

## THE BOUNDEDNESS AND UPPER SEMICONTINUITY OF THE PULLBACK ATTRACTORS FOR A 2D MICROPOLAR FLUID FLOWS WITH DELAY

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**ABSTRACT.** In this paper, two properties of the pullback attractor for a 2D non-autonomous micropolar fluid flows with delay on unbounded domains are investigated. First, we establish the  $H^1$ -boundedness of the pullback attractor. Further, with an additional regularity limit on the force and moment with respect to time  $t$ , we remark the  $H^2$ -boundedness of the pullback attractor. Then, we verify the upper semicontinuity of the pullback attractor with respect to the domains.

**1. Introduction.** The micropolar fluid model is a qualitative generalization of the well-known Navier-Stokes model in the sense that it takes into account the microstructure of fluid [7]. The model was first derived in 1966 by Eringen [4] to describe the motion of a class of non-Newtonian fluid with micro-rotational effects and inertia involved. It can be expressed by the following equations:

$$\begin{cases} \frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \operatorname{rot} \omega + (u \cdot \nabla)u + \nabla p = f, \\ \frac{\partial \omega}{\partial t} - (c_a + c_d)\Delta \omega + 4\nu_r \omega + (u \cdot \nabla)\omega \\ \quad - (c_0 + c_d - c_a)\nabla \operatorname{div} \omega - 2\nu_r \operatorname{rot} u = \tilde{f}, \\ \nabla \cdot u = 0, \end{cases} \quad (1)$$

where  $u = (u_1, u_2, u_3)$  is the velocity,  $\omega = (\omega_1, \omega_2, \omega_3)$  is the angular velocity field of rotation of particles,  $p$  represents the pressure,  $f = (f_1, f_2, f_3)$  and  $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$  stand for the external force and moment, respectively. The positive parameters  $\nu, \nu_r, c_0, c_a$  and  $c_d$  are viscous coefficients. Actually,  $\nu$  represents the usual Newtonian viscosity and  $\nu_r$  is called microrotation viscosity.

Micropolar fluid models play an important role in the fields of applied and computational mathematics. There is a rich literature on the mathematical theory of micropolar fluid model. Particularly, the existence, uniqueness and regularity of solutions for the micropolar fluid flows have been investigated in [6]. Extensive studies on long time behavior of solutions for the micropolar fluid flows have also been done. For example, in the case of 2D bounded domains: Łukaszewicz [7] established the existence of  $L^2$ -global attractors and its Hausdorff dimension and fractal dimension

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estimation. Chen, Chen and Dong proved the existence of  $H^2$ -global attractor and uniform attractor in [1] and [2], respectively. Łukaszewicz and Tarasińska [9] investigated the existence of  $H^1$ -pullback attractor. Zhao, Sun and Hsu [18] established the existence of  $L^2$ -pullback attractor and  $H^1$ -pullback attractor of solutions for a universe given by a tempered condition, respectively. For the case of 2D unbounded domains: Dong and Chen [3] investigated the existence and regularity of global attractors. Zhao, Zhou and Lian [19] established the existence of  $H^1$ -uniform attractor and further gave the inclusion relation between  $L^2$ -uniform attractor and the  $H^1$ -uniform attractor. Sun and Li [15] verified the existence of pullback attractor and further investigated the tempered behavior and upper semicontinuity of the pullback attractor. More recently, Sun, Cheng and Han [14] investigated the existence of random attractors for 2D stochastic micropolar fluid flows.

As we know, in the real world, delay terms appear naturally, for instance as effects in wind tunnel experiments (see [10]). Also the delay situations may occur when we want to control the system via applying a force which considers not only the present state but also the history state of the system. The delay of partial differential equations (PDE) includes finite delays (constant, variable, distributed, etc) and infinite delays. Different types of delays need to be treated by different approaches.

In this paper, we consider the situation that the velocity component  $u_3$  in the  $x_3$ -direction is zero and the axes of rotation of particles are parallel to the  $x_3$  axis, that is  $u = (u_1, u_2, 0)$ ,  $\omega = (0, 0, \omega_3)$ ,  $f = (f_1, f_2, 0)$ ,  $\tilde{f} = (0, 0, \tilde{f}_3)$ . Let  $\Omega \subseteq \mathbb{R}^2$  be an open set with boundary  $\Gamma$  that is not necessarily bounded but satisfies the following Poincaré inequality:

$$\text{There exists } \lambda_1 > 0 \text{ such that } \lambda_1 \|\varphi\|_{L^2(\Omega)}^2 \leq \|\nabla \varphi\|_{L^2(\Omega)}^2, \quad \forall \varphi \in H_0^1(\Omega). \quad (2)$$

Then we discuss the following 2D non-autonomous incompressible micropolar fluid flows with finite delay:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla)u + \nabla p \\ \qquad \qquad \qquad = f(t, x) + g(t, u_t), \quad \text{in } (\tau, +\infty) \times \Omega, \\ \frac{\partial \omega}{\partial t} - \bar{\alpha}\Delta \omega + 4\nu_r \omega - 2\nu_r \nabla \times u + (u \cdot \nabla)\omega \\ \qquad \qquad \qquad = \tilde{f}(t, x) + \tilde{g}(t, \omega_t), \quad \text{in } (\tau, +\infty) \times \Omega, \\ \nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \Omega, \end{array} \right. \quad (3)$$

where  $\bar{\alpha} := c_0 + 2c_d > 0$ ,  $x := (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$ ,  $u := (u_1, u_2)$ ,  $g$  and  $\tilde{g}$  stand for the external force containing some hereditary characteristics  $u_t$  and  $\omega_t$ , which are defined on  $(-h, 0)$  as follows

$$u_t(s) := u(t+s), \quad \omega_t(s) := \omega(t+s), \quad \forall t \geq \tau, \quad s \in (-h, 0).$$

where  $h$  is a positive fixed number, and

$$\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla \times \omega := \left( \frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1} \right).$$

To complete the formulation of the initial boundary value problem to system (3), we give the following initial boundary conditions:

$$(u(\tau), \omega(\tau)) = (u^{in}, \omega^{in}), (u_\tau(s), \omega_\tau(s)) = (\phi_1^{in}(s), \phi_2^{in}(s)), \quad s \in (-h, 0), \quad (4)$$

$$u = 0, \quad \omega = 0, \quad \text{on } (\tau, +\infty) \times \Gamma. \quad (5)$$

For problem (3)-(5), Sun and Liu established the existence of pullback attractor in [16], recently.

The first purpose of this work is to investigate the boundedness of the pullback attractor obtained in [16]. We remark that García-Luengo, Marín-Rubio and Real [5] proved the  $H^2$ -boundedness of the pullback attractors of the 2D Navier-Stokes equations in bounded domains. Motivated by [5] and following its main idea, we generalize their results to the 2D micropolar fluid flows with finite delay in unbounded domains. Compared with the Navier-Stokes equations ( $\omega = 0, \nu_r = 0$ ), the micropolar fluid flow consists of the angular velocity field  $\omega$ , which leads to a different nonlinear term  $B(u, w)$  and an additional term  $N(u)$  in the abstract equations (13). In addition, the time-delay term considered in this work also increases the difficulty.

The second purpose of this work is to investigate the upper semicontinuity of the pullback attractor with respect to the domain  $\Omega$ . To this end, using the arguments in [15, 17], we first let  $\{\Omega_m\}_{m=1}^\infty$  be an expanding sequence of simply connected, bounded and smooth subdomains of  $\Omega$  such that  $\bigcup_{m=1}^\infty \Omega_m = \Omega$ . Then we consider the Cauchy problem (3)-(5) in  $\Omega_m$ . We will conclude that there exists a pullback attractor  $\widehat{\mathcal{A}}_{\widehat{H}(\Omega_m)}$  for the problem (3)-(5) in each  $\Omega_m$ . Finally, we establish the upper semicontinuity by showing  $\lim_{m \rightarrow \infty} \text{dist}_{E_{\widehat{H}}^2}(\mathcal{A}_{\widehat{H}(\Omega_m)}(t), \mathcal{A}_{\widehat{H}}(t)) = 0, \forall t \in \mathbb{R}$ .

Throughout this paper, we denote the usual Lebesgue space and Sobolev space by  $L^p(\Omega)$  and  $W^{m,p}(\Omega)$  endowed with norms  $\|\cdot\|_p$  and  $\|\cdot\|_{m,p}$ , respectively. Especially, we denote  $H^m(\Omega) := W^{m,2}(\Omega)$ .

$$\mathcal{V} := \mathcal{V}(\Omega) := \{\varphi \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \mid \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0\},$$

$$\widehat{\mathcal{V}} := \widehat{\mathcal{V}}(\Omega) := \mathcal{V} \times C_0^\infty(\Omega),$$

$$H := H(\Omega) := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega) \times L^2(\Omega), \text{ with norm } \|\cdot\|_H$$

and dual space  $H^*$ ,

$$V := V(\Omega) := \text{closure of } \mathcal{V} \text{ in } H^1(\Omega) \times H^1(\Omega), \text{ with norm } \|\cdot\|_V$$

and dual space  $V^*$ ,

$$\widehat{H} := \widehat{H}(\Omega) := \text{closure of } \widehat{\mathcal{V}} \text{ in } L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \text{ with norm } \|\cdot\|_{\widehat{H}}$$

and dual space  $\widehat{H}^*$ ,

$$\widehat{V} := \widehat{V}(\Omega) := \text{closure of } \widehat{\mathcal{V}} \text{ in } H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega), \text{ with norm } \|\cdot\|_{\widehat{V}}$$

and dual space  $\widehat{V}^*$ .

$(\cdot, \cdot)$ — the inner product in  $L^2(\Omega)$ ,  $H$  or  $\widehat{H}$ ,  $\langle \cdot, \cdot \rangle$ — the dual pairing between  $V$  and  $V^*$  or between  $\widehat{V}$  and  $\widehat{V}^*$ . Throughout this article, we simplify the notations  $\|\cdot\|_2, \|\cdot\|_H$  and  $\|\cdot\|_{\widehat{H}}$  by the same notation  $\|\cdot\|$  if there is no confusion. Furthermore,

we denote

$L^p(I; X) :=$  space of strongly measurable functions on the closed interval  $I$ ,  
with values in the Banach space  $X$ , endowed with norm

$$\|\varphi\|_{L^p(I;X)} := \left( \int_I \|\varphi\|_X^p dt \right)^{1/p}, \text{ for } 1 \leq p < \infty,$$

$\mathcal{C}(I; X) :=$  space of continuous functions on the interval  $I$ , with values  
in the Banach space  $X$ , endowed with the usual norm,

$L^2_{loc}(I; X) :=$  space of locally square integrable functions on the interval  $I$ ,  
with values in the Banach space  $X$ , endowed with the usual norm,

$\text{dist}_M(X, Y) -$  the Hausdorff semidistance between  $X \subseteq M$  and  $Y \subseteq M$  defined by

$$\text{dist}_M(X, Y) = \sup_{x \in X} \inf_{y \in Y} \text{dist}_M(x, y).$$

Following the above notations, we additionally denote

$$\begin{aligned} L^2_{\widehat{H}} &:= L^2(-h, 0; \widehat{H}), & L^2_{\widehat{V}} &:= L^2(-h, 0; \widehat{V}), \\ E^2_{\widehat{H}} &:= \widehat{H} \times L^2_{\widehat{H}}, & E^2_{\widehat{V}} &:= \widehat{V} \times L^2_{\widehat{V}}, & E^2_{\widehat{H} \times L^2_{\widehat{V}}} &:= \widehat{H} \times L^2_{\widehat{V}}. \end{aligned}$$

The norm  $\|\cdot\|_X$  for  $X \in \{E^2_{\widehat{H}}, E^2_{\widehat{V}}, E^2_{\widehat{H} \times L^2_{\widehat{V}}}\}$  is defined as

$$\begin{aligned} \|(w, v)\|_{E^2_{\widehat{H}}} &:= (\|w\|_{\widehat{H}}^2 + \|v\|_{L^2_{\widehat{H}}}^2)^{1/2}, & \|(w, v)\|_{E^2_{\widehat{V}}} &:= (\|w\|^2 + \|v\|_{L^2_{\widehat{V}}}^2)^{1/2}, \\ \|(w, v)\|_{E^2_{\widehat{H} \times L^2_{\widehat{V}}}} &:= (\|w\|^2 + \|v\|_{L^2_{\widehat{V}}}^2)^{1/2}. \end{aligned}$$

The rest of this paper is organized as follows. In section 2, we make some preliminaries. In section 3, we investigate the boundedness of the pullback attractor. In section 4, we prove the upper semicontinuity of the pullback attractor with respect to the domains.

**2. Preliminaries.** In this section, for the sake of discussion, we first introduce some useful operators and put problem (3)-(5) into an abstract form. Then we recall some important known results about the non-autonomous micropolar fluid flows.

To begin with, we define the operators  $A, B(\cdot, \cdot)$  and  $N(\cdot)$  by

$$\begin{cases} \langle Aw, \varphi \rangle := (\nu + \nu_r)(\nabla u, \nabla \Phi) + \bar{\alpha}(\nabla \omega, \nabla \varphi_3), \quad \forall w = (u, \omega), \varphi = (\Phi, \varphi_3) \in \widehat{V}, \\ \langle B(u, w), \varphi \rangle := ((u \cdot \nabla)w, \varphi), \quad \forall u \in V, w = (u, \omega) \in \widehat{V}, \forall \varphi \in \widehat{V}, \\ N(w) := (-2\nu_r \nabla \times \omega, -2\nu_r \nabla \times u + 4\nu_r \omega), \quad \forall w = (u, \omega) \in \widehat{V}. \end{cases} \tag{6}$$

What follows are some useful estimates and properties for the operators  $A, B(\cdot, \cdot)$  and  $N(\cdot)$ , which have been established in works [11, 13].

**Lemma 2.1.** (1) *The operator  $A$  is linear continuous both from  $\widehat{V}$  to  $\widehat{V}^*$  and from  $D(A)$  to  $\widehat{H}$ , and so is for the operator  $N(\cdot)$  from  $\widehat{V}$  to  $\widehat{H}$ , where  $D(A) := \widehat{V} \cap (H^2(\Omega))^3$ .*



**Assumption 2.2.** Assume that  $F(t, x) \in L^2_{loc}(\mathbb{R}; \widehat{H})$ ,  $\forall t \geq \tau, \tau \in \mathbb{R}$ , and there exists a  $\gamma \in (0, 2\delta_1 - 2C_G)$  such that

$$\int_{-\infty}^t e^{\gamma\theta} \|F(\theta, x)\|_{\widehat{V}^*}^2 d\theta < +\infty. \tag{14}$$

In order to facilitate the discussion, we denote by  $\mathcal{P}(X)$  the family of all nonempty subsets of  $X$ . Let  $\mathcal{D}$  be a nonempty class of families parameterized in time  $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$ , which will be called a universe in  $\mathcal{P}(X)$ . Based on these notations, we can construct the universe  $\mathcal{D}_\gamma$  in the following.

**Definition 2.3.** (Definition of universe  $\mathcal{D}_\gamma$ ) Set

$$\mathcal{R}_\gamma := \{\rho(t) : \mathbb{R} \mapsto \mathbb{R}_+ \mid \lim_{t \rightarrow -\infty} e^{\gamma t} \rho^2(t) = 0\}.$$

We denote by  $\mathcal{D}_\gamma$  the class of all families  $\widehat{D} = \{D(t) \mid t \in \mathbb{R}\} \subseteq \mathcal{P}(E^2_{\widehat{H}})$  such that

$$D(t) \subseteq \bar{B}_{E^2_{\widehat{H}}}(0, \rho_{\widehat{D}}(t)), \text{ for some } \rho_{\widehat{D}}(t) \in \mathcal{R}_\gamma,$$

where  $\bar{B}_{E^2_{\widehat{H}}}(0, \rho_{\widehat{D}}(t))$  represents the closed ball in  $E^2_{\widehat{H}}$  centered at zero with radius  $\rho_{\widehat{D}}(t)$ .

Based on the above assumptions, we can recall the global well-posedness of solutions and the existence of pullback attractor of problem (13).

**Proposition 2.1.** (Existence and uniqueness of solution, see [13, 16]) Let Assumption 2.1 and Assumption 2.2 hold. Then for any  $(w^{in}, \phi^{in}(s)) \in E^2_{\widehat{H}}$ , there exists a unique weak solution  $w(\cdot) := w(\cdot; \tau, w^{in}, \phi^{in}(s))$  for system (13), which satisfies

$$w \in \mathcal{C}([\tau, T]; \widehat{H}) \cap L^2(\tau, T; \widehat{V}) \text{ and } w' \in L^2(\tau, T; \widehat{V}^*), \forall T > \tau.$$

**Remark 2.1.** According to Proposition 2.1, the biparametric mapping defined by

$$U(t, \tau) : (w^{in}, \phi^{in}(s)) \mapsto (w(t; \tau, w^{in}, \phi^{in}(s)), w_t(s; \tau, w^{in}, \phi^{in}(s))), \forall t \geq \tau, \tag{15}$$

generates a continuous process in  $E^2_{\widehat{H}}$  and  $E^2_{\widehat{V}}$ , respectively, which satisfies the following properties:

- (i)  $U(\tau, \tau)(w^{in}, \phi^{in}(s)) = (w^{in}, \phi^{in}(s))$ ,
- (ii)  $U(t, \theta)U(\theta, \tau)(w^{in}, \phi^{in}(s)) = U(t, \tau)(w^{in}, \phi^{in}(s))$ .

**Proposition 2.2.** (Existence of pullback attractor, see [16]) Under the Assumption 2.1 and Assumption 2.2, there exists a pullback attractor  $\widehat{\mathcal{A}}_{\widehat{H}} = \{\mathcal{A}_{\widehat{H}}(t) \mid t \in \mathbb{R}\}$  for the process  $\{U(t, \tau)\}_{t \geq \tau}$  that satisfies the following properties:

- Compactness: for any  $t \in \mathbb{R}$ ,  $\mathcal{A}_{\widehat{H}}(t)$  is a nonempty compact subset of  $E^2_{\widehat{H}}$ ;
- Invariance:  $U(t, \tau)\mathcal{A}_{\widehat{H}}(\tau) = \mathcal{A}_{\widehat{H}}(t), \forall t \geq \tau$ ;
- Pullback attracting: for any  $\widehat{B} = \{B(\theta) \mid \theta \in \mathbb{R}\} \in \mathcal{D}_\gamma$ , it holds that

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{E^2_{\widehat{H}}}(U(t, \tau)B(\tau), \mathcal{A}_{\widehat{H}}(t)) = 0, \forall t \in \mathbb{R};$$

- Minimality: the family of sets  $\widehat{\mathcal{A}}_{\widehat{H}}$  is the minimal in the sense that if

$$\widehat{O} = \{O(t) \mid t \in \mathbb{R}\} \subseteq \mathcal{P}(E^2_{\widehat{H}}) \text{ is another family of closed sets such that}$$

$$\lim_{\tau \rightarrow -\infty} \text{dist}_{E^2_{\widehat{H}}}(U(t, \tau)B(\tau), O(t)) = 0, \text{ for any } \widehat{B} = \{B(\theta) \mid \theta \in \mathbb{R}\} \in \mathcal{D}_\gamma,$$

then  $\mathcal{A}_{\widehat{H}}(t) \subseteq O(t)$  for any  $t \in \mathbb{R}$ .

Finally, we introduce a useful lemma, which plays an important role in the proof of higher regularity of the pullback attractor.

**Lemma 2.4.** (see [12]) *Let  $X, Y$  be Banach spaces such that  $X$  is reflexive, and the inclusion  $X \subset Y$  is continuous. Assume that  $\{w_n\}_{n \geq 1}$  is a bounded sequence in  $L^\infty(\tau, t; X)$  such that  $w_n \rightharpoonup w$  weakly in  $L^q(\tau, t; X)$  for some  $q \in [1, +\infty)$  and  $w \in \mathcal{C}([\tau, t]; Y)$ . Then  $w(t) \in X$  and*

$$\|w(\theta)\|_X \leq \liminf_{n \rightarrow +\infty} \|w_n(\theta)\|_{L^\infty(\tau, t; X)}, \quad \forall \theta \in [\tau, t].$$

**3. Boundedness of the pullback attractor for the universe  $\mathcal{D}_\gamma$ .** This section is devoted to investigating the boundedness of the pullback attractor for the universe  $\mathcal{D}_\gamma$  given by a tempered condition in space  $E_{\widehat{H}}^2$ . To this end, we consider the Galerkin approximation of the solution  $w(t)$  of system (13), which is denoted by

$$w_n(t) = w_n(t; \tau, w^{in}, \phi^{in}(s)) = \sum_{j=1}^n \xi_{nj}(t) e_j, \quad w_{nt}(\cdot) = w_n(t + \cdot), \quad (16)$$

where the sequence  $\{e_j\}_{j=1}^\infty$  is an orthonormal basis of  $\widehat{H}$  and formed by eigenvectors of the operator  $A$ , that is, for all  $j \geq 1$ ,

$$e_j \in D(A) \text{ and } Ae_j = \lambda_j e_j, \quad (17)$$

where the eigenvalues  $\{\lambda_j\}_{j \geq 1}$  of  $A$  are real number that we can order in such a way

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots, \quad \lambda_j \rightarrow +\infty \text{ as } j \rightarrow \infty.$$

It is not difficult to check that  $\xi_{n,j}(t)$  is the solution of the following ordinary differential equations:

$$\begin{cases} \frac{d}{dt} \langle w_n(t), e_j \rangle + \langle Aw_n(t) + B(u_n, w_n) + N(w_n(t)), e_j \rangle \\ \hspace{10em} = \langle F(t), e_j \rangle + \langle G(t, w_{nt}), e_j \rangle, \\ \langle w_n(\tau), e_j \rangle = \langle w^{in}, e_j \rangle, \quad \langle w_{n\tau}(s), e_j \rangle = \langle \phi^{in}(s), e_j \rangle, \\ \hspace{10em} s \in (-h, 0), \quad j = 1, 2, \dots, n. \end{cases} \quad (18)$$

Next we verify the following estimates of the Galerkin approximate solutions defined by (16).

**Lemma 3.1.** *Let Assumption (2.1) and Assumption (2.2) hold. Then for any  $\tau \in \mathbb{R}, \epsilon > 0, t > \tau + h + \epsilon$  and  $(w^{in}, \phi^{in}(s)) \in \mathcal{A}_{\widehat{H}}(\tau)$ , we have*

- (i) *the set  $\{w_n(r; \tau, w^{in}, \phi^{in}(s)) \mid r \in [\tau + \epsilon, t]\}_{n \geq 1}$  is bounded in  $\widehat{V}$ ;*
- (ii) *the set  $\{w_{nr}(\cdot; \tau, w^{in}, \phi^{in}(s)) \mid r \in [\tau + h + \epsilon, t]\}_{n \geq 1}$  is bounded in  $L^2_{\widehat{V}}$ ;*
- (iii) *the set  $\{w_n(\cdot; \tau, w^{in}, \phi^{in}(s))\}_{n \geq 1}$  is bounded in  $L^2(\tau + \epsilon, t; D(A))$ ;*
- (iv) *the set  $\{w'_n(\cdot; \tau, w^{in}, \phi^{in}(s))\}_{n \geq 1}$  is bounded in  $L^2(\tau + \epsilon, t; \widehat{H})$ .*

*Proof.* Multiplying (18) by  $\beta_{nj}(t)$  and summing them for  $j = 1$  to  $n$ , then using (7), (12) and Young's inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w_n(t)\|^2 + \delta_1 \|w_n(t)\|_{\widehat{V}}^2 \\ & \leq \frac{1}{2} \frac{d}{dt} \|w_n(t)\|^2 + \langle Aw_n(t), w_n(t) \rangle + \langle N(w_n(t)), w_n(t) \rangle + \langle B(u_n, w_n), w_n(t) \rangle \\ & = \langle F(t), w_n(t) \rangle + \langle G(t, w_{nt}), w_n(t) \rangle. \end{aligned}$$

Then integrating the above inequality over  $[\tau, t], t \geq \tau$ , leads to

$$\begin{aligned} & \|w_n(t)\|^2 + 2\delta_1 \int_{\tau}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta \\ & \leq \|w^{in}\|^2 + (\delta_1 - C_G) \int_{\tau}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta + \frac{1}{\delta_1 - C_G} \int_{\tau}^t \|F(\theta)\|_{\widehat{V}^*}^2 d\theta \\ & \quad + C_G \int_{\tau}^t \|w_n(\theta)\|^2 d\theta + \frac{1}{C_G} \int_{\tau}^t \|G(\theta, w_{n\theta})\|^2 d\theta \\ & \leq \|w^{in}\|^2 + (\delta_1 - C_G) \int_{\tau}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta + \frac{1}{\delta_1 - C_G} \int_{\tau}^t \|F(\theta)\|_{\widehat{V}^*}^2 d\theta \\ & \quad + C_G \int_{\tau}^t \|w_n(\theta)\|^2 d\theta + C_G \left( \int_{\tau}^t \|w_n(\theta)\|^2 d\theta + \int_{-h}^0 \|\phi^{in}(s)\|^2 ds \right), \end{aligned}$$

which implies

$$\begin{aligned} & \|w_n(t)\|^2 + (\delta_1 - C_G) \int_{\tau}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta \\ & \leq \max\{1, C_G\} \|(w^{in}, \phi^{in})\|_{E_{\widehat{H}}^2}^2 + \frac{1}{\delta_1 - C_G} \int_{\tau}^t \|F(\theta)\|_{\widehat{V}^*}^2 d\theta. \end{aligned} \tag{19}$$

Thanks to (17), multiplying (18) by  $\lambda_j \beta_{nj}(t)$  and summing the resultant equation for  $j = 1$  to  $n$  yields that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \langle Aw_n(t), w_n(t) \rangle + \|Aw_n(t)\|^2 + \langle B(u_n, w_n), Aw_n(t) \rangle + \langle N(w_n(t)), Aw_n(t) \rangle \\ & = \langle F(t), Aw_n(t) \rangle + \langle G(t, w_{nt}), Aw_n(t) \rangle. \end{aligned}$$

Observe that  $\|u_n(t)\| \leq \|w_n(t)\|$ ,  $\|\nabla w_n(t)\| \leq \|w_n(t)\|_{\widehat{V}}$ , using (10), (11) and Young's inequality, it is easy to see that

$$\begin{aligned} & |\langle B(u_n, w_n), Aw_n(t) \rangle| + |\langle N(w_n(t)), Aw_n(t) \rangle| \\ & \leq \alpha_0 \|u_n\|^{\frac{1}{2}} \|\nabla u_n\|^{\frac{1}{2}} \|\nabla w_n\|^{\frac{1}{2}} \|Aw_n\|^{\frac{1}{2}} \|Aw_n\| + \frac{1}{4} \|Aw_n(t)\|^2 + c^2(\nu_r) \|w_n(t)\|_{\widehat{V}}^2 \\ & \leq \frac{1}{2} \|Aw_n(t)\|^2 + 4^3 \alpha_0^4 \|w_n(t)\|^2 \|w_n(t)\|_{\widehat{V}}^4 + c^2(\nu_r) \|w_n(t)\|_{\widehat{V}}^2 \end{aligned}$$

and

$$\langle F(t), Aw_n(t) \rangle + \langle G(t, w_{nt}), Aw_n(t) \rangle \leq \frac{1}{4} \|Aw_n(t)\|^2 + 2\|F(t)\|^2 + 2\|G(t, w_{nt})\|^2.$$

Therefore

$$\begin{aligned} & \frac{d}{dt} \langle Aw_n(t), w_n(t) \rangle + \frac{1}{2} \|Aw_n(t)\|^2 \\ & \leq 4\|F(t)\|^2 + 4\|G(t, w_{nt})\|^2 + 128\alpha_0^4 \|w_n(t)\|^2 \|w_n(t)\|_{\widehat{V}}^4 + 2c^2(\nu_r) \|w_n(t)\|_{\widehat{V}}^2 \\ & \leq 4\|F(t)\|^2 + 4\|G(t, w_{nt})\|^2 \\ & \quad + (128c_2\alpha_0^4 \|w_n(t)\|^2 \|w_n(t)\|_{\widehat{V}}^2 + 2c_2c^2(\nu_r)) \langle Aw_n(t), w_n(t) \rangle. \end{aligned} \tag{20}$$

Set

$$\begin{aligned} H_n(\theta) & := \langle Aw_n(\theta), w_n(\theta) \rangle, \quad I_n(\theta) := 4\|F(\theta)\|^2 + 4\|G(\theta, w_{n\theta})\|^2, \\ J_n(\theta) & := 128c_2\alpha_0^4 \|w_n(\theta)\|^2 \|w_n(\theta)\|_{\widehat{V}}^2 + 2c_2c^2(\nu_r), \end{aligned}$$



then we get

$$\frac{d}{d\theta}H_n(\theta) \leq J_n(\theta)H_n(\theta) + I_n(\theta). \tag{21}$$

By Gronwall inequality, (21) yields

$$H_n(r) \leq (H_n(\tilde{r}) + \int_{r-\epsilon}^r I_n(\theta)d\theta) \cdot \exp \left\{ \int_{r-\epsilon}^r J_n(\theta)d\theta \right\}, \quad \forall \tau \leq r - \epsilon \leq \tilde{r} \leq r \leq t.$$

Integrating the above inequality for  $\tilde{r}$  from  $r - \epsilon$  to  $r$ , we obtain

$$\epsilon H_n(r) \leq \left( \int_{r-\epsilon}^r H_n(\tilde{r})d\tilde{r} + \epsilon \int_{r-\epsilon}^r I_n(\theta)d\theta \right) \cdot \exp \left\{ \int_{r-\epsilon}^r J_n(\theta)d\theta \right\}.$$

Since

$$\begin{aligned} & \int_{r-\epsilon}^r H_n(\tilde{r})d\tilde{r} + \epsilon \int_{r-\epsilon}^r I_n(\theta)d\theta \\ &= \int_{r-\epsilon}^r \langle Aw_n(\tilde{r}), w_n(\tilde{r}) \rangle d\tilde{r} + 4\epsilon \int_{r-\epsilon}^r (\|F(\theta)\|^2 + \|G(\theta, w_n\theta)\|^2) d\theta \\ &\leq \frac{1}{c_1} \int_{\tau}^t \|w_n(\theta)\|_{\hat{V}}^2 d\theta + 4\epsilon \int_{\tau}^t \|F(\theta)\|^2 d\theta \\ &\quad + 4\epsilon C_G^2 \left( \int_{\tau}^t \|w_n(\theta)\|^2 d\theta + \int_{-h}^0 \|\phi^{in}(s)\|^2 ds \right), \\ & \int_{r-\epsilon}^r J_n(\theta)d\theta = 128c_2\alpha_0^4 \int_{r-\epsilon}^r \|w_n(\theta)\|^2 \|w_n(\theta)\|_{\hat{V}}^2 d\theta + 2\epsilon c_2 c^2(\nu_r) \\ &\leq 128c_2\alpha_0^4 \max_{\theta \in [\tau, t]} \|w_n(\theta)\|^2 \int_{\tau}^t \|w_n(\theta)\|_{\hat{V}}^2 d\theta + 2\epsilon c_2 c^2(\nu_r), \end{aligned}$$

we can conclude that

$$\begin{aligned} & \|w_n(r)\|_{\hat{V}}^2 \leq c_2 H_n(r) \\ &\leq \left[ \frac{c_2}{c_1\epsilon} \int_{\tau}^t \|w_n(\theta)\|_{\hat{V}}^2 d\theta + 4c_2 \int_{\tau}^t \|F(\theta)\|^2 d\theta \right. \\ &\quad \left. + 4c_2 C_G^2 \left( \int_{\tau}^t \|w_n(\theta)\|^2 d\theta + \int_{-h}^0 \|\phi^{in}(s)\|^2 ds \right) \right] \\ &\quad \cdot \exp \left\{ 128c_2\alpha_0^4 \max_{\theta \in [\tau, t]} \|w_n(\theta)\|^2 \int_{\tau}^t \|w_n(\theta)\|_{\hat{V}}^2 d\theta + 2\epsilon c_2 c^2(\nu_r) \right\}, \end{aligned}$$

which together with (19) and Assumption 2.2 implies the assertion (i).

Now, integrating (20) over  $[\tau + \epsilon, t]$ , we obtain

$$\begin{aligned} & \int_{\tau+\epsilon}^t \|Aw_n(\theta)\|^2 d\theta \\ &\leq \frac{2}{c_1} \|w_n(\tau + \epsilon)\|_{\hat{V}}^2 + 8 \int_{\tau}^t \|F(\theta)\|^2 d\theta + 8C_G^2 \left( \int_{\tau}^t \|w_n(\theta)\|^2 d\theta + \int_{-h}^0 \|\phi^{in}(s)\|^2 ds \right) \\ &\quad + \left( 256\alpha_0^4 \max_{\theta \in [\tau+\epsilon, t]} (\|w_n(\theta)\| \|w_n(\theta)\|_{\hat{V}})^2 + 4c^2(\nu_r) \right) \int_{\tau+\epsilon}^t \|w_n(\theta)\|_{\hat{V}}^2 d\theta, \end{aligned}$$

which together with (19), Assumption 2.2 and the assertion (i) gives the assertion (iii).

In addition,

$$\begin{aligned} \int_{-h}^0 \|w_{nr}(\theta)\|_{\widehat{V}}^2 d\theta &= \int_{r-h}^r \|w_n(\theta)\|_{\widehat{V}}^2 d\theta \\ &\leq h \cdot \max_{\theta \in [\tau+\epsilon, t]} \|w_n(\theta)\|_{\widehat{V}}^2, \quad \tau + h + \epsilon \leq r \leq t, \end{aligned} \tag{22}$$

which together with the assertion (i) yields the assertion (ii).

Finally, multiplying (18) by  $\beta'_{nj}(t)$  and summing the resultant equation for  $j = 1$  to  $n$ , we obtain

$$\begin{aligned} &\|w'_n(t)\|^2 + \frac{1}{2} \frac{d}{dt} \langle Aw_n(t), w_n(t) \rangle + \langle B(u_n, w_n), w'_n(t) \rangle + \langle N(w_n(t)), w'_n(t) \rangle \\ &= (F(t), w'_n(t)) + (G(t, w_{nt}), w'_n(t)). \end{aligned} \tag{23}$$

From Assumption 2.1, it follows that

$$\begin{aligned} (F(t), w'_n(t)) + (G(t, w_{nt}), w'_n(t)) &\leq (\|F(t)\| + \|G(t, w_{nt})\|) \|w'_n(t)\| \\ &\leq 2\|F(t)\|^2 + 2\|G(t, w_{nt})\|^2 + \frac{1}{4}\|w'_n(t)\|^2. \end{aligned} \tag{24}$$

By Lemma 2.2,

$$\begin{aligned} |\langle B(u_n, w_n), w'_n(t) \rangle| &\leq \alpha_0 \|u_n\|^{\frac{1}{2}} \|\nabla u_n\|^{\frac{1}{2}} \|\nabla w_n\|^{\frac{1}{2}} \|Aw_n\|^{\frac{1}{2}} \|w'_n(t)\| \\ &\leq \alpha_0 \|w_n\|^{\frac{1}{2}} \|w_n\|_{\widehat{V}} \|Aw_n\|^{\frac{1}{2}} \|w'_n(t)\| \\ &\leq \alpha_0^2 \|w_n\| \|w_n\|_{\widehat{V}}^2 \|Aw_n\| + \frac{1}{4} \|w'_n(t)\|^2 \end{aligned} \tag{25}$$

and

$$|\langle N(w_n(t)), w'_n(t) \rangle| \leq \frac{1}{4} \|w'_n(t)\|^2 + c^2(\nu_r) \|w_n(t)\|_{\widehat{V}}^2. \tag{26}$$

Taking (23)-(26) into account, we obtain

$$\begin{aligned} &\|w'_n(t)\|^2 + 2 \frac{d}{dt} \langle Aw_n(t), w_n(t) \rangle \\ &\leq 8\|F(t)\|^2 + 8\|G(t, w_{nt})\|^2 + 4\alpha_0^2 \|w_n\|_{\widehat{V}}^3 \|Aw_n(t)\| + 4c^2(\nu_r) \|w_n(t)\|_{\widehat{V}}^2. \end{aligned}$$

Integrating the above inequality, yields

$$\begin{aligned} &\int_{\tau+\epsilon}^t \|w'_n(\theta)\| d\theta \\ &\leq 2c_1^{-1} \|w_n(\tau + \epsilon)\|_{\widehat{V}}^2 + 8 \int_{\tau+\epsilon}^t \|F(\theta)\|^2 d\theta + 8 \int_{\tau+\epsilon}^t \|G(\theta, w_{n\theta})\|^2 d\theta \\ &\quad + 4\alpha_0^2 \int_{\tau+\epsilon}^t \|w_n(\theta)\|_{\widehat{V}}^3 \|Aw_n(\theta)\| d\theta + 4c^2(\nu_r) \int_{\tau+\epsilon}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta \\ &\leq 2c_1^{-1} \|w_n(\tau + \epsilon)\|_{\widehat{V}}^2 + 8 \int_{\tau+\epsilon}^t \|F(\theta)\|^2 d\theta \\ &\quad + 8C_G^2 \left( \int_{\tau}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta + \int_{-h}^0 \|\phi^{in}(s)\|^2 ds \right) + 4c^2(\nu_r) \int_{\tau+\epsilon}^t \|w_n(\theta)\|_{\widehat{V}}^2 d\theta, \\ &\quad + 2\alpha_0^2 \max_{\theta \in [\tau+\epsilon, t]} \|w_n(\theta)\|_{\widehat{V}}^2 \int_{\tau+\epsilon}^t (\|w_n(\theta)\|_{\widehat{V}}^2 + \|Aw_n(\theta)\|^2) d\theta \end{aligned}$$

which together with (20), Assumption 2.2 and the assertions (i)-(iii) gives the assertion (iv). The proof is complete.  $\square$

With the above lemma, we are ready to conclude this section with the following  $H^1$ -boundedness of the pullback attractor  $\widehat{\mathcal{A}}_{\widehat{H}}$  for the universe  $\mathcal{D}_\gamma$ .

**Theorem 3.2.** *Let assumptions 2.1-2.2 hold and  $\widehat{\mathcal{A}}_{\widehat{H}} = \{\mathcal{A}_{\widehat{H}}(t) \mid t \in \mathbb{R}\}$  be the pullback attractor of system (13). Then for any  $\tau \in \mathbb{R}, \epsilon > 0, t > \tau + h + \epsilon$  and  $(w^{in}, \phi^{in}) \in E_{\widehat{H}}^2$ , the set  $\bigcup_{r \in [\tau+h+\epsilon, t]} U(r, \tau)\mathcal{A}_{\widehat{H}}(\tau)$  is bounded in  $E_{\widehat{V}}^2$ .*

*Proof.* Based on Lemma 3.1, following the standard diagonal procedure, there exist a subsequence (denoted still by)  $\{w_n\}_{n \geq 1}$  and a function  $w(\cdot) \in L^\infty(\tau + \epsilon, t; \widehat{V}) \cap L^2(\tau + \epsilon, t; D(A))$  with  $w'(\cdot) \in L^2(\tau + \epsilon, t; \widehat{H})$  such that, as  $n \rightarrow \infty$ ,

$$w_n(\cdot) \rightharpoonup^* w(\cdot) \text{ weakly star in } L^\infty(\tau + \epsilon, t; \widehat{V}), \tag{27}$$

$$w_n(\cdot) \rightharpoonup w(\cdot) \text{ weakly in } L^2(\tau + \epsilon, t; D(A)), \tag{28}$$

$$w'_n(\cdot) \rightharpoonup w'(\cdot) \text{ weakly in } L^2(\tau + \epsilon, t; \widehat{H}). \tag{29}$$

Furthermore, it follows from the uniqueness of the limit function that  $w(\cdot)$  is a weak solution of system (13). According to compact embedding theorem, (28) and (29) implies  $w(\cdot) \in \mathcal{C}([\tau + \epsilon, t]; \widehat{V})$ . Then Theorem 3.2 follows from (27), Lemma 2.4 and Lemma 3.1.  $\square$

**Remark 3.1.** We here point out that the boundedness of pullback attractor  $\widehat{\mathcal{A}}$  in  $E_{D(A)}^2$  can be proved by using similar proof as that in  $E_{\widehat{V}}^2$  if we improve the regularity of  $F(t)$  and  $G(t, w_{nt})$  with respect to  $t$ , where  $D(A) := \widehat{V} \cap (H^2)^3$ . Exactly, assume that

$$(I) F(t, x) \in W_{loc}^{1,2}(\mathbb{R}; \widehat{H}), \forall t \geq \tau, \tau \in \mathbb{R}, \text{ and } \int_{-\infty}^t e^{\gamma\theta} \|F'(\theta, x)\|_{\widehat{V}^*}^2 d\theta < +\infty.$$

$$(II) (G(t, \xi))' = \frac{dG}{dt}: \mathbb{R} \times L^2(-h, 0; \widehat{H}) \mapsto (L^2(\Omega))^3 \text{ satisfies:}$$

◦ For any  $\xi \in L^2(-h, 0; \widehat{H})$ , the mapping  $\mathbb{R} \ni t \mapsto G(t, \xi) \in (L^2(\Omega))^3$  is measurable.

$$\circ (G(\cdot, 0))' = (0, 0, 0).$$

◦ There exists a constant  $\tilde{L}_G > 0$  such that for any  $t \in \mathbb{R}$  and any  $\xi, \eta \in L^2(-h, 0; \widehat{H})$ ,

$$\|(G(t, \xi))' - (G(t, \eta))'\| \leq \tilde{L}_G \|\xi - \eta\|_{L^2(-h, 0; \widehat{H})}.$$

◦ There exists  $\tilde{C}_G \in (0, \delta_1)$  such that, for any  $t \geq \tau$  and any  $w, v \in L^2(\tau-h, t; \widehat{H})$ ,

$$\int_\tau^t \|(G(\theta, w_\theta))' - (G(\theta, v_\theta))'\|^2 d\theta \leq \tilde{C}_G^2 \int_{\tau-h}^t \|w(\theta) - v(\theta)\|^2 d\theta.$$

Then we can deduce that the Galerkin approximate solutions  $\{w_n(\cdot)\}_{n \geq 1}$  is bounded in  $D(A) = \widehat{V} \cap (H^2)^3$ . Moreover,  $\{w'_n(\cdot)\}_{n \geq 1}$  is bounded in  $\widehat{H}$ . Further, we can conclude the  $H^2$ -boundedness of the pullback attractor  $\widehat{\mathcal{A}}$ .

**4. Upper semicontinuity of the pullback attractor.** In this section, we concentrate on verifying the upper semicontinuity of the pullback attractor  $\widehat{\mathcal{A}}_{\widehat{H}}$  obtained in Proposition 2.2 with respect to the spatial domain. To this end, first we let  $\{\Omega_m\}_{m=1}^\infty$  be an expanding sequence of simply connected, bounded and smooth subdomains of  $\Omega$  such that  $\bigcup_{m=1}^\infty \Omega_m = \Omega$ . Then we consider the system (3) in each

$\Omega_m$  and define the operators  $A, B(\cdot, \cdot)$  and  $N(\cdot)$  as previous (in (6)) with the spatial domain  $\Omega$  replaced by  $\Omega_m$ . Further we can formulate the weak version of problem (3)-(5) as follows:

$$\begin{cases} \frac{\partial w_m}{\partial t} + Aw_m + B(u_m, w_m) + N(w_m) \\ \hspace{15em} = F(t, x) + G(t, w_{mt}), \text{ in } (\tau, +\infty) \times \Omega_m, \\ \nabla \cdot u_m = 0, \text{ in } (\tau, +\infty) \times \Omega_m, \\ w_m = (u_m, \omega_m) = 0, \text{ on } (\tau, +\infty) \times \Gamma, \\ w_m(\tau) = w_m^{in}, w_{m\tau}(s) = \phi_m^{in}(s), s \in (-h, 0). \end{cases} \tag{30}$$

On each bounded domain  $\Omega_m$ , the well-posedness of solution can be established by Galerkin method and energy method, one can refer to [7].

**Lemma 4.1.** *Suppose Assumption 2.1 and Assumption 2.2 hold, then for any given  $(w_m^{in}, \phi_m^{in}) \in E_{\widehat{H}(\Omega_m)}^2$ , system (30) has a unique weak solution  $w_m$  satisfying*

$$w_m(\cdot) \in \mathcal{C}([\tau, T]; \widehat{H}(\Omega_m)) \cap L^2(\tau, T; \widehat{V}(\Omega_m)), \quad w'_m(\cdot) \in L^2(\tau, T; \widehat{V}^*(\Omega_m)), \quad \forall T > \tau.$$

Moreover, the solution  $w_m(\cdot)$  depends continuously on the initial value  $w_m^{in}$  with respect to  $\widehat{H}(\Omega_m)$  norm.

According to Lemma 4.1, the map defined by

$$\begin{aligned} U_m(t, \tau) : (w_m^{in}, \phi_m^{in}(s)) &\mapsto U_m(t, \tau, w_m^{in}, \phi_m^{in}(s)) \\ &= (w_m(t), w_{mt}(s; \tau, w^{in}, \phi^{in}(s))), \quad \forall t \geq \tau, \end{aligned} \tag{31}$$

generates a continuous process  $\{U(t, \tau)\}_{t \geq \tau}$  in  $\widehat{H}(\Omega_m)$ . Moreover, on any smooth bounded domain  $\Omega_m$ , with similar proof as those of Lemma 3.2, Lemma 3.3 and Lemma 3.6 in [16], we can obtain the existence of pullback  $\mathcal{D}_\gamma^{\widehat{H}(\Omega_m)}$ -absorbing for the process  $\{U_m(t, \tau)\}_{t \geq \tau}$  and the pullback  $\mathcal{D}_\gamma^{\widehat{H}(\Omega_m)}$ -asymptotic compactness of the process in  $\widehat{H}(\Omega_m)$ . That is,

**Lemma 4.2.** *Under the assumptions 2.1 and 2.2, it holds that*

(1) *for any  $(w_m^{in}, \phi_m^{in}(s)) \in E_{\widehat{H}(\Omega_m)}^2$ , the family  $\widehat{\mathcal{B}}^{\widehat{H}(\Omega_m)} := \{\mathcal{B}^{\widehat{H}(\Omega_m)}(t) \mid t \in \mathbb{R}\}$  given by*

$$\mathcal{B}^{\widehat{H}(\Omega_m)}(t) = \{(w, \phi) \in E_{\widehat{H}(\Omega_m) \times L^2_{\widehat{V}(\Omega_m)}}^2 \mid \|(w, \phi)\|_{E_{\widehat{H}(\Omega_m) \times L^2_{\widehat{V}(\Omega_m)}}}^2 \leq R_1(t)\}$$

*is pullback  $\mathcal{D}_\gamma^{\widehat{H}(\Omega_m)}$ -absorbing for the process  $\{U_m(t, \tau)\}$ , where  $R_1(t)$  is bounded for all  $t \in \mathbb{R}$ .*

(2) *for any  $\epsilon > 0$ , there exist  $r_m := r_m(\epsilon, t, \widehat{\mathcal{B}}^{\widehat{H}(\Omega_m)}) > 0$ ,  $\tau_m := \tau_m(\epsilon, t, \widehat{\mathcal{B}}^{\widehat{H}(\Omega_m)}) < t$  such that for any  $r \in [r_m, m], \tau \leq \tau_m$ ,*

$$\|w_m(t, \tau, w_m^{in}, \phi_m^{in}(s))\|_{L^2(\Omega_m \setminus \Omega_r)} \leq \epsilon, \quad \forall (w_m^{in}, \phi_m^{in}(s)) \in \mathcal{B}^{\widehat{H}(\Omega_m)}(\tau).$$

(3) *the process  $\{U_m(t, \tau)\}_{t \geq \tau}$  is pullback  $\mathcal{D}_\gamma^{\widehat{H}(\Omega_m)}$ -asymptotically compact in  $\widehat{H}(\Omega_m)$ .*

Then based on the Remark 2.1 in [16], we conclude that

**Theorem 4.3.** *Let assumptions 2.1 and 2.2 hold. Then there exists a pullback attractor  $\widehat{\mathcal{A}}_{\widehat{H}(\Omega_m)} = \{\mathcal{A}_{\widehat{H}(\Omega_m)}(t) \mid t \in \mathbb{R}\}$  for the system (30) in  $\widehat{H}(\Omega_m)$ .*

In the following, we investigate the relationship between the solutions of system (30) and (13). Indeed, we devoted to proving the solutions  $w_m$  of system (30) converges to the solution of system (13) as  $m \rightarrow \infty$ . To this end, for  $w_m \in \widehat{H}(\Omega_m)$ , we extend its domain from  $\Omega_m$  to  $\Omega$  by setting

$$\tilde{w}_m = \begin{cases} w_m, & x \in \Omega_m, \\ 0, & x \in \Omega \setminus \Omega_m, \end{cases} \tag{32}$$

then it holds that

$$\|w_m\|_{\widehat{H}(\Omega_m)} = \|\tilde{w}_m\|_{\widehat{H}(\Omega_m)} = \|\tilde{w}_m\|_{\widehat{H}(\Omega)} := \|w_m\|.$$

Next, using the same proof as that of Lemma 8.1 in [8], we have

**Lemma 4.4.** *Let assumptions 2.1-2.2 hold and  $\{(w_m^{in}, \phi_m^{in}(s))\}_{m \geq 1}$  be a sequence in  $E^2_{\widehat{H}(\Omega_m) \times L^2_{\widehat{V}(\Omega_m)}}$  converging weakly to an element  $(w^{in}, \phi^{in}(s)) \in E^2_{\widehat{H} \times L^2_{\widehat{V}}}$  as  $m \rightarrow \infty$ . Then for any  $t \geq \tau$ ,*

$$w_m(t; \tau, w_m^{in}, \phi_m^{in}(s)) \rightharpoonup w(t; \tau, w^{in}, \phi^{in}(s)) \text{ weakly in } \widehat{H}, \tag{33}$$

$$w_m(\cdot; \tau, w_m^{in}, \phi_m^{in}(s)) \rightharpoonup w(\cdot; \tau, w^{in}, \phi^{in}(s)) \text{ weakly in } L^2(t-h, t; \widehat{V}). \tag{34}$$

Based on Lemma 4.4, we set out to prove the following important lemma.

**Lemma 4.5.** *Let assumptions 2.1-2.2 hold, then for any  $t \in \mathbb{R}$ , any sequence  $\{(w_m(t), w_{mt}(s))\}_{m \geq 1}$  with  $(w_m(\tau), w_{m\tau}(s)) = (w_m^{in}, \phi_m^{in}(s)) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(\tau), m = 1, 2, \dots$ , there exists  $(w(t), w_t(s)) \in \mathcal{A}_{\widehat{H}}(t)$  such that*

$$(w_m(t), w_{mt}(s)) \rightarrow (w(t), w_t(s)) \text{ strongly in } E^2_{\widehat{H}}. \tag{35}$$

*Proof.* From the compactness of pullback attractor, it follows that the sequence  $\{(w_m^{in}, \phi_m^{in}(s))\}_{m \geq 1}$  is bounded in  $E^2_{\widehat{H}}$ . Hence, there exist a subsequence (denoted still by)  $\{(w_m^{in}, \phi_m^{in}(s))\}_{m \geq 1}$  and a  $(w^{in}, \phi^{in}(s)) \in \mathcal{A}_{\widehat{H}}(\tau)$  such that

$$(w_m^{in}, \phi_m^{in}(s)) \rightharpoonup (w^{in}, \phi^{in}(s)) \text{ weakly in } E^2_{\widehat{H} \times L^2_{\widehat{V}}} \text{ as } m \rightarrow \infty. \tag{36}$$

Further, according to Lemma 4.4 and the invariance of the pullback attractor, we can conclude that for any  $t \in \mathbb{R}$ , there exist a  $(w_m(t), w_{mt}(s)) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(t)$  with  $(w_m(\tau), w_{m\tau}(s)) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(\tau)$  and a  $(w(t), w_t(s)) \in \mathcal{A}_{\widehat{H}}(t)$  with  $(w(\tau), w_\tau(s)) \in \mathcal{A}_{\widehat{H}}(\tau)$  such that

$$(w_m(t), w_{mt}(s)) \rightharpoonup (w(t), w_t(s)) \text{ weakly in } E^2_{\widehat{H} \times L^2_{\widehat{V}}} \text{ as } m \rightarrow \infty. \tag{37}$$

Then, using the same way of proof as Lemma 3.6 in [16], we can obtain that the convergence relation of (37) is strong. The proof is complete.  $\square$

With the above lemma, we are ready to state the main result of this section.

**Theorem 4.6.** *Let Assumption 2.1 and Assumption 2.2 hold, then for any  $t \in \mathbb{R}$ , it holds that*

$$\lim_{m \rightarrow \infty} \text{dist}_{E^2_{\widehat{H}}}(\mathcal{A}_{\widehat{H}(\Omega_m)}(t), \mathcal{A}_{\widehat{H}}(t)) = 0, \tag{38}$$

where  $\mathcal{A}_{\widehat{H}}(t)$  and  $\mathcal{A}_{\widehat{H}(\Omega_m)}(t)$  are the pullback attractor of system (13) and system (30), respectively.

*Proof.* Suppose the assertion (38) is false, then for any  $m = 1, 2, \dots$ , there exist  $t_0 \in \mathbb{R}, \epsilon_0 > 0$  and a sequence  $(w_m(t_0), w_{mt_0}(s)) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(t_0)$  such that

$$\text{dist}_{E_{\widehat{H}}^2}((w_m(t_0), w_{mt_0}(s)), \mathcal{A}_{\widehat{H}}(t_0)) \geq \epsilon_0. \quad (39)$$

However, it follows from Lemma 4.5 that there exists a subsequence

$$\{(w_{m_k}(t_0), w_{m_k t_0}(s))\} \subseteq \{(w_m(t_0), w_{mt_0}(s))\}$$

such that

$$\lim_{k \rightarrow \infty} \text{dist}_{E_{\widehat{H}}^2}((w_{m_k}(t_0), w_{m_k t_0}(s)), \mathcal{A}_{\widehat{H}}(t_0)) = 0,$$

which is in contradiction to (39). Therefore, (38) is true. The proof is complete.  $\square$

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