ELECTRONIC RESEARCH ARCHIVE
Volume **28**, Number **4**, December **2020**

doi:10.3934/era.2020075
pp. **1419–1438**

A COUPLED p-LAPLACIAN ELLIPTIC SYSTEM: EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOR

YICHEN ZHANG AND MEIQIANG FENG*

School of Applied Science, Beijing Information Science & Technology University
Beijing 100192, China

(Communicated by Wan-Tong Li)

ABSTRACT. We prove uniqueness, existence and asymptotic behavior of positive solutions to the system coupled by p-Laplacian elliptic equations

$$\begin{cases} -\Delta_p z_1 = \lambda_1 g_1(z_2) \ \text{ in } \Omega, \\ -\Delta_p z_2 = \lambda_2 g_2(z_1) \ \text{ in } \Omega, \\ z_1 = z_2 = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, λ_1 and λ_2 are positive parameters, Ω is the open unit ball in $\mathbb{R}^N$, $N \geq 2$.

1. **Introduction.** It is well known that p-Laplace equations are quasilinear equations when $p \neq 2$(see [22], [12]), and there are many important applications in physics, game theory and image processing (see [14], [5]). In the past few decades, a good many of results have been developed for single p-Laplace equations by different methods, for instance, see [21, 23, 19, 18] and the references cited therein. Specially, in [24], Zhang and Li considered the following p-Laplacian equation

$$\begin{cases} -\Delta_p u = g(u) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\triangle_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $N < p < \infty$, Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 1$. The authors applied differential equations theory in Banach spaces and dynamics theory to study problem (1), and obtained excellent multiple solutions and sign-changing solutions theorems of p-Laplacian.

At the same time, we notice that many authors have paid more attention to existence and uniqueness problems, for example, see Castro, Sankar and Shivaji [2], Lin [13], Hai [10], and Guo [6]. Specially, Guo and Webb [7] considered the following p-Laplacian equation

$$\begin{cases} \Delta_p u = -\lambda f(u) \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{2}$$

They obtained existence and uniqueness results to problem (2) for large λ if $f \geq 0, (f(x)/x^{p-1})' < 0$ for $x > 0$ and f satisfies some p-sublinearity conditions at ∞

2020 *Mathematics Subject Classification.* 35J60, 35J66, 35J92.

Key words and phrases. p-Laplacian equatios, positive solutions, uniqueness and asymptotic behavior, eigenvalue theory in cones, Gustafson and Schmitt's fixed-point theorem.

This work is sponsored by the National Natural Science Foundation of China (11301178) and the Beijing Natural Science Foundation of China (1163007).

* Corresponding author: Meiqiang Feng.

and 0. In [11], using sub-supersolution method together with sharp estimates near the boundary, Hai and Shivaji improved the results of (2) in a unit ball under much weaker assumptions than in [7]. Recently, Shivaji, Sim and Son [20], and Chu, Hai and Shivaji [3] generalize the study in [7] from a bounded domain to the exterior domains and obtained some excellent results.

Moreover, we notice that various of system problems have become an important area of investigation in recent years. To identify a few, we refer the reader to [4, 15, 16, 17]. In [10], Hai considered the existence and uniqueness of positive solutions to the following elliptic system

$$\begin{cases} \Delta u = -\lambda f(v) & \text{in } \Omega, \\ \Delta v = -\mu g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is the open ball in $\mathbb{R}^N$, $f,\ g : \mathbb{R}^+ \to \mathbb{R}^+$, λ and μ are positive parameters.

Inspired by the above works, we are interested in the existence and uniqueness of positive radial solutions to the following system

$$\begin{cases} -\Delta_p z_1 = \lambda_1 g_1(z_2) & \text{in } \Omega, \\ -\Delta_p z_2 = \lambda_2 g_2(z_1) & \text{in } \Omega, \\ z_1 = z_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here $\triangle_p$ denotes the singular\degenerate p-Laplace operator $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, $\lambda_1 > 0$ and $\lambda_2 > 0$ are parameters, g_1 and g_2 are continuous nonlinearities, and $\Omega = \{x \in \mathbb{R}^N : |x| < 1\},\ N \geq 2$.

We also give new existence results for system (3). Our main tool is the eigenvalue theory in cones. However, based on the idea of decoupling method we will investigate composite operators. Besides, the exactly determined intervals of positive parameters $\lambda_1 \times \lambda_2$ are established.

The rest of the paper is organized as follows. In Section 2, we present some necessary definitions, Lemmas and theorems that will be used to prove our main results, Theorem 2.4. Section 3 is devoted to proving the existence and uniqueness of positive solution to system (3). In Section 4, we establish the exactly determined intervals of positive parameter $\lambda_1 \times \lambda_2$ in which system (3) admits at least one positive solution. Section 5 verifies the existence and asymptotic behavior of positive radial solutions to system (3). Finally, in Section 6, we will give some remarks on our main results.

2. **Preliminaries and some lemmas.** In order to study the existence of the positive radial solutions for system (3), let us firstly introduce the radial coordinates form of the p-Laplacian operator. Letting $r = |x|$, and $u(r) = z_1(x), v(r) = z_2(x)$, then

$$\Delta_p z_1(x) = r^{1-N}(r^{N-1}|u'(r)|^{p-2}u'(r))',$$
$$\Delta_p z_2(x) = r^{1-N}(r^{N-1}|v'(r)|^{p-2}v'(r))'.$$

Therefore, the study of positive radial solutions of system (3) is reduced to the study of positive solutions to the following system:

$$\begin{cases} -(r^{N-1}\varphi_p(u'))' = \lambda_1 r^{N-1} g_1(v) & \text{in } 0 < r < 1, \\ -(r^{N-1}\varphi_p(v'))' = \lambda_2 r^{N-1} g_2(u) & \text{in } 0 < r < 1, \\ u'(0) = v'(0) = u(1) = v(1) = 0, \end{cases} \tag{4}$$

where $\varphi_p(s) = |s|^{p-2}s,\ (\varphi_p)^{-1} = \varphi_q,\ \frac{1}{p} + \frac{1}{q} = 1.$

Lemma 2.1. *Let $p>1, q>1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. Then $\varphi_p(s)=|s|^{p-2}s$ is odd, and*

$$s\varphi_p(s)>0 \text{ if } s\neq 0,\ \varphi_p(st)=\varphi_p(s)\varphi_p(t), \varphi_p(0)=0,\ \varphi_p(1)=1,\ \varphi_p(-1)=-1,$$

$$\varphi_p(s+t)=\begin{cases} 2^{p-1}(\varphi_p(s)+\varphi_p(t)), & \text{if } p\geq 2,\ s,t>0,\\ \varphi_p(s)+\varphi_p(t), & \text{if } 1<p<2,\ s,t>0.\end{cases}$$

On the other hand, $\varphi_p(s)$ is increasing on $[0,\infty)$, and for $a\geq 0$, $\varphi_p(s^a)=\varphi_p^a(s)$ on $[0,\infty)$.

Next we mainly analyze the existence of positive solutions for system (4). In order to get our theorems, we let $\mathbb{R}_+=[0,+\infty)$ and g_i satisfy
$(\mathbf{C_0})$ g_1 and $g_2:\mathbb{R}_+\to\mathbb{R}_+$ are continuous.
$(\mathbf{C_1})$ g_1 and $g_2:\mathbb{R}_+\to\mathbb{R}_+$ are nondecreasing, C^1 on $(0,\infty)$ and

$$\lim_{x\to 0^+}\sup xg_1'(x)<\infty,\quad \lim_{x\to 0^+}\sup xg_2'(x)<\infty.$$

$(\mathbf{C_2})$ There exist nonnegative numbers a,b,A,D, where $ab<1$ and $A,D>0$ such that

$$\lim_{x\to 0^+}\inf\frac{g_1(x)}{\varphi_p(x^a)}>0,\quad \lim_{x\to 0^+}\inf\frac{g_2(x)}{\varphi_p(x^b)}>0,$$

$$\lim_{x\to\infty}\frac{g_1(x)}{\varphi_p(x^a)}=A,\quad \lim_{x\to\infty}\frac{g_2(x)}{\varphi_p(x^b)}=D$$

and for $a_1>a$ and $b_1>b$,

$$\frac{g_1(x)}{\varphi_p(x^{a_1})} \text{ and } \frac{g_2(x)}{\varphi_p(x^{b_1})}$$

are nonincreasing for x large.

A pair of functions $u,v\in C[0,1]\cap C^1(0,1)$ with $\phi_p(u'),\phi_p(v')\in C^1(0,1)$ is called to be a positive solution of (4) if $u(t),v(t)>0$ for all $t\in(0,1)$, and u and v satisfy (4).

Let

$$E=C([0,1],\mathbb{R})\times C([0,1],\mathbb{R}).$$

Then E is a Banach space with the norm $\|(u,v)\|=\max\{\|u\|_\infty,\|v\|_\infty\}$, where $\|u\|_\infty=\max_{t\in[0,1]}|u(t)|$.

Define a cone P by

$$P=\{(u,v)\in E: u\geq 0, v\geq 0\}.$$

Define an operator $F:\ E\to E$ by

$$F(u,v)(t)=(A(u,v)(t),B(u,v)(t)),\quad t\in[0,1],$$

where

$$A(u,v)(t)=\int_t^1\varphi_q(\frac{1}{s^{N-1}}\int_0^s\lambda_1\tau^{N-1}g_1(v(\tau))d\tau)ds,$$

and

$$B(u,v)(t)=\int_t^1\varphi_q(\frac{1}{s^{N-1}}\int_0^s\lambda_2\tau^{N-1}g_2(u(\tau))d\tau)ds.$$

It is easy to check that $F:P\to P$ is completely continuous and the solution of system (4) is equivalent to the fixed point equation

$$F(u,v)=(u,v).$$

Therefore, the task of the present paper is to search nonzero fixed points of F.

The following well-known results are crucial in the proofs of our results.

Lemma 2.2. *(See Lemma 2.4 of [9], on page 131) Let P be a cone in a Banach space E and $T: P \to P$ be a completely continuous mapping satisfying*
(a) There exist $k \in K$, $\|k\| = 1$, and a number $r > 0$ such that all solutions $y \in P$ of

$$y = Ty + \theta k, \quad 0 < \theta < \infty$$

satisfy $\|y\| \neq r$.
(b) There exists $R > r$ such that all solutions $z \in P$ of

$$z = \theta Tz, \quad 0 < \theta < 1.$$

Then T has a fixed point $x \in P$, $r \leq \|x\| \leq R$.

3. **Uniqueness of positive solution.** In this section, we analyze the uniqueness of fixed point of F for $\lambda_1\lambda_2^a$ and $\lambda_1^b\lambda_2$ sufficiently large.

Lemma 3.1. *Let h be continuous on $\mathbb{R}_+$ and C^1 on $(0,\infty)$ such that*

$$\lim_{x\to 0^+} \sup xh'(x) < \infty.$$

Let M, ε, r be positive numbers with $\varepsilon < 1$. Then there is a positive constant C such that

$$|h(\gamma x) - \varphi_p(\gamma^r)h(x)| \leq C(1-\gamma)$$

for $\varepsilon \leq \gamma < 1$ and $0 \leq x \leq M$.

Proof. Let $0 \leq x \leq M$. Define $H(\gamma) = h(\gamma x) - \varphi_p(\gamma^r)h(x)$, $\varepsilon \leq \gamma < 1$. Using the mean value theorem, there is a $c \in (\gamma, 1)$ such that

$$|H(\gamma)| = |H(\gamma) - H(1)| = (1-\gamma)|xh'(cx) - r(p-1)c^{r(p-1)-1}h(x)| \leq C(1-\gamma),$$

where

$$C = \frac{\sup\{|yh'(y)| : 0 < y \leq M\}}{\varepsilon} + r(p-1)\max(\varepsilon^{r(p-1)-1}, 1)\sup\{|h(y)| : y \leq M\}.$$

□

Next we will check the upper and lower estimates for possible positive solutions of system (4).

Lemma 3.2. *Let (u,v) be a positive solution of (4). Then there exist positive constants M_i, $i \in \{1,2,3,4\}$ and M independent of u, v such that*

$$M_1(\varphi_q(\lambda_1\lambda_2^a))^{\frac{1}{1-ab}}(1-t) \leq u(t) \leq M_2(\varphi_q(\lambda_1\lambda_2^a))^{\frac{1}{1-ab}}(1-t), \ 0 < t < 1,$$

$$M_3(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}}(1-t) \leq v(t) \leq M_4(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}}(1-t), \ 0 < t < 1$$

for $\min\{\lambda_1\lambda_2^a, \lambda_2\lambda_1^b\} \geq M$.

Proof. Suppose that u and v are a pair of positive solutions for system (4). By integrating, we get

$$u(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}}\int_0^s \lambda_1\tau^{N-1}g_1(v(\tau))d\tau\right)ds,$$
$$v(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}}\int_0^s \lambda_2\tau^{N-1}g_2(u(\tau))d\tau\right)ds.$$

Next, we can denote by $c_i,\ i = 1, 2, \ldots,$ positive constants independent of $u, v, \lambda_1, \lambda_2$. Since v is decreasing, we have

$$\begin{aligned} u(\frac{1}{2}) \geq& \int_{\frac{1}{2}}^{1} \varphi_q(\frac{1}{s^{N-1}} \int_0^{\frac{1}{2}} \lambda_1 \tau^{N-1} g_1(v(\tau)) d\tau) ds \\ =&\varphi_q(\lambda_1) \int_{\frac{1}{2}}^{1} \varphi_q(\frac{1}{s^{N-1}} \int_0^{\frac{1}{2}} \tau^{N-1} g_1(v(\tau)) d\tau) ds \\ \geq&\varphi_q(\lambda_1) \int_{\frac{1}{2}}^{1} \varphi_q(\frac{1}{s^{N-1}}) ds \cdot \varphi_q(\int_0^{\frac{1}{2}} \tau^{N-1} d\tau) \cdot \varphi_q(g_1(v(\frac{1}{2}))) \\ \geq&\frac{1}{2} \varphi_q(\frac{\lambda_1}{N2^N}) \varphi_q(g_1(v(\frac{1}{2}))). \end{aligned} \tag{5}$$

Similarly we can get

$$v(\frac{1}{2}) \geq \frac{1}{2} \varphi_q(\frac{\lambda_2}{N2^N}) \varphi_q(g_2(u(\frac{1}{2}))). \tag{6}$$

By $(\mathbf{C_2})$, there are two positive constants K_1 and K_2 such that

$$g_1(x) \geq \varphi_p(K_1 x^a), \quad g_2(x) \geq \varphi_p(K_2 x^b) \text{ for } x \geq 0. \tag{7}$$

This together with (5) and (6), shows that

$$u(\tfrac{1}{2}) \geq \tfrac{1}{2} \varphi_q(\tfrac{\lambda_1}{N2^N}) K_1 [\tfrac{1}{2} \varphi_q(\tfrac{\lambda_2}{N2^N})]^a K_2^a (u(\tfrac{1}{2}))^{ab} = c_1 \varphi_q(\lambda_1 \lambda_2^a)(u(\tfrac{1}{2}))^{ab}.$$

Thus,

$$u(\frac{1}{2}) \geq (c_1)^{\frac{1}{1-ab}} \varphi_q(\lambda_1 \lambda_2^a)^{\frac{1}{1-ab}} = c_2 (\varphi_q(\lambda_1 \lambda_2^a))^{\frac{1}{1-ab}}. \tag{8}$$

Similarly, we can get

$$v(\frac{1}{2}) \geq (c_1)^{\frac{1}{1-ab}} \varphi_q(\lambda_2 \lambda_1^b)^{\frac{1}{1-ab}} = c_2' (\varphi_q(\lambda_2 \lambda_1^b))^{\frac{1}{1-ab}}. \tag{9}$$

It follows from (7), (8) and (9) that for $t \geq \frac{1}{2}$

$$\begin{aligned} -u'(t) =&\varphi_q(\frac{\lambda}{t^{N-1}} \int_0^t s^{N-1} g_1(v) ds) \\ \geq&\varphi_q(\lambda_1 \int_0^{\frac{1}{2}} s^{N-1} g_1(v) ds) \\ \geq&\varphi_q(\frac{\lambda_1}{N2^N} g_1(v(\frac{1}{2}))) \\ =&\varphi_q(\frac{\lambda_1}{N2^N}) \varphi_q(g_1(v(\frac{1}{2}))) \\ \geq&\varphi_q(\frac{\lambda_1}{N2^N}) K_1 c_2^a (\varphi_q(\lambda_1 \mu^a))^{\frac{1}{1-ab}} \\ =&c_3 (\varphi_q(\lambda_1 \lambda_2^a))^{\frac{1}{1-ab}}. \end{aligned}$$

Then by integrating, for $t \geq \frac{1}{2}$, we get

$$u(t) \geq c_3 (\varphi_q(\lambda_1 \lambda_2^a))^{\frac{1}{1-ab}} (1 - t). \tag{10}$$

Similarly,

$$v(t) \geq c_4 (\varphi_q(\lambda_2 \lambda_1^b))^{\frac{1}{1-ab}} (1 - t). \tag{11}$$

Because of u, v being decreasing, this shows the left-side inequalities for u, v in Lemma 3.2.

According to the formulas for u, v, we can see that

$$\begin{aligned} u(t) \leq & |u|_\infty \\ \leq & \int_0^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s \lambda_1 \tau^{N-1} g_1(|v|_\infty) d\tau\Big) ds \\ \leq & \int_0^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^1 \lambda_1 \tau^{N-1} g_1(|v|_\infty) d\tau\Big) ds \\ \leq & \varphi_q(\lambda_1 g_1(|v|_\infty)) \end{aligned} \tag{12}$$

and

$$|v|_\infty \leq \varphi_q(\lambda_2 g_2(|u|_\infty)). \tag{13}$$

By (7) and (11), for large $\lambda_1\lambda_2^a$ and $\lambda_2\lambda_1^b$, we get

$$|v(t)| \geq c_4(\varphi_q(\lambda_1\lambda_2^b))^{\frac{1}{1-ab}}(1-t) \gg 1 \quad (i.e.\ |v|_\infty \text{ is large}) \tag{14}$$

and

$$\begin{aligned} \lambda_1 g_1(|v|_\infty) \geq & \lambda_1 \varphi_p(K_1 |v|_\infty^a) \\ \geq & \lambda_1 \varphi_p(K_1 [c_4(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}}]^a) \\ \geq & \lambda_1 \varphi_p(K_1 [c_4^a(\varphi_q(\lambda_1\lambda_2^a))^{\frac{1}{1-ab}}]) \\ \gg & 1. \end{aligned}$$

Note that from $(\mathbf{C_2})$ it follows that

$$\begin{aligned} |v|_\infty \leq & \varphi_q(\lambda_2 g_2(|u|_\infty)) \\ \leq & \varphi_q(\lambda_2 g_2(\varphi_q(\lambda_1 g_1(|v|_\infty)))) \\ \leq & \varphi_q(\lambda_2 g_2(\varphi_q(\lambda_1 K_1)|v|_\infty^a)) \\ \leq & \varphi_q(\lambda_2 K_2 \varphi_p((\varphi_q(\lambda_1 K_1))^b (|v|_\infty)^{ab})) \\ = & \varphi_q(K_1^b K_2)\varphi_q(\lambda_2\lambda_1^b)(|v|_\infty)^{ab} \\ = & c_5 \varphi_q(\lambda_2\lambda_1^b)(|v|_\infty)^{ab}, \end{aligned}$$

or

$$|v|_\infty \leq c_5^{\frac{1}{1-ab}}(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}}. \tag{15}$$

By combining the equation of u' and (15), we can get

$$\begin{aligned} -u'(t) \leq & \varphi_q\Big(\frac{\lambda_1}{t^{N-1}}\int_0^t s^{N-1} g_1(|v|_\infty) ds\Big) \\ \leq & \varphi_q(\lambda_1 g_1(|v|_\infty)) \\ \leq & \varphi_q(\lambda_1 f(c_5^{\frac{1}{1-ab}}(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}})) \\ \leq & \varphi_q(\lambda_1) K_1 c_5^{\frac{1}{1-ab}}(\varphi_q(\lambda_1\lambda_2^a)^{\frac{1}{1-ab}}) \\ = & c_6(\varphi_q(\lambda_1\lambda_2^a)^{\frac{1}{1-ab}}), \end{aligned} \tag{16}$$

and then it follows from integrating that

$$u(t) \leq c_6(\varphi_q(\lambda_1\lambda_2^a)^{\frac{1}{1-ab}})(1-t), \quad 0 < t < 1.$$

Similarly, we can get the upper estimate for $v(t)$. This completes the proof. $\square$

Theorem 3.3. *Assume* $(\mathbf{C_0}) - (\mathbf{C_2})$ *hold. Then there is a constant* $\beta > 0$ *such that the system* (4) *admits a unique positive solution for* $\min(\lambda_1\lambda_2^a, \lambda_2\lambda_1^b) \geq \beta$.

Proof. We shall check the conditions of Lemma 2.2 to prove the existence of solution. Let $(u, v) \in P$ satisfy

$$(u, v) = F(u, v) + \theta(1, 1)$$

for some $\theta > 0$. Because of u, v are nonincreasing and $u, v > 0$ on $(0, 1)$, the proof is analogous to that of Lemma 3.2 that

$$u\left(\frac{1}{2}\right) \geq c_2(\varphi_q(\lambda_1\lambda_2^a))^{\frac{1}{1-ab}}.$$

So, $\|(u, v)\| \neq r$, where $0 < r < c_2(\varphi_q(\lambda_1\lambda_2^a))^{\frac{1}{1-ab}}$.

Next, set $(u, v) \in P$ with

$$(u, v) = \theta F(u, v)$$

for some $\theta \in (0, 1)$. Then, by (12) and (13) we get

$$|v|_\infty \leq \varphi_q(\lambda_2 g_2(\varphi_q(\lambda_1 g_1(|v|_\infty)))), \quad |u|_\infty \leq \varphi_q(\lambda_1 g_1(\varphi_q(\lambda_2 g_2(|u|_\infty)))).$$

In addition, if $|v|_\infty \to \infty$, by $(\mathbf{C_2})$ and $ab < 1$ we can see that

$$1 \leq \frac{\varphi_q(\lambda_2 g_2(\varphi_q(\lambda_1 g_1(|v|_\infty))))}{|v|_\infty} \leq \frac{c_5\varphi_q(\lambda_2\lambda_1^b)(|v|_\infty)^{ab}}{|v|_\infty} = 0,$$

which is impossible. So, there is a number $R > r$ such that $\|(u, v)\| \neq R$. Therefore, Lemma 2.2 shows that F has a fixed point (u, v) with $r \leq \|(u, v)\| \leq R$. Thus, it follows that (4) admits one positive solution, and the existence is proved.

Next, we shall show that the solution is unique. Suppose that (u, v) and (u_1, u_2) are positive solutions of system (4) and let $\min\{\lambda_1\lambda_2^a,\ \lambda_2\lambda_1^b,\}$ be large enough such that Lemma 3.2 holds. It follows from Lemma 3.2 that

$$\frac{M_1}{M_2}u_1 \leq u \leq \frac{M_2}{M_1}u_1 \quad \text{on} \quad (0, 1).$$

Let $\alpha = \sup\{d > 0 : u \geq du_1 \text{ in } (0, 1)\}$. Then obviously $d_0 \leq \alpha \leq \infty$ and $u \geq \alpha u_1$ in $(0, 1)$, where $\alpha_0 = \frac{M_1}{M_2}$. We assert that $\alpha \geq 1$. In fact, we can assume by contradiction that $\alpha < 1$. Since u, v are decreasing and

$$(t^{N-1}\varphi_p(u'))' = -\lambda_1 t^{N-1} g_1(\int_t^1 \varphi_q(\frac{1}{s^{N-1}}\int_0^s \lambda_2\tau^{N-1}g_2(u(\tau))d\tau)ds),$$

$$(t^{N-1}\varphi_p(\alpha u_1'))' = -\lambda_1 t^{N-1}\varphi_p(\alpha) g_1(\int_t^1 \varphi_q(\frac{1}{s^{N-1}}\int_0^s \lambda_2\tau^{N-1}g_2(u_1(\tau))d\tau)ds),$$

we can get

$$\begin{aligned}
&[t^{N-1}(\varphi_p(u') - \varphi_p(\alpha u_1'))]' \\
&\leq -\lambda_1 t^{N-1}[g_1(\int_t^1 \varphi_q(\frac{\lambda_2}{s^{N-1}}\int_0^s \tau^{N-1}g_2(\alpha u_1(\tau))d\tau)ds) \\
&-\varphi_p(\alpha) g_1(\int_t^1 \varphi_q(\frac{\lambda_2}{s^{N-1}}\int_0^s \tau^{N-1}g_2(u_1(\tau))d\tau)ds)].
\end{aligned} \tag{17}$$

Let $b_1 > b_2 > b,\ a_1 > a$ and $a_1b_1 < 1$. Then we assert that

$$\int_0^s \tau^{N-1}g_2(\alpha u_1(\tau))d\tau \geq \varphi_p(\alpha^{b_1})\int_0^s \tau^{N-1}g_2(u_1(\tau))d\tau. \tag{18}$$

According to $\frac{g_2(x)}{\varphi_p(x^{b_2})}$ is nonincreasing for $x \gg 1$ and $\alpha \geq \alpha_0$, we can get

$$\frac{g_2(\alpha x)}{\varphi_p((\alpha x)^{b_2})} \geq \frac{g_2(x)}{\varphi_p(x^{b_2})},$$

or

$$g_2(\alpha x) \geq \varphi_p(\alpha^{b_2}) g_2(x).$$

By Lemma 3.2, we can get

$$u(r) \geq M_1(\varphi_q(\lambda_1 \lambda_2^a))^{\frac{1}{1-ab}}(1-T) \gg 1,$$

where $T \in (\frac{1}{2}, 1)$.

Since $\alpha < 1$, then for $s \leq T$, we have

$$\int_0^s \tau^{N-1}(g_2(\alpha u_1) - \varphi_p(\alpha^{b_1}) g_2(u_1)) d\tau \geq (\varphi_p(\alpha^{b_2}) - \varphi_p(\alpha^{b_1})) \int_0^s \tau^{N-1} g_2(u_1) d\tau \geq 0.$$

For $s > T$, combined with Lemma 3.1, we have

$$\begin{aligned} &\textstyle\int_0^s \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1}) g_2(u)) d\tau \\ &= \textstyle\int_0^T \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1}) g_2(u)) d\tau + \int_T^s \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1}) g_2(u)) d\tau \\ &\geq (\varphi_p(\alpha^{b_2}) - \varphi_p(\alpha^{b_1})) \textstyle\int_0^s \tau^{N-1} g_2(u) d\tau - C(1-T)(1-\alpha). \end{aligned}$$

Because

$$\int_0^T \tau^{N-1} g_2(u) d\tau \geq \int_0^{\frac{1}{2}} \tau^{N-1} g_2(u) d\tau \geq \frac{g_2(u(\frac{1}{2}))}{N 2^N} \geq \frac{K_2}{N 2^N}$$

and since there is a positive number $l > 0$ such that

$$(\varphi_p(\alpha^{b_2}) - \varphi_p(\alpha^{b_1})) \geq l(1 - \alpha^{p-1}) \quad \text{for } \alpha_0 \leq \alpha \leq 1.$$

This follows that

$$\int_0^s \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1}) g_2(u)) d\tau > 0, s > T$$

when T is sufficiently close to 1. So, we can see that

$$\int_0^s \tau^{N-1}(g_1(\alpha u) - \varphi_p(\alpha^{b_1}) g_1(u)) d\tau > 0, \quad \text{for } s > T$$

when T is sufficiently close to 1. So, the proof of (18) is finished.

Substituting (18) into (17) and integrating gets

$$z^{N-1}(u' - \alpha u_1')(z) \leq -\lambda_1 \int_0^z G(\alpha, t) dt,$$

where

$$\begin{aligned} G(\alpha, t) = t^{N-1}[&g_1(\alpha^{b_1} \textstyle\int_t^1 \varphi_q(\frac{\lambda_2}{s^{N-1}} \int_0^s \tau^{N-1} g_2(u_1(\tau)) d\tau) ds) \\ &-\varphi_p(\alpha) g_1(\textstyle\int_t^1 \varphi_q(\frac{\lambda_2}{s^{N-1}} \int_0^s \tau^{N-1} g_2(u_1(\tau)) d\tau) ds)]. \end{aligned}$$

Applying (7) and Lemma 3.2, for $t \leq T$, we get

$$\begin{aligned} &\textstyle\int_r^1 \varphi_q(\frac{1}{s^{N-1}} \int_0^s \lambda_2 \tau^{N-1} g_2(u_1(\tau)) d\tau) ds \\ &= \varphi_q(\lambda_2) \textstyle\int_r^1 \varphi_q(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} g_2(u_1(\tau)) d\tau) ds \\ &\geq \varphi_q(\lambda_2) \textstyle\int_T^1 \varphi_q(\frac{1}{s^{N-1}} \int_0^T \tau^{N-1} g_2(u_1(\tau)) d\tau) ds \end{aligned}$$

$$\begin{aligned}
&\geq \varphi_q(g_2(u_1(T)))\varphi_q(\lambda_2)\textstyle\int_T^1 \varphi_q(\frac{1}{s^{N-1}}\int_0^T \tau^{N-1}d\tau)ds \\
&\geq (1-T)\varphi_q(\tfrac{\lambda_2 T^N}{N})\varphi_q(g_2(u_1(T))) \\
&\geq (1-T)\varphi_q(\tfrac{\lambda_2 T^N}{N})K_2 u_1^b(T) \\
&\geq (1-T)\varphi_q(\tfrac{\lambda_2 T^N}{N})K_2\left[M_1\left(\varphi_q(\lambda_1\lambda_2^a)\right)^{\frac{1}{1-ab}}(1-t)\right]^b \\
&\geq \varphi_q(\tfrac{\lambda_2 T^N}{N})K_2 M_1^b\left(\varphi_q(\lambda_1\lambda_2^a)\right)^{\frac{b}{1-ab}}(1-T)^{b+1} \\
&= h(T)(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}} \\
&\gg 1,
\end{aligned}$$

where $h(T)=\varphi_q(\frac{T^N}{N})K_2M_1^b(1-T)^{b+1}$.

Because $\frac{g_1(x)}{\varphi_p(x^{a_1})}$ is nonincreasing for $x\gg 1$, we get

$$g_1(\alpha^{b_1}x)\geq \varphi_p(\alpha^{a_1b_1})g_1(x).$$

Thus

$$\begin{aligned}
&G(\alpha,t)=t^{N-1}(\varphi_p(\alpha^{a_1b_1})-\varphi_p(\alpha))g_1\left(\int_t^1\varphi_q(\frac{1}{s^{N-1}}\int_0^s\mu\tau^{N-1}g_2(u_1(\tau))d\tau)ds\right)\\
\geq& t^{N-1}l_0(1-\alpha^{p-1})g_1(h(T)(\varphi_q(\lambda_2\lambda_1^b))^{\frac{1}{1-ab}})\\
\geq& H(T)t^{N-1}\varphi_q(\lambda_2\lambda_1^b))^{\frac{a}{1-ab}}(1-\alpha^{p-1})>0,\quad t\leq T, \qquad (19)
\end{aligned}$$

where $H(T)=l_0k^*K_2h^a(T))$ and l_0 is a positive constant so that

$$\varphi_p(\alpha^{a_1b_1})-\varphi_p(\alpha)\geq l_0(1-\alpha^{p-1}) \text{ for } \alpha_0\leq\alpha\leq 1.$$

This proves that

$$z^{N-1}(u'-\alpha u_1')(z)<0,\quad 0<z\leq T.$$

On the other hand, if $z>T$, then for large $\lambda_1\lambda_2^a$ and $\lambda_2\lambda_1^b$, and T sufficiently close to 1, it follows from Lemma 3.1 and (19) that

$$\begin{aligned}
\textstyle\int_0^z G(\alpha,r)dr &\geq \textstyle\int_0^{\frac12}G(\alpha,t)dt+\int_T^z G(\alpha,t)dt\\
&\geq \tfrac{H(\frac12)}{N2^N}\varphi_q(\lambda_2\lambda_1^b))^{\frac{a}{1-ab}}(1-\alpha^{p-1})-C(1-T)(1-\alpha^{\frac{1}{q_1}})\\
&>0.
\end{aligned}$$

Therefore, we have

$$(u'-\alpha u_1')(z)<0 \;\; for \;\; 0<z\leq 1.$$

This shows that there is a constant $\widetilde{\alpha}>\alpha$ in $(0,1)$ such that $u\geq\widetilde{\alpha}u_1$, which is a contradiction. Thus $\alpha\geq 1$ and hence $u=u_1$ in (0,1). Similarly, we can verify $v=v_1$ in $(0,1)$ and so we finish the proof of Theorem 3.3. □

4. **New existence results.** In this section, we will establish some new existence results of positive solutions for system (4). To achieve this goal, we will define a new cone P and a composite operator T.

Lemma 4.1. *(See Theorem 2.3.6 of [8], on page 99) Suppose that D is an open subset of the an infinite-dimensional real Banach space E, $\theta\in D$, and P is a cone of E. If the operator $\Gamma: P\cap D\to P$ is completely continuous with $\Gamma\theta=\theta$ and satisfies*

$$\inf_{x\in P\cap\partial D}\Gamma x>0,$$

then Γ *has a proper element on* $P \cap \partial D$ *associated with a positive eigenvalue. That is, there exist* $x_0 \in P \cap \partial D$ *and* $\mu_0 > 0$ *such that* $\Gamma x_0 = \mu_0 x_0$*.*

Let $E = C[0,1]$. Then E is a real Banach space with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{t \in J} |x(t)|.$$

Let $J = [0,1]$ and P be the cone

$$P := \left\{ v \in E : v(t) \geq 0,\ t \in J,\ v(t) \geq \frac{1}{4}\|v\|,\ t \in [\frac{1}{4}, \frac{3}{4}] \right\}. \tag{20}$$

It is easy to see that P is a normal cone of E.

For $v \in P$, define $T_i : P \to E (i = 1,2)$ as

$$(T_1 v)(t) = \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_1(v(s)) ds) d\tau, \tag{21}$$

$$(T_2 v)(t) = \varphi_q(\lambda_2) \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_2(v(s)) ds) d\tau. \tag{22}$$

It follows from Lemma 3 in [1] that $T_i (i = 1,2)$ maps P into itself. Moreover, T_1 and T_2 are completely continuous by standard arguments.

Define a composite operator $T = T_1 T_2$, which is also completely continuous from P to itself. So the operator T also maps P into P. Therefore the next task of this paper is to search nonzero fixed points of operator T.

Let

$$g_1^\infty := \lim_{v \to \infty} \frac{g_1(v)}{\varphi_p(v)},\quad g_1^0 := \lim_{v \to 0} \frac{g_1(v)}{\varphi_p(v)};$$

$$g_2^\infty := \lim_{v \to \infty} \frac{g_2(v)}{\varphi_p(v)},\quad g_2^0 := \lim_{v \to 0} \frac{g_2(v)}{\varphi_p(v)},$$

and

$$A = \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} ds = \frac{3^N - 1}{N 4^N},\ B = \int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}}) d\tau,\ B^* = \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}}) d\tau. \tag{23}$$

Theorem 4.2. *Suppose that* $(\mathbf{C_0})$ *holds. If* $0 < g_i^\infty < +\infty (i = 1,2)$*, then there exists* $\beta_0 > 0$ *such that, for every* $R > \beta_0$*, system (4) admits a pair of positive solutions* u_R, v_R *satisfying* $\|u_R\| = R$ *for any*

$$\lambda_{1R} \lambda_2 \in [\lambda_R, \bar{\lambda}_R],$$

where λ_R *and* $\bar{\lambda}_R$ *are positive finite numbers.*

Proof. Since $0 < g_i^\infty < +\infty$, there exist $0 < l_1 < l_2,\quad \mu > 0$ so that

$$l_1 \varphi_p(v) < g_1(v) < l_2 \varphi_p(v),\ \forall v \geq \mu;$$

$$l_1 \varphi_p(u) < g_2(u) < l_2 \varphi_p(u),\ \forall u \geq \mu.$$

Next, we verify that $\beta_0 = 4\mu$ is required. Letting

$$\Omega_R = \{x \in E : \|x\| < R\},$$

then $0 \in \Omega_R$ and Ω_R is a bounded open subset of Banach space E.

Since $R > \beta_0$, for any $u, v \in P \cap \partial \Omega_R$, we get

$$u(t) \geq \frac{1}{4}\|u\| = \frac{1}{4} R,\ v(t) \geq \frac{1}{4}\|v\| = \frac{1}{4} R,\quad t \in [\frac{1}{4}, \frac{3}{4}],$$

and

$$u(t) \geq \frac{1}{4}\|u\| > \frac{1}{4}\beta_0 = \mu,\ v(t) \geq \frac{1}{4}\|v\| > \frac{1}{4}\beta_0 = \mu, \quad t \in [\frac{1}{4}, \frac{3}{4}].$$

So, for any $v \in P \cap \partial\Omega_R$, we have

$$\begin{aligned}(T_1 v)(t) \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_0^{\frac{3}{4}} s^{N-1} g_1(v(s))ds)d\tau \\ \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} g_1(v(s))ds)d\tau \\ \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_1 \varphi_p(v(s))ds)d\tau \\ \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_1 \varphi_p(\frac{1}{4}\|v\|)ds)d\tau \\ = & \frac{1}{4}\|v\| \varphi_q(l_1 A)B,\ \forall t \in J.\end{aligned}$$

Analogously, for $u \in P \cap \partial\Omega_R$, we obtain

$$\begin{aligned}(T_2 u)(t) \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_0^{\frac{3}{4}} s^{N-1} g_2(u(s))ds)d\tau \\ \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} g_2(u(s))ds)d\tau \\ \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_1 \varphi_p(u(s))ds)d\tau \\ \geq & \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_1 \varphi_p(\frac{1}{4}\|u\|)ds)d\tau \\ = & \frac{1}{4}\|u\| \varphi_q(l_1 A)B,\ \forall t \in J.\end{aligned}$$

Therefore, we get

$$\begin{aligned}(Tu)(t) = & (T_1 T_2 u)(t) \\ \geq & \frac{1}{4}\|T_2 u\| \varphi_q(l_1 A)B \\ \geq & \frac{1}{16}\|u\|(\varphi_q(l_1 A)B)^2.\end{aligned}$$

This gives that

$$\inf_{u \in P \cap \partial\Omega_R} Tu \geq \frac{1}{16}\|u\|(\varphi_q(l_1 A)B)^2 > 0.$$

For any $R > \beta_0$, Lemma 4.1 yields that operator T admits a proper element $u_R \in P$ associated with the eigenvalue $\mu_{1R} > 0$, and u_R satisfies $\|u_R\| = R$.

For operator T, we can denote $v_R = T_2 u_R$, then u_R and v_R are the solutions of system (4).

Let $\lambda_{1R} = \frac{1}{\varphi_p(\mu_{1R})}$. Then we get

$$Tu_R = \mu_{1R} u_R = \frac{1}{\varphi_q(\lambda_{1R})} u_R. \tag{24}$$

It follows from the proof above that, for any $R > \beta_0$, system (4) has a pair of positive solutions u_R and v_R with $u_R \in P \cap \partial\Omega_R$ associated with $\lambda_1 = \lambda_{1R} > 0$. Thus, by (24) we get

$$u_R(t) = \varphi_q(\lambda_{1R}) T u_R,$$

and so

$$u_R(t) = \varphi_q(\lambda_{1R}) \int_t^1 \varphi_q(\tfrac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_1(v_R(s))ds)d\tau,$$
$$v_R(t) = \varphi_q(\lambda_{2R}) \int_t^1 \varphi_q(\tfrac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_2(u_R(s))ds)d\tau$$

with $\|u_R\| = R$.

On the one hand,

$$\begin{aligned} u_R(t) =&\varphi_q(\lambda_{1R}) \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_1(v_R(s))ds)d\tau \\ \leq&\varphi_q(\lambda_{1R}) \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} g_1(v_R(s))ds)d\tau \\ \leq&\varphi_q(l_2\lambda_{1R})\|v_R\| \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}})d\tau \\ =&\varphi_q(l_2\lambda_{1R})B^*\|v_R\|, \ \forall t \in J. \end{aligned}$$

Analogously,

$$v_R(t) \leq \varphi_q(l_2\lambda_2)B^*\|u_R\|, \ \forall t \in J.$$

This verifies that

$$\|u_R\| = R \leq \varphi_q(l_2^2(B^*)^2\lambda_{1R}\lambda_2)\|u_R\|,$$

and so,

$$\lambda_{1R}\lambda_2 \geq \frac{1}{l_2^2\varphi_p((B^*)^2)} = \lambda_R.$$

On the other hand,

$$\begin{aligned} (u_R)(t) \geq&\varphi_q(\lambda_{1R}) \int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^{\frac{3}{4}} s^{N-1} g_1(v_R(s))ds)d\tau \\ \geq&\varphi_q(\lambda_{1R}) \int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} g_1(v_R(s))ds)d\tau \\ \geq&\varphi_q(\lambda_{1R}) \int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_1\varphi_p(v_R(s))ds)d\tau \\ \geq&\varphi_q(\lambda_{1R}) \int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_1\varphi_p(\frac{1}{4}\|v_R\|)ds)d\tau \\ =&\frac{1}{4}\varphi_q(\lambda_{1R}l_1A)B\|v_R\|, \ \forall t \in J. \end{aligned}$$

Analogously, we can show that

$$(v_R)(t) \geq \frac{1}{4}\varphi_q(\lambda_2 l_1 A)B\|u_R\|, \ \forall t \in J.$$

Therefore, we get

$$\|u_R\| \geq \frac{1}{16}\varphi_q(\lambda_{1R}\lambda_2 l_1^2 A^2)B^2\|u_R\|,$$

and so,

$$\lambda_{1R}\lambda_2 \leq \frac{\varphi_p(16)}{l_1^2 A^2 \varphi_p(B^2)} = \bar{\lambda}_R. \tag{25}$$

We hence get $\lambda_{1R}\lambda_2 \in [\lambda_R, \bar{\lambda}_R]$. This gives the proof. □

If we define another composite operator $T^* = T_2^* T_1^*$, where

$$(T_1^* v)(t) = \varphi_q(\lambda_1) \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_1(v(s))ds)d\tau, \tag{26}$$

$$(T_2^* v)(t) = \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_2(v(s))ds)d\tau. \tag{27}$$

Corollary 1. *Let $T^* = T_2^* T_1^*$. Suppose that $(\mathbf{C_0})$ holds. If $0 < g_i^\infty < +\infty (i = 1, 2)$, then there exists $\beta_0 > 0$ such that, for every $R > \beta_0$, system (4) admits a pair of positive solutions u_R, v_R satisfying $\|v_R\| = R$ for any*

$$\lambda_1 \lambda_{2R} \in [\lambda_R, \bar{\lambda}_R], \tag{28}$$

where λ_R and $\bar{\lambda}_R$ are positive finite numbers.

Proof. Similar to the proof of Theorem 4.2, we can prove Corollary 1. □

Theorem 4.3. *Suppose that $(\mathbf{C_0})$ holds. If $0 < g_i^0 < +\infty (i = 1, 2)$, then there exists $\beta_0^* > 0$ such that, for every $0 < r < \beta_0^*$, system (4) admits a pair of positive solutions u_r, v_r satisfying $\|u_r\| = r$ for any*

$$\lambda_{1r}\lambda_2 \in [\lambda_r, \bar{\lambda}_r],$$

where λ_r and $\bar{\lambda}_r$ are positive finite numbers.

Proof. Similar to the proof of Theorem 4.2, we can prove Theorem 4.3. □

Theorem 4.4. *Suppose that $(\mathbf{C_0})$ holds. If $g_i^\infty = +\infty (i = 1, 2)$, then there exists $\bar{\beta}_0 > 0$ such that, for every $R_* > \bar{\beta}_0$, system (4) admits a pair of positive solutions u_{R_*}, v_{R_*} satisfying $\|u_{R_*}\| = R_*$ for any*

$$\lambda_{1R_*}\lambda_2 \in (0, \lambda_{R_*}], \tag{29}$$

where λ_{R_*} is a positive finite number.

Proof. Since $g_i^\infty = +\infty$, there exist $l_* > 0$, $\mu^* > 0$ so that

$$g_1(v) > l_* \varphi_p(v), \ \forall v \geq \mu^*;$$

$$g_2(u) > l_* \varphi_p(u), \ \forall u \geq \mu^*.$$

Now, we show that $\bar{\beta}_0 = 4\mu^*$ is required. Set

$$\Omega_{R_*} = \{x \in E : \|x\| < R_*\}.$$

Since $R_* > \bar{\beta}_0$, for any $u, v \in P \cap \partial\Omega_{R_*}$, we get

$$u(t) \geq \frac{1}{4}\|u\| = \frac{1}{4}R_*, \ v(t) \geq \frac{1}{4}\|v\| = \frac{1}{4}R_*, \quad t \in [\frac{1}{4}, \frac{3}{4}],$$

and

$$u(t) \geq \frac{1}{4}\|u\| > \frac{1}{4}\bar{\beta}_0 = \mu^*, \ v(t) \geq \frac{1}{4}\|v\| > \frac{1}{4}\bar{\beta}_0 = \mu^*, \quad t \in [\frac{1}{4}, \frac{3}{4}].$$

So, for any $v \in P \cap \partial\Omega_{R_*}$, we have

$$\begin{aligned}(T_1 v)(t) \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_0^{\frac{3}{4}} s^{N-1} g_1(v(s))ds)d\tau \\ \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} g_1(v(s))ds)d\tau \\ \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_* \varphi_p(v(s))ds)d\tau \\ \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_* \varphi_p(\frac{1}{4}\|v\|)ds)d\tau \\ =& \frac{1}{4}\|v\|\varphi_q(l_* A)B, \ \forall t \in J.\end{aligned}$$

Analogously, for $u \in P \cap \partial\Omega_{R_*}$, we obtain

$$\begin{aligned}(T_2 u)(t) \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_0^{\frac{3}{4}} s^{N-1} g_2(u(s))ds)d\tau \\ \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} g_2(u(s))ds)d\tau \\ \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_* \varphi_p(u(s))ds)d\tau \\ \geq& \int_{\frac{3}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}} \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} l_* \varphi_p(\frac{1}{4}\|u\|)ds)d\tau \\ =& \frac{1}{4}\|u\|\varphi_q(l_* A)B, \ \forall t \in J.\end{aligned}$$

Therefore, we get

$$\begin{aligned}(Tu)(t) =& (T_1 T_2 u)(t) \\ \geq& \frac{1}{4}\|T_2 u\|\varphi_q(l_* A)B \\ \geq& \frac{1}{16}\|u\|(\varphi_q(l_* A)B)^2.\end{aligned}$$

This gives that

$$\inf_{u \in P \cap \partial\Omega_{R_*}} Tu \geq \frac{1}{16}\|u\|(\varphi_q(l_* A)B)^2 > 0.$$

For any $R_* > \bar{\beta}_0$, Lemma 4.1 yields that operator T admits a proper element $u_{R_*} \in P$ associated with the eigenvalue $\mu_{1R_*} > 0$, and u_{R_*} satisfies $\|u_{R_*}\| = R_*$.

For operator T, we denote $v_{R_*} = T_2 u_{R_*}$, then u_{R_*} and v_{R_*} are the solutions of system (4).

Let $\lambda_{1R_*} = \frac{1}{\varphi_p(\mu_{1R_*})}$. Next, similar to the proof of (25), we can verify that (29) holds. This finishes the proof of Theorem 4.4. □

Theorem 4.5. *Suppose that* (**f**) *holds. If* $g_i^0 = +\infty (i = 1, 2)$, *then there exists* $\beta_1 > 0$ *such that, for every* $0 < r^* < \beta_1$, *system* (4) *admits a nontrivial radial* *solution* $u_{r^*} = (u_{1r^*}, u_{2r^*})$ *satisfying* $\|u_{1r^*}\| = r^*$ *for any*

$$\lambda_{1r^*}\lambda_2 \in (0, \lambda^{**}],$$

where λ^{**} *is a positive finite number.*

Proof. Similar to the proof of Theorem 4.4, we can prove Theorem 4.5. □

5. **Asymptotic behavior of positive solutions.** In this section, we study the asymptotic behavior of positive solutions for system (4).

Let P be defined as (20), and T_1^* and T_2 be respectively defined in (26) and (22). Define a composite operator $\widetilde{T_1} = T_1^* T_2$, which is completely continuous from P to itself. So the operator $\widetilde{T_1}$ also maps P into P. We also define another composite operator

$$\widetilde{T_2} = T_2 T_1^*,$$

which has the same meaning as $\widetilde{T_1}$.

Theorem 5.1. *Suppose that* $(\mathbf{C_0})$ *holds. For* $i \in \{1, 2\}$*, then we have the following two conclusions.*

$(\mathbf{C_3})$ *If* $g_i^0 = 0$ *and* $g_i^\infty = \infty$*, then for every* $\lambda_i > 0$ *system* (4) *admits a pair of positive solutions* $u_{\lambda_1}, v_{\lambda_2}$ *with*

$$\lim_{\lambda_1 \to 0^+} \|u_{\lambda_1}\| = \infty, \quad \lim_{\lambda_2 \to 0^+} \|v_{\lambda_2}\| = \infty;$$

$(\mathbf{C_4})$ *If* $g_i^0 = \infty$ *and* $g_i^\infty = 0$*, then for every* $\lambda_i > 0$ *system* (4) *admits a pair of positive solutions* $u_{\lambda_1}, v_{\lambda_2}$ *with*

$$\lim_{\lambda_1 \to 0^+} \|u_{\lambda_1}\| = 0, \quad \lim_{\lambda_2 \to 0^+} \|v_{\lambda_2}\| = 0.$$

Proof. We need only verify this theorem under condition $(\mathbf{C_3})$ because the proof is similar when $(\mathbf{C_4})$ is satisfied. For $i \in \{1, 2\}$, let $\lambda_i > 0$. Since $g_i^0 = 0$, there exists $r > 0$ such that

$$g_1(v) \le \frac{1}{\lambda_1 \varphi_p(B^*)} \varphi_p(v), \quad \forall \ \ 0 \le v \le r,$$

$$g_2(u) \le \frac{1}{\lambda_2 \varphi_p(B^*)} \varphi_p(u), \quad \forall \ \ 0 \le u \le r,$$

where B^* is defined in (23).

Thus, for $i = \{1, 2\}$ and $u, v \in P \cap \partial\Omega_r$, we get

$$\begin{aligned}
(T_1^* v)(t) =& \varphi_q(\lambda_1) \int_t^1 \varphi_q\Big(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_1(v(s)) ds\Big) d\tau \\
\le& \varphi_q(\lambda_1) \int_0^1 \varphi_q\Big(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} g_1(v(s)) ds\Big) d\tau \\
\le& \varphi_q(\lambda_1) \int_0^1 \varphi_q\Big(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} \frac{1}{\lambda_1 \varphi_p(B^*)} \varphi_p(v(s)) ds\Big) d\tau \\
\le& \varphi_q(\lambda_1) \int_0^1 \varphi_q\Big(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} \frac{1}{\lambda_1 \varphi_p(B^*)} \varphi_p(\|v\|) ds\Big) d\tau \\
\le& \|v\|, \ \ \forall t \in J,
\end{aligned}$$

and

$$\begin{aligned}(T_2u)(t) =&\varphi_q(\lambda_2)\int_t^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_0^\tau s^{N-1}g_2(u(s))ds)d\tau\\ \leq&\varphi_q(\lambda_2)\int_0^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_0^1 s^{N-1}g_2(u(s))ds)d\tau\\ \leq&\varphi_q(\lambda_2)\int_0^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_0^1 s^{N-1}\frac{1}{\lambda_2\varphi_p(B^*)}\varphi_p(u(s))ds)d\tau\\ \leq&\varphi_q(\lambda_2)\int_0^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_0^1 s^{N-1}\frac{1}{\lambda_2\varphi_p(B^*)}\varphi_p(\|u\|)ds)d\tau\\ \leq&\|u\|,\ \forall t\in J.\end{aligned}$$

So

$$\begin{aligned}\|\widetilde{T_1}u\| =&\|T_1^*T_2u\|\\ \leq&\|T_2u\|\\ \leq&\|u\|.\end{aligned}\tag{30}$$

Next, for $i=\{1,2\}$, considering $g_i^\infty=\infty$, there exists $\hat{R}$ satisfying $0<r<\hat{R}$ so that

$$g_1(v)\geq\varepsilon\varphi_p(v),\quad \forall v\geq\hat{R},$$

$$g_2(u)\geq\varepsilon\varphi_p(u),\quad \forall u\geq\hat{R},$$

where $\varepsilon>0$ satisfies

$$\varphi_q(\lambda_1\lambda_2A^2\varepsilon^2)B^2\geq 1,\tag{31}$$

where A and B are respectively defined in (23).

Let $R>4\hat{R}$. Then, for $u,v\in P\cap\partial\Omega_R$, we get

$$u(t)\geq\frac{1}{4}\|u\|\geq\hat{R},\ v(t)\geq\frac{1}{4}\|v\|\geq\hat{R},\ t\in[\frac{1}{4},\frac{3}{4}],$$

and then

$$\begin{aligned}(T_1^*v)(t) =&\varphi_q(\lambda_1)\int_t^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_0^\tau s^{N-1}g_1(v(s))ds)d\tau\\ \geq&\varphi_q(\lambda_1)\int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_0^{\frac{3}{4}} s^{N-1}g_1(v(s))ds)d\tau\\ \geq&\varphi_q(\lambda_1)\int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}g_1(v(s))ds)d\tau\\ \geq&\varphi_q(\lambda_1)\int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}\varepsilon\varphi_p(v(s))ds)d\tau\\ \geq&\varphi_q(\lambda_1)\int_{\frac{3}{4}}^1 \varphi_q(\frac{1}{\tau^{N-1}}\int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1}\varepsilon\varphi_p(\frac{1}{4}\|v\|)ds)d\tau\\ =&\varphi_q(\lambda_1A\varepsilon)B\|v\|,\ \forall t\in J.\end{aligned}$$

Similarly, we get

$$(T_2u)(t)\geq\varphi_q(\lambda_2A\varepsilon)B\|u\|,\ \forall t\in J.$$

So, by (31), we have

$$\begin{aligned}(\widetilde{T_1}v_1)(t) =&(T_1^*T_2u)(t)\\ \geq&\varphi_q(\lambda_1 A\varepsilon)B\|T_2u\|\\ \geq&\varphi_q(\lambda_1\lambda_2A^2\varepsilon^2)B^2\|u\|\\ \geq&\|u\|. \end{aligned}\tag{32}$$

From the above estimate and the fixed point theorem of cone expansion and compression of norm type, we deduce that operator $\widetilde{T_1}$ has a fixed point $u \in P \cap (\bar{\Omega}_R\backslash\Omega_r)$. Denote $v = T_2u$, then u and v are the desired solution of system (4).

Similarly, we can prove that $\widetilde{T_2}$ has a fixed point $v \in P \cap (\bar{\Omega}_R\backslash\Omega_r)$.

Next, for $i \in \{1, 2\}$, we prove that $\|u_{\lambda_1}\| \to +\infty,\ \|v_{\lambda_2}\| \to +\infty$ as $\lambda_i \to 0^+$. In deed, if not, there are a number $\varsigma_i > 0$ and a sequence $\lambda_{im} \to +\infty$ such that

$$\|u_{\lambda_{1m}}\| \leq \varsigma_1,\ \|v_{\lambda_{2m}}\| \leq \varsigma_2\ (m = 1, 2, 3, \cdots).$$

Moreover, the sequence $\{\|u_{\lambda_{1m}}\|\}$ and $\{\|v_{\lambda_{2m}}\|\}$ respectively contain a subsequence that converges to a number $\eta_i(0 \leq \eta_i \leq \varsigma_i)$. For simplicity, we suppose that $\{\|u_{\lambda_{1m}}\|\}$ itself converges to η_1, and $\{\|v_{\lambda_{2m}}\|\}$ itself converges to η_2.

If $\eta_1 > 0, \eta_2 > 0$, then $\|u_{\lambda_{1m}}\| > \frac{\eta_1}{2},\ \|v_{\lambda_{2m}}\| > \frac{\eta_2}{2}$ for sufficiently large m ($m > M$, M denotes a natural number), and so

$$\begin{aligned}\frac{1}{\varphi_q(\lambda_1 m)} =&\frac{\|\int_t^1\varphi_q(\frac{1}{\tau^{N-1}}\int_0^\tau s^{N-1}g_1(v(s))ds)d\tau\|}{\|u_{\lambda_{1m}}\|}\\ \leq&\frac{\|\int_0^1\varphi_q(\frac{1}{\tau^{N-1}}\int_0^1 s^{N-1}g_1(v(s))ds)d\tau\|}{\|u_{\lambda_{1m}}\|}\\ \leq&\frac{\varphi_q(D_1)B^*}{\|u_{\lambda_{1m}}\|}\\ <&\frac{2\varphi_q(D_1)B^*}{\eta_1}\ (m > M),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varphi_q(\lambda_2 m)} =&\frac{\|\int_t^1\varphi_q(\frac{1}{\tau^{N-1}}\int_0^\tau s^{N-1}g_2(u(s))ds)d\tau\|}{\|v_{\lambda_{2m}}\|}\\ \leq&\frac{\|\int_0^1\varphi_q(\frac{1}{\tau^{N-1}}\int_0^1 s^{N-1}g_2(u(s))ds)d\tau\|}{\|v_{\lambda_{2m}}\|}\\ \leq&\frac{\varphi_q(D_2)B^*}{\|v_{\lambda_{2m}}\|}\\ <&\frac{2\varphi_q(D_1)B^*}{\eta_2}\ (m > M),\end{aligned}$$

where,

$$D_1 = \max\left\{g_1(v),\ r \leq \|v\| \leq R\right\},$$

$$D_2 = \max\left\{g_2(u),\ r \leq \|u\| \leq R\right\}.$$

This gives a contradiction as $\lambda_{im} \to 0^+$ for $i \in \{1, 2\}$.

If $\eta_1 = 0$ and $\eta_2 = 0$, then $\|u_{\lambda_{1m}}\| \to 0, \|v_{\lambda_{2m}}\| \to 0$ for sufficiently large m $(m > M)$, and so it follows from $(\mathbf{C_3})$ that for any $\varepsilon > 0$ there is $r^* > 0$ so that

$$g_1(v_{\lambda_{2m}}) \leq \varepsilon\varphi_p(v),\ \forall\ 0 \leq v_{\lambda_{2m}} \leq r^*,$$

$$g_2(u_{\lambda_{1m}}) \leq \varepsilon\varphi_p(u),\ \forall\ 0 \leq u_{\lambda_{1m}} \leq r^*.$$

Then, for $u_{\lambda_{1m}},\ v_{\lambda_{2m}} \in P \cap \partial\Omega_{r^*}$, we have

$$\begin{aligned}\frac{1}{\varphi_q(\lambda_1 m)} &= \frac{\| \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_1(v(s))ds)d\tau\|}{\|u_{\lambda_{1m}}\|}\\ &\leq \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} g_1(v(s))ds)d\tau\|}{\|u_{\lambda_{1m}}\|}\\ &\leq \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} \varepsilon\varphi_p(v(s))ds)d\tau\|}{\|u_{\lambda_{1m}}\|}\\ &\leq \frac{\varphi_q(\varepsilon)B^*\|v\|}{\|u_{\lambda_{1m}}\|},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varphi_q(\lambda_2 m)} &= \frac{\| \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^\tau s^{N-1} g_2(u(s))ds)d\tau\|}{\|v_{\lambda_{2m}}\|}\\ &\leq \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} g_2(v(s))ds)d\tau\|}{\|v_{\lambda_{2m}}\|}\\ &\leq \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}} \int_0^1 s^{N-1} \varepsilon\varphi_p(v(s))ds)d\tau\|}{\|v_{\lambda_{2m}}\|}\\ &\leq \frac{\varphi_q(\varepsilon)B^*\|v\|}{\|v_{\lambda_{2m}}\|},\end{aligned}$$

where B^* is defined in (23). Because ε is arbitrary, for $i \in \{1, 2\}$, we get $\lambda_{im} \to +\infty$ $(m \to +\infty)$, which contradicts $\lambda_{im} \to 0^+$. The proof of Theorem 5.1 is finished. □

6. **Some remarks.** In this section, we offer some remarks and applications on the associated system (4).

Remark 6.1. The present research extends the study in Hai [10] from Laplacian system to p-Laplacian system. Meanwhile, we obtain some new existence results by defining composite operators and using the eigenvalue theory in cones. Moreover, we also analyze the asymptotic behavior of positive solutions to system (4).

Remark 6.2. In this paper, we also generalize the study in Guo [6], Guo and Webb [7], Hai and Shivaji [11], Shivaji, Sim and Son [20], and Chu, Hai and Shivaji [3] from single p-Laplacian equation to coupled p-Laplacian system. Here, we not only get the uniqueness results, but also we obtain some existence results, and we consider the asymptotic behavior of positive solutions.

Remark 6.3. The approaches to prove Theorem 3.3, Theorem 4.2-Theorem 4.5 and Theorem 5.1 can be applied to the single equation case

$$\begin{cases} -\triangle_p z = \lambda g(z) \ \text{ in } \ \Omega, \\ z = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),\ 1 < p < N$, λ is a positive parameter, Ω is the open unit ball in $\mathbb{R}^N$.

Acknowledgments. The authors also would like to thank the anonymous referees for their valuable comments which has helped to improve the paper.

REFERENCES

[1] R. P. Agarwal, H. Lü and D. O'Regan, Eigenvalues and the one-dimensional p-Laplacian, *J. Math. Anal. Appl.*, **266** (2002), 383–400.

[2] A. Castro, L. Sankar and R. Shivaji, Uniqueness of non-negative solutions for semipositone problems on exterior domains, *J. Math. Anal. Appl.*, **394** (2012), 432–437.

[3] K. D. Chu, D. D. Hai and R. Shivaji, Uniqueness of positive radial solutions for infinite semipositone p-Laplacian problems in exterior domains, *J. Math. Anal. Appl.*, **472** (2019), 510–525.

[4] L. D'Ambrosio and E. Mitidieri, Quasilinear elliptic systems in divergence form associated to general nonlinearities, *Adv. Nonlinear Anal.*, **7** (2018), 425–447.

[5] Y. Du and Z. Guo, Boundary blow-up solutions and the applications in quasilinear elliptic equations, *J. Anal. Math.*, **89** (2003), 277–302.

[6] Z. M. Guo, Existence and uniqueness of positive radial solutions for a class of quasilinear elliptic equations, *Appl. Anal.*, **47** (1992), 173–189.

[7] Z. M. Guo, J. R. L. Webb, Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large, *Proc. Roy. Soc. Edinburgh Sect. A*, **124** (1994), 189–198.

[8] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Inc., Boston, MA, 1988.

[9] G. B. Gustafson and K. Schmitt, Nonzero solutions of boundary value problems for second order ordinary and delay differential equations, *J. Differential Equations*, **12** (1972), 129–147.

[10] D. D. Hai, Uniqueness of positive solutions for a class of semilinear elliptic systems, *Nonlinear Anal.*, **52** (2003), 595–603.

[11] D. D. Hai and R. Shivaji, Existence and uniqueness for a class of quasilinear elliptic boundary value problems, *J. Differential Equations*, **193** (2003), 500–510.

[12] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Dover Publications, Inc. Mineola, New York, 2006.

[13] S.-S. Lin, On the number of positive solutions for nonlinear elliptic equations when a parameter is large, *Nonlinear Anal.*, **16** (1991), 283–297.

[14] P. Lindqvist, *Notes on the p-Laplace Equation, Report. University of Jyväskylä Department of Mathematics and Statistics*, vol. 102, *University of Jyväskylä, Jyväskylä,* 2006.

[15] Z. Liu, J. Su and Z.-Q. Wang, A twist condition and periodic solutions of Hamiltonian systems, *Adv. Math.*, **218** (2008), 1895–1913.

[16] Z. Lou, T. Weth and Z. Zhang, Symmetry breaking via Morse index for equations and systems of Hénon-Schrödinger type, *Z. Angew. Math. Phys.*, **70** (2019), Paper No. 35, 19 pp.

[17] N. Mavinga and R. Pardo, A priori bounds and existence of positive solutions for semilinear elliptic systems, *J. Math. Anal. Appl.*, **449** (2017), 1172–1188.

[18] K. Perera, R. Shivaji and I. Sim, A class of semipositone p-Laplacian problems with a critical growth reaction term, *Adv. Nonlinear Anal.*, **9** (2020), 516–525.

[19] J. Sánchez, Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-Laplacian, *J. Math. Anal. Appl.*, **292** (2004), 401–414.

[20] R. Shivaji, I. Sim and B. Son, A uniqueness result for a semipositone p-Laplacian problem on the exterior of a ball, *J. Math. Anal. Appl.*, **445** (2017), 459–475.

[21] B. Son and P. Wang, Analysis of positive radial solutions for singular superlinear p-Laplacian systems on the exterior of a ball, *Nonlinear Anal.*, **192** (2020), 111657, 15 pp.

[22] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.*, **12** (1984), 191–202.

[23] M. Xiang, B. Zhang and V. D. Rădulescu, Existence of solutions for perturbed fractional p-Laplacian equations, *J. Differential Equations*, **260** (2016), 1392–1413.

[24] Z. Zhang and S. Li, On sign-changing andmultiple solutions of the p-Laplacian, *J. Funct. Anal.*, **197** (2003), 447–468.

Received April 2020; revised June 2020.

E-mail address: zyc_bjstu@126.com
E-mail address: meiqiangfeng@sina.com