that $y^h,p^h\in V$ are intermediate variables satisfying the equations as follows

$$a(y^h,v)+(\phi(y^h),v)=(f+u_h,v),\quad \forall\ v\in V, \tag{16}$$
$$a(q,p^h)+(\phi'(y^h)p^h,q)=(y_h-y_d,q),\quad \forall\ q\in V. \tag{17}$$

Employing the Galerkin orthogonality, the approximation properties, Lemma 2.2, and $||\nabla h_{\mathcal{T}}||_\infty\le\mu$, there exist similar results, which resemble to Lemma 3.1 in [21], for nonlinear elliptic optimal control problems with sufficiently small μ that

$$||y^h-y_h||_0\le C|||h_{\mathcal{T}_h}(y^h-y_h)|||,$$
$$||p^h-p_h||_0\le C|||h_{\mathcal{T}_h}(p^h-p_h)|||.$$

It has been proved $|||y-y^h||| \le C||y-y^h||_1 \le C||u-u_h||_0$ in the Theorem 3.1 of [28] for nonlinear elliptic optimal control problems. Then associated with the Lemma 4.4 in [7], we deduce similar conclusions for nonlinear optimal control problems that

$$|||y-y^h||| \le C||u-u_h||_0. \tag{18}$$

$$|||p-p^h||| \le C(||u-u_h||_0 + ||y-y_h||_0). \tag{19}$$

By using the triangle inequality, we have

$$||y-y_h||_0 \le ||y-y^h||_0 + ||y^h-y_h||_0,$$
$$||p-p_h||_0 \le ||p-p^h||_0 + ||p^h-p_h||_0.$$

In connection with what we discussed above, the triangle inequality and Lemma 2.1, it is easy to prove the prevenient results in Lemma 3.1. □

Now we are in the position to derive a posteriori error estimation for both the control, the state and adjoint state variables.

Theorem 3.2. *Let $(u,y,p) \in U_{ad} \times H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of (7)-(9) and $(u_h,y_h,p_h) \in U_{ad}^h \times V_h \times V_h$ be the solution of (12)-(14). Then we have a posteriori error upper bound*

$$\begin{aligned}&||u-u_h||_0^2 + ||y-y_h||_0^2 + ||p-p_h||_0^2\\ \le& c(\eta^2_{1,\mathcal{T}_h}(p_h,\mathcal{T}_h) + \eta^2_{2,\mathcal{T}_h}(u_h,y_h,\mathcal{T}_h) + \eta^2_{3,\mathcal{T}_h}(y_h,p_h,\mathcal{T}_h)),\end{aligned}$$

and apparently a global lower bound

$$\begin{aligned}&C(\eta^2_{1,\mathcal{T}_h}(p_h,\mathcal{T}_h) + \eta^2_{2,\mathcal{T}_h}(u_h,y_h,\mathcal{T}_h) + \eta^2_{3,\mathcal{T}_h}(y_h,p_h,\mathcal{T}_h))\\ \le& ||u-u_h||_0^2 + ||y-y_h||_0^2 + ||p-p_h||_0^2 + osc^2_{\mathcal{T}_h}(f,\mathcal{T}_h) + osc^2_{\mathcal{T}_h}(y_h-y_d,\mathcal{T}_h),\end{aligned}$$

where c and C only depend on the shape regularity of $\mathcal{T}_h$.

Proof. In view of Lemma 7.3.1 in [27], we similarly derive that

$$||u-u_h||_0^2 \le C\eta^2_{1,\mathcal{T}_h}(p_h,\mathcal{T}_h) + C||p_h-p^h||_0^2, \tag{20}$$

where p^h is the solution of (3.2). Then we shall to deal with $||p_h-p^h||_0^2$.

Let $\xi = p^h - p_h$, and $\xi_I = \pi_h\xi$ where π_h is the standard Lagrange interpolation operator of ξ, then it follows from Lemma 2.1, Lemma 2.3 and (8) that

$$\begin{aligned}||p^h-p_h||_0^2 =& a(\nabla\xi, \nabla(p^h-p_h)) + (\phi'(y^h)(p^h-p_h),\xi)\\ =& (\nabla(\xi-\xi_I),\nabla(p_h-p^h)) + (\phi'(y_h)p_h - \phi'(y^h)p^h, \xi-\xi_I)\\ &+ (\nabla\xi_I,\nabla(p^h-p_h)) + (\phi'(y^h)p^h - \phi'(y_h)p_h,\xi_I)\\ &+ ((\phi'(y_h)-\phi'(y^h))p_h,\xi)\\ =& \sum_{T\in\mathcal{T}_h}\int_T (y_h-y_d-\phi'(y_h)p_h)(\xi-\xi_I) - \sum_{T\in\mathcal{T}_h}\int_{\partial T}[(\nabla p_h)\cdot\mathbf{n}](\xi-\xi_I)ds\\ &+ (y^h-y_h,\xi_I) + ((\phi'(y_h)-\phi'(y^h))p_h,\xi)\end{aligned}$$

$$
\begin{aligned}
\leq & C \sum_{T\in\mathcal{T}_h} h_T^2 ||y_h - y_d - \phi'(y_h)p_h||_{0,T}|\xi|_{2,T} \\
& + C \sum_{\partial T\setminus\partial\Omega} h_T^{3/2}\Big(\int_T [(\nabla p_h)\cdot \mathbf{n}]^2\Big)^{1/2} ||\xi||_{2,\partial T\setminus\partial\Omega} \\
& + ||y^h - y_h||_0||\xi||_0 + C||\phi'(y_h) - \phi'(y^h)||_0||p_h||_0||\xi||_{0,\infty} \\
\leq & C \sum_{T\in\mathcal{T}_h} h_T^4 \int_T (y_h - y_d - \phi'(y_h)p_h)^2 + C \sum_{\partial T\setminus\partial\Omega} h_T^3[(\nabla p_h)\cdot\mathbf{n}]^2 \\
& + ||y^h - y_h||_0^2 + C\delta||\xi||_2^2,
\end{aligned}
$$

where $\phi(\cdot)\in W^{2,\infty}(\Omega)$, the embedding $||v||_{0,\infty}\leq C||v||_2$ and $||p^h||_0\leq C$ have been adopted and δ is positive. Then choosing $\delta=\frac{1}{2C}$, we obtain

$$
||p_h - p^h||_0^2 \leq C\eta_{3,\mathcal{T}_h}^2(y_h, p_h, \mathcal{T}_h) + C||y_h - y^h||_0^2. \tag{21}
$$

Similarly there is going to be proved by letting $\tilde{\xi} = y^h - y_h$ that

$$
\begin{aligned}
& ||y_h - y^h||_0^2 \\
=& a(\nabla(y^h - y_h), \nabla\tilde{\xi}) + (\phi(y^h) - \phi(y_h), \tilde{\xi}) \\
=& (\nabla(y^h - y_h), \nabla(\tilde{\xi} - \tilde{\xi}_I)) + (\phi(y^h) - \phi(y_h), \tilde{\xi} - \tilde{\xi}_I) \\
=& \sum_{T\in\mathcal{T}_h}\int_T (f + u_h - \phi(y_h))(\tilde{\xi} - \tilde{\xi}_I) - \sum_{\partial T\setminus\partial\Omega}\int_T [(\nabla y_h)\cdot\mathbf{n}](\tilde{\xi} - \tilde{\xi}_I) \\
\leq& C\sum_{T\in\mathcal{T}_h} h_T^4\int_T (f + u_h - \phi(y_h))^2 + C\sum_{\partial T\setminus\partial\Omega} h_T^3\int_T [(\nabla y_h)\cdot\mathbf{n}]^2 + C\delta||\tilde{\xi}||_0^2,
\end{aligned}
$$

where $\phi(\cdot)\in W^{2,\infty}(\Omega)$ have been applied and hence we have

$$
||y_h - y^h||_0^2 \leq C\eta_{2,\mathcal{T}_h}^2(u_h, y_h, \mathcal{T}_h). \tag{22}
$$

It is easy to derive the expect upper bound by combining with (20)-(22).

Next we are going to deduce the global lower error bound through the standard bubble function [1, 19]. Similar to Lemma 3.7 in [19], it can be similarly proved that there exists polynomial $w_T\in H_0^2(T)$ such that

$$
\int_T h_T^4((y_h - y_d)_T - \phi'(y_h)p_h)^2 = \int_T h_T^4((y_h - y_d)_T - \phi'(y_h)p_h)w_T, \tag{23}
$$

and apparently

$$
||w_T||_{0,T}^2 \leq C\int_T ((y_h - y_d)_T - \phi'(y_h)p_h)^2, \tag{24}
$$

$$
ch_T^{-2}||w_T||_{0,T}^2 \leq |w_T|_{2,T}^2 \leq Ch_T^{-2}||w_T||_{0,T}^2. \tag{25}
$$

Then it follows from (23) and (24) that

$$
\begin{aligned}
& \int_T h_T^4((y_h - y_d)_T - \phi'(y_h)p_h)^2 \\
=& \int_T h_T^4((y_h - y_d)_T - \phi'(y_h)p_h)w_T, \\
=& \int_T h_T^4(y_h - y_d - \phi'(y_h)p_h)w_T + \int_T h_T^4((y_h - y_d)_T - (y_h - y_d))w_T \\
\equiv& I_1 + I_2.
\end{aligned}
$$

Combining with (2.8) and Lemma 2.4, there holds

$$\begin{aligned}I_1 =&\int_T h_T^4(p_h-p)\Delta w_T+\int_T h_T^4(y_h-y)w_T+\int_T h_T^4\phi'(y)(p-p_h)p_hw_T\\&+\int_T h_T^4((\phi'(y)-\phi'(y_h))p_hw_T\\ \leq& C(||p-p_h||_{0,T}^2+||y_h-y||_{0,T}^2+h_T^4||\phi'(y)||_{0,T}||p-p_h||_{0,T}||p_h||_{0,T}||w_T||_{0,\infty,T}\\&+h_T^4||\phi'(y)-\phi'(y_h)||_{0,T}||p_h||_{0,T}||w_T||_{0,\infty,T})+C(h_T^2||w_T||_{0,T}^2+h_T^4|w_T|_{2,T}^2)\\ \leq& C(||p-p_h||_{0,T}^2+||y_h-y||_{0,T}^2)+C\delta h_T^4||w_T||_0^2, \end{aligned}\tag{26}$$

where $\phi(\cdot)\in W^{2,\infty}(\Omega)$ have been used, the embedding $||v||_{0,\infty,T}\leq C||v||_{2,T}$ and the property $||p_h||\leq C$ have been adopted.

Similarly, we have

$$I_2 \leq C\int_T h_T^4((y_h-y_d)_T-(y_h-y_d)^2+C\delta h_T^4||w_T||_{0,T}. \tag{27}$$

Hence by using the Cauchy inequality with the help of (25)-(27), we obtain

$$\begin{aligned}&\int_T h_T^4((y_h-y_d)_T-\phi'(y_h)p_h)^2\\ \leq& C(||p-p_h||_{0,T}^2+||y_h-y||_{0,T}^2+\int_T h_T^4((y_h-y_d)_T-(y_h-y_d))^2)\\&+Ch_T^4\int_T h_T^4((y_h-y_d)_T-\phi'(y_h)p_h)^2.\end{aligned}$$

Then it brings about

$$\begin{aligned}&\int_T h_T^4((y_h-y_d)-\phi'(y_h)p_h)^2\\ \leq& C\bigg(\int_T h_T^4((y_h-y_d)_T-\phi'(y_h)p_h)^2+\int_T h_T^4((y_h-y_d)_T-(y_h-y_d))^2\bigg)\\ \leq& C\bigg(||p-p_h||_{0,T}^2+||y_h-y||_{0,T}^2+\int_T h_T^4((y_h-y_d)_T-(y_h-y_d))^2\bigg).\end{aligned}\tag{28}$$

Then we need to use the new bubble functions defined in [19] to both in deal with the jump. Similar to [18, 19], it can be similarly proved that there exists polynomial $w_{\partial T}\in H_0^2(T)$ such that

$$\int_{\partial T} h_T^3[(\nabla p_h)\cdot\mathbf{n}]^2=\int_{\partial T} h_T^3[(\nabla p_h)\cdot\mathbf{n}]w_{\partial T}, \tag{29}$$

and apparently

$$||w_{\partial T}||_{0,\partial T\backslash\partial\Omega}^2\leq C\int_{\partial T} h_T^3[(\nabla p_h)\cdot\mathbf{n}]^2, \tag{30}$$

$$ch_T^{-2}||w_{\partial T}||_{0,\partial T\backslash\partial\Omega}^2\leq ||w_{\partial T}||_{2,\partial T\backslash\partial\Omega}^2\leq Ch_T^{-2}||w_{\partial T}||_{0,\partial T\backslash\partial\Omega}^2. \tag{31}$$

And then it follows from (29) and(30) that

$$\begin{aligned}&\int_{\partial T} h_T^3[(\nabla p_h)\cdot\mathbf{n}]^2=\int_{\partial T} h_T^3[(\nabla p_h)\cdot\mathbf{n}]w_{\partial T}=\int_{\partial T}[(\nabla p_h)\cdot\mathbf{n}-(\nabla p)\cdot\mathbf{n}]w_{\partial T}\\=&\int_{\partial T} h_T^3(p-p_h)\Delta w_{\partial T}+\int_{\partial T} h_T^3(y-y_d-\phi'(y)p)w_{\partial T}\\ \equiv& I_3+I_4.\end{aligned}$$

Similarly, it can be deduced that

$$I_3 \leq C\delta h_T^3 ||w_{\partial T}||_{0,\partial T\backslash\partial\Omega}^2 + C||p-p_h||_{0,\partial T\backslash\partial\Omega}^2.$$

Combining with (13) and Lemma 2.4, there holds

$$\begin{aligned}
I_4 =& \int_{\partial T} h_T^3 (y_h - y_d - \phi'(y_h)p_h) w_{\partial T} + \int_{\partial T} h_T^3 (y-y_h) w_{\partial T} \\
&+ \int_{\partial T} h_T^3 \phi'(y)(p-p_h) w_{\partial T} + \int_{\partial T} h_T^3 \tilde{\phi}''(y)(y-y_h)p_h w_{\partial T} \\
\leq& C(||y-y_h||_{0,\partial T\backslash\partial\Omega}^2 + \int_{\partial T} h_T^3 (y_h - y_d - \phi'(y_h)p_h)^2 \\
&+ h_T^3 ||\phi'(y)||_{0,\partial T\backslash\partial\Omega} ||p-p_h||_{0,\partial T\backslash\partial\Omega} ||w_{\partial T}||_{0,\infty,\partial T\backslash\partial\Omega} \\
&+ h_T^3 ||\tilde{\phi}''(y)||_{0,\partial T\backslash\partial\Omega} ||y-y_h||_{0,\partial T\backslash\partial\Omega} ||p_h||_{0,\partial T\backslash\partial\Omega} ||w_{\partial T}||_{0,\infty,\partial T\backslash\partial\Omega}) \\
&+ C\delta (h_T^3 ||w_{\partial T}||_{0,\partial T\backslash\partial\Omega}^2 + h_T^3 |w_{\partial T}|_{2,\partial T\backslash\partial\Omega}^2) \\
\leq& C(||y-y_h||_{0,\partial T\backslash\partial\Omega}^2 + ||p-p_h||_{0,\partial T\backslash\partial\Omega}^2 + \int_{\partial T} h_T^3 (y_h - y_d - \phi'(y_h)p_h)^2) \\
&+ C\delta h_T^3 ||w_{\partial T}||_{0,\partial T\backslash\partial\Omega}^2.
\end{aligned}$$

Hence by using the Cauchy inequality with the help of (29)-(30), we have

$$\begin{aligned}
&\sum_{\partial T\backslash\partial\Omega} \int_{\partial T} h_T^3 [(\nabla p_h)\cdot \mathbf{n}]^2 \\
&\leq C(||p-p_h||_0^2 + ||y-y_h||_0^2) + C\sum_{T\in\mathcal{T}_h} \int_T h_T^4 (y_h - y_d - \phi'(y_h)p_h)^2,
\end{aligned} \tag{32}$$

where $\phi(\cdot) \in W^{2,\infty}(\Omega)$ and $w_{\partial T} \in H_0^2(\Omega)$ have been used.

In connection with (28) and (32), it is easy to get that

$$\begin{aligned}
&\eta_{3,\mathcal{T}_h}^2(y_h, p_h, \mathcal{T}_h) \\
=& \sum_{T\in\mathcal{T}_h} \int_T h_T^4 (y_h - y_d - \phi'(y_h)p_h)^2 + \sum_{\partial T\backslash\partial\Omega} \int_{\partial T} h_T^3 ([\nabla p_h]\cdot\mathbf{n})^2 \\
\leq& C\|p_h - p\|_0^2 + C\|y-y_h\|_0^2 + C\sum_{T\in\mathcal{T}_h} \int_T h_T^4 ((y_h - y_d) - (y_h - y_d)_T)^2 \\
=& C\|p_h - p\|_0^2 + C\|y-y_h\|_0^2 + Cosc_{\mathcal{T}_h}^2(y_h - y_d, \mathcal{T}_h).
\end{aligned}$$

It can also be deduced that

$$\begin{aligned}
\eta_{2,\mathcal{T}_h}^2(u_h, y_h, \mathcal{T}_h) =& \sum_{T\in\mathcal{T}_h} \int_T h_T^4 (f + u_h - \phi(y_h))^2 \\
&+ \sum_{\partial T\backslash\partial\Omega} \int_{\partial T} h_T^3 ([\nabla y_h]\cdot\mathbf{n})^2 \\
\leq& C\|y_h - y\|_0^2 + C\|u-u_h\|_0^2 + C\sum_{T\in\mathcal{T}_h} \int_T h_T^4 (f - f_T)^2 \\
=& C\|y_h - y\|_0^2 + C\|u-u_h\|_0^2 + Cosc_{\mathcal{T}_h}^2(f, \mathcal{T}_h).
\end{aligned}$$

Above-mentioned results tell the proof of Theorem 3.2 is accomplished. □

Theorem 3.2 gives a reliable and efficient posteriori error estimations for the sum of the L^2-norms errors for the control, the state and the co-state variables. Then we introduce an adaptive finite element algorithm to explain what we mainly investigate in this paper.

Algorithm 3.1. Adaptive finite element algorithm for nonlinear optimal control problems:

(o) Given an initial mesh $\mathcal{T}_{h_0}$ and construct finite element space $U_{ad}^{h_0}$ and V_{h_0}. Select marking parameter $0 < \theta \leq 1$ and set $k := 0$.

(1) Solve the discrete nonlinear optimal control problem (12)-(14), then obtain approximate solution $(u_{h_k}, y_{h_k}, p_{h_k})$ with respect to $\mathcal{T}_{h_k}$.

(2) Compute the local error estimator $\eta_{\mathcal{T}_{h_k}}(T)$ for all $T \in \mathcal{T}_{h_k}$.

(3) Select a minimal subset $\mathcal{M}_{h_k}$ of $\mathcal{T}_{h_k}$ such that

$$\eta^2_{\mathcal{T}_{h_k}}(\mathcal{M}_{h_k}) \geq \theta \eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}),$$

where $\eta^2_{\mathcal{T}_{h_k}}(\omega) = \eta^2_{1,\mathcal{T}_h}(p_h, \omega) + \eta^2_{2,\mathcal{T}_h}(u_h, y_h, \omega) + \eta^2_{3,\mathcal{T}_h}(y_h, p_h, \omega)$ for all $\omega \subset \mathcal{T}_{h_k}$.

(4) Refine $\mathcal{M}_{h_k}$ by bisecting $b \geq 1$ times in passing from $\mathcal{T}_{h_k}$ to $\mathcal{T}_{h_{k+1}}$ and generally additional elements are refined in the process in order to ensure that $\mathcal{T}_{h_{k+1}}$ is conforming.

(5) Solve the discrete nonlinear optimal control problem (12)-(14), then obtain approximate solution $(u_{h_{k+1}}, y_{h_{k+1}}, p_{h_{k+1}})$ with respect to $\mathcal{T}_{h_{k+1}}$.

(6) Set $k = k + 1$ and go to step (2).

4. Convergence analysis.

In this section, we will do our best to demonstrate the convergence while we first give some properties which take vital significance to the proof of the convergence and even the quasi-optimality for the error indicators and the data oscillations before we begin to show the convergence analysis.

Lemma 4.1. *Let $(u, y, p) \in U_{ad} \times H^1_0(\Omega) \times H^1_0(\Omega)$ be the solution of (7)-(9) and $(u_h, y_h, p_h) \in U^h_{ad} \times V_h \times V_h$ be the solution of (12)-(14). Then we have a posteriori upper bound*

$$\begin{aligned}
&||u - u_h||_0^2 + |||h_{\mathcal{T}_h}(y - y_h)|||^2 + |||h_{\mathcal{T}_h}(p - p_h)|||^2 \\
\leq &C(\eta^2_{1,\mathcal{T}_h}(p_h, \mathcal{T}_h) + \eta^2_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{T}_h) + \eta^2_{3,\mathcal{T}_h}(y_h, p_h, \mathcal{T}_h)),
\end{aligned}$$

where C only depends on the shape regularity of $\mathcal{T}_h$.

Proof. By applying (18), the triangle inequality, and Lemma 2.1, we obtain

$$\begin{aligned}
|||h_{\mathcal{T}_h}(y - y_h)|||^2 \leq &|||h_{\mathcal{T}_h}(y - y^h)|||^2 + |||h_{\mathcal{T}_h}(y^h - y_h)|||^2 \\
\leq &(||\nabla h_{\mathcal{T}_h}||_\infty |||y - y^h|||)^2 + |||h_{\mathcal{T}_h}(y^h - y_h)|||^2 \\
\leq &C||u - u_h||_0^2 + |||h_{\mathcal{T}_h}(y^h - y_h)|||^2. \qquad (33)
\end{aligned}$$

Similar to the proof of Lemma 3.3 in [21], we deduce that

$$|||h_{\mathcal{T}_h}(y^h - y_h)|||^2 \leq \eta^2_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{T}_h). \qquad (34)$$

Analogously, the following conclusions can be drawn

$$|||h_{\mathcal{T}_h}(p - p_h)|||^2 \leq C||u - u_h||_0^2 + C||y - y_h||^2 + |||h_{\mathcal{T}_h}(p^h - p_h)|||^2, \qquad (35)$$

and apparently

$$|||h_{\mathcal{T}_h}(p^h - p_h)|||^2 \leq \eta^2_{3,\mathcal{T}_h}(y_h, p_h, \mathcal{T}_h). \qquad (36)$$

It is obvious to get the expected result in Lemma 4.1 via using (33)-(36), and the upper bound in Theorem 3.2. □

Next, we gives a stability result for error indicators which is can be found in Lemma 3.4 in [21], Lemma 4.1 in [23], and even Proposition 3.3 in [8], and so on.

Lemma 4.2. *For $T \in \mathcal{T}_h,\ \mathcal{T}_h \in \mathbb{T}$, let $u_{h_1}, u_{h_2} \in U_{ad}^h$, $y_{h_1}, y_{h_2} \in V_h$ and $p_{h_1}, p_{h_2} \in V_h$, we have*

$$\begin{aligned}
&\eta_{1,\mathcal{T}_h}(p_{h_1},T) - \eta_{1,\mathcal{T}_h}(p_{h_2},T)\\
\leq & C(|||h_{\mathcal{T}_h}(p_{h_1}-p_{h_2})|||_T + ||\nabla h_{\mathcal{T}_h}||_{\infty,T}||p_{h_1}-p_{h_2}||_{0,T}), \qquad (37)\\
&\eta_{2,\mathcal{T}_h}(u_{h_1},y_{h_1},T) - \eta_{2,\mathcal{T}_h}(u_{h_2},y_{h_2},T)\\
\leq & C(h_T^2||u_{h_1}-u_{h_2}||_{0,T} + |||h_{\mathcal{T}_h}(y_{h_1}-y_{h_2})|||_{\omega_T}\\
&+ ||\nabla h_{\mathcal{T}_h}||_{\infty,T}||y_{h_1}-y_{h_2}||_{0,\omega_T}), \qquad (38)\\
&\eta_{3,\mathcal{T}_h}(y_{h_1},p_{h_1},T) - \eta_{3,\mathcal{T}_h}(y_{h_2},p_{h_2},T)\\
\leq & C(h_T^2||y_{h_1}-y_{h_2}||_{0,T} + |||h_{\mathcal{T}_h}(p_{h_1}-p_{h_2})|||_{\omega_T}\\
&+ ||\nabla h_{\mathcal{T}_h}||_{\infty,T}||p_{h_1}-p_{h_2}||_{0,\omega_T}), \qquad (39)\\
&osc_{\mathcal{T}_h}(y_{h_1}-y_d,T) - osc_{\mathcal{T}_h}(y_{h_2}-y_d,T)\\
\leq & C(|||h_{\mathcal{T}_h}(y_{h_1}-y_{h_2})|||_T + ||\nabla h_{\mathcal{T}_h}||_{\infty,T}||y_{h_1}-y_{h_2}||_{0,T}), \qquad (40)
\end{aligned}$$

where ω_T denotes the patch of elements that share an edge with T.

Proof. We first prove (37) while (40) can be just proved similarly. Consulting the literatures [1, 21, 25], namely the trace inequality, there exists $T \in \mathcal{T}_h, \mathcal{T}_h \in \mathbb{T}$ such that

$$||v||_{0,\partial T} \leq C(h_T^{-1/2}||v||_{0,T} + h_T^{1/2}||v||_{1,T}), \qquad (41)$$

for arbitrary $v \in H^1(\Omega)$. In connection with the inverse estimates and (41), we have

$$||[\nabla(p_{h_1}-p_{h_2})]\cdot \mathbf{n}||_{0,\partial T\setminus\partial\Omega} \leq Ch_T^{-1/2}|||p_{h_1}-p_{h_2}|||_{\omega_T}. \qquad (42)$$

Recalling (1) in Lemma 1, we know that

$$\begin{aligned}
h_T|||p_{h_1}-p_{h_2}|||_{\omega_T} \leq & C||h_{\mathcal{T}_h}\nabla(p_{h_1}-p_{h_2})||_{0,\omega_T}\\
\leq & C(|||h_{\mathcal{T}_h}\nabla(p_{h_1}-p_{h_2})|||_{\omega_T} + ||(p_{h_1}-p_{h_2})\nabla h_{\mathcal{T}_h}||_{0,\omega_T}). \qquad (43)
\end{aligned}$$

Recalling the definition of $\eta_{1,\mathcal{T}_h}(p_h,T)$, we employ the triangle inequality to calculate for $T \in \mathcal{T}_{h_k}$ that

$$\eta_{1,\mathcal{T}_h}(p_{h_1},T) \leq \eta_{1,\mathcal{T}_h}(p_{h_2},T) + h_T^{3/2}||[\nabla(p_{h_1}-p_{h_2})]\cdot\mathbf{n}||_{0,\partial T\setminus\partial\Omega}. \qquad (44)$$

Then it is easy to derive the desired result (37) by adopting (42)-(44).

Next we are to prove (38) while (39) can be proved similarly. We calculate while applying the inequality (15) in Lemma 2.4 for the edge $T \cap T'$ to obtain that

$$\begin{aligned}
&h_T^{3/2}||\nabla(y_{h_1}-y_{h_2})||_{0,T\cap T'}\\
\leq & h_T^{3/2}(||\nabla(y_{h_1}-y_{h_2})_T||_{0,T\cap T'} + ||\nabla(y_{h_1}-y_{h_2})_{T'}||_{0,T\cap T'})\\
\leq & Ch_T^{3/2}\big(h_T^{-1/2}||\nabla(y_{h_1}-y_{h_2})||_{0,T} + h_T^{1/2}|\nabla(y_{h_1}-y_{h_2})|_{0,T}\\
&+ h_T^{-1/2}||\nabla(y_{h_1}-y_{h_2})||_{0,T'} + h_T^{1/2}|\nabla(y_{h_1}-y_{h_2})|_{0,T'}\big)\\
\leq & C(|||h_{\mathcal{T}_h}(y_{h_1}-y_{h_2})|||_{0,T\cup T'} + ||(y_{h_1}-y_{h_2})\nabla h_{\mathcal{T}_h}||_{0,T\cup T'}. \qquad (45)
\end{aligned}$$

Recalling the definition $\eta_{2,\mathcal{T}_h}(u_h, y_h, T)$, Lemma 2.3 and Lemma 3.1, we adopt the triangle inequality to calculate for $T \in \mathcal{T}_{h_k}$ that

$$\begin{aligned}
&\eta_{2,\mathcal{T}_h}(u_{h_1}, y_{h_1}, T)\\
\leq&\eta_{2,\mathcal{T}_h}(u_{h_2}, y_{h_2}, T) + \big(h_T^3||[\nabla(y_{h_1} - y_{h_2})]||_{0,\partial T\backslash\partial\Omega} + h_T^4||\phi(y_{h_1}) - \phi(y_{h_2})||_{0,T}^2\big)^{\frac{1}{2}}\\
\leq&\eta_{2,\mathcal{T}_h}(u_{h_2}, y_{h_2}, T) + \big(h_T^4||\Delta(y_{h_1} - y_{h_2})||_{0,T}^2\\
&+ h_T^3||[\nabla(y_{h_1} - y_{h_2})]||_{0,\partial T\backslash\partial\Omega} + h_T^4||\phi'(y_{h_1})||_{0,T}||y_{h_1} - y_{h_2}||_{0,T}^2\big)^{\frac{1}{2}}\\
\leq&\eta_{2,\mathcal{T}_h}(u_{h_2}, y_{h_2}, T) + \big(h_T^3||[\nabla(y_{h_1} - y_{h_2})]||_{0,\partial T\backslash\partial\Omega}\\
&+ h_T^4||u_{h_1} - u_{h_2}||_{0,T}^2 + h_T^4|||y_{h_1} - y_{h_2}|||_T\big)^{\frac{1}{2}}. \qquad (46)
\end{aligned}$$

Then it is easy to deduce the expected result (38) by connecting (45) into (46). □

Lemma 4.3. *For $\mathcal{T}_h \in \mathbb{T}$, let $\mathcal{M}_h \subset \mathcal{T}_h$ be the set of marked elements and let $\tilde{\mathcal{T}}_h \in \mathbb{T}$ be the refinement of $\mathcal{T}_h$ so that we have*

$$\begin{aligned}
&\eta_{1,\tilde{\mathcal{T}}_h}^2(\tilde{p}, \tilde{\mathcal{T}}_h) - (1+\sigma)\big[\eta_{1,\mathcal{T}_h}^2(p, \mathcal{T}_h) - \lambda\eta_{1,\mathcal{T}_h}^2(p, \mathcal{M}_h)\big]\\
\leq&C(1+\frac{1}{\sigma})\bigg(\sum_{T\in\tilde{\mathcal{T}}_h} h_T^4||u_h - \tilde{u}_h||_{0,T}^2 + |||h_{\tilde{\mathcal{T}}_h}(y_h - \tilde{y}_h)|||\bigg), \qquad (47)\\
&\eta_{2,\tilde{\mathcal{T}}_h}^2(\tilde{u}_h, \tilde{y}_h, \tilde{\mathcal{T}}_h) - (1+\sigma)\big[\eta_{2,\mathcal{T}_h}^2(u_h, y_h, \mathcal{T}_h) - \lambda\eta_{2,\mathcal{T}_h}^2(u_h, y_h, \mathcal{M}_h)\big]\\
\leq&C(1+\frac{1}{\sigma})\bigg(\sum_{T\in\tilde{\mathcal{T}}_h} h_T^4||u_h - \tilde{u}_h||_{0,T}^2 + |||h_{\tilde{\mathcal{T}}_h}(y_h - \tilde{y}_h)|||^2\\
&+ ||\nabla h_{\tilde{\mathcal{T}}_h}||_\infty^2||y_h - \tilde{y}_h||_0^2\bigg), \qquad (48)\\
&\eta_{3,\tilde{\mathcal{T}}_h}^2(\tilde{y}_h, \tilde{p}_h, \tilde{\mathcal{T}}_h) - (1+\sigma)\big[\eta_{3,\mathcal{T}_h}^2(y_h, p_h, \mathcal{T}_h) - \lambda\eta_{3,\mathcal{T}_h}^2(y_h, p_h, \mathcal{M}_h)\big]\\
\leq&C(1+\frac{1}{\sigma})\bigg(\sum_{T\in\tilde{\mathcal{T}}_h} h_T^4||u_h - \tilde{u}_h||_{0,T}^2 + |||h_{\tilde{\mathcal{T}}_h}(p_h - \tilde{p}_h)|||^2\\
&+ ||\nabla h_{\tilde{\mathcal{T}}_h}||_\infty^2||p_h - \tilde{p}_h||_0^2\bigg), \qquad (49)\\
&osc_{\mathcal{T}_h}^2(y_h - y_d, \mathcal{T}_h \cap \tilde{\mathcal{T}}_h) - 2osc_{\tilde{\mathcal{T}}_h}^2(\tilde{y}_h - y_d, \mathcal{T}_h \cap \tilde{\mathcal{T}}_h)\\
\leq&2C\big(|||h_{\mathcal{T}_h}(y_h - \tilde{y}_h)|||^2 + ||\nabla h_{\tilde{\mathcal{T}}_h}||_\infty^2||y_h - \tilde{y}_h||_0^2\big), \qquad (50)
\end{aligned}$$

where $u_h \in U_{ad}^h, \tilde{u}_h \in U_{ad}^{\tilde{h}}, y_h, p_h \in V_h, \tilde{y}_h, \tilde{p}_h \in V_{\tilde{h}}, \sigma \in (0, 1]$ and $\lambda = 1 - 2^{-\frac{3b}{2}}$.

Proof. We just prove (49) and (50). The proofs of (47) and (48) are similar with (49). Employing the Young's inequality with parameter σ and (39), we obtain

$$\begin{aligned}
&\eta_{3,\tilde{\mathcal{T}}_h}^2(\tilde{y}_h, \tilde{p}_h, \tilde{\mathcal{T}}_h) - \eta_{3,\tilde{\mathcal{T}}_h}^2(y_h, p_h, \tilde{\mathcal{T}}_h) \leq C(1+\frac{1}{\sigma})\bigg(\sum_{T\in\mathcal{T}_h} h_T^4\|u_h - \tilde{u}_h\|_{0,T}^2\\
&+ |||h_{\tilde{\mathcal{T}}_h}(p_h - \tilde{p}_h|||^2 + ||\nabla h_{\tilde{\mathcal{T}}_h}||_\infty^2||p_h - \tilde{p}_h||_0^2\bigg) + \delta\eta_{3,\tilde{\mathcal{T}}_h}^2(y_h, p_h, \tilde{\mathcal{T}}_h). \qquad (51)
\end{aligned}$$

For a marked element $T' \in \mathcal{M}_h \subset \mathcal{T}_h$, let $\tilde{\mathcal{T}}_{h_{T'}} = \{T \in \tilde{\mathcal{T}}_h : T \subset T'\}$. Note that the jump $[\nabla p] = 0$ as $u \in U_{ad}^h \subset U_{ad}^{\tilde{h}}$, and $y, p \in V_{\mathcal{T}_h} \subset V_{\tilde{\mathcal{T}}_h}$. According to the definition of h_T, we can obtain $h_T = |T|^{\frac{1}{2}} \leq (2^{-b}|T'|)^{\frac{1}{2}} \leq 2^{-\frac{b}{2}} h'_T$, inferring that

$$\sum_{T\in\tilde{\mathcal{T}}_{h_{T'}}} \eta^2_{3,\tilde{\mathcal{T}}_h}(y,p,T) \leq 2^{-\frac{3b}{2}} \eta^2_{3,\mathcal{T}_h}(y,p,T').$$

For $T' \in \mathcal{T}_h \backslash \mathcal{M}_h$, we deduce that

$$\eta^2_{3,\tilde{\mathcal{T}}_h}(y,p,T') \leq \eta^2_{3,\mathcal{T}_h}(y,p,T'),$$

which can use to derive that

$$\begin{aligned} \eta^2_{3,\tilde{\mathcal{T}}_h} \leq & 2^{-\frac{3b}{2}} \eta^2_{3,\mathcal{T}_h}(y,p,\mathcal{M}_h) + \eta^2_{2,\mathcal{T}_h}(y,p,\mathcal{T}_h\backslash\mathcal{M}_h) \\ = & \eta^2_{3,\mathcal{T}_h}(y,p,\mathcal{T}_h) - \lambda\eta^2_{3,\mathcal{T}_h}(y,p,\mathcal{M}_h). \end{aligned} \tag{52}$$

Then adding (52) into (51) and rearranging the terms can obtain the expected result (49).

Next, for arbitrary $T \in \mathcal{T}_h \cap \tilde{\mathcal{T}}_h$ by using (50) in Lemma 4.2, we have

$$\begin{aligned} & osc^2_{\mathcal{T}_h}(y_h - y_d, T) - 2osc^2_{\mathcal{T}_h}(\tilde{y}_h - y_d, T) \\ \leq & 2C(|||h_{\mathcal{T}_h}(y_h - \tilde{y}_h)|||_T + ||\nabla h_{\mathcal{T}_h}||_{\infty,T}||y_h - \tilde{y}_h||_{0,T}), \end{aligned} \tag{53}$$

where the Young's inequality have been applied and $osc_{\mathcal{T}_h}(y - y_d, T) = osc_{\tilde{\mathcal{T}}_h}(y - y_d, T)$. Then summing over $T \in \mathcal{T}_h \cap \tilde{\mathcal{T}}_h$ for (53), we can easy to derive the desired result (50). □

In order to facilitate computation, we introduce the following new notation

$$\begin{aligned} e^2_{h_k} &= |||h_{\mathcal{T}_{h_k}}(y - y_{h_k})|||^2 + |||h_{\mathcal{T}_{h_k}}(p - p_{h_k})|||^2, \\ E^2_{h_k} &= |||h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}} - y_{h_k})|||^2 + |||h_{\mathcal{T}_{h_{k+1}}}(p_{h_{k+1}} - p_{h_k})|||^2, \\ \eta^2_{\mathcal{T}_{h_k}}(\omega) &= \eta^2_{1,\mathcal{T}_{h_k}}(p_{h_k},\omega) + \eta^2_{2,\mathcal{T}_{h_k}}(u_{h_k}, y_{h_k},\omega) + \eta^2_{3,\mathcal{T}_{h_k}}(y_{h_k}, p_{h_k},\omega), \\ osc^2_{\mathcal{T}_{h_k}}(\omega) &= osc^2_{\mathcal{T}_{h_k}}(f,\omega) + osc^2_{\mathcal{T}_{h_k}}(y_{h_k} - y_d,\omega), \end{aligned}$$

for $\omega \in \mathcal{T}_{h_k}$. As to the proof of the convergence, one of the main obstacle is that there do not have the orthogonality while it is vital of the proof for the convergence. Thus getting back to the second place we transfer proof of the quasi-orthogonality. The latter is popularly adopted in the adaptive mixed and the nonconforming adaptive finite element methods [24]. Apparently it is true for the following basic relationships with $\mathcal{T}_{h_k}, \mathcal{T}_{h_{k+1}} \in \mathbb{T}$ and $\mathcal{T}_{h_k} \subset \mathcal{T}_{h_{k+1}}$ that

$$\begin{aligned} |||h_{\mathcal{T}_{h_{k+1}}}(v - v_{h_{k+1}})||| = & |||h_{\mathcal{T}_{h_{k+1}}}(v - v_{h_k})||| - |||h_{\mathcal{T}_{h_{k+1}}}(v_{h_{k+1}} - v_{h_k})||| \\ & - 2a(h_{\mathcal{T}_{h_{k+1}}}(v - v_{h_{k+1}}), h_{\mathcal{T}_{h_{k+1}}}(v_{h_{k+1}} - v_{h_k})), \end{aligned} \tag{54}$$

so that we obtain the quasi-orthogonality in Lemma 4.4 by estimating the last term of (54).

Lemma 4.4. *For any $\epsilon > 0, \mathcal{T}_{h_k}, \mathcal{T}_{h_{k+1}} \in \mathbb{T}$, there holds*

$$\begin{aligned} & (1-\epsilon)|||h_{\mathcal{T}_{h_{k+1}}}(y - y_{h_{k+1}})|||^2 - (1+\epsilon)|||h_{\mathcal{T}_{h_k}}(y - y_{h_k})|||^2 \\ & + |||h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}} - y_{h_k})|||^2 \end{aligned}$$

$$\leq\frac{C}{\epsilon}(\mu^2+h_0^2)\big(||u_h-u_{h_{k+1}}||_0^2+||y_h-y_{h_k}||_0^2+||y-y_{h_{k+1}}||_0^2\big), \quad (55)$$

$$(1-\epsilon)|||h_{\mathcal{T}_{h_{k+1}}}(p-p_{h_{k+1}})|||^2-(1+\epsilon)|||h_{\mathcal{T}_{h_k}}(p-p_{h_k})|||^2$$
$$+|||h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k})|||^2$$
$$\leq\frac{C}{\epsilon}(\mu^2+h_0^2)\big(||u_h-u_{h_{k+1}}||_0^2+||p_h-p_{h_k}||_0^2+||p-p_{h_{k+1}}||_0^2\big), \quad (56)$$

where $h_0=\max_{T\in\mathcal{T}_{h_0}} h_T$.

Proof. We just prove (55) while (56) can be proved in a similar way. Let $y^{h_{k+1}}$ satisfying (16) with $f+u_{h_{k+1}}$, then we have

$$a(h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}}),h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k}))$$
$$=a(h_{\mathcal{T}_{h_{k+1}}}(y-y^{h_{k+1}}),h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k}))$$
$$+a(h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_{k+1}}),h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k})). \quad (57)$$

Similar to the proof of Lemma 4.3 in [23], we can estimate the second term of the right side of (57) as shown below

$$a(h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_{k+1}}),h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k}))$$
$$\leq\frac{\epsilon}{8}(|||h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_{k+1}})|||^2+|||h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_k}|||^2)$$
$$+C(1+\frac{1}{\epsilon})\mu^2(||y^{h_{k+1}}-y_{h_{k+1}}||_0^2+||y^{h_{k+1}}-y_{h_k}||_0^2).$$

Next we will subdivide the proof. By applying (2), (18), and the triangle inequality, we have

$$|||h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_{k+1}})|||$$
$$\leq|||h_{\mathcal{T}_{h_{k+1}}}(y-y^{h_{k+1}})|||+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}})|||$$
$$\leq\mu||y-y^{h_{k+1}}||_0+|||h_{\mathcal{T}_{h_{k+1}}}\nabla(y-y^{h_{k+1}})|||+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}})|||$$
$$\leq C(\mu+h_0)||u-u_{h_{k+1}}||_0+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}})|||,$$

in which we use the same way to derive that

$$|||h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_k})|||\leq C(\mu+h_0)||u-u_{h_{k+1}}||_0+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_k})|||.$$

It follows from the triangle inequality that

$$||y^{h_{k+1}}-y_{h_{k+1}}||_0\leq C(||u-u_{h_{k+1}}||_0+||y-y_{h_{k+1}}||_0),$$

and apparently

$$||y^{h_{k+1}}-y_{h_k}||_0\leq C(||u-u_{h_{k+1}}||_0+||y-y_{h_k}||_0).$$

Then in connection with what we discuss above to deduce that

$$a(h_{\mathcal{T}_{h_{k+1}}}(y^{h_{k+1}}-y_{h_{k+1}}),h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k}))$$
$$\leq\frac{\epsilon}{4}(|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}})|||^2+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_k})|||^2)$$
$$+\frac{C}{\epsilon}(\mu^2+h_0^2)(||u-u_{h_{k+1}}||_0^2+||y-y_{h_k}||_0^2+|||y-y_{h_k}|||^2). \quad (58)$$

For the first term of the right side of (57), we obtain

$$\begin{aligned}
&a(h_{\mathcal{T}_{h_{k+1}}}(y-y^{h_{k+1}}), h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k}))\\
=&((y-y^{h_{k+1}})\nabla h_{\mathcal{T}_{h_{k+1}}}, \nabla(h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k})))\\
&+(h_{\mathcal{T}_{h_{k+1}}}\nabla(y-y^{h_{k+1}}), \nabla(h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k})))\\
\leq&\frac{\epsilon}{4}(|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}})|||^2+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_k})|||^2)\\
&+\frac{C}{\epsilon}(\mu^2+h_0^2)||u-u_{h_{k+1}}||_0^2.
\end{aligned} \tag{59}$$

Contacting (57), (58) and (59) to gain

$$\begin{aligned}
&a(h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}}), h_{\mathcal{T}_{h_{k+1}}}(y_{h_{k+1}}-y_{h_k}))\\
\leq&\frac{C}{\epsilon}(\mu^2+h_0^2)(||u-u_{h_{k+1}}||_0^2+||y-y_{h_k}||_0^2+|||y-y_{h_k}|||^2)\\
&+\frac{\epsilon}{2}(|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_{k+1}})|||^2+|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_k})|||^2).
\end{aligned} \tag{60}$$

Because of $h_{\mathcal{T}_{h_{k+1}}} \leq h_{\mathcal{T}_{h_k}}$, we have the following result

$$\begin{aligned}
|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_k})||| \leq& |||h_{\mathcal{T}_{h_{k+1}}}\nabla(y-y_{h_k})|||+\mu||y-y_{h_k}||_0\\
\leq& |||h_{\mathcal{T}_{h_k}}(y-y_{h_k})|||+2\mu||y-y_{h_k}||_0,
\end{aligned}$$

such that

$$|||h_{\mathcal{T}_{h_{k+1}}}(y-y_{h_k})|||^2 \leq (1+\epsilon)|||h_{\mathcal{T}_{h_k}}(y-y_{h_k})|||^2+4\mu^2(1+\frac{1}{\epsilon})||y-y_{h_k}||_0^2. \tag{61}$$

Then we can obtain (55) by combing with (54), (60), and (61). □

Theorem 4.5. *Let $(u,y,p) \in U_{ad} \times H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of (7)-(9) and $(u_h, y_h, p_h) \in U_{ad}^h \times V_h \times V_h$ be the solution of (12)-(14) generated by the adaptive finite element algorithm 3.1. Then exist $\gamma > 0$ and $\alpha \in (0,1]$ depending only on the shape of regularity of initial $\mathcal{T}_{h_0}, b, \Omega$ and the marking parameter $\theta \in (0,1]$ such that*

$$e_{h_{k+1}}^2+\gamma\eta_{h_{k+1}}^2(\mathcal{T}_{h_{k+1}}) \leq \alpha(e_{h_k}^2+\gamma\eta_{h_k}^2(\mathcal{T}_{h_k})), \tag{62}$$

apparently providing $h_0 \ll 1$ and sufficiently small μ.

Proof. Taking Lemma 3.1, Lemma 4.3, and Lemma 4.4 into account, we can easy to derive that

$$e_{h_k}^2 \leq C\eta_{\mathcal{T}_{h_k}}^2(\mathcal{T}_{h_k}), \tag{63}$$

$$\begin{aligned}
(1-\epsilon)e_{h_{k+1}}^2 \leq& (1+\epsilon)^2 e_{h_k}^2-E_{h_k}^2+\frac{C}{\epsilon}(\mu^2+h_0^2)\Big(||u-u_{h_{k+1}}||_0^2+||y-y_{h_k}||_0^2\\
&+||y-y_{h_{k+1}}||_0^2+||p-p_{h_k}||_0^2+||p-p_{h_{k+1}}||_0^2\Big),
\end{aligned} \tag{64}$$

$$\begin{aligned}
\eta_{\mathcal{T}_{h_{k+1}}}^2(\mathcal{T}_{h_{k+1}}) \leq& (1+\sigma)\big(\eta_{\mathcal{T}_{h_k}}^2(\mathcal{T}_{h_k})-\lambda\eta_{\mathcal{T}_{h_k}}^2(\mathcal{M}_{h_k})\big)+C(1+\frac{1}{\sigma})\Big[(\mu^2\\
&+h_0^2)\Big(||u-u_{h_k}||_0^2+||u-u_{h_{k+1}}||_0^2+||y-y_{h_k}||_0^2+||y-y_{h_{k+1}}||_0^2\\
&+||p-p_{h_k}||_0^2+||p-p_{h_{k+1}}||_0^2\Big)+E_{h_k}^2\Big].
\end{aligned} \tag{65}$$

Simplifying (64) and (65) by employing Lemma 3.1 and Lemma 4.1, we have

$$\begin{aligned}
&(1-\epsilon)e^2_{h_{k+1}}\\
\leq&(1+\epsilon)^2e^2_{h_k}-E^2_{h_k}+\frac{C}{\epsilon}(\mu^2+h_0^2)\Big(\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k})+\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\Big),
\end{aligned}\tag{66}$$

$$\begin{aligned}
&\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\\
\leq&(1+\sigma)\Big(\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k})-\lambda\eta^2_{\mathcal{T}_{h_k}}(\mathcal{M}_{h_k})\Big)\\
&+C(1+\frac{1}{\sigma})\Big[(\mu^2+h_0^2)\Big(\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k})+\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\Big)+E^2_{h_k}\Big].
\end{aligned}\tag{67}$$

Multiplying (67) with $\gamma_1=\frac{1}{C(1+\sigma^{-1})}$ and adding the result to (66), we can easy to deduce that

$$\begin{aligned}
&(1-\epsilon)e^2_{h_{k+1}}+\gamma_1\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\\
\leq&(1+\epsilon)^2e^2_{h_k}+\gamma_1(1+\sigma)\Big(\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k})-\lambda\eta^2_{\mathcal{T}_{h_k}}(\mathcal{M}_{h_k})\Big)\\
&+\gamma_2(\mu^2+h_0^2)\Big(\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k})+\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\Big),
\end{aligned}$$

where $\gamma_2=1+\frac{C}{\epsilon}$. Applying the marking strategy in Algorithm in 3.1 and (63), we have

$$\begin{aligned}
&(1-\epsilon)e^2_{h_{k+1}}+\big[\gamma_1-\gamma_2(\mu^2+h_0^2)\big]\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\\
\leq&\Big((1+\epsilon)^2-\frac{\gamma_3\gamma_1(1+\sigma)\lambda\theta}{C}\Big)e^2_{h_k}+\big[\gamma_1(1+\sigma)(1-(1-\gamma_3)\lambda\theta)\\
&+\gamma_2(\mu^2+h_0^2)\big]\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}),
\end{aligned}$$

where $\gamma_3\in(0,1)$. By simple calculation and letting

$$\begin{aligned}
\gamma=&\frac{\gamma_1-\gamma_2(\mu^2+h_0^2)}{1-\epsilon},\\
\alpha_1=&1-\frac{\frac{1}{C}\gamma_3\gamma_1(1+\sigma)\lambda\theta-\epsilon(3+\epsilon)}{1-\epsilon},\\
\alpha_2=&\frac{\gamma_1(1+\sigma)(1-(1-\gamma_3)\lambda\theta)+\gamma_2(\mu^2+h_0^2)}{\gamma_1-\gamma_2(\mu^2+h_0^2)},
\end{aligned}$$

where $\alpha_1\in(0,1)$ if choosing ϵ,σ small enough and where $\alpha_2\in(0,1)$ if electing μ,h_0 sufficiently small, then here holds

$$e^2_{h_{k+1}}+\gamma\eta^2_{\mathcal{T}_{h_{k+1}}}(\mathcal{T}_{h_{k+1}})\leq\alpha_1e^2_{h_k}+\alpha_2\gamma\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}),$$

which tells a contraction property, namely (62), in Algorithm 3.1 if selecting $\alpha=\max\{\alpha_1,\alpha_2\}$. □

Theorem 4.6. *Let $(u,y,p)\in U_{ad}\times H_0^1(\Omega)\times H_0^1(\Omega)$ be the solution of (7)-(9) and $(u_h,y_h,p_h)\in U_{ad}^h\times V_h\times V_h$ be the solution of (12)-(14) generated by the adaptive finite element algorithm 3.1 with the other conditions being same with Theorem 4.5, then there holds*

$$\|u-u_{h_k}\|_0^2+\|y-y_{h_k}\|_0^2+\|p-p_{h_k}\|_0^2\to0\quad\text{as}\quad k\to\infty.$$

Proof. It follows from Lemma 4.1 and Theorem 4.5 that we can obviously get

$$||u - u_{h_k}||_0 \to 0 \quad \text{as} \quad k \to \infty.$$

Then combining with Lemma 3.1, it is distinct to get the desired result in Theorem 4.6. $\square$

5. **Quasi-optimality analysis.** In this section, we consider the quasi-optimality for the adaptive finite element method. Firstly we give the notations interpretation. For any $\mathcal{T}_h, \tilde{\mathcal{T}}_h \in \mathbb{T}$, let $\#\mathcal{T}_h$ be the number of elements in $\mathcal{T}_h$, and $\mathcal{T}_h \oplus \tilde{\mathcal{T}}_h$ be the smallest common conforming refinement of $\mathcal{T}_h$ and $\tilde{\mathcal{T}}_h$ satisfying [15, 30, 31] the property

$$\#(\mathcal{T}_h \oplus \tilde{\mathcal{T}}_h) \le \#\mathcal{T}_h + \#\tilde{\mathcal{T}}_h - \#\mathcal{T}_{h_0}. \tag{68}$$

According to [14, 21, 23, 24], we need to define a function approximation class

$$\mathcal{A}^s := \{(u, y, p, y_d) \in L^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) : |(u, y, p, y_d)|_s < +\infty\},$$

where

$$\begin{aligned} |(u, y, p, y_d)|_s := \sup_{N>0} N^s \inf_{\mathcal{T}_h \in \mathbb{T}_N} \inf_{(u_h, y_h, p_h) \in U_{ad}^h \times V_h \times V_h} \{\|u - u_h\|_0^2 \\ + \|y - y_h\|_0^2 + \|p - p_h\|_0^2 + osc_{\mathcal{T}_h}^2(\mathcal{T}_h)\}^{\frac{1}{2}}, \end{aligned}$$

and

$$\mathbb{T}_N := \{\mathcal{T}_h \in \mathbb{T} : \#\mathcal{T}_h - \#\mathcal{T}_{h_0} \le N_0\}.$$

We need a local upper bound for the distance between two nested solutions consulting [8] in order to illustrate the quasi-optimality of an adaptive finite element method due to the errors here can only be estimated by using refined element indices without buffer.

Lemma 5.1. *Let $(u, y, p) \in U_{ad} \times H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of (7)-(9). Given sufficiently small μ, let $\mathcal{T}_h \in \mathbb{T}_\mu$ and $\mathcal{T}_h \subset \tilde{\mathcal{T}}_h \in \mathbb{T}$, $(u_h, y_h, p_h) \in U_{ad}^h \times V_h \times V_h$ and $(\tilde{u}_h, \tilde{y}_h, \tilde{p}_h) \in U_{ad}^{\tilde{h}} \times V_{\tilde{h}} \times V_{\tilde{h}}$ be the solution of (12)-(14) on $\mathcal{T}_h$ and $\tilde{\mathcal{T}}_h$. Then there holds*

$$||u_h - \tilde{u}_h||_0^2 + ||y_h - \tilde{y}_h||_0^2 + ||p_h - \tilde{p}_h||_0^2 \le C\eta_{\mathcal{T}_h}^2(\mathcal{R}_h), \tag{69}$$

where $\mathcal{R}_h := \mathcal{R}_{\mathcal{T}_h \to \tilde{\mathcal{T}}_h}$ is the subset of elements that are refined from $\mathcal{T}_h$ to $\tilde{\mathcal{T}}_h$.

Proof. According to the optimal condition (14), we obtain

$$\begin{aligned} (\alpha u_h + p_h, v - u_h) \ge 0, \quad \forall\, v \in V_h, \\ (\alpha \tilde{u}_h + \tilde{p}_h, v - \tilde{u}_h) \ge 0, \quad \forall\, v \in V_{\tilde{h}}, \end{aligned}$$

thus getting

$$\begin{aligned} \alpha||u_h - \tilde{u}_h||_0^2 =& (\alpha u_h, u_h - \tilde{u}_h) - (\alpha \tilde{u}_h, u_h - \tilde{u}_h) \\ \le& (\tilde{p}_h - p_h, u_h - \tilde{u}_h) + (\alpha u_h + p_h, u_h - \tilde{u}_h). \end{aligned} \tag{70}$$

For $(\tilde{p}_h - p_h, u_h - \tilde{u}_h)$ of (70), we have

$$\begin{aligned}
&(\tilde{p}_h - p_h, u_h - \tilde{u}_h)\\
=&(S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+\tilde{u}_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d), u_h - \tilde{u}_h)\\
=&(S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+\tilde{u}_h) - y_d) - S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d), u_h - \tilde{u}_h)\\
&+ (S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d), u_h - \tilde{u}_h)\\
=&(S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(\tilde{u}_h - u_h)), u_h - \tilde{u}_h)\\
&+ (S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d), u_h - \tilde{u}_h)\\
=&(S_{\tilde{\mathcal{T}}_h}(\tilde{u}_h - u_h), S_{\tilde{\mathcal{T}}_h}(u_h - \tilde{u}_h))\\
&+ (S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d), u_h - \tilde{u}_h)\\
\leq&(S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d), u_h - \tilde{u}_h).
\end{aligned}$$

In order to estimate the right side of the above inequality, we suppose that $\varphi \in H_0^1(\Omega)$ is the solution of the following equation

$$a(\varphi, v) = (S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d), v), \quad \forall\, v \in H_0^1(\Omega).$$

By applying the duality arguments, we can gain that

$$\begin{aligned}
&||S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d)||_0^2\\
=&a(\varphi, S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d))\\
=&a(\varphi - \xi_{\mathcal{T}_h}\varphi, S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d))\\
&+ (S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h), \xi_{\mathcal{T}_h}\varphi - \varphi)\\
&+ (S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h), \varphi),
\end{aligned} \tag{71}$$

in which $\xi_{\mathcal{T}_h}$ is the standard Lagrange interpolator onto V_h. For the first term of right side of (71), we have

$$\begin{aligned}
&a(\varphi - \xi_{\mathcal{T}_h}\varphi, S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d))\\
=&a(\varphi - \xi_{\mathcal{T}_h}\varphi, S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f+u_h) - y_d) - S^*_{\tilde{\mathcal{T}}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d))\\
&+ a(\varphi - \xi_{\mathcal{T}_h}\varphi, S^*_{\tilde{\mathcal{T}}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d))\\
\leq&C(||\varphi||_2||S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h)||_0\\
&+ ||h^{-1}_{\mathcal{T}_h}\nabla(\varphi - \xi_{\mathcal{T}_h}\varphi)||_0||h_{\mathcal{T}_h}\nabla(S^*_{\tilde{\mathcal{T}}_h}((S_{\mathcal{T}_h}(f+u_h) - y_d)\\
&- S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d)))||_0)\\
\leq&C||\varphi||_2(||S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h)||_0\\
&+ ||h_{\mathcal{T}_h}\nabla(S^*_{\tilde{\mathcal{T}}_h}((S_{\mathcal{T}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d)))||_0).
\end{aligned} \tag{72}$$

Similar to Lemma 2 in [14], we infer that

$$||S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h)||_0 \leq C\eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h), \tag{73}$$

$$\begin{aligned}
&||h_{\mathcal{T}_h}\nabla(S^*_{\tilde{\mathcal{T}}_h}((S_{\mathcal{T}_h}(f+u_h) - y_d) - S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f+u_h) - y_d)))||_0\\
\leq&C\eta_{3,\mathcal{T}_h}(y_h, p_h, \mathcal{R}_h).
\end{aligned} \tag{74}$$

By using the similar way, we deduce that

$$(S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h), \xi_{\mathcal{T}_h}\varphi - \varphi) \leq C\eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h), \tag{75}$$

$$(S_{\tilde{\mathcal{T}}_h}(f+u_h) - S_{\mathcal{T}_h}(f+u_h), \varphi) \leq C\eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h). \tag{76}$$

Combining (72)-(76) and Lemma 2.2 with H^2-regularity, we derive that

$$(\tilde{p}_h - p_h, u_h - \tilde{u}_h) \leq C(\eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h) + \eta_{3,\mathcal{T}_h}(y_h, p_h, \mathcal{R}_h)). \tag{77}$$

Next we are going to estimate the second term on the right side of (70). Assume that

$$(\alpha u_h + p_h, u_h - \tilde{u}_h) \leq (\alpha u_h + p_h, v_h - \tilde{u}_h), \quad \forall v_h \in U_{ad}^h.$$

Then we set $v_h = \pi_{\mathcal{T}_h}\tilde{u}_h$ for which $\pi_{\mathcal{T}_h}$ is L^2-projection onto $\mathbb{P}_0(\mathcal{T}_h)$, thus obtaining

$$(\alpha u_h + p_h, v_h - \tilde{u}_h)_T = ((I_{\mathcal{T}_h} - \pi_{\mathcal{T}_h})(\alpha u_h + p_h), (\pi_{\mathcal{T}_h} - I_{\mathcal{T}_h})(v_h - \tilde{u}_h)),$$

for arbitrary $T \in \mathcal{T}_h$ and where $I_{\mathcal{T}_h}$ is the identical L^2-projection onto $\mathbb{P}_0(\mathcal{T}_h)$. Owing to $T \in \mathcal{T}_h \backslash \mathcal{R}_h \subset \mathcal{T}_h$ such that $(\pi_{\mathcal{T}_h} - I_{\mathcal{T}_h})(\tilde{u}_h - u_h) = 0$, then here holds

$$(\alpha u_h + p_h, v_h - \tilde{u}_h) \leq C\eta_{1,\mathcal{T}_h}(p_h, \mathcal{R}_h)||u_h - \tilde{u}_h||_0. \tag{78}$$

In connection with (70), (77) and (78), we infer that

$$||u_h - \tilde{u}_h||_0 \leq C\big(\eta_{1,\mathcal{T}_h}(p_h, \mathcal{R}_h) + \eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h) + \eta_{3,\mathcal{T}_h}(y_h, p_h, \mathcal{R}_h)\big). \tag{79}$$

Employing (73)-(76) and the triangle inequality, we deduce that

$$\begin{aligned}
||p_h - \tilde{p}_h||_0 =&||S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f + u_h) - y_d) - S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f + \tilde{u}_h) - y_d)||_0 \\
\leq&||S^*_{\mathcal{T}_h}(S_{\mathcal{T}_h}(f + u_h) - y_d) - S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f + u_h) - y_d)||_0 \\
&+ ||S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f + u_h) - y_d) - S^*_{\tilde{\mathcal{T}}_h}(S_{\tilde{\mathcal{T}}_h}(f + \tilde{u}_h) - y_d)||_0 \\
\leq&C\big(||u_h - \tilde{u}_h||_0 + \eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h) + \eta_{3,\mathcal{T}_h}(y_h, p_h, \mathcal{R}_h)\big).
\end{aligned} \tag{80}$$

It is similar to Lemma 2 in [14] that we infer that

$$\begin{aligned}
||y_h - \tilde{y}_h||_0 =&||S_{\mathcal{T}_h}(f + u_h) - S_{\tilde{\mathcal{T}}_h}(f + \tilde{u}_h)||_0 \\
\leq&||S_{\mathcal{T}_h}(f + u_h) - S_{\tilde{\mathcal{T}}_h}(f + u_h)||_0 + ||S_{\tilde{\mathcal{T}}_h}(f + u_h) - S_{\tilde{\mathcal{T}}_h}(f + \tilde{u}_h)||_0 \\
\leq&C\big(||u_h - \tilde{u}_h||_0 + \eta_{2,\mathcal{T}_h}(u_h, y_h, \mathcal{R}_h)\big).
\end{aligned} \tag{81}$$

To sum up, the proof is finished by adopting (79)-(81). □

Next lemma tells the Dörfler property on the set $\mathcal{R}_h = \mathcal{R}_{\mathcal{T}_{h_k} \to \mathcal{T}_h}$ in order to bound the number of marked elements.

Lemma 5.2. *We assume that the marking parameter $\theta \in (0, \theta^*)$ with $\theta^* = \frac{C}{1+4C}$ and let $(u_h, y_h, p_h) \in U_{ad}^h \times V_h \times V_h$ and $(u_{h_k}, y_{h_k}, p_{h_k}) \in U_{ad}^{h_k} \times V_{h_k} \times V_{h_k}$ be the solution of (12)-(14) on $\mathcal{T}_h$ and $\mathcal{T}_{h_k}$. There exists a constant $\delta = \frac{1}{2}(1 - \frac{\theta}{\theta^*})$ such that*

$$e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}) \leq \delta(e^2_{\mathcal{T}_h} + osc^2_{\mathcal{T}_h}(\mathcal{T}_h)), \tag{82}$$

where $e^2_{\mathcal{T}_h} = ||u - u_h||_0^2 + ||y - y_h||_0^2 + ||p - p_h||_0^2$, similarly for $e^2_{\mathcal{T}_{h_k}}$. Then there holds

$$\eta^2_{\mathcal{T}_h}(\mathcal{R}_h) \geq \theta\eta^2_{\mathcal{T}_h}(\mathcal{T}_h).$$

Proof. Combining with (82) and the upper bound in Theorem 4.5, we can obtain

$$\begin{aligned}
C(1 - 2\delta)\eta^2_{\mathcal{T}_h}(\mathcal{T}_h) \leq&(1 - 2\delta)(e^2_{\mathcal{T}_h} + osc^2_{\mathcal{T}_h}(\mathcal{T}_h) \\
\leq&e^2_{\mathcal{T}_h} - 2e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_h}(\mathcal{T}_h) - 2osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}).
\end{aligned}$$

Employing the triangle inequality and the Young's inequality, here holds

$$||u-u_h||_0^2 \le 2(||u-u_{h_k}||_0^2 + ||u_h-u_{h_k}||_0^2),$$
$$||y-y_h||_0^2 \le 2(||y-y_{h_k}||_0^2 + ||y_h-y_{h_k}||_0^2),$$
$$||p-p_h||_0^2 \le 2(||p-p_{h_k}||_0^2 + ||p_h-p_{h_k}||_0^2).$$

Thus obtaining the result

$$\eta^2_{\mathcal{T}_h}(\mathcal{T}_h) - 2\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}) \le 2C\eta^2_{\mathcal{T}_h}(\mathcal{R}_h), \tag{83}$$

with Lemma 5.1. By applying the dominance property which is similar to Remark 2.1 in [8], we infer that

$$osc^2_{\mathcal{T}_h}(T) \le \eta^2_{\mathcal{T}_h}(T), \tag{84}$$

for $T\in\mathcal{R}_h$ and apparently

$$osc^2_{\mathcal{T}_h}(f,T) \le \eta^2_{\mathcal{T}_{h_k}}(f,T), \tag{85}$$

for $T\in\mathcal{T}_h\backslash\mathcal{R}_h$. For $T\in\mathcal{T}_h\cap\tilde{\mathcal{T}}_h$, we can get the following result by employing (50) of Lemma 4.3 and the inverse estimates

$$osc^2_{\mathcal{T}_h}(\mathcal{T}_h\backslash\mathcal{R}_h) - 2osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_h\backslash\mathcal{R}_h) \le 2C(||y_h-y_{h_k}||_0^2 + ||p_h-p_{h_k}||_0^2). \tag{86}$$

Then it can be derived via using (83), (84) and (86) that

$$C(1-2\delta)\eta^2_{\mathcal{T}_h}(\mathcal{T}_h) \le (1+4C)\eta^2_{\mathcal{T}_h}(\mathcal{R}_h), \tag{87}$$

which tells the proof. □

Lemma 5.3. *Assume that the marking parameter θ satisfies the the conditions in Lemma 5.2. Let $(u,y,p)\in U_{ad}\times H_0^1(\Omega)\times H_0^1(\Omega)$ be the solution of (7)-(9) and $(u_{h_k},y_{h_k},p_{h_k})\in U_{ad}^{h_k}\times V_{h_k}\times V_{h_k}$ be the solution of (12)-(14) generated by Algorithm 3.1. Then the number of marked elements $\mathcal{M}_{h_k}\subset\mathcal{T}_{h_k}$ satisfies*

$$\#\mathcal{M}_{h_k} \le C\beta^{\frac{1}{2s}}\delta^{-\frac{1}{2s}}|(u,y,p,y_d)|_s^{\frac{1}{s}}(e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}))^{-\frac{1}{2s}},$$

if $(u,y,p,y_d)\in\mathcal{A}^s$ for μ being small enough.

Proof. Let $\nu^2 = \delta\beta^{\frac{1}{2s}}(e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}))$, where δ is defined in Lemma 5.2 and β is to be defined as follows. Then there exists a $\mathcal{T}_{h_\nu}\in\mathbb{T}$ and $(u_{h_\nu},y_{h_\nu},p_{h_\nu})\in U_{ad}^{h_\nu}\times V_{h_\nu}\times V_{h_\nu}$ such that

$$\#\mathcal{T}_{h_\nu} - \#\mathcal{T}_{h_0} \le C|(u,y,p,y_d)|_s^{\frac{1}{s}}\nu^{-\frac{1}{s}}, \tag{88}$$

$$\begin{aligned}&||u-u_{h_\nu}||_0^2 + ||y-y_{h_\nu}||_0^2 + ||p-p_{h_\nu}||_0^2 + osc^2_{\mathcal{T}_{h_\nu}}(f,\mathcal{T}_{h_\nu})\\ &+ osc^2_{\mathcal{T}_{h_\nu}}(y_{h_\nu}-y_d,\mathcal{T}_{h_\nu}) \le \nu^2,\end{aligned} \tag{89}$$

for any $(u,y,p,y_d)\in\mathcal{A}^s$. Next we suppose $\mathcal{T}_{h_\epsilon} = \mathcal{T}_{h_\nu}\oplus\mathcal{T}_{h_k}$ is the smallest common refinement of $\mathcal{T}_{h_\nu}$ and $\mathcal{T}_{h_k}$, and let $(u_{h_\epsilon},y_{h_\epsilon},p_{h_\epsilon})$ be the solution of (12)-(14). Then we give the inequalities as follows in order to deduce the expect result we wanted

$$e^2_{\mathcal{T}_{h_\epsilon}} + osc^2_{\mathcal{T}_{h_\epsilon}}(\mathcal{T}_{h_\epsilon}) \le \beta(e^2_{\mathcal{T}_{h_\nu}} + osc^2_{\mathcal{T}_{h_\nu}}(\mathcal{T}_{h_\nu})), \tag{90}$$

where there are some notations been defined by

$$e^2_{\mathcal{T}_{h_\nu}} = ||u-u_{h_\nu}||_0^2 + ||y-y_{h_\nu}||_0^2 + ||p-p_{h_\nu}||_0^2,$$
$$osc^2_{\mathcal{T}_{h_\nu}}(\mathcal{T}_{h_\nu}) = osc^2_{\mathcal{T}_{h_\nu}}(f,\mathcal{T}_{h_\nu}) + osc^2_{\mathcal{T}_{h_\nu}}(y_{h_\nu},\mathcal{T}_{h_\nu}),$$

and apparently $e^2_{\mathcal{T}_{h_\epsilon}}$ and $osc^2_{\mathcal{T}_{h_\epsilon}}(\mathcal{T}_{h_\epsilon})$ can be defined similarly. Just as obviously, here holds

$$||v - v_{h_\nu}||_0^2 = ||v - v_{h_\epsilon}||_0^2 + ||v_{h_\epsilon} - v_{h_\nu}||_0^2 + 2(v - v_{h_\epsilon}, v_{h_\epsilon} - v_{h_\nu}), \tag{91}$$

for all $v \in U_{ad}, v_{h_\nu} \in U_{ad}^{h_\nu}$, and $v_{h_\epsilon} \in U_{ad}^{h_\epsilon}$. Applying the Young's inequality and (91), we have

$$\begin{aligned}(u - u_{h_\epsilon}, u_{h_\epsilon} - u_{h_\nu}) =&(u - u_{h_\nu}, u_{h_\epsilon} - u_{h_\nu}) + (u_{h_\nu} - u_{h_\epsilon}, u_{h_\epsilon} - u_{h_\nu})\\ \leq&(u - u_{h_\nu}, u_{h_\epsilon} - u_{h_\nu})\\ \leq&||u - u_{h_\nu}||_0^2 + \frac{1}{4}||u_{h_\epsilon} - u_{h_\nu}||_0^2,\end{aligned} \tag{92}$$

and apparently in the same way

$$(y - y_{h_\epsilon}, y_{h_\epsilon} - y_{h_\nu}) = ||y - y_{h_\nu}||_0^2 + \frac{1}{4}||y_{h_\epsilon} - y_{h_\nu}||_0^2, \tag{93}$$

$$(p - p_{h_\epsilon}, p_{h_\epsilon} - p_{h_\nu}) = ||p - p_{h_\nu}||_0^2 + \frac{1}{4}||p_{h_\epsilon} - p_{h_\nu}||_0^2. \tag{94}$$

Hence combining (91)-(94) to get

$$\begin{aligned}&||u - u_{h_\epsilon}||_0^2 + ||u_{h_\epsilon} - u_{h_\nu}||_0^2 + ||y - y_{h_\epsilon}||_0^2\\ &+ ||y_{h_\epsilon} - y_{h_\nu}||_0^2 + ||p - p_{h_\epsilon}||_0^2 + ||p_{h_\epsilon} - p_{h_\nu}||_0^2\\ \leq&6(||u - u_{h_\nu}||_0^2 + ||y - y_{h_\nu}||_0^2 + ||p - p_{h_\nu}||_0^2).\end{aligned} \tag{95}$$

For all $T' \in \mathcal{T}_{h_\nu}$, assuming that $\mathcal{T}_{h_{T'}} := \{T \in \mathcal{T}_{h_\epsilon} : T \in T'\}$, then we derive that

$$\begin{aligned}\sum_{T\in\mathcal{T}_{h_{T'}}} ||f - f_T^2||_{0,T}^2 =& \sum_{T\in\mathcal{T}_{h_{T'}}} \left(\int_T f^2 - \frac{(\int_T f)^2}{|T|}\right)\\ =&\int_{T'} f^2 - \sum_{T\in\mathcal{T}_{h_{T'}}} \frac{(\int_T f)^2}{|T|}\\ \leq&\int_{T'} f^2 - \frac{\sum_{T\in\mathcal{T}_{h_{T'}}}(\int_T f)^2}{|T'|}\\ \leq&C\int_{T'} f^2 - \frac{(\int_{T'} f)^2}{|T'|}\\ =&C||f - f_T||_{0,T'}^2,\end{aligned}$$

which tells that

$$osc^2_{\mathcal{T}_{h_\epsilon}}(f, \mathcal{T}_{h_\epsilon}) \leq Cosc^2_{\mathcal{T}_{h_\nu}}(f, \mathcal{T}_{h_\nu}). \tag{96}$$

By using the same way, we have

$$osc^2_{\mathcal{T}_{h_\epsilon}}(y_{h_\epsilon} - y_d, \mathcal{T}_{h_\epsilon}) \leq Cosc^2_{\mathcal{T}_{h_\nu}}(y_{h_\nu} - y_d, \mathcal{T}_{h_\nu}). \tag{97}$$

Therefore, we can gain (90) via employing (95)-(97). Based on the definition of ν^2 and (89), we deduce that

$$e^2_{\mathcal{T}_{h_\epsilon}} + osc^2_{\mathcal{T}_{h_\epsilon}}(\mathcal{T}_{h_\epsilon}) \leq \beta\nu^2 = \delta(e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k})).$$

According to Lemma 5.2, we find that the subset $\mathcal{R}_{\mathcal{T}_{h_k}\to\mathcal{T}_{h_\epsilon}}$ verifies the marking property for $\theta \leq \theta^*$, then we infer that

$$\#\mathcal{M}_{h_k} \leq \#\mathcal{R}_{\mathcal{T}_{h_k}\to\mathcal{T}_{h_\epsilon}} \leq \#\mathcal{T}_{h_\epsilon} - \#\mathcal{T}_{h_k} \leq \#\mathcal{T}_{h_\nu} - \mathcal{T}_{h_0}. \tag{98}$$

Hence we can obtain the desired result by contacting with (88), (98), and the definition of ν^2. □

Next we derive a equivalent property of the indicator dominates oscillation by concluding Theorem 3.2, Lemma 3.1, and Lemma 4.1 as follows

$$e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}} \approx e^2_{h_k} + \gamma\eta^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}), \tag{99}$$

which is of vital importance for the proof of quasi-optimality.

Theorem 5.4. *Assume that $\mathcal{T}_{h_0}$ satisfies the condition (b) of Section 4 in [32]. Let $(u, y, p) \in U_{ad} \times H^1_0(\Omega) \times H^1_0(\Omega)$ be the solution of (7)-(9) and $(u_{h_k}, y_{h_k}, p_{h_k}) \in U^{h_k}_{ad} \times V_{h_k} \times V_{h_k}$ be the solution of (12)-(14) generated by Algorithm 3.1. Then there holds*

$$\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0} \leq C|(u, y, p, y_d)|^{\frac{1}{s}}_{s}(e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}))^{-\frac{1}{2s}},$$

provided $h_0 \ll 1$ if $(u, y, p, y_d) \in \mathcal{A}^s$ for μ being small enough.

Proof. Similar to the Theorem 4 in [14], we infer that

$$\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0} \leq C\sum_{i=0}^{k-1}\mathcal{M}_{h_k}. \tag{100}$$

Employing (100) and Lemma 5.3 to get that

$$\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0} \leq C\sum_{i=0}^{k-1}\mathcal{M}_{h_k} \leq C\chi\sum_{1=0}^{k-1}(e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}))^{-\frac{1}{2s}}, \tag{101}$$

where $\chi = \beta^{\frac{1}{2s}}|(u, y, p, y_d)|^{\frac{1}{s}}_{s}\delta^{-\frac{1}{2s}}$. Combining with (99), (101), and Theorem 4.5, we deduce that

$$\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0} \leq C\chi(e^2_{\mathcal{T}_{h_k}} + osc^2_{\mathcal{T}_{h_k}}(\mathcal{T}_{h_k}))^{-\frac{1}{2s}}\sum_{i=0}^{k}\alpha^{\frac{i}{k}},$$

which yields the proof of Theorem 5.4. □

6. **Numerical examples.**

In this section, we firstly present the adaptive iteration method where the purpose is to provide empirical analysis for our theory.

Algorithm 6.1. Given an initial control $u^0_h \in U^h_{ad}$, then for $k = 1, 2, \cdots$, seek (y^k_h, p^k_h, u^k_h) such that

$$\begin{aligned}
&a\big(y^k_h, w_h\big) + \big(\phi(y^k_h), w_h\big) = \big(f + u^{k-1}_h, w_h\big), \quad \forall\ w_h \in V_h,\\
&a\big(q_h, p^{k-1}_h\big) + \big(\phi(y^k_h)p^{k-1}_h, q_h\big) = \big(y^k_h - y_d, q_h\big), \quad \forall\ q_h \in V_h,\\
&\big(\alpha u^k_h + p^{k-1}_h, v_h - u^k_h\big) \geq 0, \quad \forall\ v^k_h \in U^h_{ad},
\end{aligned}$$

and apparently

$$u^k_h = \frac{1}{\alpha}\big(-\mathcal{P}_h p^k_h + \max\big(0, \bar{p}^k_h\big)\big),$$

where $\mathcal{P}_h$ is the L^2-projection from $L^2(\Omega)$ to U^h and $\bar{p}^k_h = \frac{\int_\Omega p^k_h}{|\Omega|}$.

Example 1. We consider the nonlinear optimal control problem subject to the state equation

$$-\Delta y + y^3 = f + u, \quad -\Delta p + 3y^2 p = y - y_d,$$

where we choose $\alpha = 0.5$ and $\Omega = [0,1] \times [0,1]$ and apparently exact solution

$$\begin{aligned} u =& \frac{1}{\alpha}(\max(0, \bar{p}) - p), \\ p =& -(\sin(\pi x_1) + \sin(\pi x_2)), \\ y =& \sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

By simple calculation we have $\int_\Omega p dx = -\frac{4}{\pi}$ which satisfies $u \in U_{ad}$.

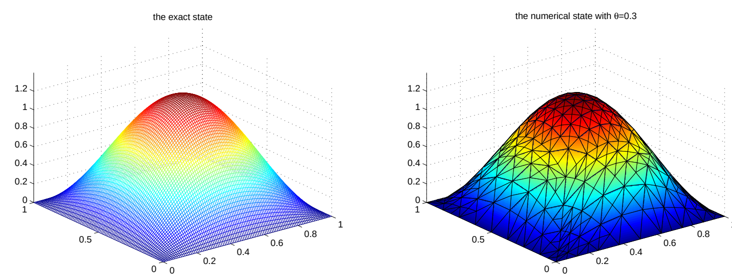

FIGURE 1. The profiles of the exact state and the numerical state on adaptively refined grids with $\theta = 0.3$ and 15 adaptive loops for Example 1.

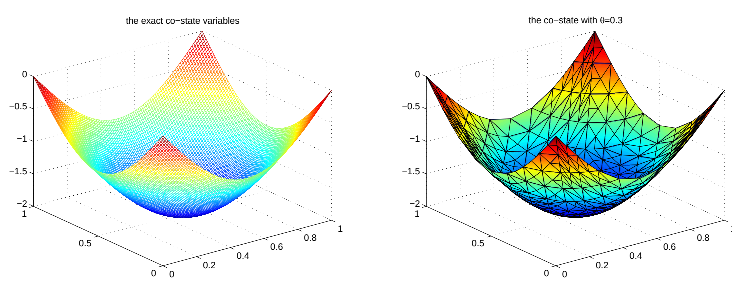

FIGURE 2. The profiles of the exact state , the numerical state, the exact co-state variables and the co-state variables on adaptively refined grids with $\theta = 0.3$ and 15 adaptive loops for Example 1.

In terms of the same error and actuarial accuracy, the adaptive refinement process saves time than the uniform refinement process. In Figures 1-2, we provide the profiles of the exact state variables, the numerical state variables, the exact co-state and the co-state on adaptively refined grids with $\theta = 0.3$ and 15 adaptive loops for Example 1 generated by Algorithm 3.1 and then we plot the adaptive girds after 5 steps and 13 steps with $\theta = 0.3$ and 15 adaptive iterations for Example 1 in Figure 3. It is easy to observe that lager gradients exist in some certain regions but the

FIGURE 3. The adaptive girds after 5 steps and 13 steps with $\theta = 0.3$ and 15 adaptive loops for Example 1 generated by Algorithm 3.1.

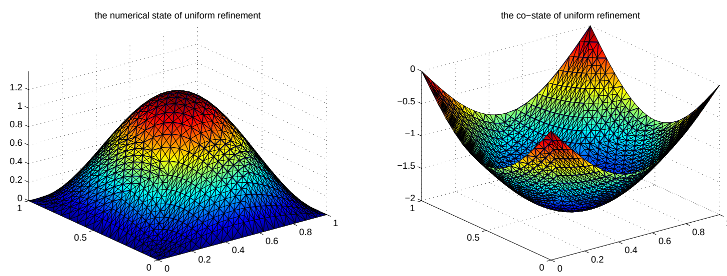

FIGURE 4. The profiles of the numerical state and the co-state variables on uniformly refined grids ($\theta = 1$) and 15 adaptive loops for Example 1.

solutions are smooth. Moreover, the grid refinement at the center of the domain, while the solutions may have lager gradients near the boundary.

In Figure 4, we plot the profiles of the numerical state and the co-state variables on uniformly refined grids ($\theta = 1$) and 15 adaptive loops for Example 1 generated by Algorithm 3.1, and then the adaptively refined triangulations after 5 adaptive iterations with $\theta = 0.4$ and uniformly refined triangulations after 5 uniform iterations of 15 loops are performed in Figure 5. Obviously, it is not difficult to find that the uniformly refined grids seem to have a better encryption effect, but in connection with Figures 1-2 and Figure 4 to observe that the adaptive finite element method may deliver even much smaller errors compared to uniformly refined method. Combining Figure 3 and Figure 5, the adaptive iteration is more efficient and effective than the uniform iteration whereas the uniform iteration needs to be solved at a higher cost of grids, thus increasing iteration time. So the adaptive finite element method has some advantages in the numerical approximation process.

In Figure 6, we show the convergence history of the total error estimate indicators, where we plot the adaptive triangle iterations of 15 adaptive loops with the coefficient $\theta = 0.3$ and $\theta = 0.4$, even more there provides a convergence of the total error estimation indicators for the uniform triangle iterations ($\theta = 1$). We can see an error reduction with slope -1 that is the optimal convergence rate what we expect

FIGURE 5. The adaptive girds after 5 steps with $\theta = 0.3$ and the uniform refinement ($\theta = 1$) after 5 steps for Example 1 generated by Algorithm 3.1.

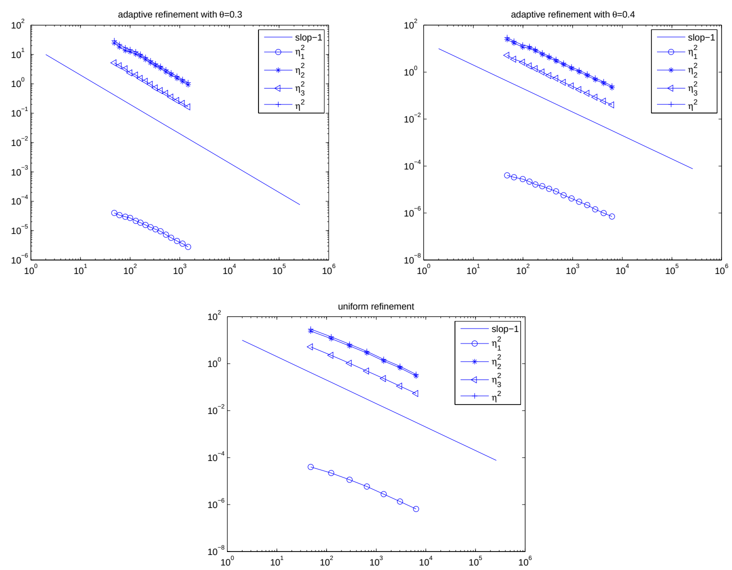

FIGURE 6. The error estimations of adaptively refined grids with $\theta = 0.3$, $\theta = 0.4$, and the error estimates of uniformly refined grids for Example 1.

via applying linear finite elements from the upper two pictures in Figure 6. Meanwhile, we give the comparisons of the error estimations, for which we know that the optimal second order convergence for the error reductions of error estimations.

Example 2. We consider the same nonlinear optimal control problem as Example 1 with $\alpha = 0.1$, $\Omega = (0,1) \times (0,1)$, and apparently the exact solution

$$y = \begin{cases} 5 \times 10^{10} e^{\frac{1}{m}}, & m < 0, \\ 0, & m \geq 0, \end{cases}$$

$$p = \begin{cases} -7 \times 10^{10} e^{\frac{1}{m}}, & m < 0, \\ 0, & m \geq 0, \end{cases}$$

where $m = (x_1 - 0.2)^2 + (x_2 - 0.6)^2 - 0.04$ and absolutely $\int_\Omega u dx > 0$ via simply calculating.

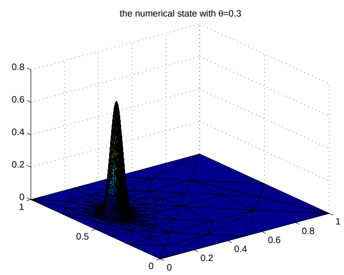

FIGURE 7. The profiles of the numerical state and the co-state on adaptively refined grids with $\theta = 0.3$ and 27 adaptive loops for Example 2.

FIGURE 8. The adaptive girds after 15 steps and 25 steps with $\theta = 0.3$ for Example 2 generated by Algorithm 3.1.

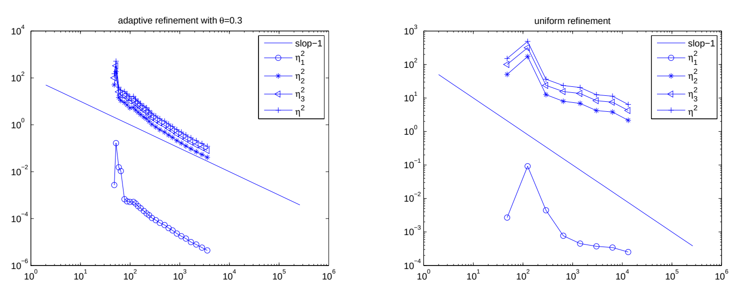

FIGURE 9. The error estimations of adaptively refined grids with $\theta = 0.3$ and the error estimations of uniformly refined grids for Example 2.

In Figure 7, we plot the numerical state and the co-state on adaptively refined grids with $\theta = 0.3$ and 27 adaptive loops for Example 2 generated by Algorithm

3.1. Obviously, we can observe the lager gradients concentrate on the certain regions as the adaptive finite element method may deliver much smaller errors compared with the uniform refinement. In Figure 8, we present the adaptive grids after 15 and 25 adaptively refined by choosing Dörfler's marking parameter $\theta = 0.3$ with 27 adaptive loops. Apparently, the grids gather around the regions where there exist much lager gradients. Therefor they just validate the phenomenon in Figure 7. In addition, the grides focus on the points as f and y_d have singularities near these points. Hence when dealing with singular points, adaptive encryption has better effect.

We give the comparisons of convergence history of Example 2 in Figure 9. The left plot in Figure 9 is adaptively refined with Dörfler's marking parameter $\theta = 0.3$ and 27 adaptive loops while the right one is uniform refinement ($\theta = 1$). With the optimal L^2-norms convergence we desired, we can see the errors reduction for adaptive refinement. Moreover, the reduced orders only can be found in uniform refinement because of the singularity of the solutions.

Acknowledgments. The authors express their thanks to the referees for their helpful suggestions, which led to improvements of the presentation.

REFERENCES

[1] M. Ainsworth and J. T. Oden, A posteriori error estimators in finite element analysis, *Comput. Methods Appl. Mech. Engrg.*, **142** (1997), 1–88.
[2] I. Babuška and W. C. Rheinboldt, Error estimates for adaptive finite computations, *SIAM J. Numer. Anal.*, **15** (1978), 736–754.
[3] R. Becker, S. Mao and Z. Shi, A convergent noncomforming adaptive finite element method with quasi-optimal complexity, *SIAM J. Numer. Anal.*, **47** (2010), 4639–4659.
[4] P. Binev, W. Dahmen and R. DeVore, Adaptive finite element methods with convergence rates, *Numer. Math.*, **97** (2004), 219–268.
[5] D. Braess, C. Carstensen and R. H. W. Hoppe, Convergence analysis of a conforming adaptive finite element method for an obstacle problem, *Numer. Math.*, **107** (2007), 455–471.
[6] C. Carstensen and R. H. W. Hoppe, Error reduction and convergence for an adaptive mixed finite element method, *Math. Comput.*, **75** (2006), 1033–1042.
[7] E. Casas, Error estimations for the numerical approximation of semilinear elliptic control problems with finitely many state constraints, *ESAIM Control Optim. Calc. Var.*, **8** (2002), 345–374.
[8] J. M. Cascon, C. Kreuzer, R. Nochetto and K. G. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, *SIAM J. Numer. Anal.*, **46** (2008), 2524–2550.
[9] Z. Chen and J. Feng, An adaptive finite element algorithm with reliable and efficient error control for linear parabolic problems, *Math. Comput.*, **73** (2004), 1167–1193.
[10] H. Chen, X. Gong, L. He and A. Zhou, Adaptive finite element approximations for a class of nonlinear eigenvalue problems in quantum physics, *Adv. Appl. Math. Mech.*, **3** (2011), 493–518.
[11] Y. Chen and Z. Lu, Error estimates of fully discrete mixed finite element methods for semi-linear quadratic parabolic optimal control problem, *Comput. Methods Appl. Mech. Engrg.*, **199** (2010), 1415–1423.
[12] Y. Chen and Z. Lu, *High Efficient and Accuracy Numerical Methods for Optimal Control Problems*, Science Press, Beijing, 2015.
[13] X. Dai, J. Xu and A. Zhou, Convergence and optimal complexity of adaptive finite element eigenvalue computations, *Numer. math.*, **110** (2008), 313–355.
[14] A. Demlow and R. Stevenson, Convergence and quasi-optimality of an adaptive finite element method for controlling L^2 errors, *Numer. Math.*, **117** (2011), 185–218.
[15] W. Dörfler, A convergent adaptive algorithm for Poisson's equation, *SIAM J. Numer. Anal.*, **33** (1996), 1106–1124.
[16] K. Eriksson and C. Johnson, An adaptive finite element method for linear elliptic problems, *Math. Comput.*, **50** (1988), 361–383.

[17] A. Gaevskaya, R. H. W. Hoppe, Y. llish and M. Kieweg, Convergence analysis of an adaptive finite element method for distributed control problems with control constrains, *Inter. Ser. Numer. Math.*, **155** (2007), 47–68.

[18] L. Ge, W. Liu and D. Yang, L^2 norm equivalent a posteriori error for a constraint optimal control problem, *Inter. J. Numer. Anal. Model.*, **6** (2009), 335–353.

[19] L. Ge, W. Liu and D. Yang, Adaptive finite element approximation for a constrained optimal control problem via multi-meshes, *J. Sci. Comput.*, **41** (2009), 238–255.

[20] W. Gong and N. Yan, Adaptive finite element method for elliptic optimal control problems: Convergence and optimality, *Numer. Math.*, **135** (2017), 1121–1170.

[21] W. Gong, N. Yan and Z. Zhou, Convergence of L^2-norm based adaptive finite element method for elliptic optimality control problem, arXiv:1608.08699.

[22] K. Kohls, K. Siebert and A. Rösch, Convergence of adaptive finite elements for optimal control problems with control constraints, *Inter. Ser. Numer. Math.*, **165** (2014), 403–419.

[23] H. Leng and Y. Chen, Convergence and quasi-optimality of an adaptive finite element method for optimal control problems on L^2-errors, *J. Sci. Comput.*, **73** (2017), 438–458.

[24] H. Leng and Y. Chen, Convergence and quasi-optimality of an adaptive finite element method for optimal control problems with integral control constraint, *Adv. Comput. Math.*, **44** (2018), 367–394.

[25] R. Li, W. Liu, H. Ma and T. Tang, Adaptive finite element approximation for distributed elliptic optimal control problems, *SIAM J. Control Optim.*, **41** (2002), 1321–1349.

[26] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin, 1971.

[27] W. Liu and N. Yan, *Adaptive Finite Element Methods for Optimal Control Governed by PDEs*, Science Press, Beijing, 2008.

[28] W. Liu and N. Yan, A posteriori error estimates for control problems governed by nonlinear elliptic equation, *Appl. Numer. Math.*, **47** (2003), 173–187.

[29] K. Mekchay and R. H. Nochetto, Convergence of adaptive finite element methods for general second order linear elliptic PDEs, *SIAM J. Numer. Anal.*, **43** (2005), 1803–1827.

[30] P. Morin, R. H. Nochetto and K. G. Siebert, Data oscillation and convergence of adaptive FEM, *SIAM J. Numer. Anal.*, **38** (2000), 466–488.

[31] P. Morin, R. H. Nochetto and K. G. Siebert, Convergence of adaptive finite element methods, *SIAM Rev.*, **44** (2002), 631–658.

[32] R. Stevenson, Optimality of a standard adaptive finite element method, *Found. Comput. Math.*, **7** (2007), 245–269.

Received December 2019; 1st revision April 2020; final revision June 2020.

E-mail address: zulianglu@sanxiau.edu.cn
E-mail address: feihuangc@126.com
E-mail address: xiankuiwu@126.com
E-mail address: linligx@126.com
E-mail address: liushangw@163.com