ELECTRONIC RESEARCH ARCHIVE
Volume **28**, Number **4**, December **2020**
doi:10.3934/era.2020074
pp. **1395–1418**

# THE STRUCTURE AND STABILITY OF PULLBACK ATTRACTORS FOR 3D BRINKMAN-FORCHHEIMER EQUATION WITH DELAY

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(Communicated by Alain Miranville)

Abstract. This paper concerns the stability of pullback attractors for 3D Brinkman-Forchheimer equation with delays. By some regular estimates and the variable index to deal with the delay term, we get the sufficient conditions for asymptotic stability of trajectories inside the pullback attractors for a fluid flow model in porous medium by generalized Grashof numbers.

1. **Introduction.** The delay effect originates from the boundary controllers in engineering. The dynamics of a system with boundary delay could be described mathematically by a differential equation with delay term subject to boundary value condition such as [20]. There are many results available in literatures on the well-posedness and pullback dynamics of fluid flow models with delays especially the 2D Navier-Stokes equations, which can be seen in [1], [2], [8] and references therein. Inspired by these works, in this paper, we study the stability of pullback attractors for 3D Brinkman-Forchheimer (BF) equation with delay, which is also a continuation of our previous work in [6]. The existence and structure of attractors are significant to understand the large time behavior of solutions for non-autonomous

2020 *Mathematics Subject Classification.* 35Q30, 35B40, 35B41, 76D03, 76D05.

*Key words and phrases.* 3D Brinkman-Forchheimer equations, the generalized Grashof number, complete trajectory.

Research partly supported by the Fund of Young Backbone Teacher in Henan Province (No. 2018GGJS039).

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evolutionary equations. Furthermore, the asymptotic stability of trajectories inside invariant sets determines many important properties of trajectories. The 3D Brinkman-Forchheimer equation with delay is given below:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + \alpha u + \beta|u|u + \gamma|u|^2u + \nabla p = f(t,u_t) + g(x,t), \\ \nabla\cdot u = 0, \\ u(t,x)|_{\partial\Omega} = 0, \\ u|_{t=\tau} = u_\tau(x),\ x\in\Omega, \\ u_\tau(\theta,x) = u(\tau+\theta,x) = \phi(\theta),\ \theta\in(-h,0),\ h>0. \end{cases} \tag{1}$$

Here, $(x,t)\in\Omega\times\mathbb{R}^+$ with $\Omega\subset\mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. $u=(u_1,u_2,u_3)$ is the velocity vector field, $p$ is the pressure, $\nu>0$ and $\alpha>0$ denotes the Brinkman kinematic viscosity and the Darcy coefficients respectively, $\beta>0$ and $\gamma>0$ are the Forchheimer coefficients, the external force $g(x,t)\in L^2_{loc}(\mathbb{R};H)$ is a locally square integrable function and the delay term is considered either

(1). a general delay $f(t,u_t)$ with $u_t:[-h,0]\to H$ defined as $u_t=u(t+s)$ which denotes constant, variable and distributed delays, see Caraballo and Real [1], [2].

or

(2). the special application of $f(t,u_t)$ as a sub-linear operator

$$f(t,u_t) = F(u(t-\rho(t))) \tag{2}$$

for a smooth function $\rho(\cdot)$ defined in Section 4, which satisfies subadditive and positive homogeneous property with second variable component, see Marín-Rubio and Real [8].

The BF equation describes the conservation law of fluid flow in a porous medium that obeys the Darcy's law. The physical background of 3D BF model can be seen in [14], [9], [18], [19]. For the dynamic systems of problem (1) without delay, i.e., $f(t,u_t)=0$, we can refer to [6], [10], [15], [16], [21] and literature therein for the existence of global weak solution and attractors. For the problem (1) with delay $f(t,u_t)$, the global existence of mild solution, continuous dependence on initial data and the minimal family of pullback attractors have been obtained in [6]. In addition, the upper semi-continuous property of pullback attractors as delay vanishes has been proved by virtue of some regular estimates. Furthermore, as a special application of the delay $f(t,u_t)$, the pullback dynamics of problem (1) with sub-linear operator (2) has also been shown. However, the asymptotic stability of trajectories inside pullback attractors is still open. Motivated by [3] and [17], applying regularity for weak solution and iteration technique with variable indices, we present some sufficient conditions with the generalized Grashof number to achieve the stability of pullback attractors in this paper. The main features and results can be summarized as follows.

(a) For problem (1) with delay $f(t,u_t)$, we use the regular estimate to achieve an upper bound of Grashof number, which implies the exponential stability of trajectories inside pullback attractors. The proof does not depend on initial data with more regularity, see Section 3. Here we use, the delay $f(t,u_t)$ in problem (1) can be the constant, variable and distributed delays $F(u(t-h))$, $F(u(t-\rho(t)))$ and $\int_{-h}^0 k(t,s)u(t+s)ds$ respectively, here $F(\cdot)$ is an appropriate function, see [1], [2].

(b) For problem (1) with special application of $f(t,u_t)$ as the sub-linear operator, the variable indices have been introduced to deal with nonlinear term $\beta|u|u+\gamma|u|^2u$ and sub-linear operator by iterative argument, see Section 4.

(c) The asymptotic stability of trajectories inside pullback attractors is further research of the results established in [6]. However, the stability of pullback attractors for (1) with infinite delay is still unknown.

## 2. Preliminary.

In this section, we give some notations and the equivalent abstract form of (1) in this section.

Denoting $E := \{u|u \in (C_0^\infty(\Omega))^3, divu = 0\}$, $H$ is the closure of $E$ in $(L^2(\Omega))^3$ topology, $\|\cdot\|_H$ and $(\cdot,\cdot)$ denote the norm and inner product in $H$ respectively. $V$ is the closure of set $E$ in $(H^1(\Omega))^3$ topology, $\|\cdot\|_V$ and $((\cdot,\cdot))$ denote the norm and inner product in $V$ respectively. $H'$ and $V'$ are dual spaces of $H$ and $V$ respectively. Clearly, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, $H'$ and $V'$ are dual spaces of $H$ and $V$ respectively, where the injection is dense, continuous. The norm $\|\cdot\|_*$ denotes the norm in $V'$, $\langle\cdot,\cdot\rangle$ be the dual product in $V$ and $V'$. Let $P$ be the Helmholz-Leray orthogonal projection from $(L^2(\Omega))^3$ onto the space $H$, we define $A := -P\Delta$ as the Stokes operator with domain $D(A) = (H^2(\Omega))^3 \bigcap V$ and $\lambda$ is the first eigenvalue of $A$, the sequence $\{\omega_j\}_{j=1}^\infty$ is an orthonormal system of eigenfunctions of $A$, and $\{\lambda_j\}_{j=1}^\infty$ $(0 < \lambda_1 \le \lambda_2 \le \cdots)$ are eigenvalues of $A$ corresponding to the eigenfunctions $\{\omega_j\}_{j=1}^\infty$, see more details in [13].

By the Helmholz-Leray projection defined above, (1) can be transformed to the abstract equivalent form

$$\begin{cases} \frac{\partial u}{\partial t} + \nu Au + P(\alpha u + \beta|u|u + \gamma|u|^2u) = Pf(t,u_t) + Pg(t,x), \\ u|_{\partial\Omega} = 0, \\ u|_{t=\tau} = u_\tau(x), \\ u_\tau(\theta,x) = \phi(\theta,x) \text{ for } \theta \in (-h,0), \end{cases} \tag{3}$$

then we show our results for (3) with $f(t,u_t)$ as either general case or its special case $F(u(t-\rho(t)))$ in Sections 3 and 4, respectively.

We also define some Banach spaces on delayed interval as $C_H = C([-h,0];H)$, $C_V = C([-h,0];V)$ with the norms

$$\|\phi\|_{C_H} = \sup_{\theta\in[-h,0]} \|\phi(\theta)\|_H, \quad \|\phi\|_{C_V} = \sup_{\theta\in[-h,0]} \|\phi(\theta)\|_V,$$

respectively. The Lebesgue integrable spaces on delayed interval can be denoted as $L_H^p = L^p(-h,0;H)$, $L_V^p = L^p(-h,0;V)$. The product space is defined as $X_H = C([\tau-h,T];H) \times C([\tau,T];H)$ and $M_H = H \times (C_H \cap L_V^2)$ for purpose of phase space in next sections.

## 3. The asymptotic stability of trajectories inside pullback attractors for (1) with delay $f(t,u_t)$.

### 3.1. Assumptions.

Some assumptions on the external forces and parameters which will be imposed in our main results are the following:

$(H_f)$ The function $f : \mathbb{R} \times C_H \to H$ satisfies:

$(a)$ For any $\xi \in C_H$, the function $f(\cdot,\xi)$ is measurable and $f(\cdot,0) \equiv 0$.

$(b)$ There exists a $L_f > 0$ such that

$$\|f(t,\xi) - f(t,\eta)\|_H \le L_f\|\xi-\eta\|_{C_H}, \text{ for } \xi,\eta \in C_H.$$

($c$) There exists $C_f > 0$, such that for all $u, v \in C([\tau - h, t]; H)$

$$\int_\tau^t \|f(r, u_r) - f(r, v_r)\|_H^2 dr \le C_f^2 \int_{\tau-h}^t \|u(r) - v(r)\|_H^2 dr, \text{ for } \tau \le t. \tag{4}$$

($H_g$) The function $g(\cdot, \cdot) \in L^2_{loc}(\mathbb{R}, V')$ satisfies that there exists $\eta > 0$ such that

$$\int_{-\infty}^t e^{\eta s} \|g(s, \cdot)\|_{V'}^2 ds < \infty. \tag{5}$$

holds for any $t \ge \tau$.

($H_0$) The coefficients satisfy $\alpha + 3\beta + 2\gamma - \eta - \frac{2C_f^2}{\alpha} e^{\eta h} \ge 0$.

### 3.2. Some retarded integral inequalities.

**Lemma 3.1.** *(The Gronwall inequality with differential form) Let $m(\cdot) \in C^1[\mathbb{R}^+, \mathbb{R}^+]$, $v(\cdot)$, $h(\cdot) \in C[\mathbb{R}^+, \mathbb{R}^+]$ and*

$$\frac{d}{dt} m(t) \le v(t) m(t) + h(t), \ m(t = \tau) = m_\tau, \ t \ge \tau. \tag{6}$$

*Then*

$$m(t) \le m_\tau e^{\int_\tau^t v(s) ds} + \int_\tau^t h(s) e^{\int_s^t v(\sigma) d\sigma} ds, \ t \ge \tau. \tag{7}$$

In this part, we shall present some retarded integral inequalities from Li, Liu and Ju [5]. Consider the following retarded integral inequalities:

$$\|y(t)\|_X \le E(t, \tau)\|y_\tau\|_X + \int_\tau^t K_1(t, s)\|y_s\|_X ds + \int_t^\infty K_2(t, s)\|y_s\|_X ds + \rho, \quad \forall\, t \ge \tau, \tag{8}$$

where $E$, $K_1$ and $K_2$ are non-negative measurable functions on $\mathbb{R}^2$, $\rho \ge 0$ denotes a constant. Let $X$ be a Banach space with spatial variable, based on the retarded Banach space above, then we use $\|\cdot\|_{C_X}$ denotes the norm of space $C([-h, 0]; X)$ for some $h \ge 0$, $y(t) \ge 0$ is a continuous function defined on $C([-h, T]; X)$, $y_t(s) = y(t + s)$ for $s \in [-h, 0]$.

Let $\mathcal{L}(E, K_1, K_2, \rho) = \{y \in C([-h, T]; X) | y \ge 0$ and satisfies the inequality (8)$\}$, and

$$\kappa(K_1, K_2) = \sup_{t \ge \tau} \Big( \int_\tau^t K_1(t, s) ds + \int_t^\infty K_2(t, s) ds \Big).$$

We assume that

$$\lim_{t \to +\infty} E(t + s, s) = 0 \tag{9}$$

uniformly with respect to $s \in \mathbb{R}^+$. Moreover, we suppose that $\kappa(K_1, K_2) < +\infty$.

**Lemma 3.2.** *(The retarded Gronwall inequality) Denoting $\vartheta = \sup_{t \ge s \ge \tau} E(t, s)$ and $\kappa = \kappa(K_1, K_2)$, then we have the following estimates:*

*(1) If $\kappa < 1$, then for any $R$, $\varepsilon > 0$, there exists $\tilde{T} > 0$ such that*

$$\|y_t\|_X < \mu\rho + \varepsilon, \tag{10}$$

*for $t > \tilde{T}$ and all bounded functions $y \in \mathcal{L}(E, K_1, K_2, \rho)$ with $\|y_0\|_X \le R$, where $\mu = \frac{1}{1-\kappa}$.*

*(2) If $\kappa < \frac{1}{1+\vartheta}$, then there exist $M$, $\lambda > 0$ which are independent on $\rho$ such that*

$$\|y_t\|_X \le M \|y_0\|_X e^{-\lambda t} + \gamma\rho, \quad t \ge \tau \tag{11}$$

*for all bounded functions* $y \in \mathcal{L}(E, K_1, K_2, \rho)$, *where* $\gamma = \frac{\mu+1}{1-\kappa c}$ *and* $c = \max\{\frac{\vartheta}{1-\kappa}, 1\}$.

*(3) If* $\kappa < \frac{1}{1+\vartheta}$, *then the solution reduces to trivial for the occasion* $\kappa c < 1$.

*Proof.* See Li, Liu and Ju [5]. □

**Remark 1.** (The special case: $K_2 = 0$) Denote $(K_1, K_2) = (K_1, 0)$ and let $\vartheta$, $\kappa$, $\mu$, $\gamma$ be the constants defined in Lemma 3.2. Then we have the similar estimates as in Lemma 3.2.

3.3. **Well-posedness.** The minimal family of pullback attractors will be stated here in preparation for our main result.

**• Some inequalities**

**Lemma 3.3.** *(1) (See [7], [11]) Assume that* $\beta \geq 2$, *then for any* $a, b \in \mathbb{R}^n$, *we have*

$$(|a|^{\beta-2}a - |b|^{\beta-2}b) \cdot (a - b) \geq \gamma_0 |a - b|^\beta,$$

*where* $\gamma_0 > 0$ *is a constant which is determined by the volume of domain and its dimension, such as* $\min \gamma_0 = \frac{1}{2}6^{-\frac{\beta}{2}}$ *in a 3-dimensional smooth domain.*

*(2) The following* $\mathrm{C_q}$*-inequality holds*

$$|x^q - y^q| \leq Cq(|x|^{q-1} + |y|^{q-1})|x - y|$$

*for the integer* $q \geq 2$.

**• Well-posedness**

**Theorem 3.4.** *Assume that the external forces* $g(t, x)$ *and* $f(t, u_t)$ *satisfy the hypothesis* $(H_g)$ *and* $(H_f)$, *the initial data* $(u_\tau, \phi) \in M_H = H \times (C_H \cap L^2_V)$ *and* $(H_0)$ *are also true. Then there exists a unique global weak solution* $u = u(\cdot, \tau, u_\tau, \phi) \in C([\tau - h, T]; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; \mathbf{L}^4(\Omega))$ *of equation* (1) *on* $[\tau - h, T]$.

*Proof.* **Step 1.** Existence of local approximate solution.

By the property of the Stokes operator $A$, the sequence of eigenfunctions $\{w_i,\ i = 1, 2, \cdots\}$ of the Stokes operator is an orthonormal complete basis of $H$ formed by elements of $V \cap (H^2(\Omega))^3$ such that

$$Aw_i = \lambda_i w_i,\ i = 1, 2, \cdots . \tag{12}$$

Let $H_m = span\{w_1, w_2, \cdots, w_m\}$, $P_m : H \to H_m$ be a projection, then the approximate solutions can be written as $u_m(t) = \sum_{j=1}^{m} h_{jm}(t) w_j$ (where $h_{jm}(t) = (v_m(t), w_j)$ is to be determined) which solve the problem

$$\begin{cases} (\partial_t u_m, w_j) + \nu(\nabla u_m, \nabla w_j) + (\alpha u_m + \beta |u_m| u_m + \gamma |u_m|^2 u_m, w_j) \\ \qquad\qquad = (f(t, u_{mt}), w_j) + \langle g, w_j \rangle, \\ u_m(\tau) = P_m u_\tau = u_{\tau m}, \\ u_{m\tau}(\theta, x) = P_m \phi(\theta) = \phi_m(\theta) \text{ for } \theta \in [-h, 0], \end{cases} \tag{13}$$

Then it is easy to check that (13) is equivalent to an ordinary differential equations with unknown variable function $h_{jm}(t)$. By the Cauchy-Peano Theorem of ordinary differential equation, the problem (13) possesses a local solution over the time interval $[0, t_m]$.

**Step 2.** Uniform estimates of approximate solutions.

Multiplying (13) by $h_{jm}(t)$, and then summing from $j = 1$ to $m$, it yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m\|_H^2 + \nu\|u_m\|_V^2 + \alpha\|u_m\|_H^2 + \beta\|u_m\|_{\mathbb{L}^3(\Omega)}^3 + \gamma\|u_m\|_{\mathbb{L}^4(\Omega)}^4 \\
\leq\ & |(g(t) + f(s, u_{mt}(s)), u_m)| \\
\leq\ & \alpha\|u_m\|_H^2 + \frac{\nu}{2}\|u_m\|_V^2 + \frac{1}{2\nu}\|g(t)\|_{V'}^2 + \frac{1}{4\alpha}\|f(t, u_{mt})\|_H^2. \qquad (14)
\end{aligned}$$

Integrating in time, using the hypotheses on $f(\cdot,\cdot)$ and $g(t,x)$, by the Young inequality, we get

$$\begin{aligned}
&\|u_m\|_H^2 + \nu\int_\tau^t \|u_m\|_V^2 ds + 2\beta\int_\tau^t \|u_m\|_{\mathbb{L}^3(\Omega)}^3 ds + 2\gamma\int_\tau^t \|u_m\|_{\mathbb{L}^4(\Omega)}^4 ds \\
\leq\ & \|u_\tau\|_H^2 + \frac{C_f^2}{4\alpha}\int_{-h}^0 \|\phi(s)\|_H^2 ds + \frac{1}{2\nu}\int_\tau^t \|g(s)\|_{V'}^2 ds + \frac{C_f^2}{4\alpha}\int_\tau^t \|u_m\|_H^2 ds.
\end{aligned} \qquad (15)$$

Using the Gronwall Lemma of integrable form, we conclude that

$$\{u_m\} \text{ is bounded in the space } L^\infty(\tau, T; H) \cap L^2(\tau - h, T; V) \cap L^3(\tau, T; \mathbb{L}^3(\Omega)) \cap L^4(\tau, T; \mathbb{L}^4(\Omega)).$$

**Step 3.** Compact argument and passing to limit for deriving the global weak solutions.

In this step, we shall prove $\{u_m\}$ has a strong convergence subsequence by the Aubin-Lions Lemma along with the uniformly bounded estimate of $\frac{du_m}{dt}$ in $L^2(0, T; V')$. By the estimates of $u_m$ above step and continuous embedding $V \hookrightarrow \mathbb{L}^p(\Omega)$ with $p \in [1, 6]$ for three dimension, we can obtained that $|u_m|u_m \in L^2(\tau, T; H)$ and $|u_m|^2 u_m \in L^2(\tau, T; H)$. From the equation

$$\frac{du_m}{dt} = -\nu A u_m - \alpha u_m - \beta|u_m|u_m - \gamma|u_m|^2 u_m + P(g(t) + f(t, u_{mt}) \qquad (16)$$

and assumptions $(H_f)$ and $(H_g)$, we can see that $\{du_m/dt\}$ is bounded in $L^2(\tau, T; V')$.

By virtue of the Aubin-Lions Lemma, we obtain that $\{u_m\}$ has a strong convergent subsequence (also denoted as $\{u_m\}$ without confusion) with $u \in L^2(\tau - h, T; V)$ and $du/dt \in L^2(\tau, T; V')$ such that

$$\begin{cases}
u_m(t) \rightharpoonup u(t) \text{ weakly * in } L^\infty(\tau, T; H), \\
u_m(t) \to u(t) \text{ stongly in } L^2(\tau, T; H), \\
u_m(t) \rightharpoonup u(t) \text{ weakly in } L^2(\tau, T; V), \\
du_m/dt \rightharpoonup du/dt \text{ weakly in } L^2(\tau, T; V'), \\
f(\cdot, u_{m\cdot}) \rightharpoonup f(\cdot, u_\cdot) \text{ weakly in } L^2(\tau, T; H), \\
u_m \rightharpoonup u(t) \text{ weakly in } L^3(\tau, T; \mathbb{L}^3(\Omega)), \\
u_m \rightharpoonup u(t) \text{ weakly in } L^4(\tau, T; \mathbb{L}^4(\Omega))
\end{cases} \qquad (17)$$

which coincides with the initial data $u_m(\tau) = P_m u_\tau \to u(\tau) = u_\tau$ and $\phi_m(s) \to \phi(s)$.

For the purpose of passing to limit in (13), denoting $v = u_m - u$, we point out that we can deal with the nonlinear terms as the following novelty. Since $w_j$ is an eigenfunction of Stokes operator, we claim that

$$\begin{aligned}&\int_\tau^T (\beta|u_m|u_m - \beta|u|u, w_j)ds\\ \leq\ & C_{\lambda_1}\beta\|u_m\|^4_{L^4(\tau,T;\mathbb{L}^4(\Omega))}\|u_m - u\|^4_{L^4(\tau,T;\mathbb{L}^4(\Omega))}\\ &+C\beta\|u_m - u\|_{L^\infty(\tau,T;H)}\|u\|^2_{L^2(\tau-h,T;H)}\end{aligned}$$

and

$$\begin{aligned}&\int_\tau^T (\gamma|u_m|^2u_m - \gamma|u|^2u, w_j)ds\\ \leq\ & C\gamma\|u_m\|^2_{L^2(\tau,T;V)}\|u_m - u\|^4_{L^4(\tau,T;\mathbb{L}^4(\Omega))}\\ &+C\gamma\|u_m - u\|^4_{L^4(\tau,T;\mathbb{L}^4(\Omega))}(\|u\|^2_{L^2(\tau-h,T;V)} + \|u_m\|^4_{L^4(\tau,T;\mathbb{L}^4(\Omega))}) \qquad (18)\end{aligned}$$

and the convergence of delayed external force $f(t, u_{mt})$ can be verified by the hypotheses.

Thus, passing to the limit of (13), we conclude that $u$ is at least one of global weak solutions for problem (1). □

• **The regularity**

**Proposition 1.** *Assume that the external forces $g(t)$ and $f(t, u_t)$ satisfy the hypothesis $(H_g)$ and $(H_f)$, the initial data $(u_\tau, \phi) \in M_H = H \times (C_H \cap L^2_V)$ and $(H_0)$ are also true. Then the global weak solution $u$ in Theorem 3.4 has the regular boundedness in $L^\infty(\tau, T; V)$.*

*Proof.* Taking inner product of (3) with $Au$, it yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{1/2}u\|^2_H + \nu\|Au\|^2_H + \alpha\|A^{1/2}u\|^2_H\\ &+\beta\int_\Omega |u|u\cdot Audx + \gamma\int_\Omega |u|^2u\cdot Audx\\ =\ & (f(t, u_t), Au) + (g(t), Au). \qquad (19)\end{aligned}$$

According to Lemma 3.3, the nonlinear terms have the following estimates

$$|\beta(|u|u, Au)| \leq \frac{\nu}{2}\|Au\|^2_H + \frac{\beta}{4\nu}\|u\|^4_{\mathbf{L}^4} \qquad (20)$$

and

$$\gamma\int_\Omega |u|^2u\cdot Audx = \frac{\gamma}{2}\int_\Omega |\nabla(|u|^2)|^2dx + \gamma\int_\Omega |u|^2|\nabla u|^2dx \qquad (21)$$

and

$$(f(t, u_t), Au) + (g(t), Au) \leq \frac{1}{2\nu}\|f(t, u_t)\|^2_H + \frac{1}{2\nu}\|g(t)\|^2_H + \frac{\nu}{2}\|Au\|^2_H, \qquad (22)$$

hence, we conclude that

$$\begin{aligned}&\frac{d}{dt}\|A^{1/2}u\|^2_H + 2\alpha\|A^{1/2}u\|^2_H + \gamma\int_\Omega |\nabla(|u|^2)|^2dx + 2\gamma\int_\Omega |u|^2|\nabla u|^2dx\\ \leq\ & \frac{\beta}{2\nu}\|u\|^4_{\mathbf{L}^4} + \frac{1}{\nu}\|f(t, u_t)\|^2_H + \frac{1}{\nu}\|g(t)\|^2_H. \qquad (23)\end{aligned}$$

Letting $t-1\le s\le t$, neglecting the third and fourth terms on the left hand side of (23), integrating (23) with time variable from $s$ to $t$, it yields

$$\begin{aligned}&\|A^{1/2}u(t)\|_H^2+2\alpha\int_s^t\|A^{1/2}u(r)\|_H^2dr\\ \le\;&\|A^{1/2}u(s)\|_H^2+\frac{\beta}{2\nu}\int_s^t\|u(r)\|_{\mathbf{L}^4}^4dr\\ &+\frac{2}{\nu}\int_s^t\|f(r,u_r)\|_H^2dr+\frac{2}{\nu}\int_s^t\|g(r)\|_H^2dr\end{aligned}\tag{24}$$

and

$$\int_s^t\|f(r,u_r)\|_H^2dr\le L_f^2\|\phi(\theta)\|_{L_H^2}^2+L_f^2\int_s^t\|u(r)\|_H^2dr.\tag{25}$$

Then integrating with $s$ from $t-1$ to $t$, using the uniform boundedness of $u$ in Theorem 3.4, we deduce that

$$\begin{aligned}\|A^{1/2}u(t)\|_H^2\;\le\;&\int_{t-1}^t\|A^{1/2}u(s)\|_H^2ds+\frac{\beta}{2\nu}\int_{t-1}^t\|u(r)\|_{\mathbf{L}^4}^4dr\\ &+\frac{2L_f^2}{\nu}\|\phi(\theta)\|_{L_H^2}^2+\frac{2L_f^2}{\nu}\int_\tau^t\|u(r)\|_H^2dr+\frac{2}{\nu}\int_{t-1}^t\|g(r)\|_H^2dr\\ \le\;&C\Big[\|\phi\|_{L_H^2}^2+\|u_\tau\|_H^2\Big]+C\int_\tau^t\|g\|_H^2ds\\ &+\frac{2L_f^2}{\nu\lambda_1}\int_\tau^t\|u(r)\|_V^2dr,\end{aligned}\tag{26}$$

which means the uniform boundedness of the global weak solution $u$ in $L^\infty(\tau,T;V)$. The proof has been finished. □

• **Uniqueness**

**Proposition 2.** *Assume the hypotheses in Theorem 3.4 hold. Then the global weak solution $u$ is unique.*

*Proof.* Using the same energy estimates as above, we can deduce the uniqueness easily, here we skip the details. □

3.4. **Pullback attractors.** To description of pullback attractors, the functional space $M_H=H\times(C_H\cap L_V^2)$ is used as our phase space equipped with the norm $\|(\xi,\zeta)\|_{M_H}=\|\xi\|_H+\|\zeta\|_{L_V^2}+\|\zeta\|_{C_H}$ for $(\xi,\zeta)\in M_H$. Based on the well-posedness, we shall verify the pullback dissipation and asymptotic compactness for the process to achieve the existence of pullback attractors, which also needs the following assumption:
$(H_1)$ For every $u\in L^2(\tau-h,T;V)$, there exists a $\eta\in(0,\nu\lambda_1)$ which is independent on $u$ such that

$$\int_\tau^t e^{\eta s}\|f(s,u_s)\|_H^2ds<C_f^2\int_{\tau-h}^t e^{\eta s}\|u(s)\|_H^2ds.\tag{27}$$

for any $t \le T$.

• **The continuous process**

**Proposition 3.** *For given $f : \mathbb{R} \times C_H \to H$ and $g \in L^2_{loc}(\mathbb{R}; V')$ satisfying $(H_f), (H_g)$, $(H_1)$ and $(H_0)$. Then, the solution of problem (1) generates a biparametric family of mappings $U(t,\tau) : M_H \to M_H$ by $U(t,\tau)(u_\tau, \phi) = (u(t), u_t)$, which is a continuous process.*

• **Pullback dissipation**

**Lemma 3.5.** *Assume that $f : \mathbb{R} \times C_H \to H$ and $g \in L^2_{loc}(\mathbb{R}; V')$ satisfying $(H_f), (H_g)$, $(H_1)$ and $(H_0)$. Then, for any $(u_\tau, \phi) \in M_H$, the solution $u$ of (1) satisfies the estimates*

$$\begin{aligned} \|u(t)\|_H^2 \quad &\le \quad e^{-\frac{8\eta C_f}{\alpha}(t-\tau)}\Big(\|u_\tau\|_H^2 + C_f\|\phi(r)\|^2_{L^2_H}\Big) \\ &\quad + \frac{e^{-\frac{8\eta C_f}{\alpha}t}}{\nu - \eta\lambda^{-1}}\int_\tau^t e^{\eta r}\|g(r)\|^2_{V'}dr \end{aligned} \tag{28}$$

*and*

$$\begin{aligned} \nu\int_s^t \|u(r)\|_V^2 dr \quad &\le \quad \|u(s)\|_H^2 + \frac{8C_f}{\alpha}\|u_s\|^2_{L^2_H} \\ &\quad + \frac{1}{\nu}\int_s^t \|g(r)\|^2_{V'}dr + \frac{8C_f}{\alpha}\int_s^t \|u(r)\|_H^2 dr, \end{aligned} \tag{29}$$

$$\begin{aligned} \beta\int_s^t \|u(r)\|^3_{\mathbb{L}^3(\Omega)} dr \quad &\le \quad \|u(s)\|_H^2 + \frac{8C_f}{\alpha}\|u_s\|^2_{L^2_H} \\ &\quad + \frac{1}{\nu}\int_s^t \|g(r)\|^2_{V'}dr + \frac{8C_f}{\alpha}\int_s^t \|u(r)\|_H^2 dr, \end{aligned} \tag{30}$$

$$\begin{aligned} \gamma\int_s^t \|u(r)\|^4_{\mathbb{L}^4(\Omega)} dr \quad &\le \quad \|u(s)\|_H^2 + \frac{8C_f}{\alpha}\|u_s\|^2_{L^2_H} \\ &\quad + \frac{1}{\nu}\int_s^t \|g(r)\|^2_{V'}dr + \frac{8C_f}{\alpha}\int_s^t \|u(r)\|_H^2 dr. \end{aligned} \tag{31}$$

*Proof.* By the energy estimate of (1) and using Young's inequality, we arrive at

$$\begin{aligned} &\frac{d}{dt}\|u\|_H^2 + 2\nu\|u\|_V^2 + 2\alpha\|u\|_H^2 + 2\beta\|u\|^3_{\mathbb{L}^3(\Omega)} + 2\gamma\|u\|^2_{\mathbb{L}^4(\Omega)} \\ &\le \frac{1}{\nu - \eta\lambda^{-1}}\|g\|^2_{V'} + (\nu - \eta\lambda^{-1})\|u\|_V^2 + 2\alpha\|u\|_H^2 + \frac{8}{\alpha}\|f(t,u_t)\|_H^2, \end{aligned} \tag{32}$$

where $\eta \in (0, \nu\lambda_1)$.

Multiplying the above inequality by $e^{\eta t}$, we obtain

$$\begin{aligned} &\frac{d}{dt}(e^{\eta t}\|u\|_H^2) + e^{\eta t}\nu\lambda_1\|u\|_H^2 + 2\beta e^{\eta t}\|u\|^3_{\mathbb{L}^3(\Omega)} + 2\gamma e^{\eta t}\|u\|^2_{\mathbb{L}^4(\Omega)} \\ &\le \frac{1}{\nu - \eta\lambda^{-1}}e^{\eta t}\|g\|^2_{V'} + \frac{8C_f}{\alpha}e^{\eta t}\|f(t,u_t)\|_H^2. \end{aligned}$$

Thus integrating with respect to time variable, it yields

$$\begin{aligned} & e^{\eta t}\|u\|_H^2 + \nu\lambda_1 \int_\tau^t e^{\eta r}\|u(r)\|_H^2 dr \\ \leq\ & e^{\eta\tau}\Big(\|u_\tau\|_H^2 + C_f\int_{-h}^0 \|\phi(r)\|_H^2 dr\Big) + \frac{1}{\nu-\eta\lambda^{-1}}\int_\tau^t e^{\eta r}\|g(r)\|_{V'}^2 dr \\ & + \frac{8C_f}{\alpha}\int_\tau^t e^{\eta r}\|u(r)\|_H^2 dr \end{aligned} \tag{33}$$

and by the Gronwall Lemma, we can derive the estimate in our theorem.

Using the energy estimate of (1) again, we can check that

$$\begin{aligned} & \frac{d}{dt}\|u\|_H^2 + 2\nu\|u\|_V^2 + 2\alpha\|u\|_H^2 + 2\beta\|u\|_{\mathbb{L}^3(\Omega)}^3 + 2\gamma\|u\|_{\mathbb{L}^4(\Omega)}^2 \\ & \leq \frac{1}{\nu}\|g\|_{V'}^2 + \nu\|u\|_V^2 + 2\alpha\|u\|_H^2 + \frac{8}{\alpha}\|f(t,u_t)\|_H^2, \end{aligned} \tag{34}$$

Integrating from $s$ to $t$, using the estimate of $u$ in $H$, we can derive the desired result. □

Based on Lemma 3.5, we can present the pullback dissipation based on the following universes for the tempered dynamics.

**Definition 3.6.** (Universe). (1) We will denote by $\mathcal{D}_\eta^{M_H}$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H)$ such that

$$\lim_{\tau\to-\infty}\Big(e^{\eta\tau}\sup_{(\xi,\zeta)\in D(\tau)}\|(\xi,\zeta)\|_{M_H}^2\Big) = 0. \tag{35}$$

(2) $\mathcal{D}_F^{M_H}$ denotes the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with $D$ a fixed nonempty bounded subset in $M_H$.

**Remark 2.** The universes $\mathcal{D}_\eta^{M_H}$ and $\mathcal{D}_\eta^{M_H}$ satisfy include closed property.

**Proposition 4.** *(The $\mathcal{D}_\eta^{M_H}$ and $\mathcal{D}_F^{M_H}$ pullback absorbing sets in $M_H$) For given $f : \mathbb{R}\times C_H \to H$ and $g \in L^2_{loc}(\mathbb{R}; V')$ satisfying $(H_f), (H_g)$, $(H_1)$ and $(H_0)$ holds. Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset M_H$ is defined by*

$$D_0(t) = \overline{B}_H(0,\rho_H(t)) \times \Big(\overline{B}_{L_V^2}(0,\rho_{L_H^2}(t)) \cap \overline{B}_{C_H}(0,\rho_{C_H}(t))\Big)$$

*is the pullback $\mathcal{D}_\eta^{M_H}$-absorbing set for the process $U(t,\tau)$ on $M_H$ and $\widehat{D}_0 \in \mathcal{D}_\eta^{M_H}$, where the balls is defined as centered in the point zero and measured by the radius*

$$\begin{aligned} \rho_H^2(t) &= 1 + \frac{e^{-\frac{8\eta C_f}{\alpha}(t-h)}}{\nu-\eta\lambda^{-1}}\int_{-\infty}^t e^{\eta r}\|g(r)\|_{V'}^2 dr, \\ \rho_{L_V^2}^2(t) &= \frac{1}{\nu}\Big[1 + \|u_\tau\|_H^2 + \frac{8C_f}{\alpha}\|\phi\|_{L_H^2}^2 \\ &\qquad + \frac{\|g(r)\|_{L^2(t-h,t;V')}^2}{\nu} + \frac{8C_f h}{\alpha}\rho_H^2(t)\Big]. \end{aligned}$$

*Moreover, the pullback $\mathcal{D}_F^{M_H}$-absorbing set can be defined as the same technique.*

*Proof.* Using the estimates in Lemma 3.5, choosing any $\hat{D} \in \mathcal{D}_\eta^{M_H}(t)$, there exists a pullback time $\tau(\hat{D}, t) \leq t-h$ such that

$$\|u(t,\tau;u_\tau,\phi)\|_H^2 \leq \rho_H^2(t) = 1 + \frac{e^{-\frac{8\eta C_f}{\alpha}(t-h)}}{\nu-\eta\lambda^{-1}}\int_{-\infty}^t e^{\eta r}\|g(r)\|_{V'}^2 dr \tag{36}$$

holds for any $\tau \leq \tau(\hat{D}, t)$ and $(u_\tau, \phi) \in D(\tau)$. Moreover, in particular, it yields that $\|u_t\|^2_{C_H} \leq \rho^2_H(t)$. By the similar technique and estimate in Lemma 3.5, we derive that $\|u_t\|^2_{L^2_V} \leq \rho^2_{L^2_V}(t)$. Combining the above estimate and the definition of universe, we conclude that $\widehat{D}_0 \in \mathcal{D}^{M_H}_\eta$. The proof has been finished. □

• **Pullback asymptotic compactness**

**Theorem 3.7.** *Assume that $f : \mathbb{R} \times C_H \to H$ and $g \in L^2_{loc}(\mathbb{R}; H)$ satisfying $(H_f), (H_g)$, $(H_1)$ and $(H_0)$ holds. Then, the processes $U(t,\tau) : M_H \to M_H$ generated by the solution of problem (1) is $\mathcal{D}^{M_H}_\eta$-pullback asymptotically compact.*

*Proof.* **Step 1.** Weak convergence of the sequence $\{u^n(t,x)\}$ in the interval $[t-h, t]$ for arbitrary $t \geq \tau$ and weak convergence of $\{u^n(t)\}$ in $H$.

For arbitrary fixed $t \geq \tau$, consider a family $\hat{D} \in \mathcal{D}^{M_H}_\eta$, let $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$ and $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ be two sequences for all $n$, then we denote $\{(u^n, u^n_t)\} \in \hat{D}$ as a sequence with $u^n(\cdot) = u(\cdot; \tau_n, u^{\tau_n}, \phi^n)$.

By using the similar energy estimate in Theorem 3.4 and technique in Proposition 4, there exists a pullback time $\tau(\hat{D}, t) \leq t - 3h - 1$, such that the sequence $\{u^n\}$ with $\tau \leq \tau(\hat{D}, t)$ is bounded in $L^\infty(t-3h-1, t; H) \cap L^2(t-2h-1, t; V) \cap L^3(t-2h-1, t; \mathbb{L}^3(\Omega)) \cap L^4(t-2h-1, t; \mathbb{L}^4(\Omega))$. From the equation, we can check that

$$\begin{aligned}
\|(u^n)'\|_{L^2(t-h-1,t;V')} \quad \leq \quad & \nu\|u^n\|_{L^2(t-h-1,t;V)} + \alpha\lambda_1^{-1}\|u_n\|_{L^2(t-h-1,t;V)} \\
& + \beta\|u^n\|_{L^4(t-h-1,t;L^4(\Omega))} + C_{\lambda_1,|\Omega|}\gamma\|u^n\|_{L^2(t-h-1,t;V)} \\
& + C_\alpha\|f(t,u^n_t)\|_{L^2(t-h-1,t;H)} + \frac{C}{\nu}\|g\|_{L^2(t-h-1,t;V')}. \quad (37)
\end{aligned}$$

From the hypotheses $(H_f)$, $f(t, u^n_t)$ is bounded in $L^2(t-h-1, t; H)$, which implies $\{(u^n)'\}$ is bounded in $L^2(t-h-1, t; V')$. Hence, by the Aubin-Lions Lemma and the diagonal procedure, there exists a subsequence (relabeled also as $\{u^n\}$) such that $u^n(t) \to u(t)$ strongly in $L^2(t-h-1, t; H)$. Combining the uniform boundedness of sequence above, it yields that

$$\begin{cases}
u^n \rightharpoonup u \text{ weakly * } \text{ in } L^\infty(t-3h-1, t; H), \\
u^n \rightharpoonup u \text{ weakly in } L^2(t-2h-1, t; V), \\
(u^n)' \rightharpoonup u' \text{ weakly in } L^2(t-h-1, t; V'), \\
u_m \rightharpoonup u(t) \text{ weakly in } L^3(t-2h-1, t; \mathbb{L}^3(\Omega)), \\
u_m \rightharpoonup u(t) \text{ weakly in } L^4(t-2h-1, t; \mathbb{L}^4(\Omega)), \\
u^n \to u \text{ stongly } \text{ in } L^2(t-h-1, t; H), \\
u^n(s) \to u(s) \text{ stongly } \text{ in } H, \text{ a.e. } s \in (t-h-1, t).
\end{cases} \quad (38)$$

By Theorem 3.4, from the hypothesis on $f$, it follows that

$$f(\cdot, u^n_\cdot) \rightharpoonup f(\cdot, u_\cdot) \text{ weakly } \text{ in } L^2(t-h-1, t; H). \quad (39)$$

Thus, from (38) and (39), we can conclude that $u \in C([t-h-1, t]; H)$ is a weak solution for problem (1) with the initial data of $u(\cdot, x)$ at the initial time $t-h-1$ denoted as $u_{t-h-1}$.

From the uniform bounded estimate of $u^n$ by Proposition 1 in $L^\infty(t-h-1, t; V)$ and $(u^n)'$ is uniform bounded in $L^2(t-h-1, t; V')$, using the Aubin-Lions-Simon Lemma (see [12] ), we can derive that

$$u^n \to u \text{ strongly in } C([t-h-1, t]; H). \quad (40)$$

Therefore, we can conclude that

$$u^n(s_n) \rightharpoonup u(s) \quad \text{weakly in } H \tag{41}$$

for any $\{s_n\} \subset [t-h-1, t]$, $s_n \to s \in [t-h-1, t]$, which implies

$$\liminf_{n\to\infty} \|u^n(s_n)\|_H \geq \|u(s)\|_H. \tag{42}$$

**Step 2.** The strong convergence of corresponding sequences via energy equation method: $u^n(s_n) \to u(s)$ strongly in $C([t-h,t]; H)$.

The asymptotic compactness of sequence $u^n$ in $H$ will be presented in sequel, i.e.,

$$\|u^n(s_n) - u(s)\|_H \to 0 \text{ as } n \to +\infty, \tag{43}$$

which is equivalent to prove (42) combining with

$$\limsup_{n\to\infty} \|u^n(s_n)\|_H \leq \|u(s)\|_H \tag{44}$$

for a sequence $\{s_n\} \subset [t-h, t]$ and $s_n \to s$ as $n \to +\infty$, which will be proved next.

Using the energy estimate to all $u^n$ and $u$, we obtain that for all $t-h-1 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} &\|u^n(s_2)\|_H^2 + \nu \int_{s_1}^{s_2} \|u^n(r)\|_V^2 dr + 2\beta \int_{s_1}^{s_2} \|u^n(r)\|_{\mathbb{L}^4(\Omega)}^3 dr + 2\gamma \int_{s_1}^{s_2} \|u^n(r)\|_{\mathbb{L}^4(\Omega)}^4 \\ \leq\ & \frac{2C_f^2}{\alpha} \int_{s_2}^{s_2} \|u_r^n\|_H^2 dr + \frac{8}{\nu} \int_{s_1}^{s_2} \|g(r)\|_{V'}^2 dr \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\|u(s_2)\|_H^2 + \nu \int_{s_1}^{s_2} \|u(r)\|_V^2 dr + 2\beta \int_{s_1}^{s_2} \|u(r)\|_{\mathbb{L}^4(\Omega)}^3 dr + 2\gamma \int_{s_1}^{s_2} \|u(r)\|_{\mathbb{L}^4(\Omega)}^4 \\ \leq\ & \frac{2C_f^2}{\alpha} \int_{s_2}^{s_2} \|u_r\|_H^2 dr + \frac{8}{\nu} \int_{s_1}^{s_2} \|g(r)\|_{V'}^2 dr. \end{aligned} \tag{46}$$

Then, we define the functionals $J_n(s)$ and $J(s)$ defined for $s \in [t-h-1, t]$ as following

$$J_n(s) = \frac{1}{2}\|u^n\|_H^2 - \int_{t-h-1}^{s} \langle g(r), u^n(r)\rangle dr - \int_{t-h-1}^{s} (f(r, u_r^n), u^n(r)) dr \tag{47}$$

and

$$J(t) = \frac{1}{2}\|u(s)\|_H^2 - \int_{t-h-1}^{s} \langle g(r), u(r)\rangle dr - \int_{t-h-1}^{s} (f(r, u_r), u(r)) dr. \tag{48}$$

Combining the convergence in (38), observing that $J_n(s)$ and $J(s)$ are continuous and non-increasing in $[t-h-1, t]$, we derive that

$$\int_{t-h-1}^{t} \langle g(r), u^n(r)\rangle dr \to 2\int_{t-h-1}^{t} \langle g(r), u(r)\rangle dr \tag{49}$$

and

$$\int_{t-h-1}^{t} (f(r, u_r^n), u^n(r)) dr \to 2\int_{t-h-1}^{t} (f(r, u_r), u(r)) dr \tag{50}$$

as $n \to +\infty$, which implies that

$$J_n(s) \to J(s) \quad \text{a.e.} s \in (t-h-1, t), \tag{51}$$

i.e., for $\forall\ \varepsilon > 0$, there exists a $n_k \in \mathbb{N}$, for all $n \geq n_k$ and $s_k \in [t-h-1, t]$, such that

$$|J_n(s_k) - J(s_k)| \leq \frac{\varepsilon}{2}. \tag{52}$$

Since $J(s)$ is continuous and $J_n(s)$ is uniformly continuous with respect to time $s$, then for any $\varepsilon > 0$, there exists $\tilde{n}_k \in \mathbb{N}$ such that for the sequence $\{s_k\} \subset [t-h-1, t]$ with $s_k \to s$ for all $n \geq \tilde{n}_k$,

$$|J(s_k) - J(s)| \leq \frac{\varepsilon}{2}, \tag{53}$$

Choosing $\bar{n}_k = \max\{n_k, \tilde{n}_k\}$, then for all $n > \bar{n}_k$, it yields that

$$|J_n(s_n) - J(s)| \leq |J_n(s_n) - J(s_n)| + |J(s_n) - J(s)| < \varepsilon. \tag{54}$$

Therefore, for any $\{s_n\} \subset [t-h-1, t]$, we have

$$\limsup_{n\to\infty} J_n(s_n) \leq J(s), \tag{55}$$

which implies

$$\limsup_{n\to\infty} \|u^n(s_n)\|_H \leq \|u(s)\|_H. \tag{56}$$

we conclude the strong convergence $u^n(s_n) \to u(s)$ in $C([t-h, t]; H)$.

**Step 3.** The strong convergence: $u^n(s_n) \to u(s)$ strongly in $L^2(t-h, t; V)$.

Combining the energy estimates in (45) and (46), noting the energy functionals $J_n(\cdot)$ and $J(\cdot)$, using the convergence in (38), we can deduce the norm convergence

$$\|u^n(s)\|_{L^2(t-h,t;V)} \to \|u(s)\|_{L^2(t-h,t;V)}. \tag{57}$$

Hence jointing with the weak convergence in (38), we can derive that $u^n(s_n) \to u(s)$ strongly in $L^2(t-h, t; V)$.

**Step 4.** The $\mathcal{D}_\eta^{M_H}$-pullback asymptotic compactness.

By using the results from Steps 2 to 4 and noting the definition of universe, we can conclude that the processes is $\mathcal{D}_\eta^{M_H}$-pullback asymptotic compact in $M_H$, which means the proof has been finished. □

**Remark 3.** Using the similar technique, we can derive the processes $U(t,\tau) : M_H \to M_H$ generated by the solution of problem (1) is $\mathcal{D}_F^{M_H}$-pullback asymptotic compact.

**Theorem 3.8.** *Assume that $f : \mathbb{R} \times C_H \to H$ and $g \in L^2_{loc}(\mathbb{R}; H)$ satisfying $(H_f), (H_g)$, $(H_1)$ and $(H_0)$ holds. Then, the process $U(t,\tau) : M_H \to M_H$ generated by the solution of problem (1) possess the minimal pullback attractors $\mathcal{A}_{\mathcal{D}_\eta^{M_H}}(t)$ and $\mathcal{A}_{\mathcal{D}_F^{M_H}}(t)$ in $M_H$, which satisfy the following relation*

$$\mathcal{A}_{\mathcal{D}_F^{M_H}}(t) \subset \mathcal{A}_{\mathcal{D}_\eta^{M_H}}(t). \tag{58}$$

*Proof.* From Proposition 3, we observe that the process $U(t,\tau)$ is continuous in $M_H$. The $\mathcal{D}_F^{M_H}$ and $\mathcal{D}_\eta^{M_H}$ pullback absorbing sets are established by Proposition 4. By Theorems 3.7 and Remark 3 give the $\mathcal{D}_\eta^{M_H}$ and $\mathcal{D}_F^{M_H}$ pullback asymptotic compactness of the processes. Using the existence theory of pullback attractors in [3] or [4], we can conclude our desired results.

Based on the universes defined in Definition 3.6, the relation between $\mathcal{A}_{\mathcal{D}_F^{M_H}}(t)$ $\mathcal{A}_{\mathcal{D}_\eta^{M_H}}(t)$ holds easily. □

3.5. **Main result: The sufficient condition of asymptotic stability of trajectories inside pullback attractors of (1) with $f(t,u_t)$.**

**Definition 3.9.** The pullback attractors is asymptotically stable if the trajectories inside attractor reduces to a single orbit as $\tau \to -\infty$.

**Theorem 3.10.** *Assume that $2\nu\lambda_1 - \frac{L_f^2}{\alpha} > 0$, the external forces $g \in L^2_{loc}(\mathbb{R};H)$ and $f(t,u_t)$ satisfy the hypothesis $(H_f)$, $(H_g)$ and $(H_1)$, the initial data $(\phi, u_\tau) \in M_H$ and $(H_0)$ holds. Then the trajectories inside pullback attractors $\mathcal{A}_{\mathcal{D}_\eta^{M_H}}(t)$ is asymptotically stable if*

$$G(t) \le K_0,$$

*where $G^2(t) = \frac{\langle \|g\|_H^2 \rangle|_{\le t}}{\nu^2\lambda_1}$ is a generalized Grashof number for the fluid flow, and*

$$K_0 = \Big\{[\nu^2\lambda_1(2\nu\lambda_1+\alpha)]\Big/\Big[4C_{|\Omega|}\beta\Big(\frac{L_f^2}{\alpha}\frac{2-\frac{L_f^2}{\alpha}}{1-\frac{2L_f^2}{\alpha}}+\frac{1}{\alpha}\Big)\Big]\Big\}^{1/2},$$

*here $C_{|\Omega|} > 0$ is a constant which depends on the volume of $\Omega$.*

*Proof.* Let $u(t)$ and $v(t)$ be two weak solutions of problem (3) with delay $f(t,u_t)$ which subject to initial data

$$u(\tau+\theta)|_{\theta\in[-h,0]} = \phi(\theta), \quad u|_{t=\tau} = u_\tau \tag{59}$$

and

$$v(\tau+\theta)|_{\theta\in[-h,0]} = \tilde{\phi}(\theta), \quad v|_{t=\tau} = \tilde{u}_\tau \tag{60}$$

respectively. Denoting

$$(u,u_t) = U(t,\tau)(u_\tau,\varphi) \ \text{ and } \ (v,v_t) = U(t,\tau)(\tilde{u}_\tau,\tilde{\varphi}) \tag{61}$$

as two trajectories inside the pullback attractors, letting $w = u(t) - v(t)$ and $w_t = u_t - v_t$, then it is easy to check that $w$ satisfies the following problem

$$\begin{cases} \frac{\partial w}{\partial t} + \nu A w + P\Big(\alpha w + \beta(|u|u - |v|v) + \gamma(|u|^2u - |v|^2v)\Big) \\ \qquad\qquad\qquad\qquad = P(f(t,u_t) - f(t,v_t)), \\ w|_{\partial\Omega} = 0, \\ w(t=\tau) = u_\tau - \tilde{u}_\tau, \\ w(\tau+\theta) = \phi(\theta) - \tilde{\phi}(\theta), \ \theta\in[-h,0]. \end{cases} \tag{62}$$

Taking inner product of (62) with $w$ in $H$, using Poincaré's inequality and Lemma 3.3, it follows

$$\gamma(|u|^2u - |v|^2v, u-v) \ge \gamma\gamma_0\|u-v\|^4_{\mathbf{L}^4} \tag{63}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w\|_H^2 + \nu\|w\|_V^2 + \alpha\|w\|_H^2 + \gamma\gamma_0\|w\|^4_{\mathbf{L}^4} \\ \le\ & \Big|\beta(|u|u - |v|v, w)\Big| + \Big|(f(t,u_t) - f(t,v_t), w)\Big| \\ \le\ & \beta\Big(\int_\Omega |u|^2|w|dx + \int_\Omega |w||v|^2 dx\Big) + \frac{\alpha}{2}\|w\|_H^2 + \frac{L_f^2}{2\alpha}\|w_t\|_H^2 \end{aligned}$$

$$\begin{aligned} &\leq \beta(\|u\|_{\mathbf{L}^4}^2 + \|v\|_{\mathbf{L}^4}^2)\|w\|_H^2 + \frac{\alpha}{2}\|w\|_H^2 + \frac{L_f^2}{2\alpha}\|w_t\|_H^2 \\ &\leq C_{|\Omega|}\beta(\|u\|_V^2 + \|v\|_V^2)\|w\|_H^2 + \frac{\alpha}{2}\|w\|_H^2 + \frac{L_f^2}{2\alpha}\|w_t\|_H^2. \end{aligned} \tag{64}$$

Using the Poincaré inequality and Lemma 3.1, noting that if

$$2\nu\lambda_1 + \alpha - 2C_{|\Omega|}\beta(\|u\|_V^2 + \|v\|_V^2) > 0, \tag{65}$$

then we can obtain

$$\begin{aligned} \|w\|_H^2 \quad &\leq \quad e^{\int_\tau^t [2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)-(2\nu\lambda_1+\alpha)]ds}\Big[\|u_\tau - \tilde{u}_\tau\|_H^2 + \\ &+\frac{L_f^2}{\alpha}\int_\tau^t e^{-\int_s^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]d\sigma}\|w_t\|_H^2 ds\Big]. \end{aligned} \tag{66}$$

Denoting

$$E(t,\tau) = e^{-\int_\tau^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]ds} \tag{67}$$

and

$$K_1(t,s) = \frac{L_f^2}{\alpha}e^{-\int_s^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]d\sigma} \tag{68}$$

and

$$\Theta = \sup_{t\geq s\geq\tau} E(t,s), \qquad \kappa(K_1,0) = \sup_{t\geq\tau}\int_\tau^t K_1(t,s)ds, \tag{69}$$

by virtue of Lemma 3.2, choosing $\kappa(K_1,0) < \frac{1}{1+\Theta}$, then there exists $M > 0$ and $\lambda > 0$, such that we can obtain the estimate

$$\|w_t\|_H^2 \quad \leq \quad M\|u_\tau - \tilde{u}_\tau\|_H^2 e^{-\lambda(t-\tau)}. \tag{70}$$

Substituting (70) into (64), using Lemma 3.1 again, we can conclude the following estimate

$$\begin{aligned} \|w\|_H^2 \quad \leq \quad &\|u_\tau - \tilde{u}_\tau\|_H^2 e^{-\int_\tau^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]ds} \\ &+\frac{L_f^2}{\alpha}M\|u_\tau - \tilde{u}_\tau\|_H^2 e^{-\lambda(t-\tau)}\int_\tau^t e^{-\int_s^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]d\sigma}ds. \end{aligned} \tag{71}$$

From (70) and (71), if we fixed $u_\tau$ and $\tilde{u}_\tau$ and let $\tau \to -\infty$, then we can conclude that the trajectories inside pullback attractors reduce to a point, which implies the pullback attractors is asymptotically stable provided that

$$2\nu\lambda_1 + \alpha > 2C_{|\Omega|}\beta\langle\|u\|_V^2 + \|v\|_V^2\rangle_{\leq t}, \tag{72}$$

where

$$\langle h\rangle_{\leq t} = \limsup_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t h(r)dr. \tag{73}$$

Since $u$ and $v$ are the global weak solutions for problem (3), we will use some delicate estimates to make (72) more explicit next. Multiplying (3) with $u$ and integrating by parts over $\Omega$, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|u\|_H^2 + \nu\|A^{1/2}u\|_H^2 + \alpha\|u\|_H^2 + \beta\|u\|_{\mathbf{L}^3}^3 + \gamma\|u\|_{\mathbf{L}^4}^4 \\ \leq\ & \alpha\|u\|_H^2 + \frac{1}{2\alpha}\Big[\|f(t,u_t)\|_H^2 + \|g\|_H^2\Big] \\ \leq\ & \alpha\|u\|_H^2 + \frac{L_f^2}{2\alpha}\|u_t\|_H^2 + \frac{1}{2\alpha}\|g\|_H^2. \end{aligned} \tag{74}$$

Using the Poincaré inequality and Lemma 3.1, then we can obtain

$$\begin{aligned} \|u\|_H^2 \leq\ & e^{-2\nu\lambda_1(t-\tau)}\|u_\tau\|_H^2 + \\ & + \frac{L_f^2}{\alpha}\int_\tau^t e^{-2\nu\lambda_1(t-s)}\|u_s\|_H^2 ds + \frac{1}{\alpha}\int_\tau^t e^{-2\nu\lambda_1(t-s)}\|g\|_H^2 ds. \end{aligned} \tag{75}$$

Denoting

$$E(t,\tau) = e^{-2\nu\lambda_1(t-\tau)} \tag{76}$$

and

$$K_1(t,s) = \frac{L_f^2}{\alpha}e^{-2\nu\lambda_1(t-s)} \tag{77}$$

and

$$\rho = \frac{1}{\alpha}\int_\tau^t e^{-2\nu\lambda_1(t-s)}\|g\|_H^2 ds, \tag{78}$$

letting

$$\Theta = \sup_{t\geq s\geq\tau} E(t,s), \qquad \kappa(K_1,0) = \sup_{t\geq\tau}\int_\tau^t K_1(t,s)ds, \tag{79}$$

by virtue of Lemma 3.2, choosing $\kappa(K_1,0) < \frac{1}{1+\Theta}$, then there exists $\hat{M} > 0$ and $\hat{\lambda} > 0$, such that we can obtain the estimate

$$\begin{aligned} \|u_t\|_H^2 \leq\ & \hat{M}\|u_\tau\|_H^2 e^{-\lambda(t-\tau)} + \frac{2 - \frac{L_f^2}{\alpha}}{1 - \frac{2L_f^2}{\alpha}}\int_\tau^t e^{-2\nu\lambda_1(t-s)}\|g\|_H^2 ds \\ \leq\ & \hat{M}\|u_\tau\|_H^2 e^{-\lambda(t-\tau)} + \frac{2 - \frac{L_f^2}{\alpha}}{1 - \frac{2L_f^2}{\alpha}}\int_\tau^t \|g\|_H^2 ds. \end{aligned} \tag{80}$$

Substituting (80) into (75), using Lemma 3.1 again, we can conclude the following estimate

$$\|u\|_H^2 \leq C\|u_\tau\|_H^2 e^{-\lambda(t-\tau)} + \Big(\frac{L_f^2}{\alpha}\frac{2 - \frac{L_f^2}{\alpha}}{1 - \frac{2L_f^2}{\alpha}} + \frac{1}{\alpha}\Big)\int_\tau^t \|g\|_H^2 ds. \tag{81}$$

Integrating (74) from $\tau$ to $t$, it follows

$$\begin{aligned} & \|u\|_H^2 + 2\nu\int_\tau^t \|u\|_V^2 ds + 2\beta\int_\tau^t \|u\|_{\mathbf{L}^3}^3 ds + 2\gamma\int_\tau^t \|u\|_{\mathbf{L}^4}^4 ds \\ \leq\ & \Big[\frac{1}{\alpha}\|\phi\|_{L_H^2}^2 + \|u_\tau\|_H^2\Big] + \frac{L_f^2}{\alpha}\int_\tau^t \|u_t(s)\|_H^2 ds + \frac{1}{\alpha}\int_\tau^t \|g\|_H^2 ds. \end{aligned} \tag{82}$$

By the estimate of (80) and (81), we derive

$$\int_\tau^t \|u(r)\|_{\mathbf{L}^4}^4 dr \le C\Big[\frac{1}{\alpha}\|\phi\|_{L_H^2}^2 + \|u_\tau\|_H^2\Big] + \Big(\frac{L_f^2}{\alpha}\frac{2-\frac{L_f^2}{\alpha}}{1-\frac{2L_f^2}{\alpha}} + \frac{1}{\alpha}\Big)\int_\tau^t \|g\|_H^2 ds \tag{83}$$

and

$$\int_\tau^t \|u(r)\|_V^2 dr \le C\Big[\frac{1}{\alpha}\|\phi\|_{L_H^2}^2 + \|u_\tau\|_H^2\Big] + \Big(\frac{L_f^2}{\alpha}\frac{2-\frac{L_f^2}{\alpha}}{1-\frac{2L_f^2}{\alpha}} + \frac{1}{\alpha}\Big)\int_\tau^t \|g\|_H^2 ds. \tag{84}$$

Combining (72), (73) with (84), we conclude that

$$\langle \|u\|_V^2 + \|v\|_V^2\rangle|_{\le t} \le 2\Big(\frac{L_f^2}{\alpha}\frac{2-\frac{L_f^2}{\alpha}}{1-\frac{2L_f^2}{\alpha}} + \frac{1}{\alpha}\Big)\langle\|g\|_H^2\rangle|_{\le t} \tag{85}$$

and hence the asymptotic stability holds provided that

$$4C_{|\Omega|}\beta\Big(\frac{L_f^2}{\alpha}\frac{2-\frac{L_f^2}{\alpha}}{1-\frac{2L_f^2}{\alpha}} + \frac{1}{\alpha}\Big)\langle\|g\|_H^2\rangle|_{\le t} \le 2\nu\lambda_1 + \alpha. \tag{86}$$

If we define the generalized Grashof number as $G(t) = \Big(\frac{\langle\|g\|_H^2\rangle|_{\le t}}{\nu^2\lambda_1}\Big)^{1/2}$, then a sufficient condition for the asymptotic stability of trajectories inside pullback attractors can be conclude as

$$G(t) \le \Big\{(2\nu\lambda_1 + \alpha)\Big/\Big[4C_{|\Omega|}\nu^2\beta\lambda_1\Big(\frac{L_f^2}{\alpha}\frac{2-\frac{L_f^2}{\alpha}}{1-\frac{2L_f^2}{\alpha}} + \frac{1}{\alpha}\Big)\Big]\Big\}^{1/2} = K_0, \tag{87}$$

which completes the proof for our first result. □

**Remark 4.** Theorem 3.10 is a further research for the existence of pullback attractor in [6].

## 4. Application: The asymptotic stability of trajectories inside pullback attractors for (1) with sub-linear operator $f(t, u_t) = F(u(t - \rho(t)))$.

### 4.1. Assumptions.

We first state some hypothesis on the external forces and sub-linear operator.

$(\mathbf{H_f})$ The function $f(t, u_t) = F(u(t-\rho(t)))$ satisfies the following assumptions.

$(\mathbf{H_f}) - (\mathbf{a})$ The function $\rho \in C^1([0, +\infty); [0, h])$, and there exists a constant $\rho^*$ satisfying

$$\Big|\frac{d\rho}{dt}\Big| \le \rho^* < 1, \quad \forall t \ge 0.$$

$(\mathbf{H_f}) - (\mathbf{b})$ The external force $f(\cdot, y) : [\tau, +\infty) \times H \to H$ is measurable for all $y \in H$ and $f(\cdot, 0) = 0$. In addition, there exist functions $a, b : [\tau, +\infty) \to [0, +\infty)$, with $a \in L^q_{loc}(\mathbb{R})$ and $b \in L^1_{loc}(\mathbb{R})$ for all $T \ge \tau$ and $1 \le q \le +\infty$ with $\limsup_{\tau\to-\infty}\int_\tau^t b(s)ds = \tilde{b}_0 \in (0, +\infty)$, such that

$$\|F(y)\|_H^2 \le a(t)\|y\|_H^2 + b(t), \quad \forall t \ge \tau, y \in H. \tag{88}$$

$(\mathbf{H_f}) - (\mathbf{c})$ There exist $\kappa \in L^\infty(\tau, T)$, $L(R) > 0$ and $R > 0$, such that

$$\|F(u) - F(v)\|_H \le L(R)\kappa^{\frac{1}{2}}(t)\|u - v\|_H, \ u, v \in H. \tag{89}$$

holds for $\|u\|_H \le R, \|v\|_H \le R$ and $t \in [\tau, T]$.

($\mathbf{H_g}$) The function $g(\cdot,\cdot) \in L^2_{loc}(\mathbb{R}, H)$, which satisfies that there exists a $m > 0$, such that

$$\int_{-\infty}^{t} e^{ms}\|g(s,\cdot)\|_H^2 ds < \infty, \quad \forall t \in \mathbb{R}. \tag{90}$$

($\widetilde{\mathbf{H}}_\mathbf{0}$) When $a \in L^q_{loc}(\mathbb{R})$, it holds

$$\frac{\nu}{2} - \frac{\|a\|_{L^q_{loc}(\mathbb{R})}}{1-\rho^*} > 0. \tag{91}$$

4.2. **Well-posedness and pullback attractors of (1) with $f(t, u_t) = F(u(t-\rho(t)))$.** In this part, the well-posedness and pullback attractors for problem (1) with sub-linear operator will be stated for our discussion in sequel.

- **Well-posedness**

Assume that the initial date $u_\tau \in H$ and $\phi \in C_H \cap L^{2q'}(-h, 0; H)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ and recall that (1) with sub-linear operator has the following abstract form:

$$\begin{cases} u(t) + \int_\tau^t P(\nu Au + \alpha u + \beta|u|u + \gamma|u|^2 u)ds \\ \qquad\qquad = u(\tau) + \int_\tau^t P\Big(F\big(u(s-\rho(s))\big) + g(s,x)\Big)ds, \\ w|_{\partial\Omega} = 0, \\ u(t=\tau) = u_\tau, \\ u(\tau+t) = \phi(t), \ t \in [-h, 0], \end{cases} \tag{92}$$

which possesses a global mild solution as the following theorem.

**Theorem 4.1.** *Assume that the external forces $g(t)$ and $f(t, u_t) = F(u(t-\rho(t)))$ satisfy the hypothesis ($\mathbf{H_f}$) and ($\mathbf{H_g}$), the initial data $u_\tau \in H$ and $\phi \in C_H \cap L^{2q'}_H$ and ($\widetilde{\mathbf{H}}_\mathbf{0}$) are also true. Then there exists a unique global mild solution $u = u(\cdot, \tau, u_\tau, \phi) \in L^\infty(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; \mathbf{L}^4(\Omega))$ of problem (1) for special case of $f(t, u_t) = F(t-\rho(t))$, such that it satisfies (92) in distributed sense and the following energy equality*

$$\begin{aligned} &\|u(t)\|_H^2 + 2\nu\int_\tau^t \|u(s)\|_V^2 ds + 2\alpha\int_\tau^t \|u(s)\|_H^2 ds \\ &+2\beta\int_\tau^t \|u(s)\|_{\mathbf{L}^3}^3 ds + 2\gamma\int_\tau^t \|u(s)\|_{\mathbf{L}^4}^4 ds \\ = \ &\|u_\tau\|_H^2 + 2\int_\tau^t \Big[\big(F(u(s-\rho(s))), u(s)\big) + 2(g(s,x), u(s))\Big]ds. \end{aligned} \tag{93}$$

*Moreover, we can define a continuous process $\{U(t,\tau)|t \geq \tau, \tau \in \mathbb{R}\}$ as $U(t,\tau) : (C_H \cap L^{2q'}_H) \times H \to C_H \times H$, i.e., $U(t,\tau)(\phi, u_\tau) = (u_t, u)$.*

*Proof.* Using the Galerkin method and compact argument as in Section 3.3, we can easily derive the result. □

- **The pullback dynamics**

After obtaining the existence of the global well-posedness, we establish the existence of the pullback attractors to (1) with sub-linear operator.

**Theorem 4.2.** *(The pullback attractors in $H$) Assume that ($\mathbf{H_f}$) and ($\mathbf{H_g}$) hold, the initial data $u_\tau \in H$ and $\phi \in C_H \cap L^{2q'}_H$ and ($\widetilde{\mathbf{H}}_\mathbf{0}$) are also true. then the process*

*$U(t,\tau)$ associated to problem (1) with $f(t,u_t) = F(u(t-\rho(t)))$ has two families of pullback attractors $\mathcal{A}_{C_H\times H}(t)$ similar as in Theorem 3.7.*

*Proof.* Using the similar technique as in Section3.3, we can obtain the existence of pullback attractors, here we skip the details. □

### 4.3. The asymptotic stability of trajectories inside pullback attractors of (1) with special case $f(t,u_t) = F(u(t-\rho(t)))$.

**Theorem 4.3.** *We assume that the external forces $g(t)$ and $f(t,u_t) = F(u(t-\rho(t)))$ satisfy the hypothesis $(\mathbf{H_f})$ and $(\mathbf{H_g})$, the initial data $(\phi, u_\tau) \in (C_H \cap L_H^{2q'}) \times H$ and $(\widetilde{\mathbf{H}}_\mathbf{0})$ holds true.*

*Then the trajectories inside pullback attractors $\mathcal{A}_{C_H}$ is asymptotically stable if*

$$G(t) \le \tilde{K}_0, \tag{94}$$

*where $G^2(t) = \frac{\langle \|g\|_H^2\rangle|_{\le t}}{\nu^2\lambda_1}$ is defined as the generalized Grashof number for the fluid flow, $\langle \|g\|_H^2\rangle|_{\le t} = \lim\limits_{\tau\to-\infty}\dfrac{1}{t-\tau}\displaystyle\int_\tau^t \|g(s)\|_H^2 ds$ and*

$$\tilde{K}_0 = \Big\{(2\nu\lambda_1+\alpha)\Big/\Big[2C_{|\Omega|}\beta\nu\lambda_1\big(\frac{1}{\alpha^2(1-\rho^*)} + \|\tilde{a}(t)\|_{L^1}\big)\Big]\Big\}^{1/2} > 0,$$

*here $C_{|\Omega|} > 0$ is a constant dependent on the volume of $\Omega$.*

*Proof.* **Step 1.** The inequality for asymptotic stability of trajectories.

Let $u(t)$ and $v(t)$ be two solutions of problem (92) with initial data

$$u(\theta+\tau)|_{\theta\in[-h,0]} = \phi(\theta)|_{\theta\in[-h,0]}, \quad u|_{t=\tau} = u_\tau \tag{95}$$

and

$$v(\theta+\tau)|_{\theta\in[-h,0]} = \tilde{\phi}(\theta)|_{\theta\in[-h,0]}, \quad v|_{t=\tau} = \tilde{u}_\tau \tag{96}$$

respectively, then $u, v$ can be represented by

$$(u, u_t) = (U(t,\tau)u_\tau, U(t,\tau)\phi), \quad (v, v_t) = (U(t,\tau)\tilde{u}_\tau, U(t,\tau)\tilde{\phi}). \tag{97}$$

If we denote $w = u(t) - v(t)$ and $w(t-\rho(t)) = u(t-\rho(t)) - v(t-\rho(t))$, then it is easy to check that $w$ satisfies the following problem

$$\begin{cases} \frac{\partial w}{\partial t} + \nu Aw + P\Big(\alpha w + \beta(|u|u - |v|v) + \gamma(|u|^2u - |v|^2 v)\Big) \\ \qquad\qquad\qquad = P\Big(F\big(u(t-\rho(t))\big) - F\big(v(t-\rho(t))\big)\Big), \\ w|_{\partial\Omega} = 0, \\ w(t=\tau) = u_\tau - \tilde{u}_\tau, \\ w(\tau+\theta) = \phi(\theta) - \tilde{\phi}(\theta), \ \theta\in[-h,0]. \end{cases} \tag{98}$$

Multiplying (98) with $w$ and integrating by parts in $\Omega$, using the Poincaré and Young inequalities, from (63) we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_H^2+\nu\|w\|_V^2+\alpha\|w\|_H^2+\gamma\gamma_0\|w\|_{\mathbf{L}^4}^4\\
\leq\ & |\beta(|u|u-|v|v,w)|+\Big|\Big(F\big(u(t-\rho(t))\big)-F\big(v(t-\rho(t))\big),w\Big)\Big|\\
\leq\ & C_{|\Omega|}\beta(\|u\|_V^2+\|u\|_V^2)\|w\|_H^2+\frac{\alpha}{2}\|w\|_H^2\\
&+\frac{1}{\alpha}\|F\big(u(t-\rho(t))\big)-F\big(v(t-\rho(t))\big)\|_H^2\\
\leq\ & C_{|\Omega|}\beta(\|u\|_V^2+\|u\|_V^2)\|w\|_H^2+\frac{\alpha}{2}\|w\|_H^2\\
&+\frac{L^2(R)\kappa(t)}{\alpha}\|w(t-\rho(t))\|_H^2.
\end{aligned}\tag{99}$$

Using the Poincaré inequality and Lemma 3.1, noting that if

$$2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)>0,\tag{100}$$

then we can obtain

$$\begin{aligned}
\|w\|_H^2\ \leq\ & e^{\int_\tau^t[2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)-(2\nu\lambda_1+\alpha)]ds}\Big[\|u_\tau-\tilde{u}_\tau\|_H^2+\\
&+\frac{L^2(R)\|\kappa(t)\|_{L^\infty}}{\alpha}\int_\tau^t e^{-\int_s^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]d\sigma}\\
&\times\|w(t-\rho(t))\|_H^2ds\Big].
\end{aligned}\tag{101}$$

Denoting

$$E(t,\tau)=e^{-\int_\tau^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]ds}\tag{102}$$

and

$$K_1(t,s)=\frac{L^2(R)\|\kappa(t)\|_{L^\infty}}{\alpha}e^{-\int_s^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]d\sigma}\tag{103}$$

and

$$\Theta=\sup_{t\geq s\geq\tau}E(t,s),\qquad \kappa(K_1,0)=\sup_{t\geq\tau}\int_\tau^t K_1(t,s)ds,\tag{104}$$

by virtue of Lemma 3.2, choosing $\kappa(K_1,0)<\frac{1}{1+\Theta}$, then there exists $\tilde{M}>0$ and $\tilde{\lambda}>0$, such that we can obtain the estimate

$$\|w(t-\rho(t))\|_H^2\ \leq\ \tilde{M}\|u_\tau-\tilde{u}_\tau\|_H^2e^{-\tilde{\lambda}(t-\tau)}.\tag{105}$$

Substituting (105) into (99), using Lemma 3.1 again, we can conclude the following estimate

$$\begin{aligned}
\|w\|_H^2\ \leq\ & \|u_\tau-\tilde{u}_\tau\|_H^2e^{-\int_\tau^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]ds}\\
&+\frac{L^2(R)\|\kappa(t)\|_{L^\infty}}{\alpha}\tilde{M}\|u_\tau-\tilde{u}_\tau\|_H^2e^{-\tilde{\lambda}(t-\tau)}\\
&\times\int_\tau^t e^{-\int_s^t[2\nu\lambda_1+\alpha-2C_{|\Omega|}\beta(\|u\|_V^2+\|v\|_V^2)]d\sigma}ds.
\end{aligned}\tag{106}$$

From the result in last section, we can find that the pullback attractors is asymptotically stable as $\tau\to-\infty$ provided that

$$2\nu\lambda_1+\alpha>2C_{|\Omega|}\beta\langle\|u\|_V^2+\|v\|_V^2\rangle_{\leq t}.\tag{107}$$

**Step 2.** Some energy estimate for (1) with sub-linear operator.

Multiplying (3) with $u$ in $H$, and then integrating by parts over $\Omega$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|_H^2+\nu\|A^{1/2}u\|_H^2+\alpha\|u\|_H^2+\beta\|u\|_{\mathbf{L}^3}^3+\gamma\|u\|_{\mathbf{L}^4}^4\\ \leq\ &\alpha\|u\|_H^2+\frac{1}{2\alpha}\Big[\|f\big(t,u(t-\rho(t))\big)\|_H^2+\|g\|_H^2\Big].\end{aligned}\tag{108}$$

Moreover, let $\theta=s-\rho(s)$, then it yields

$$d\theta=(1-\rho'(s))ds,\ \ a(t)\to\tilde{a}(\bar{t})\in L^p(\tau,T),\tag{109}$$

which means $\tilde{a}\in L^q(\tau-h,\tau)$ and

$$\begin{aligned}&\int_\tau^t\|f(s,u(s-\rho(s)))\|_H^2ds\\ \leq&\int_\tau^t a(s)\|u(s-\rho(s))\|_H^2ds+\int_\tau^T b(s)ds\\ \leq&\frac{1}{1-\rho^*}\int_{\tau-\rho(\tau)}^{t-\rho(t)}\tilde{a}(s)\|u(s)\|_H^2ds+\int_\tau^t b(s)ds\\ \leq&\frac{1}{1-\rho^*}\left(\int_{-\rho(\tau)}^0\tilde{a}(t+\tau)\|\phi(t)\|_H^2dt+\int_\tau^t\tilde{a}(s)\|u(s)\|_H^2ds\right)+\int_\tau^t b(s)ds\\ \leq&\frac{1}{1-\rho^*}\left(\|\phi(t)\|_{L_H^{2q}}^2\|\tilde{a}\|_{L^q(\tau-h,\tau)}+\int_\tau^t\tilde{a}(s)\|u(s)\|_H^2ds\right)+\int_\tau^t b(s)ds,\end{aligned}\tag{110}$$

Integrating (108) with time variable from $\tau$ to $t$, we conclude that

$$\begin{aligned}&\|u\|_H^2+2\nu\int_\tau^t\|u\|_V^2ds+2\beta\int_\tau^t\|u\|_{\mathbf{L}^3}^3ds+2\gamma\int_\tau^t\|u\|_{\mathbf{L}^4}^4ds\\ \leq\ &\frac{\|\tilde{a}\|_{L^q(\tau-h,\tau)}}{\alpha(1-\rho^*)}\|\phi(t)\|_{L_H^{2q}}^2+\|u_\tau\|_H^2+\frac{1}{\alpha(1-\rho^*)}\int_\tau^t\tilde{a}(s)\|u(s)\|_H^2ds\\ &+\frac{1}{\alpha}\int_\tau^t\|g\|_H^2ds+\frac{1}{\alpha}\int_\tau^t b(s)ds,\end{aligned}\tag{111}$$

then we can achieve that

$$\begin{aligned}\|u(t)\|_H^2\ \ \leq\ \ &\Big[\frac{\|\tilde{a}\|_{L^q(\tau-h,\tau)}}{\alpha(1-\rho^*)}\|\phi(t)\|_{L_H^{2q}}^2+\|u_\tau\|_H^2\Big]e^{-\chi_\sigma(t,\tau)}\\ &+\frac{1}{\alpha}\int_\tau^t\|g\|_H^2e^{-\chi_\sigma(t,s)}ds+\frac{1}{\alpha}\int_\tau^t b(s)e^{-\chi_\sigma(t,s)}ds,\end{aligned}\tag{112}$$

where the new variable index $\chi_\sigma(\cdot,\cdot)$ is introduced as

$$\chi_\sigma(t,s)=(2\nu\lambda_1-\sigma)(t-s)-\frac{1}{\alpha(1-\rho^*)}\int_s^t\tilde{a}(r)dr,\tag{113}$$

which satisfies the relations

$$\chi_\sigma(0,t)-\chi_\sigma(0,s)=-\chi_\sigma(t,s)\tag{114}$$

and

$$\chi_\sigma(0,r)\leq\chi_\sigma(0,t)+\Big(2\nu\lambda_1-\delta\Big)h,\quad\text{if }2\nu\lambda_1+\alpha-\delta>0\tag{115}$$

for $r\in[t-h,t]$.

Moreover, using the variable index introduced above, we can conclude that

$$\begin{aligned} & 2\nu \int_\tau^t \|u(r)\|_V^2 dr \\ \leq\ & \frac{\|\tilde{a}\|_{L^q(\tau-h,\tau)}}{\alpha(1-\rho^*)}\|\phi(t)\|^2_{L^{2q}_H} + \|u_\tau\|_H^2 \\ & + \frac{1}{\alpha}\int_\tau^t \|g\|_H^2 ds + \frac{1}{\alpha}\int_\tau^t b(s)ds \\ & + \frac{1}{\alpha(1-\rho^*)}\Big[\frac{\|\tilde{a}\|_{L^q(\tau-h,\tau)}}{\alpha(1-\rho^*)}\|\phi(t)\|^2_{L^{2q}_H} + \|u_\tau\|_H^2\Big]\int_\tau^t \tilde{a}(s)e^{-\chi_\sigma(s,\tau)}ds \\ & + \frac{1}{\alpha^2(1-\rho^*)}\int_\tau^t \|g(s)\|_H^2 ds \int_\tau^t \tilde{a}(s)ds + \frac{\|b\|_{L^1(\tau,T)}}{\alpha^2(1-\rho^*)}\int_\tau^t \tilde{a}(s)ds. \qquad (116) \end{aligned}$$

**Step 3.** The sufficient condition for asymptotic stability of trajectories inside pullback attractors.

Combining (107) with (116), we conclude that

$$\begin{aligned} & 2C_{|\Omega|}\beta\langle \|u\|_V^2 + \|v\|_V^2\rangle|_{\leq t} \\ \leq\ & \frac{2C_{|\Omega|}\beta}{\nu}\Big[\Big(\frac{1}{\alpha^2(1-\rho^*)} + \int_\tau^t \tilde{a}(s)ds\Big)\langle \|g(t)\|_H^2\rangle|_{\leq t} \\ & + \frac{1}{\alpha}\langle \|b(t)\|_{L^1}\rangle|_{\leq t} + \frac{\|b\|_{L^1(\tau,T)}}{\alpha^2(1-\rho^*)}\langle \|\tilde{a}(t)\|_{L^1}\rangle|_{\leq t}\Big]. \qquad (117) \end{aligned}$$

and hence the asymptotic stability holds provided that

$$\begin{aligned} & \Big(\frac{1}{\alpha^2(1-\rho^*)} + \|\tilde{a}(t)\|_{L^1}\Big)\langle \|g(t)\|_H^2\rangle|_{\leq t} + \frac{1}{\alpha}\langle \|b(t)\|_{L^1}\rangle|_{\leq t} + \frac{\|b\|_{L^1(\tau,T)}}{\alpha^2(1-\rho^*)}\langle \|\tilde{a}(t)\|_{L^1}\rangle|_{\leq t} \\ & \leq \frac{\nu(2\nu\lambda_1+\alpha)}{2C_{|\Omega|}\beta}. \qquad (118) \end{aligned}$$

If we define the generalized Grashof number as $G(t) = \Big(\frac{\langle \|g\|_H^2\rangle|_{\leq t}}{\nu^2\lambda_1}\Big)^{1/2}$, neglecting the positive terms on the left-hand side of (118), then a sufficient condition for the asymptotic stability of trajectories inside pullback attractors can be conclude as

$$G(t) \leq \Big\{(2\nu\lambda_1+\alpha)\Big/\Big[2C_{|\Omega|}\beta\nu\lambda_1\big(\frac{1}{\alpha^2(1-\rho^*)} + \|\tilde{a}(t)\|_{L^1}\big)\Big]\Big\}^{1/2} = \tilde{K}_0, \qquad (119)$$

which completes the proof for our first result. □

**Remark 5.** If we denote

$$\limsup_{\tau\to-\infty} \frac{1}{t-\tau}\int_\tau^t b(r)dr = b_0 \in [0,+\infty) \qquad (120)$$

and

$$\limsup_{\tau\to-\infty} \frac{1}{t-\tau}\int_\tau^t \tilde{a}(r)dr = \tilde{a}_0 \in [0,+\infty), \qquad (121)$$

such that there exists some $\sigma > 0$, the following assumption

$$\frac{\nu(2\nu\lambda_1+\alpha)}{2C_{|\Omega|}\beta} > \frac{b_0}{\alpha} + \frac{\|b\|_{L^1(\tau,T)}\tilde{\alpha}_0}{\alpha^2(1-\rho^*)} + \delta \qquad (122)$$

holds. Then more precise sufficient condition for the asymptotic stability of pullback attractors is

$$G(t) \leq \Big[ \frac{\frac{\nu(2\nu\lambda_1+\alpha)}{2C_{|\Omega|}\beta} - \frac{b_0}{\alpha} - \frac{\|b\|_{L^1(\tau,T)}\tilde{\alpha}_0}{\alpha^2(1-\rho^*)}}{\nu^2\lambda_1(\frac{1}{\alpha^2(1-\rho^*)} + \|\tilde{a}(t)\|_{L^1})} \Big]^{1/2} \tag{123}$$

which has smaller upper boundedness than (119).

5. **Further research.** The structure and stability of 3D BF equations with delay are investigated in this paper. A future research in the pullback dynamics of (1) is to study the geometric property of pullback attractors, such as the fractal dimension.

**Acknowledgments.** Xin-Guang Yang was partially supported by the Fund of Young Backbone Teacher in Henan Province (No. 2018GGJS039) and Henan Overseas Expertise Introduction Center for Discipline Innovation (No. CXJD2020003). Xinjie Yan was partly supported by Excellent Innovation Team Project of "Analysis Theory of Partial Differential Equations" in China University of Mining and Technology (No. 2020QN003). Ling Ding was partly supported by NSFC of China (Grant No. 1196302).

The authors want to express their most sincere thanks to refrees for the improvement of this manuscript. The authors also want to thank Professors Tomás Caraballo (Universidad de Sevilla), Desheng Li (Tianjin University) and Shubin Wang (Zhengzhou University) for fruitful discussion on this subject.

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Received March 2020; 1st revision March 2020; final revision June 2020.

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