ELECTRONIC RESEARCH ARCHIVE
Volume **28**, Number **4**, December **2020**

doi:10.3934/era.2020073
pp. **1375–1393**

THE REGULARIZED BOUSSINESQ EQUATIONS WITH PARTIAL DISSIPATIONS IN DIMENSION TWO

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(Communicated by Ronghua Pan)

Abstract. The incompressible Boussinesq system plays an important role in modelling geophysical fluids and studying the Raleigh-Bernard convection. We consider the regularized model (also named as Boussinesq-α model) to the Boussinesq equations. We consider the Cauchy problem of a two-dimensional regularized Boussinesq model with vertical dissipation in the horizontal regularized velocity equation and horizontal dissipation in the vertical regularized velocity equation and prove that this system has a unique global classical solution. Next, we consider a two-dimensional Boussinesq-α model with only vertical thermal diffusion and establish a Beale-Kato-Majda type regularity condition of smooth solution for this system.

1. **Introduction.** The Boussinesq equations describe the influence of the convection (or convection-diffusion) phenomenon in a viscous or inviscid fluid, which are used as modelling many geophysical flows, such as atmospheric fronts and ocean circulations [18, 22]. The standard two-dimensional Boussinesq equations read as:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla P = \theta e_2, \\ \partial_t \theta + (u\cdot\nabla)\theta - \kappa\Delta\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,y,0) = u_0, \quad \theta(x,y,0) = \theta_0, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification.* Primary: 35Q35; Secondary: 76D03.

Key words and phrases. Boussinesq-α equations, global existence, partial dissipation.

The first author is partially supported by National Natural Science Foundation of China (Grant No. 11126266), Natural Science Foundation of GuangDong Province (Grant No. 2016A030313390) and SCAU Fund for High-level University Building. The second author is partially supported by the National Natural Science Foundation of China (Grant Nos. 11971496 and 11431015). Part of this work was done when H. Qiu was visiting Department of Mathematics at Oklahoma State University in 2018. H. Qiu appreciates the hospitality of Department of Mathematics at OSU and would like to thank Professor Jiahong Wu for his helpful discussion and encouragement. H. Qiu's research is supported partly by China Scholarship Council. This work is also supported in part by the Visiting Program of Chern Institute of Mathematics (CIM) at Nankai University. We appreciate the hospitality of CIM.

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where $u = (u_1(x, y, t), u_2(x, y, t))$ denotes the fluid velocity vector field, $P = P(x, y, t)$ is the scalar pressure, $\theta = \theta(x, y, t)$ is the scalar temperature, the nonnegative μ and κ denote respectively the viscosity and the thermal diffusivity, $e_2 = (0, 1)$ is the unit vector, while u_0 and θ_0 are the given initial velocity and initial temperature respectively, with $\nabla \cdot u_0 = 0$. In order to model anisotropic flows with different diffusion properties in the horizontal and vertical directions, the system (1) can be generalized to the following form:

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 + \partial_x P = \mu_1 \partial_{xx} u_1 + \mu_2 \partial_{yy} u_1, \\ \partial_t u_2 + u_1 \partial_x u_2 + u_2 \partial_y u_2 + \partial_y P = \nu_1 \partial_{xx} u_2 + \nu_2 \partial_{yy} u_2 + \theta, \\ \partial_t \theta + u_1 \partial_x \theta + u_2 \partial_y \theta = \kappa_1 \partial_{xx} \theta + \kappa_2 \partial_{yy} \theta, \\ \partial_x u_1 + \partial_y u_2 = 0, \end{cases} \tag{2}$$

where u_1 and u_2 are the horizontal and vertical components of u. In fact, when $\mu_1 = \mu_2 = \nu_1 = \nu_2$ and $\kappa_1 = \kappa_2$, the system (2) reduces to the system (1). When all six parameters are zero, the system (2) reduces to the inviscid Boussinesq equations. Yet, whether or not the solution of the inviscid Boussinesq can develop finite time singularities remains open since the 2D inviscid Boussinesq equations are identical to the Euler equations for 3D axisymmetric swirling flows [19]. The global regularity has been obtained for the case that all four parameters are positive [5]. On the other hand, for the intermediate cases between two extreme cases (namely, the full dissipation case and the inviscid case), the Boussiesq equations (2) with partial viscosities or partial diffusivity have been attracting much more attention in the recent few years and important progress has been made (see, for example [7, 8, 9, 13, 14, 15, 23] and the references therein).

Most of the kinetic energy in turbulent fluid flows lies in the large scales, whereas the mathematical and computational difficulties lie in understanding the dynamical interaction between the significantly wide range of relevant scales in this multiscale phenomenon. To overcome this obstacle, much effort is being made to produce reliable turbulence models which parameterize the effect of the active small scales in terms of the large scales. Over the last years, the viscous Camassa-Holm equations have been proposed as a subgrid turbulence model (also known as the Lagrangian-averaged Navier-Stokes-α (LANS-α) model). Motivated by the remarkable performance of LANS-α model as a closure model of turbulence in infinite channels and pipes, whose solutions give excellent agreement with empirical data for a wide range of large Reynolds numbers, the alpha subgrid scale models of turbulence have been extensively studied in recent years [11, 12, 21].

An extension of the LANS-α model to the Boussinesq equations is given in [24], which is named as the Lagrangian averaged Boussinesq equations:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_{j=1}^n v_j \nabla u_j + \nabla P - \mu \Delta v = \theta e_n, \\ \partial_t \theta + (u \cdot \nabla)\theta - \kappa \Delta \theta = 0, \\ v = (1 - \alpha^2 \Delta) u, \\ \nabla \cdot v = \nabla \cdot u = 0, \\ (v(x, 0), u(x, 0), \theta(x, 0)) = (v_0, u_0, \theta_0), \quad x \in \mathbb{R}^n, n = 2, 3, \end{cases} \tag{3}$$

where $e_2 = (0, 1)$ and $e_3 = (0, 0, 1)$, $\nabla \cdot v_0 = \nabla \cdot u_0 = 0$ and the constant $\alpha > 0$ is a length scale parameter which represents the width of the filter. In [24], the inviscid case (namely, $\mu = \kappa = 0$) to the system (3) is considered and the regularity criterion $\nabla u \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^n))$ $(n = 2, 3)$ is obtained. As mentioned above that the system (2) is the generalized version of(1), the 2D Lagrangian averaged Boussinesq equations in (3) can be generalized to the following form, where we call it the anisotropic Lagrangian averaged Boussinesq equations (also named as Boussinesq-α equations):

$$\begin{cases} \partial_t v_1 + (u\cdot\nabla)v_1 + \sum_{j=1}^2 v_j\partial_x u_j + \partial_x P = \mu_1\partial_{xx}v_1 + \mu_2\partial_{yy}v_1, \\ \partial_t v_2 + (u\cdot\nabla)v_2 + \sum_{j=1}^2 v_j\partial_y u_j + \partial_y P = \nu_1\partial_{xx}v_2 + \nu_2\partial_{yy}v_2 + \theta, \\ \partial_t\theta + (u\cdot\nabla)\theta = \kappa_1\partial_{xx}\theta + \kappa_2\partial_{yy}\theta, \\ v = (1-\alpha^2\Delta)u, \\ \nabla\cdot v = \nabla\cdot u = 0, \\ (v(x,y,0), u(x,y,0), \theta(x,y,0)) = (v_0, u_0, \theta_0), \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}^+, \end{cases} \tag{4}$$

where $v = (v_1, v_2)$ and $u = (u_1, u_2)$ with $\nabla\cdot v_0 = \nabla\cdot u_0 = 0$. When $\mu_1 = \mu_2 = \nu_1 = \nu_2$, $\kappa_1 = \kappa_2$ and $\alpha \searrow 0^+$, the system (4) reduces to (1) in dimension two.

In this paper, we consider the anisotropic Lagrangian averaged Boussinesq equations with partial dissipations. Firstly, we study the Cauchy problem of the Boussinesq-α model with vertical dissipation in the horizontal regularized velocity equation and horizontal dissipation in the vertical regularized velocity equation, namely

$$\begin{cases} \partial_t v_1 + (u\cdot\nabla)v_1 + \sum_{j=1}^2 v_j\partial_x u_j + \partial_x P = \mu_2\partial_{yy}v_1, \\ \partial_t v_2 + (u\cdot\nabla)v_2 + \sum_{j=1}^2 v_j\partial_y u_j + \partial_y P = \nu_1\partial_{xx}v_2 + \theta, \\ \partial_t\theta + (u\cdot\nabla)\theta = 0, \\ v = (1-\alpha^2\Delta)u, \\ \partial_x v_1 + \partial_y v_2 = \partial_x u_1 + \partial_y u_2 = 0, \\ (u(x,y,0), \theta(x,y,0)) = (u_0, \theta_0), \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}^+. \end{cases} \tag{5}$$

We obtain the global existence of smooth solution to the above system (5), which is stated as follows.

Theorem 1.1. *Consider the 2D Boussinesq-α system (5). Assume $(v_0, \theta_0) \in H^3(\mathbb{R}^2)\times H^3(\mathbb{R}^2)$ with $\nabla\cdot v_0 = \nabla\cdot u_0 = 0$. Then the system (5) has a unique global classical solution (v,θ) satisfying*

$$(v,\theta) \in L^\infty([0,T]; H^3(\mathbb{R}^2)) \times L^\infty([0,T]; H^3(\mathbb{R}^2)),$$
$$v \in L^2([0,T]; H^4(\mathbb{R}^2)),$$

for any given $T > 0$.

Remark 1. As $\alpha \searrow 0^+$, the Lagrangian averaged Boussinesq equations (3) (namely, the Boussinesq-α) reduce to the standard Boussinesq equations (1). And the anisotropic Lagrangian averaged Boussinesq equations (5) reduce to the anisotropic Boussinesq equations (2) with $\mu_1 = \nu_2 = 0$, $\mu_2 > 0$ and $\nu_1 > 0$, namely,

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 + \partial_x P = \mu_2\partial_{yy}u_1, \\ \partial_t u_2 + u_1\partial_x u_2 + u_2\partial_y u_2 + \partial_y P = \nu_1\partial_{xx}u_2 + \theta, \\ \partial_t\theta + u_1\partial_x\theta + u_2\partial_y\theta = \kappa_1\partial_{xx}\theta + \kappa_2\partial_{yy}\theta, \\ \partial_x u_1 + \partial_y u_2 = 0, \end{cases}$$

which is considered in [1] and the global regularity of classical solution is established. From this standpoint, our result in Theorem 1.1 can be viewed as an improvement for the regularization version of the Boussinesq equations to the anisotropic Boussinesq equations.

Remark 2. For the proof of Theorem 1.1, we adapt the approach of "weakly nonlinear" energy estimate approach introduced by Lei and Zhou in [17].

Next, let us revisit (3). When $\mu = \kappa = 0$, Zhou-Fan [24] proved the following regularity criterion of smooth solution to the inviscid Boussinesq-α system (3) with $\mu = 0$ and $\kappa = 0$:

$$\nabla\theta \in L^1(0,T;L^\infty(\mathbb{R}^2)). \tag{6}$$

Now we consider the following 2D anisotropic Boussinesq-α equations with only one dissipation term $\kappa_2\partial_{yy}\theta$:

$$\begin{cases} \partial_t v_1 + (u\cdot\nabla)v_1 + \sum_{j=1}^2 v_j\partial_x u_j + \partial_x P = 0, \\ \partial_t v_2 + (u\cdot\nabla)v_2 + \sum_{j=1}^2 v_j\partial_y u_j + \partial_y P = \theta, \\ \partial_t\theta + (u\cdot\nabla)\theta = \kappa_2\partial_{yy}\theta, \\ v = (1-\alpha^2\Delta)u, \\ \nabla\cdot v = \nabla\cdot u = 0, \\ (v(x,y,0),u(x,y,0),\theta(x,y,0)) = (v_0,u_0,\theta_0), \quad (x,y,t)\in\mathbb{R}^2\times\mathbb{R}^+. \end{cases} \tag{7}$$

We establish a Beale-Kata-Majda type regularity criterion to the system (7). More precisely, our second main result in this paper is stated as follows:

Theorem 1.2. *Assume $(v_0,\theta_0)\in H^3(\mathbb{R}^2)\times H^3(\mathbb{R}^2)$ with $\nabla\cdot v_0 = \nabla\cdot u_0 = 0$ and let (v,θ) be a smooth solution to the system (7) on $[0,T_0]$ for some positive time $T_0>0$. If, for $T>T_0$, θ satisfies*

$$M(T) \equiv \int_0^T \|\partial_x\theta(t)\|_{L^\infty}dt < \infty, \tag{8}$$

then the solution can be extended to $[0,T]$.

Remark 3. Compared to (6), our result in (8) is dependent of the derivative of x for θ. The reason is that this comes from the contribution of $\partial_{yy}\theta$, which can help lead to the cancellation between the ∂_y-type terms.

Remark 4. As $\alpha \searrow 0^+$, the model (7) reduces to the corresponding anisotropic Boussinesq system, which was introduced in [1]. Theorem 1.2 can be seen as a generalization of their work.

2. Preliminaries.

In this section, we provide some notation and basic facts used in the proof.

Notation. Throughout the paper, C stands for some real positive constants which may be different in each occurrence and independent of the initial data unless we give some special explanation. For sake of simplicity, we denote $\int_{\mathbb{R}^2} dxdy$ by $\int_{\mathbb{R}^2} dx$ for $(x,y)\in\mathbb{R}^2$ and write $\sum_j$ for $\sum_{j=1}^2$.

Now, we start with the well-known Gagliardo-Nirenberg inequality.

Lemma 2.1. *Suppose that $f\in L^q(\mathbb{R}^n)\cap W^{m,r}(\mathbb{R}^n), 1\le q,r\le\infty$. Then for $0\le j\le m$, $\frac{j}{m}\le\theta\le 1, 1\le p\le\infty$ and*

$$\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r}-\frac{m}{n}) + (1-\theta)\frac{1}{q},$$

there exists a constant C such that

$$\|\nabla^j f\|_{L^p} \le C\|f\|_{L^q}^{1-\theta}\|\nabla^m f\|_{L^r}^{\theta}.$$

Let us introduce the well-known commutator estimates proved by Kato-Ponce [16].

Lemma 2.2. *Let $s>0, 1<p<\infty$, and suppose that $f,g\in\mathcal{S}(\mathbb{R}^n)$. Then we have*

$$\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}}+\|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}),$$

$$\|\Lambda^s(fg)\|_{L^p}\leq C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{q_1}}+\|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}),$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2},\Lambda=(-\Delta)^{\frac{1}{2}}$, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwarz class of rapidly decreasing functions.

We recall the well-known Calderon-Zygmund operators, which will be used to get the control between the gradient of velocity and the vorticity (see [10]).

Lemma 2.3. *(Biot-Savart law). There exists a universally positive constant C such that for every $1<p<\infty$, holding*

$$\|\nabla u\|_{L^p}\leq C\frac{p^2}{p-1}\|\nabla\times u\|_{L^p}.$$

Next, let us recall the following logarithmic Sobolev inequality (see [3, 4]).

Lemma 2.4. *Let $n\geq 2$ and $s>1+\frac{n}{p}$. The following logarithmic Sobolev embedding theorem holds for all divergence free vector fields f with $f\in L^2(\mathbb{R}^n)\cap\dot{W}^{s,p}(\mathbb{R}^n)$:*

$$\|\nabla f\|_{L^\infty(\mathbb{R}^n)}\leq C\left(1+\|f\|_{L^2(\mathbb{R}^n)}+\|\Delta f\|_{L^2(\mathbb{R}^n)}\log(1+\|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)})\right).$$

We also need the following well-known Osgood lemma [2], which will be a crucial ingredient to establish the global in time a priori estimates in the proof of Theorem 1.1.

Lemma 2.5. *Let ϕ be a measurable, positive function, γ a positive locally integrable function, and μ a continuous and nondecreasing function. Assume that, for some nonnegative real number c, the function ϕ satisfies*

$$\phi(t)\leq c+\int_{t_0}^t\gamma(s)\mu(\phi(s))ds.$$

If c is positive, then we have

$$-\mathcal{M}(\phi(t))+\mathcal{M}(c)\leq\int_{t_0}^t\gamma(s)ds \text{ with } \mathcal{M}(s)=\int_s^1\frac{dr}{\mu(r)}.$$

If c is zero and $\mu(s)$ satisfies $\int_0^1\frac{dr}{\mu(r)}=+\infty$, then $\phi=0$.

The following lemma, which has been firstly introduced in [6] to deal with the 2D MHD equations with partial viscosity, will play an important role in the proof of our second main result.

Lemma 2.6. *Assume that $f,g,\partial_y g,h$ and $\partial_x h$ are all in $L^2(\mathbb{R}^2)$. Then,*

$$\int_{\mathbb{R}^2}|fgh|dx\leq C\|f\|_2\|g\|_2^{\frac{1}{2}}\|\partial_y g\|_2^{\frac{1}{2}}\|h\|_2^{\frac{1}{2}}\|\partial_x h\|_2^{\frac{1}{2}}.$$

3. **Proof of Theorem 1.1.**

Proof. This section is devoted to the proof of Theorem 1.1. For simplicity, without loss of generality, we assume $\mu_2 = \nu_1 = 1$ in what follows.

Firstly, for the third equation of (5), according to the divergence free condition $\nabla \cdot u = 0$, we obtain immediately

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall p \in [1, \infty], \tag{9}$$

for any $t \in [0, \infty]$.

Multiplying both sides of the first and second equations of (5) by u_1 and u_2 respectively, it is easy to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \alpha^2\|\nabla u(t)\|_{L^2}^2) + \|\partial_y u_1\|_{L^2}^2 + \|\partial_x u_2\|_{L^2}^2 \\ +\alpha^2\|\nabla\partial_y u_1\|_{L^2}^2 + \alpha^2\|\nabla\partial_x u_2\|_{L^2}^2 \le \|\theta\|_{L^2}\|u_2\|_{L^2},\end{aligned} \tag{10}$$

where we used the fact:

$$\int_{\mathbb{R}^2} (u \cdot \nabla)u \cdot u dx = 0,$$

$$\int_{\mathbb{R}^2} (u \cdot \nabla)v \cdot u dx + \sum_j \int_{\mathbb{R}^2} v_j \nabla u_j \cdot u dx = 0,$$

by the incompressible condition $\nabla \cdot u = 0$.

Integrating in time and using (9), then for any $T > 0$ and $t \le T$, one have from (10) that

$$\begin{aligned}\|u(t)\|_{L^2}^2 + \alpha^2\|\nabla u(t)\|_{L^2}^2 + \int_0^t (\|\partial_y u_1(\tau)\|_{L^2}^2 + \|\partial_x u_2(\tau)\|_{L^2}^2 \\ +\alpha^2\|\nabla\partial_y u_1(\tau)\|_{L^2}^2 + \alpha^2\|\nabla\partial_x u_2(\tau)\|_{L^2}^2) d\tau \le C,\end{aligned} \tag{11}$$

where $C = C(T, u_0, \theta_0, \alpha)$.

So we obtain

$$\int_0^T \int_{\mathbb{R}^2} |v|^2 dx dt \le C. \tag{12}$$

The first and second equations of (5) can be rewritten as:

$$\partial_t v + (u \cdot \nabla)v + \sum_j v_j \nabla u_j - \begin{pmatrix}\partial_{yy} v_1 \\ \partial_{xx} v_2\end{pmatrix} + \nabla P = \begin{pmatrix}0 \\ \theta\end{pmatrix}. \tag{13}$$

Applying the operator *curl* $= \nabla\times$ to the first equation of (13), and using the fact

$$\text{curl}(u \cdot \nabla)v + \text{curl}(\sum_j v_j \nabla u_j) = (u \cdot \nabla)\text{curl}v,$$

then we have

$$\partial_t \text{curl}v + (u \cdot \nabla)\text{curl}v - (\partial_{xxx} v_2 - \partial_{yyy} v_1) = \partial_x \theta. \tag{14}$$

Testing (14) by curl and using the incompressible condition $\nabla \cdot u = 0$, then we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\text{curl}v(t)\|_{L^2}^2 &= \int_{\mathbb{R}^2} (\partial_{xxx} v_2 - \partial_{yyy} v_1) \cdot \text{curl}v dx + \int_{\mathbb{R}^2} \partial_x \theta \text{curl}v dx \\ &= I_1 + I_2.\end{aligned} \tag{15}$$

For the first term I_1 in (15), integrating by parts and using $-\Delta v = \text{curlcurl}v$ and $\partial_x v_1 + \partial_y v_2 = 0$,

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} \text{curl}\begin{pmatrix} \partial_{yy} v_1 \\ \partial_{xx} v_2 \end{pmatrix} \cdot \text{curl} v dx \\
&= \int_{\mathbb{R}^2} \begin{pmatrix} \partial_{yy} v_1 \\ \partial_{xx} v_2 \end{pmatrix} \cdot \text{curlcurl} v dx \\
&= -\int_{\mathbb{R}^2} \begin{pmatrix} \partial_{yy} v_1 \\ \partial_{xx} v_2 \end{pmatrix} \cdot \Delta v dx \\
&= -\int_{\mathbb{R}^2} \partial_{yy} v_1 \Delta v_1 dx - \int_{\mathbb{R}^2} \partial_{xx} v_2 \Delta v_2 dx \\
&= -\int_{\mathbb{R}^2} \partial_{yy} v_1 \partial_{xx} v_1 dx - \int_{\mathbb{R}^2} \partial_{yy} v_1 \partial_{yy} v_1 dx - \int_{\mathbb{R}^2} \partial_{xx} v_2 \partial_{xx} v_2 dx \\
&\quad - \int_{\mathbb{R}^2} \partial_{xx} v_2 \partial_{yy} v_2 dx \\
&= -\int_{\mathbb{R}^2} (\partial_{xy} v_1)^2 dx - \int_{\mathbb{R}^2} (\partial_{yy} v_1)^2 dx - \int_{\mathbb{R}^2} (\partial_{xx} v_2)^2 dx - \int_{\mathbb{R}^2} (\partial_{xy} v_2)^2 dx \\
&= -\int_{\mathbb{R}^2} (\partial_{yy} v_2)^2 dx - \int_{\mathbb{R}^2} (\partial_{yy} v_1)^2 dx - \int_{\mathbb{R}^2} (\partial_{xx} v_2)^2 dx - \int_{\mathbb{R}^2} (\partial_{xx} v_1)^2 dx,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
-I_1 &= \int_{\mathbb{R}^2} (\partial_{yy} v_2)^2 dx + \int_{\mathbb{R}^2} (\partial_{yy} v_1)^2 dx + \int_{\mathbb{R}^2} (\partial_{xx} v_2)^2 dx + \int_{\mathbb{R}^2} (\partial_{xx} v_1)^2 dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^2} (\partial_{xx} v_1 + \partial_{yy} v_1)^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_{xx} v_2 + \partial_{yy} v_2)^2 dx \\
&= \frac{1}{2} \|\Delta v\|_{L^2}^2.
\end{aligned}
$$

For the second term I_2, similarly,

$$
\begin{aligned}
I_2 = \int_{\mathbb{R}^2} \text{curl}\begin{pmatrix} 0 \\ \theta \end{pmatrix} \cdot \text{curl} v dx &= \int_{\mathbb{R}^2} \begin{pmatrix} 0 \\ \theta \end{pmatrix} \cdot \text{curlcurl} v dx \\
&\leq \|\theta\|_{L^2} \|curl\text{curl}v\|_{L^2} \\
&\leq \frac{1}{4} \|\Delta v\|_{L^2}^2 + C\|\theta\|_{L^2}^2.
\end{aligned}
$$

Inserting the above estimates for I_1 and I_2 into (15), one has

$$
\frac{d}{dt} \|\text{curl} v(t)\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 \leq C\|\theta_0\|_{L^2}^2. \tag{16}
$$

Integrating in time, then it follows from (16) that

$$
\|\text{curl} v(T)\|_{L^2}^2 + \int_0^T \|\Delta v(t)\|_{L^2}^2 dt \leq C.
$$

Hence, according to Lemma 2.3, at last we have

$$
\|v\|_{L^\infty([0,T];H^1)} + \|v\|_{L^2([0,T];H^2)} \leq C, \tag{17}
$$

$$
\|u\|_{L^\infty([0,T];H^3)} + \|u\|_{L^2([0,T];H^4)} \leq C. \tag{18}
$$

Here, we should point out that we can't obtain a global bound for the H^1-norm of θ at this stage. In fact, using the Hölder inequality, the incompressible condition $\nabla \cdot u = 0$, Lemma 2.2 and (18), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \nabla[(u\cdot\nabla)\theta]\cdot\nabla\theta dx \\
&= \int_{\mathbb{R}^2} ((u\cdot\nabla)\cdot\nabla\theta - \nabla[(u\cdot\nabla)\theta])\cdot\nabla\theta \\
&\le \|\nabla[(u\cdot\nabla)\theta] - (u\cdot\nabla)\cdot\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\cdot\|\nabla\theta\|_{L^2}^2 \\
&\le C\|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{19}$$

Here we used the following Gagliardo-Nirenberg inequality in Lemma 2.1

$$\|u\|_{L^\infty} \le C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \tag{20}$$

in the third inequality of (19). Then we get

$$\|\nabla\theta(t)\|_{L^2} \le e^{Ct}\|\nabla\theta_0\|_{L^2} = C(t,\theta_0).$$

Now, we apply "weakly nonlinear" energy estimate approach introduced firstly by Lei and Zhou [17] to obtain the higher global regularity. For any $T > 0$, we assume that the solution $(v(t), \theta(t))$ is regular for $t < T$ and show that it remains regular at $t = T$. More precisely, we define

$$\Phi(t) = \sup_{0\le\tau\le t}(\|\nabla^3 v(\tau)\|_{L^2}^2 + \|\nabla^3\theta(\tau)\|_{L^2}^2) < \infty,$$

and assume that $\Phi(t) < \infty$ for $t < T$ and show that

$$\Phi(t) < \infty. \tag{21}$$

It follows from the equation of $\nabla\theta$ that, for any $0 \le s < t$,

$$\|\nabla\theta(t)\|_{L^\infty} \le C\|\nabla\theta(s)\|_{L^\infty}\exp\left(\int_s^t \|\nabla u(\tau)\|_{L^\infty}d\tau\right). \tag{22}$$

Then, choose T_0 close enough to T ($T_0 < T$) and let $T_0 < t < T$, we have from Lemma 2.4 and (12) that

$$\begin{aligned}
\|\nabla\theta(t)\|_{L^\infty} &\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\left(C\int_{T_0}^t(1+\|u\|_{L^2}+\|\Delta u\|_{L^2}\log(1+\|\nabla^3 u\|_{L^2}))\right) \\
&\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\left(C\int_{T_0}^t\|\Delta u(\tau)\|_{L^2}\log(1+\|\nabla^3 v(\tau)\|_{L^2})d\tau\right) \\
&\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\left(C\int_{T_0}^t\|\Delta u(\tau)\|_{L^2}\log(1+\Phi(t))d\tau\right) \\
&\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\left(C\int_{T_0}^t\|\Delta u(\tau)\|_{L^2}d\tau\log(1+\Phi(t))\right).
\end{aligned} \tag{23}$$

According to (18), we know that

$$\int_0^T \|\Delta u(t)\|_{L^2}^2 dt < \infty,$$

then we can choose T_0 close enough to T such that, for small $\varepsilon > 0$,

$$\int_{T_0}^{t} \|\Delta u(t)\|_{L^2}^2 dt < \varepsilon.$$

It then follows from (23) that, for $T_0 \le t < T$,

$$\|\nabla\theta(t)\|_{L^\infty} \le C(1+\Phi(t))^{\varepsilon}. \tag{24}$$

Now applying the operator Δ on both sides of (14), multiplying the resulting equation by $\Delta\text{curl}v$ and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\text{curl}v(t)\|_{L^2} =& \int_{\mathbb{R}^2} \Delta(\partial_{xxx}v_2 - \partial_{yyy}v_1)\cdot\Delta\text{curl}v dx \\ &- \int_{\mathbb{R}^2}\Delta[(u\cdot\nabla)\text{curl}v]\cdot\Delta\text{curl}v dx + \int_{\mathbb{R}^2}\Delta\partial_x\cdot\Delta\text{curl}v dx \\ =& II_1 + II_2 + II_3.\end{aligned} \tag{25}$$

For the first term II_1 in (25), integrating by parts and using $-\Delta v = \text{curlcurl}v$, one has

$$\begin{aligned} II_1 =& \int_{\mathbb{R}^2} \Delta(\partial_{xxx}v_2 - \partial_{yyy}v_1)\cdot\Delta\text{curl}v dx \\ =& \int_{\mathbb{R}^2}\text{curl}\Delta\begin{pmatrix}\partial_{yy}v_1\\ \partial_{xx}v_2\end{pmatrix}\cdot\Delta\text{curl}v dx \\ =& \int_{\mathbb{R}^2}\Delta\begin{pmatrix}\partial_{yy}v_1\\ \partial_{xx}v_2\end{pmatrix}\cdot\Delta\text{curlcurl}v dx \\ =& -\int_{\mathbb{R}^2}\Delta\begin{pmatrix}\partial_{yy}v_1\\ \partial_{xx}v_2\end{pmatrix}\cdot\Delta^2 v dx \\ =& -\int_{\mathbb{R}^2}\Delta\partial_{yy}v_1\Delta^2 v_1 dx - \int_{\mathbb{R}^2}\Delta\partial_{xx}v_2\Delta^2 v_2 dx \\ =& II_{11} + II_{12}.\end{aligned} \tag{26}$$

Integrating by parts and using the condition $\partial_x v_1 + \partial_y v_2 = 0$, we have

$$\begin{aligned} II_{11} =& -\int_{\mathbb{R}^2}\partial_{yy}\Delta v_1(\partial_{xx}\Delta v_1 + \partial_{yy}\Delta v_1)dx \\ =& -\int_{\mathbb{R}^2}(\partial_{yy}\Delta v_1)^2 dx - \int_{\mathbb{R}^2}\partial_{xx}\Delta v_1\partial_{yy}\Delta v_1 dx \\ =& -\int_{\mathbb{R}^2}(\partial_{yy}\Delta v_1)^2 dx - \int_{\mathbb{R}^2}\partial_{xy}\Delta v_1\partial_{xy}\Delta v_1 dx \\ =& -\int_{\mathbb{R}^2}(\partial_{yy}\Delta v_1)^2 dx - \int_{\mathbb{R}^2}\partial_{yy}\Delta v_2\partial_{yy}\Delta v_2 dx \\ =& -\int_{\mathbb{R}^2}(\partial_{yy}\Delta v_1)^2 dx - \int_{\mathbb{R}^2}(\partial_{yy}\Delta v_2)^2 dx\end{aligned} \tag{27}$$

and

$$\begin{aligned} II_{12} =& -\int_{\mathbb{R}^2}\partial_{xx}\Delta v_2(\partial_{xx}\Delta v_2 + \partial_{yy}\Delta v_2)dx \\ =& -\int_{\mathbb{R}^2}(\partial_{xx}\Delta v_2)^2 dx - \int_{\mathbb{R}^2}\partial_{xx}\Delta v_2\partial_{yy}\Delta v_2 dx\end{aligned}$$

$$\begin{aligned} &= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v_2)^2 dx - \int_{\mathbb{R}^2} \partial_{xy}\Delta v_2 \partial_{xy}\Delta v_2 dx \\ &= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v_2)^2 dx - \int_{\mathbb{R}^2} \partial_{xx}\Delta v_1 \partial_{xx}\Delta v_1 dx \\ &= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v_2)^2 dx - \int_{\mathbb{R}^2} (\partial_{xx}\Delta v_1)^2 dx. \end{aligned} \tag{28}$$

Inserting (27) and (28) into (26) implies that

$$-II_1 \geq \frac{1}{2}\|\Delta^2 v\|_{L^2}^2. \tag{29}$$

Next we estimate the term II_2. By the incompressible condition $\nabla \cdot u = 0$, Lemma 2.2 and the Young inequality,

$$\begin{aligned} II_2 =& -\int_{\mathbb{R}^2} \Delta[(u\cdot\nabla)\text{curl}v]\cdot \Delta\text{curl}v dx \\ =& \int_{\mathbb{R}^2} ((u\cdot\nabla)\Delta\text{curl}v - \Delta[(u\cdot\nabla)\text{curl}v])\cdot\Delta\text{curl}v dx \\ \leq& \|(u\cdot\nabla)\Delta\text{curl}v - \Delta[(u\cdot\nabla)\text{curl}v]\|_{L^2}\|\Delta\text{curl}v\|_{L^2} \\ \leq& C\|\nabla u\|_{L^\infty}\|\nabla^2\text{curl}v\|_{L^2}^2 + C\|\nabla^2 u\|_{L^\infty}\|\nabla\text{curl}v\|_{L^2}\|\Delta\text{curl}v\|_{L^2} \\ \leq& C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\cdot\|\nabla^2\text{curl}v\|_{L^2}^2 \\ &+ C\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^4 u\|_{L^2}^{\frac{1}{2}}\cdot\|\nabla\text{curl}v\|_{L^2}\|\Delta\text{curl}v\|_{L^2} \\ \leq& C\|\nabla^2\text{curl}v\|_{L^2}^2 + C\|\nabla^2\text{curl}v\|_{L^2}. \end{aligned} \tag{30}$$

Here we used the Gagliardo-Nirenberg inequality (20) and (18) in the third and fourth inequalities of (30), respectively.

For the term II_3, by the Young inequality, we have

$$II_3 \leq \|\Delta\partial_x\theta\|_{L^2}\|\Delta\text{curl}v\|_{L^2} \leq C(\|\nabla^3\theta\|_{L^2}^2 + \|\Delta\text{curl}v\|_{L^2}^2). \tag{31}$$

Inserting the estimates (29)-(31) for $II_1 - II_3$, it gives that

$$\frac{d}{dt}\|\Delta\text{curl}v(t)\|_{L^2}^2 + \|\Delta^2 v\|_{L^2}^2 \leq C(\|\nabla^3\theta\|_{L^2}^2 + \|\Delta\text{curl}v\|_{L^2}^2) + C\|\Delta\text{curl}v\|_{L^2}. \tag{32}$$

Applying the operator ∇^3 on both sides of the third equation of (5), by the incompressible condition $\nabla\cdot u = 0$, Lemma 2.2 and the Young inequality, one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla^3\theta(t)\|_2^2 =& -\int_{\mathbb{R}^2}\nabla^3[(u\cdot\nabla)\theta]\cdot\nabla^3\theta dx \\ =& \int_{\mathbb{R}^2}((u\cdot\nabla)\nabla^3\theta - \nabla^3[(u\cdot\nabla)\theta])\cdot\nabla^3\theta dx \\ \leq& C(\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^3\theta\|_{L^2} \\ \leq& C\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}\cdot\|\nabla u\|_{L^2}^{\frac{1}{3}}\|\nabla^4 u\|_{L^2}^{\frac{2}{3}}\cdot\|\nabla^3\theta\|_{L^2} \\ \leq& \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^3\theta\|_{L^2}^{\frac{3}{2}} + C\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 \\ \leq& \frac{1}{4}\|\nabla^4 v\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^3\theta\|_{L^2}^{\frac{3}{2}} \\ \leq& \frac{1}{4}\|\nabla^4 v\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla^3\theta\|_{L^2}^{\frac{3}{2}}, \end{aligned} \tag{33}$$

where we used the following Gagliardo-Nirenberg inequality of Lemma 2.1 in the first inequality of (33):

$$\|\nabla^3 f\|_{L^2} \le C\|\nabla f\|_{L^2}^{\frac{1}{3}}\|\nabla^4 f\|_{L^2}^{\frac{2}{3}}.$$

Combining (32) with (33), it gives that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Delta \mathrm{curl} v(t)\|_{L^2}^2+\|\nabla^3\theta(t)\|_2^2\right)+\|\Delta^2 v\|_{L^2}^2\\ \le& C(1+\|\nabla u\|_{L^\infty})(\|\Delta \mathrm{curl} v\|_{L^2}^2+\|\nabla^3\theta\|_2^2)+C\|\Delta \mathrm{curl} v\|_{L^2}+C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla^3\theta\|_{L^2}^{\frac{3}{2}}.\end{aligned}$$

Thanks to Lemma 2.3 and Lemma 2.4 and by the definition of $\Phi(t)$, it infers from the above inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Delta \mathrm{curl} v(t)\|_{L^2}^2+\|\nabla^3\theta(t)\|_2^2\right)+\|\Delta^2 v\|_{L^2}^2\\ \le& C(1+\|\nabla u\|_{L^\infty})\Phi(t)+C\Phi(t)^{\frac{1}{2}}+C(1+\Phi(t))^{\frac{3\varepsilon}{2}}\Phi(t)^{\frac{3}{4}}\\ \le& C(1+\|u\|_{L^2}+\|\Delta u\|_{L^2}\log(1+\Phi(t)))\Phi(t)+C\Phi(t)^{\frac{1}{2}}\\ &+C(1+\Phi(t))^{\frac{3\varepsilon}{2}}\Phi(t)^{\frac{3}{4}}\\ \le& C(1+\|u\|_{L^2}+\|\Delta u\|_{L^2}\log(1+\Phi(t)))\Phi(t)+C(1+\Phi(t))^{\frac{3\varepsilon}{2}}\Phi(t)^{\frac{3}{4}}\\ \le& C(1+\|u\|_{L^2}+\|\Delta u\|_{L^2}\log(1+\Phi(t)))(1+\Phi(t))\\ \le& C(1+\|\Delta u\|_{L^2}\log(1+\Phi(t)))(1+\Phi(t)),\end{aligned} \tag{34}$$

where we have taken ε to be enough small (smaller than $\frac{1}{6}$) and used the assumption $\Phi(t)\ge 1$ (otherwise, if $\Phi(t)<1$, we here have got (21)).

Integrating in time over (T_0,t) and noticing that the function $\Phi(t)$ is monotonically increasing, one can get from (34) that

$$1+\Phi(t)\le C+C\int_{T_0}^t(1+\|\Delta(\tau)\|_{L^2}\log(1+\Phi(\tau)))(1+\Phi(\tau))d\tau.$$

Using Lemma 2.5, namely the Osgood inequality, one can conclude that

$$\Phi(T)\le C\exp\exp(C\varepsilon)-1<\infty,$$

which (21) holds. This completes the proof of Theorem 1.1. □

4. **Proof of Theorem 1.2.**

Proof. This section is devoted to the proof of Theorem 1.2. For simplicity, without loss of generality, we assume $\kappa_2=1$ in what follows.

The existence and uniqueness of local smooth solutions can be done without any difficulty as in the case of the Euler (see, e.g., [19]), thus it is sufficient to establish a priori estimates for (v,θ), namely, for $t\le T$,

$$\|(v(t),\theta(t))\|_{H^s(\mathbb{R}^2)}\le C,$$

under the condition (8). Here the constant C depends on $T, v_0, \theta_0, M(T)$.

Firstly, multiplying both sides of the first, second and third equations of (7) by u_1, u_2 and θ, respectively, it is easy to get that

$$\|\theta(t)\|_{L^2}^2+\int_0^t\|\partial_y\theta(\tau)\|_{L^2}^2d\tau=\|\theta_0\|_{L^2}^2 \tag{35}$$

and

$$\|u(t)\|_{L^2}^2+\alpha^2\|\nabla u(t)\|_{L^2}^2\le\|u_0\|_{L^2}^2+\alpha^2\|\nabla u_0\|_{L^2}^2+t\|\theta_0\|_{L^2}. \tag{36}$$

Applying the operator curl $= \nabla\times$ to the first and second equations of (7), namely,

$$\partial_t v + (u\cdot\nabla)v + \sum_j v_j \nabla u_j + \nabla P = \begin{pmatrix} 0 \\ \theta \end{pmatrix},$$

then, one has

$$\partial_t \mathrm{curl} v + (u\cdot\nabla)\mathrm{curl} v = \partial_x \theta, \tag{37}$$

where we used the fact that

$$\mathrm{curl}[(u\cdot\nabla)v] + \mathrm{curl}(\sum_j v_j\nabla u_j) = (u\cdot\nabla)\mathrm{curl} v.$$

Then, from (37) and under the condition (8), we have that, for $t \le T$,

$$\|\mathrm{curl} v(t)\|_{L^\infty} \le \|\mathrm{curl} v_0\|_{L^\infty} + \int_0^t \|\partial_x\theta(\tau)\|_{L^\infty} d\tau < \infty. \tag{38}$$

Using Lemma 2.1, Lemma 2.3 and (36), one has

$$\|v\|_{L^\infty} \le \|v\|_2^{\frac{1}{2}} \|\mathrm{curl} v\|_{L^\infty}^{\frac{1}{2}} < \infty. \tag{39}$$

As a consequence, according to (39),

$$\|u\|_{L^\infty} \le \|v\|_{L^\infty}. \tag{40}$$

Multiplying both sides of (37) by curl$v$, integrating by parts in $\mathbb{R}^2$ and using the incompressible condition $\nabla\cdot u = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl} v(t)\|_{L^2}^2 \le \|\mathrm{curl} v\|_{L^2}\|\partial_x\theta\|_{L^2} \le \|\mathrm{curl} v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2. \tag{41}$$

On the other hand, we multiplying the third equation by $\partial_{xx}\theta$ and $\partial_{yy}\theta$, respectively, and adding the resulting equations up, then integrating by parts in $\mathbb{R}^2$ gives that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\partial_x\theta(t)\|_{L^2}^2 + \|\partial_y\theta(t)\|_{L^2}^2\right) + \|\partial_{xy}\theta\|_{L^2}^2 + \|\partial_{yy}\theta\|_{L^2}^2 \\ =&\int_{\mathbb{R}^2}(u\cdot\nabla)\theta\cdot\partial_{xx}\theta dx + \int_{\mathbb{R}^2}(u\cdot\nabla)\theta\cdot\partial_{yy}\theta dx \\ =&-\int_{\mathbb{R}^2}\partial_x[(u\cdot\nabla)\theta]\partial_x\theta dx - \int_{\mathbb{R}^2}\partial_y[(u\cdot\nabla)\theta]\partial_y\theta dx \\ =&-\int_{\mathbb{R}^2}\partial_x u_1(\partial_x\theta)^2 dx - \int_{\mathbb{R}^2}\partial_x u_2\partial_y\theta\partial_x\theta dx \\ &-\int_{\mathbb{R}^2}\partial_y u_1\partial_y\theta\partial_x\theta dx - \int_{\mathbb{R}^2}\partial_y u_2(\partial_y\theta)^2 dx \\ =&J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{42}$$

Using the incompressible condition $\partial_x u_1 + \partial_y u_2 = 0$ and integrating by parts, by the Hölder and Young inequalities, one has

$$\begin{aligned} J_1 =& -\int_{\mathbb{R}^2}\partial_x u_1(\partial_x\theta)^2 dx = \int_{\mathbb{R}^2}\partial_y u_2(\partial_x\theta)^2 dx = -2\int_{\mathbb{R}^2} u_2\partial_{xy}\theta\partial_x\theta dx \\ \le& C\|u_2\|_{L^\infty}\|\partial_{xy}\theta\|_{L^2}\|\partial_x\theta\|_{L^2} \\ \le& \frac{1}{4}\|\partial_{xy}\theta\|_{L^2}^2 + C\|u_2\|_{L^\infty}^2\|\partial_x\theta\|_{L^2}^2 \\ \le& \frac{1}{4}\|\partial_{xy}\theta\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\partial_x\theta\|_{L^2}^2, \end{aligned} \tag{43}$$

$$
\begin{aligned}
J_2 = &-\int_{\mathbb{R}^2} \partial_x u_2 \partial_y \theta \partial_x \theta dx \le \|\partial_x \theta\|_{L^\infty} \|\partial_x u_2\|_{L^2} \|\partial_y \theta\|_{L^2} \\
\le &C\|\partial_x \theta\|_{L^\infty} \|\mathrm{curl} u\|_{L^2} \|\partial_y \theta\|_{L^2} \\
\le &C\|\partial_x \theta\|_{L^\infty} \|\mathrm{curl} v\|_{L^2} \|\partial_y \theta\|_{L^2} \\
\le &C\|\partial_x \theta\|_{L^\infty} (\|\mathrm{curl} v\|_{L^2}^2 + \|\partial_y \theta\|_{L^2}^2),
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
J_3 = &-\int_{\mathbb{R}^2} \partial_y u_1 \partial_y \theta \partial_x \theta dx \le \|\partial_x \theta\|_{L^\infty} \|\partial_y u_1\|_{L^2} \|\partial_y \theta\|_{L^2} \\
\le &C\|\partial_x \theta\|_{L^\infty} \|\mathrm{curl} u\|_{L^2} \|\partial_y \theta\|_{L^2} \\
\le &C\|\partial_x \theta\|_{L^\infty} \|\mathrm{curl} v\|_{L^2} \|\partial_y \theta\|_{L^2} \\
\le &C\|\partial_x \theta\|_{L^\infty} (\|\mathrm{curl} v\|_{L^2}^2 + \|\partial_y \theta\|_{L^2}^2).
\end{aligned} \tag{45}
$$

For the last term J_4 in (42), using Lemma 2.6 and the Young inequality, we have

$$
\begin{aligned}
J_4 = &-\int_{\mathbb{R}^2} \partial_y u_2 (\partial_y \theta)^2 dx \\
\le &C\|\partial_y u_2\|_{L^2} \|\partial_y \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{yx} \theta\|_{L^2}^{\frac{1}{2}} \|\partial_y \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} \theta\|_{L^2}^{\frac{1}{2}} \\
\le &C\|\mathrm{curl} u\|_{L^2} \|\partial_y \theta\|_{L^2} \cdot \|\partial_{xy} \theta\|_{L^2}^{\frac{1}{2}} \cdot \|\partial_{yy} \theta\|_{L^2}^{\frac{1}{2}} \\
\le &\frac{1}{4}\|\partial_{xy} \theta\|_{L^2}^2 + \frac{1}{2}\|\partial_{yy} \theta\|_{L^2}^2 + C\|\mathrm{curl} u\|_{L^2}^2 \|\partial_y \theta\|_{L^2}^2 \\
\le &\frac{1}{4}\|\partial_{xy} \theta\|_{L^2}^2 + \frac{1}{2}\|\partial_{yy} \theta\|_{L^2}^2 + C\|\mathrm{curl} v\|_{L^2}^2 \|\partial_y \theta\|_{L^2}^2.
\end{aligned} \tag{46}
$$

Inserting the above estimates (43)-(46) into (42), it gives that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\mathrm{curl} v(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2\right) + \|\partial_y \nabla \theta\|_{L^2}^2 \\
\le &C(1 + \|\partial_x \theta\|_{L^\infty} + \|u\|_{L^\infty} + \|\partial_y \theta\|_{L^2}^2)(\|\mathrm{curl} v\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2).
\end{aligned} \tag{47}
$$

Using the Gronwall inequality and according to (8), (35) and (39), it follows from (47) that, for any $t \in [0, T]$,

$$
\|\mathrm{curl} v(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2 + \int_0^t \|\partial_y \nabla \theta(\tau)\|_{L^2}^2 < \infty. \tag{48}
$$

Next, we establish the H^3−norm bound for (v, θ), namely, for any $t \in [0, T]$,

$$
\|\Delta \mathrm{curl} v(t)\|_{L^2}^2 + \|\nabla^3 \theta(t)\|_{L^2}^2 < \infty.
$$

Applying the operator Δ to (37), multiplying the resulting equation by $\Delta \mathrm{curl} v$, integrating by parts in $\mathbb{R}^2$, we have

$$
\frac{1}{2}\frac{d}{dt}\|\Delta \mathrm{curl} v(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta[(u \cdot \nabla)\mathrm{curl} v] \cdot \Delta \mathrm{curl} v dx + \int_{\mathbb{R}^2} \Delta \partial_x \theta \cdot \Delta \mathrm{curl} v dx = K_1 + K_2. \tag{49}
$$

Using the incompressible condition $\nabla \cdot u = 0$, the Hölder inequality and Lemma 2.2, one can get

$$
\begin{aligned}
K_1 = &\int_{\mathbb{R}^2} [(u \cdot \nabla)\Delta \mathrm{curl} v - \Delta((u \cdot \nabla)\mathrm{curl} v)] \cdot \Delta \mathrm{curl} v dx \\
\le &\|\Delta((u \cdot \nabla)\mathrm{curl} v) - (u \cdot \nabla)\Delta \mathrm{curl} v\|_{L^2} \|\Delta \mathrm{curl} v\|_{L^2} \\
\le &C\|\nabla u\|_{L^\infty} \|\Delta \mathrm{curl} v\|_{L^2}^2 + C\|\Delta u\|_{L^\infty} \|\Delta \mathrm{curl} v\|_{L^2}^2
\end{aligned}
$$

$$\begin{aligned}&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\cdot\|\Delta \mathrm{curl} v\|_{L^2}^2+C\|v\|_{L^\infty}\|\Delta \mathrm{curl} v\|_{L^2}^2\\&\le C\|\Delta \mathrm{curl} v\|_{L^2}^2,\end{aligned}\tag{50}$$

where we used the following Gagliardo-Nirenberg inequality in Lemma 2.1

$$\|f\|_{L^\infty}\le C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2}^{\frac{1}{2}}$$

in the third inequality, and (36), (40) and (48) in the fourth inequality.

For the term K_2, we have

$$\begin{aligned}K_2\le&\|\partial_x\Delta\theta\|_{L^2}\|\Delta \mathrm{curl} v\|_{L^2}\le C(\|\partial_x\Delta\theta\|_{L^2}^2+\|\Delta \mathrm{curl} v\|_{L^2}^2)\\ \le& C(\|\nabla^3\theta\|_{L^2}^2+\|\Delta \mathrm{curl} v\|_{L^2}^2).\end{aligned}\tag{51}$$

Plugging (50) and (51) into (49), one has

$$\frac{d}{dt}\|\Delta \mathrm{curl} v(t)\|_{L^2}^2\le C(\|\nabla^3\theta\|_{L^2}^2+\|\Delta \mathrm{curl} v\|_{L^2}^2).\tag{52}$$

Applying the operator ∇^3 to the third equation of (7), multiplying the resulting equation with $\nabla^3\theta$ and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^3\theta(t)\|_{L^2}^2+\|\partial_y\nabla^3\theta\|_{L^2}^2\\=&-\int_{\mathbb{R}^2}\nabla^3[(u\cdot\nabla)\theta]\cdot\nabla^3\theta dx\\=&-\int_{\mathbb{R}^2}\partial_x\Delta[(u\cdot\nabla)\theta]\partial_x\Delta\theta dx-\int_{\mathbb{R}^2}\partial_y\Delta[(u\cdot\nabla)\theta]\partial_y\Delta\theta dx\\=&L_1+L_2.\end{aligned}\tag{53}$$

The term L_1 can be written as

$$\begin{aligned}L_1=&-\int_{\mathbb{R}^2}\partial_x\Delta(u_1\partial_x\theta+u_2\partial_y\theta)\partial_x\Delta\theta dx\\=&-\int_{\mathbb{R}^2}\partial_x\Delta u_1\partial_x\theta\partial_x\Delta\theta dx-\int_{\mathbb{R}^2}\partial_x\Delta u_2\partial_y\theta\partial_x\Delta\theta dx\\&-\int_{\mathbb{R}^2}\Delta u_1\partial_{xx}\theta\partial_x\Delta\theta dx-\int_{\mathbb{R}^2}\Delta u_2\partial_{xy}\theta\partial_x\Delta\theta dx\\&-2\int_{\mathbb{R}^2}\partial_{xx}u_1\partial_{xx}\theta\partial_x\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_{xy}u_1\partial_{xy}\theta\partial_x\Delta\theta dx\\&-2\int_{\mathbb{R}^2}\partial_{xx}u_2\partial_{xy}\theta\partial_x\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_{xy}u_2\partial_{yy}\theta\partial_x\Delta\theta dx\\&-\int_{\mathbb{R}^2}[2(\partial_x u_1\partial_{xxx}\theta+\partial_y u_1\partial_{xxy}\theta+\partial_x u_2\partial_{xxy}\theta+\partial_y u_2\partial_{xyy}\theta)\\&+\partial_x u_1\partial_x\Delta\theta+\partial_x u_2\partial_y\Delta\theta]\partial_x\Delta\theta dx\\=&M_1+\cdots+M_9.\end{aligned}\tag{54}$$

Now we bound the terms of L_1 one by one. Using the Hölder inequality, we have

$$\begin{aligned}M_1=&-\int_{\mathbb{R}^2}\partial_x\Delta u_1\partial_x\theta\partial_x\Delta\theta dx\\ \le& C\|\partial_x\theta\|_{L^\infty}\|\Delta\partial_x u_1\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}\\ \le& C\|\partial_x\theta\|_{L^\infty}\|\Delta\nabla u\|_{L^2}\|\nabla^3\theta\|_{L^2}\end{aligned}$$

$$\begin{aligned}&\le C\|\partial_x\theta\|_{L^\infty}\|\Delta \mathrm{curl} v\|_{L^2}\|\nabla^3\theta\|_{L^2}\\&\le C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2).\end{aligned}\tag{55}$$

Thanks to Lemma 2.6 and the Young inequality, it gives that

$$\begin{aligned}M_2=&-\int_{\mathbb{R}^2}\partial_x\Delta u_2\partial_y\theta\partial_x\Delta\theta dx\\&\le C\|\Delta\partial_x u_2\|_{L^2}\|\partial_y\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12}\|\partial_x\Delta\theta\|_{L^2}^{\frac12}\|\partial_{xy}\Delta\theta\|_{L^2}^{\frac12}\\&\le C\|\Delta \mathrm{curl} u\|_{L^2}\|\partial_y\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12}\|\partial_x\Delta\theta\|_{L^2}^{\frac12}\cdot\|\partial_{xy}\Delta\theta\|_{L^2}^{\frac12}\\&\le\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2+C\|\partial_y\theta\|_{L^2}^{\frac23}\|\partial_{xy}\theta\|_{L^2}^{\frac23}\cdot\|\Delta \mathrm{curl} u\|_{L^2}^{\frac43}\|\partial_x\Delta\theta\|_{L^2}^{\frac23}\\&\le\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2+C\|\partial_y\theta\|_{L^2}^{\frac23}\|\partial_{xy}\theta\|_{L^2}^{\frac23}(\|\Delta \mathrm{curl} u\|_{L^2}^2+\|\partial_x\Delta\theta\|_{L^2}^2)\\&\le\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2+C\|\partial_y\theta\|_{L^2}^{\frac23}\|\partial_{xy}\theta\|_{L^2}^{\frac23}(\|\Delta \mathrm{curl} v\|_{L^2}^2+\|\partial_x\Delta\theta\|_{L^2}^2).\end{aligned}\tag{56}$$

Noting that by $\partial_x u_1+\partial_y u_2=0$

$$\Delta u_1=\partial_{xx}u_1+\partial_{yy}u_1=-\partial_{xy}u_2+\partial_{yy}u_1=-\partial_y \mathrm{curl} u,$$

then, integrating by parts and using the Hölder inequality and Lemma 2.6, one has

$$\begin{aligned}M_3=&-\int_{\mathbb{R}^2}\Delta u_1\partial_{xx}\theta\partial_x\Delta\theta dx\\&=\int_{\mathbb{R}^2}\partial_y \mathrm{curl} u\partial_{xx}\theta\partial_x\Delta\theta dx\\&=-\int_{\mathbb{R}^2}\mathrm{curl} u\partial_{xxy}\theta\partial_x\Delta\theta dx-\int_{\mathbb{R}^2}\mathrm{curl} u\partial_{xx}\theta\partial_{xy}\Delta\theta dx\\&\le C\|\mathrm{curl} u\|_{L^\infty}\|\partial_y\partial_{xx}\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}+C\|\mathrm{curl} u\|_{L^\infty}\|\partial_{xx}\theta\|_{L^2}\|\partial_{xy}\Delta\theta\|_{L^2}\\&\le\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2+C\|\mathrm{curl} u\|_{L^\infty}^2\|\nabla^3\theta\|_{L^2}^2+C\|\mathrm{curl} u\|_{L^\infty}^2\|\nabla^2\theta\|_{L^2}^2\\&\le\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2+C\|\mathrm{curl} u\|_{L^\infty}^2\|\nabla^3\theta\|_{L^2}^2+C\|\mathrm{curl} v\|_{L^\infty}^2\|\theta\|_{H^3}^2\end{aligned}\tag{57}$$

and

$$\begin{aligned}M_4=&-\int_{\mathbb{R}^2}\Delta u_2\partial_{xy}\theta\partial_x\Delta\theta dx\\&=\int_{\mathbb{R}^2}\partial_y\Delta u_2\partial_x\theta\partial_x\Delta\theta dx+\int_{\mathbb{R}^2}\Delta u_2\partial_x\theta\partial_{xy}\Delta\theta dx\\&\le C\|\partial_x\theta\|_{L^\infty}\|\Delta\partial_y u_2\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}\\&\quad+C\|\partial_{xy}\Delta\theta\|_{L^2}\|\Delta u_2\|_{L^2}^{\frac12}\|\partial_x\Delta u_2\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12}\\&\le C\|\partial_x\theta\|_{L^\infty}\|\Delta \mathrm{curl} u\|_{L^2}\|\nabla^3\theta\|_{L^2}+\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2\\&\quad+C\|\Delta u_2\|_{L^2}\|\Delta \mathrm{curl} u\|_{L^2}\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}\\&\le C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2)+\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2\\&\quad+C\|v\|_{L^2}\|\Delta \mathrm{curl} v\|_{L^2}\cdot\|\nabla\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2}\\&\le C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2)+\frac{1}{32}\|\partial_{xy}\Delta\theta\|_{L^2}^2\\&\quad+C\|\nabla\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2}\|\mathrm{curl} v\|_{H^2}^2.\end{aligned}\tag{58}$$

For the term M_5, integrating by parts and using $\partial_x u_1 + \partial_y u_2 = 0$ and Lemma 2.6, we have

$$
\begin{aligned}
M_5 =& -2\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_{xx}\theta\partial_x\Delta\theta dx \\
=& -2\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_{xx}\theta\partial_{xxx}\theta dx - 2\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_{xx}\theta\partial_{xyy}\theta dx \\
=& -\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_x(\partial_{xx}\theta)^2 dx - 2\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_{xx}\theta\partial_{xyy}\theta dx \\
=& \int_{\mathbb{R}^2} \partial_{xxx}u_1(\partial_{xx}\theta)^2 dx + 2\int_{\mathbb{R}^2} (\partial_{xxx}u_1\partial_x\theta\partial_{xyy}\theta + \partial_{xx}u_1\partial_x\theta\partial_{xxyy}\theta) dx \\
=& -\int_{\mathbb{R}^2} \partial_{xxy}u_2(\partial_{xx}\theta)^2 dx + 2\int_{\mathbb{R}^2} (\partial_{xxx}u_1\partial_x\theta\partial_{xyy}\theta + \partial_{xx}u_1\partial_x\theta\partial_{xxyy}\theta) dx \\
=& 2\int_{\mathbb{R}^2} \partial_{xx}u_2\partial_{xx}\theta\partial_{xxy}\theta dx \\
& + 2\int_{\mathbb{R}^2} \partial_{xxx}u_1\partial_x\theta\partial_{xyy}\theta dx + 2\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_x\theta\partial_{xxyy}\theta dx \\
=& -2\int_{\mathbb{R}^2} \partial_{xxx}u_2\partial_x\theta\partial_{xxy}\theta dx - 2\int_{\mathbb{R}^2} \partial_{xx}u_2\partial_x\theta\partial_{xxxy}\theta dx \\
& + 2\int_{\mathbb{R}^2} \partial_{xxx}u_1\partial_x\theta\partial_{xyy}\theta dx + 2\int_{\mathbb{R}^2} \partial_{xx}u_1\partial_x\theta\partial_{xxyy}\theta dx \\
=& M_{51} + M_{52} + M_{53} + M_{54}.
\end{aligned}
\tag{59}
$$

Using Lemma 2.3, Lemma 2.6, the Hölder inequality and the Young inequality, we have

$$
\begin{aligned}
M_{51} \le& C\|\partial_x\theta\|_{L^\infty}\|\partial_{xx}\mathrm{curl}u\|_{L^2}\|\partial_{xxy}\theta\|_{L^2} \\
\le& C\|\partial_x\theta\|_{L^\infty}(\|\partial_{xx}\mathrm{curl}v\|_{L^2}^2 + \|\partial_{xxy}\theta\|_{L^2}^2) \\
\le& C\|\partial_x\theta\|_{L^\infty}(\|\Delta\mathrm{curl}v\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2),
\end{aligned}
$$

$$
\begin{aligned}
M_{52} \le& C\|\partial_{xxxy}\theta\|_{L^2}\|\partial_{xx}u_2\|_{L^2}^{\frac12}\|\partial_{xxx}u_2\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12} \\
\le& \frac{1}{64}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\Delta u_2\|_{L^2}\|\Delta\mathrm{curl}u\|_{L^2}\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2} \\
\le& \frac{1}{64}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|v\|_{L^2}\|\Delta\mathrm{curl}v\|_{L^2}\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2} \\
\le& \frac{1}{64}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}\|\mathrm{curl}v\|_{H^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
M_{53} \le& C\|\partial_x\theta\|_{L^\infty}\|\partial_{xx}\nabla u\|_{L^2}\|\partial_{xyy}\theta\|_{L^2} \\
\le& C\|\partial_x\theta\|_{L^\infty}\|\partial_{xx}\mathrm{curl}u\|_{L^2}\|\nabla^3\theta\|_{L^2} \\
\le& C\|\partial_x\theta\|_{L^\infty}\|\Delta\mathrm{curl}v\|_{L^2}\|\nabla^3\theta\|_{L^2} \\
\le& C\|\partial_x\theta\|_{L^\infty}(\|\Delta\mathrm{curl}v\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2),
\end{aligned}
$$

$$
\begin{aligned}
M_{54} \le& C\|\partial_y(\partial_{xyy}\theta)\|_{L^2}\|\partial_{xx}u_1\|_{L^2}^{\frac12}\|\partial_{xxx}u_1\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12} \\
\le& C\|\partial_y\nabla^3\theta\|_{L^2}\|\Delta u_1\|_{L^2}^{\frac12}\|\partial_{xx}\nabla u\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12} \\
\le& \frac{1}{64}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\Delta u_1\|_{L^2}\|\partial_{xx}\mathrm{curl}u\|_{L^2}\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{64}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|v\|_{L^2}\|\Delta \mathrm{curl} v\|_{L^2}\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}\\
&\le \frac{1}{64}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}\|\mathrm{curl} v\|_{H^2}^2.
\end{aligned}
$$

Inserting the above estimates into (59), we have

$$
\begin{aligned}
M_5 \le& \frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2)\\
&+ C\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}\|\mathrm{curl} v\|_{H^2}^2\\
\le& \frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2)\\
&+ C\|\nabla\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2}\|\mathrm{curl} v\|_{H^2}^2.
\end{aligned} \tag{60}
$$

For the terms $M_6 - M_8$, integrating by parts and using the Hölder inequality and Young inequality and Lemma 2.6, one has

$$
\begin{aligned}
M_6 =& -2\int_{\mathbb{R}^2} \partial_{xy}u_1\partial_{xy}\theta\partial_x\Delta\theta dx\\
=& 2\int_{\mathbb{R}^2} \partial_y u_1\partial_{xyy}\theta\partial_x\Delta\theta dx + 2\int_{\mathbb{R}^2} \partial_y u_1\partial_{xy}\theta\partial_y\partial_x\Delta\theta dx\\
\le& C\|\mathrm{curl} u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\mathrm{curl} u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}\|\partial_y\nabla^3\theta\|_{L^2}\\
\le& \frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\mathrm{curl} u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\mathrm{curl} u\|_{L^\infty}^2\|\nabla^2\theta\|_{L^2}^2\\
\le& \frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\mathrm{curl} u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\mathrm{curl} u\|_{L^\infty}^2\|\theta\|_{H^3}^2,
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
M_7 =& -2\int_{\mathbb{R}^2} \partial_{xx}u_2\partial_{xy}\theta\partial_x\Delta\theta dx\\
=& 2\int_{\mathbb{R}^2} \partial_{xxy}u_2\partial_x\theta\partial_x\Delta\theta dx + 2\int_{\mathbb{R}^2} \partial_{xx}u_2\partial_x\theta\partial_y\partial_x\Delta\theta dx\\
\le& C\|\partial_x\theta\|_{L^\infty}\|\partial_{xx}\mathrm{curl} u\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}\\
&+ C\|\partial_y\partial_x\Delta\theta\|_{L^2}\|\partial_x\mathrm{curl} u\|_{L^2}^{\frac12}\|\partial_{xx}\mathrm{curl} u\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12}\\
\le& C\|\partial_x\theta\|_{L^\infty}\|\Delta \mathrm{curl} v\|_{L^2}\|\nabla^3\theta\|_{L^2}\\
&+ C\|\partial_y\nabla^3\theta\|_{L^2}\|v\|_{L^2}^{\frac12}\|\Delta \mathrm{curl} v\|_{L^2}^{\frac12}\|\partial_x\theta\|_{L^2}^{\frac12}\|\partial_{xy}\theta\|_{L^2}^{\frac12}\\
\le& \frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2)\\
&+ C\|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2}\|\mathrm{curl} v\|_{H^2}^2\\
\le& \frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2 + C\|\partial_x\theta\|_{L^\infty}(\|\Delta \mathrm{curl} v\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2)\\
&+ C\|\nabla\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2}\|\mathrm{curl} v\|_{H^2}^2
\end{aligned} \tag{62}
$$

and

$$
\begin{aligned}
M_8 =& -2\int_{\mathbb{R}^2} \partial_{xy}u_2\partial_{yy}\theta\partial_x\Delta\theta dx\\
=& 2\int_{\mathbb{R}^2} \partial_x u_2\partial_{yyy}\theta\partial_x\Delta\theta dx + 2\int_{\mathbb{R}^2} \partial_x u_2\partial_{yy}\theta\partial_y\partial_x\Delta\theta dx\\
\le& C\|\mathrm{curl} u\|_{L^\infty}\|\partial_{yyy}\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}\\
&+ C\|\partial_y\partial_x\Delta\theta\|_{L^2}\|\partial_x u_2\|_{L^2}^{\frac12}\|\partial_{xx}u_2\|_{L^2}^{\frac12}\|\partial_{yy}\theta\|_{L^2}^{\frac12}\|\partial_{yyy}\theta\|_{L^2}^{\frac12}
\end{aligned}
$$

$$\begin{aligned}&\le C\|\mathrm{curl}u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2+C\|\partial_y\nabla^3\theta\|_{L^2}\|\mathrm{curl}v\|_{L^2}^{\frac12}\|v\|_{L^2}^{\frac12}\|\theta\|_{H^3}\\&\le\frac{1}{32}\|\partial_y\nabla^3\theta\|_{L^2}^2+C\|\mathrm{curl}v\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2+C\|\mathrm{curl}v\|_{L^2}\|v\|_{L^2}\|\theta\|_{H^3}^2.\end{aligned}\tag{63}$$

It is easy to estimate the term M_9 that, by Lemma 2.4, we can get

$$\begin{aligned}M_9\le&C\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\\\le&C(1+\|u\|_{L^2}+\|\Delta u\|_{L^2}\log(1+\|\nabla^3u\|_{L^2}))\|\nabla^3\theta\|_{L^2}\\\le&C(1+\|u\|_{L^2}+\|v\|_{L^2}\log(1+\|\Delta\mathrm{curl}v\|_{L^2}))\|\nabla^3\theta\|_{L^2}.\end{aligned}\tag{64}$$

Inserting the above estimates (55)-(58) and (60)-(64) into (54) gives that

$$\begin{aligned}L_1\le&\frac14\|\partial_y\nabla^3\theta\|_{L^2}^2+C[1+\|\partial_x\theta\|_{L^\infty}+\|\mathrm{curl}v\|_{L^\infty}+\|\nabla\theta\|_{L^2}^{\frac23}\|\partial_y\nabla\theta\|_{L^2}^{\frac23}+\|u\|_{L^2}\\&+\|v\|_{L^2}\log(1+\|\Delta\mathrm{curl}v\|_{L^2})](\|\Delta\mathrm{curl}v\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2)\\&+C(\|\mathrm{curl}v\|_{L^\infty}^2+\|\nabla\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2}+\|\mathrm{curl}v\|_{L^2}\|v\|_{L^2})(\|\mathrm{curl}v\|_{H^2}^2+\|\theta\|_{H^3}^2)\\\le&\frac14\|\partial_y\nabla^3\theta\|_{L^2}^2+C[1+\log(1+\|\Delta\mathrm{curl}v\|_{L^2})](\|\Delta\mathrm{curl}v\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2)\\&+C(\|\mathrm{curl}v\|_{H^2}^2+\|\theta\|_{H^3}^2),\end{aligned}\tag{65}$$

where we used (8), (36), (38), (40) and (48).

The term L_2 can be further written as

$$\begin{aligned}L_2=&-\int_{\mathbb{R}^2}\partial_y\Delta u_1\partial_x\theta\partial_y\Delta\theta dx-\int_{\mathbb{R}^2}\partial_y\Delta u_2\partial_y\theta\partial_y\Delta\theta dx\\&-\int_{\mathbb{R}^2}\Delta u_1\partial_{xy}\theta\partial_y\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_{xy}u_1\partial_{xx}\theta\partial_y\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_{yy}u_1\partial_{xy}\theta\partial_y\Delta\theta dx\\&-\int_{\mathbb{R}^2}\partial_{xx}u_2\partial_{xy}\theta\partial_y\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_{xy}u_2\partial_{xy}\theta\partial_y\Delta\theta dx-3\int_{\mathbb{R}^2}\partial_{yy}u_2\partial_{yy}\theta\partial_y\Delta\theta dx\\&-2\int_{\mathbb{R}^2}\partial_xu_1\partial_{xyy}\theta\partial_y\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_yu_1\partial_{xyy}\theta\partial_y\Delta\theta dx\\&-2\int_{\mathbb{R}^2}\partial_xu_2\partial_{xyy}\theta\partial_y\Delta\theta dx-2\int_{\mathbb{R}^2}\partial_yu_2\partial_{yyy}\theta\partial_y\Delta\theta dx\\&-\int_{\mathbb{R}^2}\partial_yu_1\partial_x\Delta\theta\partial_y\Delta\theta dx-\int_{\mathbb{R}^2}\partial_yu_2\partial_y\Delta\theta\partial_y\Delta\theta dx.\end{aligned}\tag{66}$$

Due to the existence of the "favorable" derivative ∂_y, the estimates of L_2 is simpler compared with L_1. Thus, we can obtain the estimates similar to L_1. Therefore, we omit the details here.

Now, combining (65) and the estimates of (66) with (53), it follows that

$$\begin{aligned}&\frac12\frac{d}{dt}\left(\|\mathrm{curl}v(t)\|_{H^2}^2+\|\theta(t)\|_{H^3}^2\right)+\|\partial_y\nabla^3\theta\|_{L^2}^2\\\le&C(1+\log(1+\|\Delta\mathrm{curl}v\|_{L^2}^2))(\|\Delta\mathrm{curl}v\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2)\\&+C(\|\mathrm{curl}v\|_{H^2}^2+\|\theta\|_{H^3}^2).\end{aligned}$$

This together with the Gronwall inequality yields that

$$\|\mathrm{curl}v\|_{H^2}^2+\|\theta\|_{H^3}^2<\infty,$$

which completes the proof of Theorem 1.2. □

Acknowledgments. The authors thank the editor and reviewer for valuable comments and suggestions.

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Received for publication January 2020.

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