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RECENT PROGRESS ON THE MATHEMATICAL STUDY OF ANOMALOUS LOCALIZED RESONANCE IN ELASTICITY

Hongjie Li*

Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong SAR, China

Abstract. We consider the anomalous localized resonance induced by negative elastic metamaterials and its application in invisibility cloaking. We survey the recent mathematical developments in the literature and discuss two mathematical strategies that have been developed for tackling this peculiar resonance phenomenon. The first one is the spectral method, which explores the anomalous localized resonance through investigating the spectral system of the associated Neumann-Poincaré operator. The other one is the variational method, which considers the anomalous localized resonance via calculating the nontrivial kernels of a non-elliptic partial differential operator. The advantages and the relationship between the two methods are discussed. Finally, we propose some open problems for the future study.

1. **Introduction.** Plasmon materials, also known as negative materials, are the artificially engineered exotic materials. The materials do not exist in nature and could exhibit negative parameters. There are many important applications for this plasmon materials, such as plasmon resonance, superlens and absorber. Theoretical analysis of the negative materials was firstly studied by Veselago [42] in 1968. Smith et. al. [41] was the first one to realize the negative material in laboratory. The existence of such negative materials can be found in [23] for the acoustic system, [39, 40] for Maxwell system and [43] for the elastic system. Generally speaking, the exotic materials were fabricated by placing arrays of small physical units. Then for the frequency in a certain regime, the small structure could have the resonance phenomenon, which then could induce the negative properties for the corresponding materials. Such research can be found in [5, 23, 30, 39, 43].

Anomalous localized resonance (ALR) is associated with the approach to an essential singularity, which is different from the usual resonance. The ALR has the following characteristic features. Firstly, the corresponding wave field oscillates more and more highly as the loss of the material goes to certain value depending on the plasmonic configuration. Moreover, the oscillation only exists in a certain region and outside the region, the field converges to a smooth field. Thirdly, the resonance heavily depends on the location of the source term. Indeed, there is a critical radius. When the source is located inside the critical radius, then the ALR could occur. Otherwise, there is no such the resonance phenomenon. Due to these

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* Corresponding author: Hongjie Li.

distinctive characteristics, the ALR could induce the cloaking effect; that is when the phenomenon of ALR occurs, then both the plasmonic configuration and the source term are invisible with respect to the observation outside certain region. This cloaking phenomenon is referred to as cloaking due to anomalous localized resonance (CALR). CALR was first observed and rigorously justified by Milton and Nicorovici in [32] and was further studied by Ammari et al in [3]. The CALR has been extensively investigated. We refer to [4, 6, 9, 11, 15, 20, 21, 29] for the relevant study in acoustics, [8, 10, 17, 18, 19, 27, 28, 24] for elastic system and [3, 2, 1, 7, 13, 14, 15, 25, 26, 31, 32, 33, 34, 36, 37, 38] for the Maxwell system.

In this paper, we provide an overview of the recent progress on the mathematical study of anomalous localized resonance in linear elasticity. Mathematically, the ALR is caused by nontrivial kernels of a non-elliptic partial differential operator (PDO), which is the Lamé operator that governs the elastic wave propagation. The presence of the negative parameters of the plasmon material breaks the ellipticity of the corresponding PDO. Thus the nontrivial solutions of the non-elliptic PDE arise, which then induce the phenomenon of ALR. The nontrivial solutions are referred to as perfect polariton waves in the literature. Indeed, finding such nontrivial solutions is equivalent to investigating the spectrum of the boundary integral operator, called the Neumann-Poincaré (N-P) operator. Thus there are mainly two methods to explore the anomalous localized resonance. The first one is the spectral method (cf. [8, 10, 12, 17, 19, 24]). With the help of potential theory, the wave field can be expressed by the boundary integral operators. Then by matching the transmission condition on the boundary, the problem is reduced to investigate the spectral system of the N-P operator. The other one is the variational approach (cf. [27, 28]). One should first construct the variation principle for the original problem and then should find the nontrivial solutions of the corresponding non-elliptic PDE, namely the perfect plasmon waves. The two methods have their own advantages. For the spectral method, the CALR can occur for a general source $\mathbf{f}$ as long as it is located inside a critical radius. However, the parameters in the core and the shell should be a constant. For the variational method, the shape of the core could be arbitrary and the parameters in the core could be any bounded function. However, to induce the ALR, the source term $\mathbf{f}$ should be supported on a circle. Next, we present the mathematical formulation for our subsequent discussion.

Let $\mathbf{C}(\mathbf{x}) := (\mathrm{C}_{ijkl}(\mathbf{x}))_{i,j,k,l=1}^N$, $\mathbf{x} \in \mathbb{R}^N$ with $N = 2, 3$, be a four-rank elastic material tensor defined by

$$\mathrm{C}_{ijkl}(\mathbf{x}) := \lambda(\mathbf{x})\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu(\mathbf{x})(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}), \quad \mathbf{x} \in \mathbb{R}^N, \tag{1.1}$$

where $\boldsymbol{\delta}$ is the Kronecker delta. In (1.1), λ and μ are two scalar functions and referred to as the Lamé parameters. For a regular elastic material, the Lamé parameters satisfy the following two strong convexity conditions,

$$\text{i). } \mu > 0 \qquad \text{and} \qquad \text{ii). } N\lambda + 2\mu > 0. \tag{1.2}$$

Let $D, \Omega \subset \mathbb{R}^N$ with $D \subset \Omega$ be two bounded domains with connected Lipschitz boundaries. Assume that the domain $\mathbb{R}^N\backslash\overline{\Omega}$ is occupied by an elastic material parameterized by the Lamé constants (λ, μ) satisfying the strong convexity condition (1.2). The shell $\Omega\backslash\overline{D}$ is occupied by a metamaterial whose Lamé parameters are given by $(\hat{\lambda}, \hat{\mu})$, where $(\hat{\lambda}, \hat{\mu}) \in \mathbb{C}^2$ with $\Im\hat{\lambda} > 0, \Im\hat{\mu} > 0$, which shall be properly chosen in what follows. For the inner core D, the elastic material parameters are $(\breve{\lambda}, \breve{\mu})$ fulfilling the condition (1.2). Denote by $\mathbf{C}_{\mathbb{R}^N\backslash\overline{\Omega},\lambda,\mu}$ to specify the dependence of the elastic tensor on the domain $\mathbb{R}^N\backslash\overline{\Omega}$ and the Lamé parameters (λ,μ). The same notation also applies for the tensors $\mathbf{C}_{\Omega\backslash\overline{D},\hat{\lambda},\hat{\mu}}$ and $\mathbf{C}_{D,\breve{\lambda},\breve{\mu}}$. Now we introduce the following elastic tensor

$$\mathbf{C}_0 = \mathbf{C}_{\mathbb{R}^N\backslash\overline{\Omega},\lambda,\mu} + \mathbf{C}_{\Omega\backslash\overline{D},\hat{\lambda},\hat{\mu}} + \mathbf{C}_{D,\breve{\lambda},\breve{\mu}}. \tag{1.3}$$

$\mathbf{C}_0$ describes an elastic material configuration of a core-shell-matrix structure with the metamaterial located in the shell. Let $\mathbf{f} \in H^{-1}(\mathbb{R}^N)^N$ signify an excitation elastic source that is compactly supported in $\mathbb{R}^N\backslash\overline{\Omega}$. The induced elastic displacement field $\mathbf{u} = (u_i)_{i=1}^N \in \mathbb{C}^N$ corresponding to the configurations described above is governed by the following PDE system

$$\begin{cases} \nabla\cdot\mathbf{C}_0\nabla^s\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = \mathbf{f} \quad \text{in } \ \mathbb{R}^N, \\ \mathbf{u}(\mathbf{x}) \text{ satisfies the radiation condition,} \end{cases} \tag{1.4}$$

where $\omega \in \mathbb{R}_+$ is the angular frequency, and the operator ∇^s is the symmetric gradient given by

$$\nabla^s\mathbf{u} := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right), \tag{1.5}$$

with $\nabla\mathbf{u}$ denoting the matrix $(\partial_j u_i)_{i,j=1}^N$ and the superscript t signifying the matrix transpose. In (1.4), the radiation condition designates the following condition as $|\mathbf{x}| \to +\infty$ (cf. [22]),

$$\begin{aligned} (\nabla\times\nabla\times\mathbf{u})(\mathbf{x}) \times \frac{\mathbf{x}}{|\mathbf{x}|} - \mathrm{i}k_s\nabla\times\mathbf{u}(\mathbf{x}) =& \mathcal{O}(|\mathbf{x}|^{1-N}), \\ \frac{\mathbf{x}}{|\mathbf{x}|}\cdot[\nabla(\nabla\cdot\mathbf{u})](\mathbf{x}) - \mathrm{i}k_p\nabla\mathbf{u}(\mathbf{x}) =& \mathcal{O}(|\mathbf{x}|^{1-N}), \end{aligned} \tag{1.6}$$

where $\mathrm{i} = \sqrt{-1}$ and

$$k_s = \omega/\sqrt{\mu}, \quad k_p = \omega/\sqrt{\lambda+2\mu}, \tag{1.7}$$

with λ and μ defined in (1.3).

Next we introduce the following functional for $\mathbf{w},\mathbf{v} \in \left(H^1(\Omega\backslash\overline{D})\right)^N$,

$$\begin{aligned} P_{\hat{\lambda},\hat{\mu}}(\mathbf{w},\mathbf{v}) &= \int_{\Omega\backslash\overline{D}} \nabla^s\mathbf{w} : \mathbf{C}_0\overline{\nabla^s\mathbf{v}(\mathbf{x})}d\mathbf{x} \\ &= \int_{\Omega\backslash\overline{D}} \left(\hat{\lambda}(\nabla\cdot\mathbf{w})\overline{(\nabla\cdot\mathbf{v})}(\mathbf{x}) + 2\hat{\mu}\nabla^s\mathbf{w} : \overline{\nabla^s}\mathbf{v}(\mathbf{x})\right) d\mathbf{x}, \end{aligned} \tag{1.8}$$

where $\mathbf{C}_0$ and ∇^s are defined in (1.3) and (1.5), respectively. In (1.8) and also in what follows, $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^3 a_{ij}b_{ij}$ for two matrices $\mathbf{A} = (a_{ij})_{i,j=1}^3$ and $\mathbf{B} = (b_{ij})_{i,j=1}^3$. Henceforth, we define

$$E(\mathbf{u}) = \Im P_{\hat{\lambda},\hat{\mu}}(\mathbf{u},\mathbf{u}), \tag{1.9}$$

which signifies the energy dissipation exists energy of the elastic system (1.4). We are now in a position to present the definition of CALR. We say that polariton resonance occurs if for any $M \in \mathbb{R}_+$,

$$E(\mathbf{u}) \geq M, \tag{1.10}$$

where $\mathbf{u}$ depends on the Lamé parameters $(\hat{\lambda},\hat{\mu})$. In addition to (1.10), if the displacement field $\mathbf{u}$ further satisfies the following boundedness condition,

$$|\mathbf{u}| \leq C, \quad \text{when} \quad |\mathbf{x}| > \tilde{R}, \tag{1.11}$$

for a certain $\tilde{R} \in \mathbb{R}_+$, which does not depend on the Lamé parameters $(\hat{\lambda}, \hat{\mu})$, then we say that CALR occurs.

To ensure the phenomenon of CALR, the resonance condition (1.10) is crucial. For the bounded condition (1.11), the core-shell-matrix structure could generally fulfill this condition. However, if there is no core in the configuration, namely $D = \emptyset$ in (1.3), then the bounded condition is not satisfied. Thus in the rest of the paper, we focus ourself on the resonance condition (1.10). Two methods are discussed to show the resonance results. The first one is the spectral method. Through investigating the spectral system of the N-P operator, one can determine the metamaterial in the shell such that the ALR can occur. Another one is the variational method, via establishing the primal variational principle and dual variational principle, one can show the ALR result by finding perfect plasmon waves. At last, we present some open problems for future discussing.

2. **Spectral method.** To give a better description of the spectral method, we first present some preliminary knowledge for the elastic system. Set Y_n^m with $n \in \mathbb{N}_0, -n \leq m \leq n$ to be the spherical harmonic functions. Let $\mathbb{S}_R$ be the surface of the ball B_R and denote by $\mathbb{S}$ for $R = 1$ for simplicity. We also denote the surface gradient by the operators $\nabla_{\mathbb{S}}$. Let $j_n(t)$ and $h_n(t)$, $n \in \mathbb{N}_0$, denote the spherical Bessel and Hankel functions of the first kind of order n, respectively (cf. [16]). The elastostatic operator $\mathcal{L}_{\lambda,\mu}$ associated to the Lamé constants (λ, μ) is defined by,

$$\mathcal{L}_{\lambda,\mu}\mathbf{w} := \mu\triangle\mathbf{w} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{w}, \tag{2.1}$$

for $\mathbf{w} \in \mathbb{C}^3$. The traction (the conormal derivative) of $\mathbf{w}$ on $\partial\Omega$ is defined to be

$$\partial_{\boldsymbol{\nu}}\mathbf{w} = \lambda(\nabla\cdot\mathbf{w})\boldsymbol{\nu} + 2\mu(\nabla^s\mathbf{w})\boldsymbol{\nu}, \tag{2.2}$$

where ∇^s is defined in (1.5) and $\boldsymbol{\nu}$ is the outward unit normal to the boundary $\partial\Omega$. From [22], the fundamental solution $\mathbf{\Gamma}^\omega$ for the operator $\mathcal{L}_{\lambda,\mu} + \omega^2$ can be decomposed into shear and pressure components

$$\mathbf{\Gamma}^\omega(\mathbf{x}) = \mathbf{\Gamma}_s^\omega(\mathbf{x}) + \mathbf{\Gamma}_p^\omega(\mathbf{x}), \tag{2.3}$$

where

$$\mathbf{\Gamma}_p^\omega(\mathbf{x}) = -\frac{1}{\mu k_s^2}\partial_i\partial_j\Gamma_p^\omega(\mathbf{x}),$$

and

$$\mathbf{\Gamma}_p^\omega(\mathbf{x}) = \frac{1}{\mu k_s^2}(k_s^2\mathbf{I} + \partial_i\partial_j)\Gamma_p^\omega(\mathbf{x}),$$

with k_s and k_p defined in (1.7). The function

$$\Gamma_\alpha^\omega(\mathbf{x}) = \Gamma^\omega(k_\alpha\mathbf{x})$$

with $\alpha = p$, or s, and

$$\Gamma^\omega(\mathbf{x}) = \begin{cases} -\frac{\mathrm{i}}{4}H_0^{(1)}(k_\alpha|\mathbf{x}|), & N=2, \\ -\frac{e^{\mathrm{i}k_\alpha|\mathbf{x}|}}{4\pi|\mathbf{x}|}, & N=3, \end{cases} \tag{2.4}$$

where $H_0^{(1)}(k_\alpha|\mathbf{x}|)$ is the Hankel function of the first kind of order 0.

Then the single layer potential associated with the fundamental solution $\mathbf{\Gamma}^\omega$ is defined as

$$\mathbf{S}_{\partial\Omega}^\omega[\boldsymbol{\varphi}](\mathbf{x}) = \int_{\partial\Omega}\mathbf{\Gamma}^\omega(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^N, \tag{2.5}$$

for $\boldsymbol{\varphi} \in L^2(\partial\Omega)^N$. On the boundary $\partial\Omega$, the conormal derivative of the single layer potential satisfies the following jump formula

$$\frac{\partial \mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}]}{\partial \boldsymbol{\nu}}|_{\pm}(\mathbf{x}) = \left(\pm\frac{1}{2}\mathbf{I} + (\mathbf{K}^{\omega}_{\partial\Omega})^*\right)[\boldsymbol{\varphi}](\mathbf{x}) \quad \mathbf{x} \in \partial\Omega, \tag{2.6}$$

where

$$(\mathbf{K}^{\omega}_{\partial\Omega})^*[\boldsymbol{\varphi}](\mathbf{x}) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \boldsymbol{\Gamma}^{\omega}}{\partial \boldsymbol{\nu}(\mathbf{x})}(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \tag{2.7}$$

with p.v. standing for the Cauchy principal value and the subscript $\pm$ indicating the limits from outside and inside Ω, respectively. The operator $(\mathbf{K}^{\omega}_{\partial\Omega})^*$ is called to be the Neumann-Poincaré (N-P) operator.

Thus the elastic system (1.4) can be expressed as the following equation system

$$\begin{cases} \mathcal{L}_{\breve{\lambda},\breve{\mu}}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = 0, & \text{in } D, \\ \mathcal{L}_{\hat{\lambda},\hat{\mu}}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = 0, & \text{in } \Omega\backslash\overline{D}, \\ \mathcal{L}_{\lambda,\mu}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = \mathbf{f}, & \text{in } \mathbb{R}^N\backslash\overline{\Omega}, \\ \mathbf{u}|_- = \mathbf{u}|_+, \quad \partial_{\breve{\boldsymbol{\nu}}}\mathbf{u}|_- = \partial_{\hat{\boldsymbol{\nu}}}\mathbf{u}|_+ & \text{on } \partial D, \\ \mathbf{u}|_- = \mathbf{u}|_+, \quad \partial_{\hat{\boldsymbol{\nu}}}\mathbf{u}|_- = \partial_{\boldsymbol{\nu}}\mathbf{u}|_+ & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

In (2.8) and also in what follows, $\mathcal{L}_{\hat{\lambda},\hat{\mu}}$ and $\partial_{\hat{\boldsymbol{\nu}}}$ denote the Lamé operator and the traction operator associated with the Lamé parameters $\hat{\lambda}$ and $\hat{\mu}$, and the same notations hold for the single-layer potential operator $\hat{\mathbf{S}}^{\omega}_{\Omega}$ and the N-P operator $(\hat{\mathbf{K}}^{\omega}_{\partial\Omega})^*$.

With the help of the potential theory introduced before, the solution to the equation system (2.8) can be represented by

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \breve{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_1](\mathbf{x}), & \mathbf{x} \in D, \\ \hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2](\mathbf{x}) + \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_3](\mathbf{x}), & \mathbf{x} \in \Omega\backslash\overline{D}, \\ \mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_4](\mathbf{x}) + \mathbf{F}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^N\backslash\overline{\Omega}, \end{cases} \tag{2.9}$$

where $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in L^2(\partial D)^N$, $\boldsymbol{\varphi}_3, \boldsymbol{\varphi}_4 \in L^2(\partial\Omega)^N$ and $\mathbf{F}$ is the Newtonian potential of the source $\mathbf{f}$ defined by

$$\mathbf{F}(\mathbf{x}) = \int_{\mathbb{R}^N} \boldsymbol{\Gamma}^{\omega}(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^N.$$

One can easily see that the solution given (2.9) satisfies the first three condition in (2.8) and the last two conditions on the boundary yield that

$$\begin{cases} \breve{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_1] = \hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_3], & \text{on } \partial D, \\ \partial_{\breve{\boldsymbol{\nu}}}\breve{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_1|_- = \partial_{\hat{\boldsymbol{\nu}}}(\hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_3])|_+, & \text{on } \partial D, \\ \hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_3] = \mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_4] + \mathbf{F}, & \text{on } \partial\Omega, \\ \partial_{\hat{\boldsymbol{\nu}}}(\hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_3])|_- = \partial_{\boldsymbol{\nu}}(\mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_4] + \mathbf{F})|_+, & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

With the help of the jump formual in (2.6), one has that the equation system (2.10) is equivalent to the following integral system,

$$\mathbf{A}^{\omega} \begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\varphi}_3 \\ \boldsymbol{\varphi}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{F} \\ \partial_{\boldsymbol{\nu}}\mathbf{F} \end{bmatrix}, \tag{2.11}$$

where $\partial_{\hat{\boldsymbol{\nu}}_i}$ and $\partial_{\hat{\boldsymbol{\nu}}_e}$ signify the conormal derivatives on the boundaries of D and Ω, respectively.

$$\mathbf{A}^\omega = \begin{bmatrix} \breve{\mathbf{S}}^\omega_{\partial D} & -\hat{\mathbf{S}}^\omega_{\partial D} & -\hat{\mathbf{S}}^\omega_{\partial\Omega} & 0 \\ -\frac{1}{2} + (\breve{\mathbf{K}}^\omega_{\partial D})^* & -\frac{1}{2} - (\hat{\mathbf{K}}^\omega_{\partial\Omega})^* & \partial_{\hat{\boldsymbol{\nu}}_i}\hat{\mathbf{S}}^\omega_{\partial\Omega} & 0 \\ 0 & \hat{\mathbf{S}}^\omega_{\partial D} & \hat{\mathbf{S}}^\omega_{\partial\Omega} & -\mathbf{S}^\omega_{\partial\Omega} \\ 0 & \partial_{\hat{\boldsymbol{\nu}}_e}\hat{\mathbf{S}}^\omega_{\partial D} & -\frac{1}{2} + (\hat{\mathbf{K}}^\omega_{\partial\Omega})^* & -\frac{1}{2} - (\mathbf{K}^\omega_{\partial\Omega})^* \end{bmatrix}$$

From the equation (2.11), one can conclude that if the spectral system of the N-P operator $(\mathbf{K}^\omega_{\partial\Omega})^*$ is determined, then the eigensystem of the operator $\mathbf{A}^\omega$ is determined. Furthermore, by appropriately choosing the metamaterial in the shell $\Omega\backslash\overline{D}$, namely the Lamé parameters $(\hat{\lambda}, \hat{\mu})$, such that 0 is an essential spectrum of the operator $\mathbf{A}^\omega$, one can show that the resonance condition (1.10) is satisfied. For the boundedness condition (1.11), since the configuration is a core-shell-matrix structure, this condition is generally easy to prove; see [3, 17]. Thus in the following, we focus ourself on investigating the spectral system of the N-P operator $(\mathbf{K}^\omega_{\partial\Omega})^*$.

For the spectral method, Ammari et al [3] firstly apply this method to show the phenomenon of CALR in electrostatics governed by the Laplace equation in two dimensions. In this case, the corresponding N-P operator is compact and by introducing a new inner product, one can show that the corresponding N-P operator is symmetric in the new Hilbert space. Thus Hilbert-Schmidt theorem could be applied to investigate the spectrum of the N-P operator. However, for the elastostatic, the N-P operator $(\mathbf{K}^\omega_{\partial\Omega})^*$ with $\omega = 0$ is no longer compact and is only polynomial compact ([8, 12]). More precisely, in two dimensions the following polynomial operator is compact

$$\left((\mathbf{K}^0_{\partial\Omega})^*\right)^2 - k_0^2\mathbf{I}, \tag{2.12}$$

where

$$k_0 = \frac{\mu}{2(\lambda + 2\mu)}. \tag{2.13}$$

Here we would like to mention that the elastostatic denotes the case that the size of the scatter is small compared with the wavelength of the associated wave field, namely

$$\omega \cdot \mathrm{diam}(\Omega) \ll 1. \tag{2.14}$$

We also call this quasi-static approximation. By the coordinate transformation, the quasi-static approximation is equivalent to the situation where the scatter Ω is of the regular size and the frequency $\omega \ll 1$. In many researches [8, 10, 17, 27, 28], the ALR is considered by directly taking $\omega = 0$. In [8], the eigensystem of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$ is derived in two dimensions when Ω is a disk and an ellipse. Indeed, when Ω is a disk, the eigenvalues of the N-P operator are

$$\frac{1}{2}, \quad -\frac{\lambda}{2(\lambda+2\mu)}, \quad \pm k_0,$$

where k_0 is given in (2.13). The associated eigenfunctions are given as follows

1. $1/2$:

$$(1,0)^T, \quad (0,1)^T, \quad (x_2, -x_1)^T$$

2. $-\frac{\lambda}{2(\lambda+2\mu)}$:

$$(x_1, x_2)^T,$$

3. k_0

$$\begin{bmatrix} \cos m\theta \\ \sin m\theta \end{bmatrix}, \quad \begin{bmatrix} -\sin m\theta \\ \cos m\theta \end{bmatrix}, \quad m = 2, 3, \cdots,$$

4. $-k_0$

$$\begin{bmatrix} \cos m\theta \\ -\sin m\theta \end{bmatrix}, \quad \begin{bmatrix} \sin m\theta \\ \cos m\theta \end{bmatrix}, \quad m = 1, 2, 3, \cdots.$$

When the domain Ω is ellipse, the expression for the eigenvalues of the N-P operator is complicated and please refer to [8].

Whereas in three dimensions, the spectral system of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$ is more complex. Nevertheless, the paper [17] exactly shows the spectral system of N-P when the integral domain is a sphere and strictly verifies the phenomenon of CALR. Specifically, when Ω is a ball in three dimensions, the eigenvalues of the of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$ are given as follows

$$\begin{aligned} \xi_1^n =& \frac{3}{4n+2}, \\ \xi_2^n =& \frac{3\lambda - 2\mu(2n^2 - 2n - 3)}{2(\lambda + 2\mu)(4n^2 - 1)}, \\ \xi_3^n =& \frac{-3\lambda + 2\mu(2n^2 + 2n - 3)}{2(\lambda + 2\mu)(4n^2 - 1)}, \end{aligned} \tag{2.15}$$

where $n \geq 1$ are nature numbers. The corresponding eigenfunctions are respectively $\mathcal{T}_n^m$, $\mathcal{M}_n^m$ and $\mathcal{N}_n^m$, where

$$\begin{aligned} \mathcal{T}_n^m(\mathbf{x}) =& \nabla_{\mathbb{S}} Y_n^m(\hat{\mathbf{x}}) \times \boldsymbol{\nu}_{\mathbf{x}}, \\ \mathcal{M}_n^m(\mathbf{x}) =& \nabla_{\mathbb{S}} Y_n^m(\hat{\mathbf{x}}) + n Y_n^m(\hat{\mathbf{x}}) \boldsymbol{\nu}_{\mathbf{x}}, \\ \mathcal{N}_n^m(\mathbf{x}) =& \frac{a_n^m}{2n-1}(-\nabla_{\mathbb{S}} Y_{n-1}^m(\hat{\mathbf{x}}) + n Y_{n-1}^m(\hat{\mathbf{x}}) \boldsymbol{\nu}_{\mathbf{x}}). \end{aligned} \tag{2.16}$$

From [35], one has that the function $\mathcal{T}_n^m$, $\mathcal{M}_n^m$ and $\mathcal{N}_n^m$ form an complete basis on $L^2(\mathbb{S})^N$. From the expressions of the eigenvalues of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$ in (2.15), one can conclude that the polynomial operator given in (2.12) is no longer compact in three dimensions, since the eigenvalues

$$\xi_1^n = \frac{3}{4n+2} \to 0, \quad \text{as} \quad n \to \infty.$$

Thus one may suspect that the corresponding polynomial compact operator in three dimensions should have the following form

$$(\mathbf{K}^0_{\partial\Omega})^* \left(\left((\mathbf{K}^0_{\partial\Omega})^* \right)^2 - k_0^2 \mathbf{I} \right),$$

with k_0 given in (2.13). Indeed, this conclusion has been verified in [12]. Here, we briefly introduce how to calculate the spectrum of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$ in three dimensions. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors on $\partial\Omega$. From the definition of the fundamental solution $\mathbf{\Gamma}^0(\mathbf{x})$ in (2.3) and straightforward computations one can show that

$$\partial_{\boldsymbol{\nu}_{\mathbf{x}}} \mathbf{\Gamma}^0(\mathbf{x} - \mathbf{y}) = -b_1 \mathbf{K}_1(\mathbf{x}, \mathbf{y}) + \mathbf{K}_2(\mathbf{x}, \mathbf{y}), \tag{2.17}$$

where

$$\begin{aligned} \mathbf{K}_1(\mathbf{x}, \mathbf{y}) =& \frac{\boldsymbol{\nu}_{\mathbf{x}} (\mathbf{x} - \mathbf{y})^T - (\mathbf{x} - \mathbf{y}) \boldsymbol{\nu}_{\mathbf{x}}^T}{4\pi |\mathbf{x} - \mathbf{y}|^3}, \\ \mathbf{K}_2(\mathbf{x}, \mathbf{y}) =& b_1 \frac{(\mathbf{x} - \mathbf{y}) \cdot \boldsymbol{\nu}_{\mathbf{x}}}{4\pi |\mathbf{x} - \mathbf{y}|^3} \mathbf{I}_3 + b_2 \frac{(\mathbf{x} - \mathbf{y}) \cdot \boldsymbol{\nu}_{\mathbf{x}}}{4\pi |\mathbf{x} - \mathbf{y}|^5} (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y})^T, \end{aligned} \tag{2.18}$$

with

$$b_1 = \frac{\mu}{2\mu+\lambda} \quad \text{and} \quad b_2 = \frac{3(\mu+\lambda)}{2\mu+\lambda}. \tag{2.19}$$

Then by the definition of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$ in (2.7), one has that

$$\begin{aligned}(\mathbf{K}^0_{\partial\Omega})^*[\boldsymbol{\varphi}](\mathbf{x}) =& -b_1\int_{\partial D}\mathbf{K}_1(\mathbf{x},\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) + \int_{\partial D}\mathbf{K}_2(\mathbf{x},\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) \\ :=& L_1 + L_2.\end{aligned} \tag{2.20}$$

Since Ω is a central ball, for $\mathbf{x},\mathbf{y}\in\partial\Omega$, one has that

$$(\boldsymbol{\nu}_{\mathbf{x}} - \boldsymbol{\nu}_{\mathbf{y}})(\mathbf{x}-\mathbf{y})^t = (\mathbf{x}-\mathbf{y})(\boldsymbol{\nu}_{\mathbf{x}} - \boldsymbol{\nu}_{\mathbf{y}})^t$$

and thus

$$\begin{aligned}\mathbf{K}_1(\mathbf{x},\mathbf{y}) =& \frac{\boldsymbol{\nu}_{\mathbf{x}}(\mathbf{x}-\mathbf{y})^t - (\mathbf{x}-\mathbf{y})\boldsymbol{\nu}_{\mathbf{x}}^t}{4\pi|\mathbf{x}-\mathbf{y}|^3}, \\ =& \frac{(\boldsymbol{\nu}_{\mathbf{x}} - \boldsymbol{\nu}_{\mathbf{y}} + \boldsymbol{\nu}_{\mathbf{y}})(\mathbf{x}-\mathbf{y})^t - (\mathbf{x}-\mathbf{y})(\boldsymbol{\nu}_{\mathbf{x}} - \boldsymbol{\nu}_{\mathbf{y}} + \boldsymbol{\nu}_{\mathbf{y}})^t}{4\pi|\mathbf{x}-\mathbf{y}|^3}, \\ =& \frac{\boldsymbol{\nu}_{\mathbf{y}}(\mathbf{x}-\mathbf{y})^t - (\mathbf{x}-\mathbf{y})\boldsymbol{\nu}_{\mathbf{y}}^t}{4\pi|\mathbf{x}-\mathbf{y}|^3}.\end{aligned} \tag{2.21}$$

Next, one can verify that

$$\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}_{\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3} = -\frac{1}{2r_0}\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{2.22}$$

By using vector calculus identity, (2.20) and (2.22), one can obtain that

$$\begin{aligned}L_1 =& -b_1\int_{\partial\Omega}\nabla_{\mathbf{x}}\Gamma^0(\mathbf{x}-\mathbf{y})\times\boldsymbol{\nu}_{\mathbf{y}}\times\boldsymbol{\varphi}(\mathbf{y}) + \frac{1}{2r_0}\Gamma^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi} - \\ & \nabla_{\mathbf{x}}\Gamma^0(\mathbf{x}-\mathbf{y})(\boldsymbol{\nu}\cdot\boldsymbol{\varphi})ds(\mathbf{y}) \\ =& -b_1\left(\nabla\times\mathcal{S}_\Omega[\boldsymbol{\nu}\times\boldsymbol{\varphi}](\mathbf{x}) + \frac{1}{2r_0}\mathcal{S}_\Omega[\boldsymbol{\varphi}](\mathbf{x}) - \nabla\mathcal{S}_\Omega[\boldsymbol{\nu}\cdot\boldsymbol{\varphi}](\mathbf{x})\right),\end{aligned} \tag{2.23}$$

where

$$\mathcal{S}_\Omega[\phi](\mathbf{x}) := \int_{\partial\Omega}\Gamma^0(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})ds_{\mathbf{y}},$$

with $\Gamma^0(\mathbf{x})$ defined in (2.4). Then direct calculation shows that

$$\begin{aligned}\mathbf{K}_2(\mathbf{x},\mathbf{y}) =& -\frac{b_1}{2r_0}\Gamma^0(\mathbf{x}-\mathbf{y})\mathbf{I}_3 + \frac{b_2}{2r_0}\frac{(\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y})^t}{4\pi|\mathbf{x}-\mathbf{y}|^3} \\ =& -\frac{b_2}{2r_0\alpha_2}\mathbf{\Gamma}^0(\mathbf{x}-\mathbf{y}) + (\frac{b_2\alpha_1}{2r_0\alpha_2} - \frac{b_1}{2r_0})\Gamma^0(\mathbf{x}-\mathbf{y})\mathbf{I}_3.\end{aligned} \tag{2.24}$$

Hence, there holds

$$\begin{aligned}L_2 =& -\frac{b_2}{2r_0\alpha_2}\int_{\partial\Omega}\mathbf{\Gamma}^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) + \\ & \left(\frac{b_2\alpha_1}{2r_0\alpha_2} - \frac{b_1}{2r_0}\right)\int_{\partial\Omega}\Gamma^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}) \\ =& -\frac{b_2}{2r_0\alpha_2}\mathbf{S}_\Omega[\boldsymbol{\varphi}](\mathbf{x}) + (\frac{b_2\alpha_1}{2r_0\alpha_2} - \frac{b_1}{2r_0})\mathcal{S}_\Omega[\boldsymbol{\varphi}](\mathbf{x}).\end{aligned} \tag{2.25}$$

Finally, by combining (2.23) and (2.25), we have

$$(\mathbf{K}^0_{\partial\Omega})^*[\boldsymbol{\varphi}](\mathbf{x}) = -\, b_1 \left(\nabla \times \mathcal{S}_\Omega[\boldsymbol{\nu} \times \boldsymbol{\varphi}](\mathbf{x}) - \nabla \mathcal{S}_\Omega[\boldsymbol{\nu} \cdot \boldsymbol{\varphi}](\mathbf{x})\right) \\ - \frac{b_2}{2r_0\alpha_2}\mathbf{S}_\Omega[\boldsymbol{\varphi}](\mathbf{x}) + (\frac{b_2\alpha_1}{2r_0\alpha_2} - \frac{b_1}{r_0})\mathcal{S}_\Omega[\boldsymbol{\varphi}](\mathbf{x}). \tag{2.26}$$

Moreover, the eigensystem of the operator $\mathcal{S}_\Omega$ has the following expression

$$\begin{aligned} \mathcal{S}_\Omega[\mathcal{T}_n^m] &= -\frac{r_0}{2n+1}\mathcal{T}_n^m, \\ \mathcal{S}_\Omega[\mathcal{M}_n^m] &= -\frac{r_0}{(2n-1)}\mathcal{M}_n^m, \\ \mathcal{S}_\Omega[\mathcal{N}_{n+1}^m] &= -\frac{r_0}{2n+3}\mathcal{N}_{n+1}^m, \end{aligned} \tag{2.27}$$

where $\mathcal{T}_n^m$, $\mathcal{M}_n^m$ and $\mathcal{N}_n^m$ are given in (2.16). Then with the help of the jump formula defined in (2.6), one can finally derive the spectrum of the N-P operator $(\mathbf{K}^0_{\partial\Omega})^*$.

As mentioned before, [8, 10, 17, 27, 28] consider the static case by directly taking $\omega \equiv 0$. Moreover, to induce the ALR, these research break off both the two strong convexity conditions shown in (1.2). Later, the paper [24] strictly verifies the ALR for the quasi-static approximation case, namely $\omega \ll 1$. Besides, in [24], only one of the two strong convexity conditions in (1.2) is required to be violated in order to induce the ALR. This extensively extends the restriction on the matematerial in the shell and makes the fabrication of the matematerial comparatively easier.

Recently, the paper [19] considers the CALR for the elastic system in three dimensions within finite frequency beyond the quasi-static approximation; that is the quasi-static approximation $\omega \ll 1$ is not required. The spectral method is utilized to show the CALR. In this case, the N-P operator $(\mathbf{K}^\omega_{\partial\Omega})^*$ is neither compact or symmetric in any Hilbert space. Thus, showing the ALR is difficult for the elastic system within finite frequency beyond the quasi-static approximation. The critical point is again the spectrum of the N-P operator. In [19], the eigensystem of N-P operator $(\mathbf{K}^\omega_{\partial\Omega})^*$ is explicitly derived in three dimensions when Ω is a ball. In detail, the eigensystem of the N-P operator $(\mathbf{K}^\omega_{\partial\Omega})^*$ have the following expression,:

$$(\mathbf{K}^\omega_{\partial\Omega})^*\left[\mathcal{T}_n^m\right] = \lambda_{1,n}\mathcal{T}_n^m, \tag{2.28}$$

$$(\mathbf{K}^\omega_{\partial\Omega})^*\left[\mathcal{U}_n^m\right] = \lambda_{2,n}\mathcal{U}_n^m, \tag{2.29}$$

$$(\mathbf{K}^\omega_{\partial\Omega})^*\left[\mathcal{V}_n^m\right] = \lambda_{3,n}\mathcal{V}_n^m, \tag{2.30}$$

where

$$\lambda_{1,n} = \mathfrak{b}_n - 1/2,$$

and if $\mathfrak{d}_{1n} \neq 0$,

$$\begin{aligned} \lambda_{2,n} &= \frac{\mathfrak{c}_{1n} + \mathfrak{d}_{2n} - 1 + \sqrt{(\mathfrak{d}_{2n} - \mathfrak{c}_{1n})^2 + 4\mathfrak{d}_{1n}\mathfrak{c}_{2n}}}{2}, \\ \lambda_{3,n} &= \frac{\mathfrak{c}_{1n} + \mathfrak{d}_{2n} - 1 - \sqrt{(\mathfrak{d}_{2n} - \mathfrak{c}_{1n})^2 + 4\mathfrak{d}_{1n}\mathfrak{c}_{2n}}}{2}, \\ \mathcal{U}_n^m &= \left(\mathfrak{c}_{1n} - \mathfrak{d}_{2n} + \sqrt{(\mathfrak{d}_{2n} - \mathfrak{c}_{1n})^2 + 4\mathfrak{d}_{1n}\mathfrak{c}_{2n}}\right)\mathcal{M}_{n-1}^m + 2\mathfrak{d}_{1n}\mathcal{N}_{n+1}^m, \\ \mathcal{V}_n^m &= \left(\mathfrak{c}_{1n} - \mathfrak{d}_{2n} - \sqrt{(\mathfrak{d}_{2n} - \mathfrak{c}_{1n})^2 + 4\mathfrak{d}_{1n}\mathfrak{c}_{2n}}\right)\mathcal{M}_{n-1}^m + 2\mathfrak{d}_{1n}\mathcal{N}_{n+1}^m; \end{aligned}$$

if $\mathfrak{d}_{1n} = 0$,

$$\lambda_{2,n} = \mathfrak{c}_{1n} - 1/2, \quad \lambda_{3,n} = \mathfrak{d}_{2n} - 1/2,$$
$$\mathcal{U}_n^m = \mathcal{M}_{n-1}^m, \quad \mathcal{V}_n^m = \mathfrak{c}_{2n}\mathcal{M}_{n-1}^m + (\mathfrak{d}_{2n} - \mathfrak{c}_{1n})\mathcal{N}_{n+1}^m,$$

with $\mathcal{T}_n^m$, $\mathcal{M}_n^m$ and $\mathcal{N}_n^m$ given in (2.16), and the parameters $\mathfrak{b}_n$, $\mathfrak{c}_{1n}$, $\mathfrak{d}_{1n}$, $\mathfrak{c}_{2n}$ and $\mathfrak{d}_{2n}$ defined with the help of the the spherical Bessel functions $j_n(t)$ and Hankel functions of the first kind $h_n(t)$. Then the phenomenon of CALR could be demonstrated with the help of the explicit expression of the spectral system of the N-P operator $(\mathbf{K}_{\partial\Omega}^{\omega})^*$.

As aforementioned, the CALR results are different for the spectral method and the variational method. For the spectral method, the CALR results can be summarized for both the quasi-static approximation and beyond the quasi-static approximation as follows. Consider the configuration $(\mathbf{C}_0, \mathbf{f})$ given in (1.4). Suppose that the source $\mathbf{f}$ is compactly supported in $\mathbb{R}^N\backslash\overline{\Omega}$. Then if the parameters $(\breve{\lambda}, \breve{\mu})$ in the core D and the parameters $(\hat{\lambda}, \hat{\mu})$ in the shell $\Omega\backslash\overline{D}$ are chosen appropriately, then both the configuration and source are invisible provided the source $\mathbf{f}$ is located inside the critical radius r_*. If the source $\mathbf{f}$ is located outside the critical radius r_*, the ALR will not occur. We would like to mention that the parameters $(\breve{\lambda}, \breve{\mu})$ in the core D as well as the parameters $(\hat{\lambda}, \hat{\mu})$ in the shell $\Omega\backslash\overline{D}$ should be constant, and the CALR can occur for a general source $\mathbf{f}$ as long as it is located inside a critical radius.

3. **Variational method.** In this section, we discuss the anomalous localized resonance for the linear elastic system from the variational perspective. The papers [27] and [28] apply this method to explore the ALR in two and three dimensions. To utilize the variational method, one needs to first establish the variational principles. For that purpose, the configuration $\mathbf{C}_0$ in (1.3) should be modified accordingly. More precisely, the parameters should be chose as follows

$$\big(A(x) + \mathrm{i}\delta\big)(\lambda, \mu), \quad \mathbf{x} \in \mathbb{R}^N, \quad N = 2, 3, \tag{3.1}$$

where $\delta \in \mathbb{R}_+$ denotes a loss parameter and (λ, μ) are two Lamé constants satisfying the strong convexity condition (1.2). In (3.1), the function $A(x)$ has a matrix-shell-core representation in the following form

$$A(x) = \begin{cases} +1, & \mathbf{x} \in D, \\ c, & \mathbf{x} \in \Omega\backslash\overline{D}, \\ +1, & \mathbf{x} \in \mathbb{R}^N\backslash\overline{\Omega}, \end{cases} \tag{3.2}$$

where c is a negative constant denoting the metamaterial in the shell. Thus the dissipation energy defined in (1.9) becomes

$$E(\mathbf{u}) = \delta P_{\lambda,\mu}(\mathbf{u}, \mathbf{u}),$$

where $P_{\lambda,\mu}(\mathbf{u}, \mathbf{u})$ is defined in (1.8) with integration domain replaced by $\mathbb{R}^N$. Furthermore, we decompose the wave field of the system (1.4) into the real part and imaginary part, namely,

$$\mathbf{u} = \mathbf{v} + \mathrm{i}\frac{1}{\delta}\mathbf{w}.$$

Then the system (1.4) is equivalent to solve the following equation system

$$\mathcal{L}_{\lambda_A,\mu_A}\mathbf{v} - \mathcal{L}_{\lambda,\mu}\mathbf{w} = \mathbf{f}, \tag{3.3}$$

$$\mathcal{L}_{\lambda_A,\mu_A}\mathbf{w} + \delta^2\mathcal{L}_{\lambda,\mu}\mathbf{v} = \mathbf{0}, \tag{3.4}$$

where

$$(\lambda_A(\mathbf{x}), \mu_A(\mathbf{x})) := A(\mathbf{x})(\lambda, \mu), \quad \mathbf{x} \in \mathbb{R}^N \tag{3.5}$$

with A is given in (3.2). Based on the equations (3.3) and (3.4), one can construct the primal variational principle and dual variational principle for the system (1.4). In detail, the primal variational principle is established by treating the equation (3.3) as a constrain and doing the variation for the equation (3.4). Conversely, the dual variational principle is established by treating the equation (3.4) as a constrain and doing the variation for the equation (3.3). Next, we explain this in details. First, we introduce the following Banach space

$$\mathcal{S} := \{\mathbf{u} \in H^1_{\text{loc}}(\mathbb{R}^N)^N;\ \nabla\mathbf{u} \in L^2(\mathbb{R}^N)^{N\times N} \text{ and } \int_{B_{R_0}} \mathbf{u} = 0\}, \tag{3.6}$$

endowed with the Sobolev norm for $\mathbf{u} = (u_i)_{i=1}^N$,

$$\|\mathbf{u}\|_{\mathcal{S}} := \left(\int_{\mathbb{R}^N} \sum_{i=1}^N \|\nabla u_i\|^2\, dV + \int_{B_{R_0}} \|\mathbf{u}\|^2\, dV\right)^{1/2}. \tag{3.7}$$

Furthermore, we define the following two energy functionals

$$\mathbf{I}_\delta(\mathbf{v}, \mathbf{w}) := \frac{\delta}{2} P_{\lambda,\mu}(\mathbf{v}, \mathbf{v}) + \frac{1}{2\delta} P_{\lambda,\mu}(\mathbf{w}, \mathbf{w}) \quad \text{for} \quad (\mathbf{v}, \mathbf{w}) \in \mathcal{S}\times\mathcal{S}, \tag{3.8}$$

$$\mathbf{J}_\delta(\mathbf{v}, \boldsymbol{\psi}) := \int_{\mathbb{R}^3} \mathbf{f}\cdot\boldsymbol{\psi} - \frac{\delta}{2} P_{\lambda,\mu}(\mathbf{v}, \mathbf{v}) - \frac{\delta}{2} P_{\lambda,\mu}(\boldsymbol{\psi}, \boldsymbol{\psi}) \text{ for } (\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{S}\times\mathcal{S}. \tag{3.9}$$

Then, we consider the following optimization problems:

$$\begin{aligned} &\text{Minimize } \mathbf{I}(\mathbf{v}, \mathbf{w}) \text{ over all pairs } (\mathbf{v}, \mathbf{w}) \in \mathcal{S}\times\mathcal{S} \\ &\text{subject to the PDE constraint } \mathcal{L}_{\lambda_A,\mu_A}\mathbf{v} - \mathcal{L}_{\lambda,\mu}\mathbf{w} = \mathbf{f}; \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\text{Maximize } \mathbf{J}(\mathbf{v}, \boldsymbol{\psi}) \text{ over all pairs } (\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{S}\times\mathcal{S} \\ &\text{subject to the PDE constraint } \mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi} + \delta\mathcal{L}_{\lambda,\mu}\mathbf{v} = \mathbf{0}. \end{aligned} \tag{3.11}$$

The optimization problems are referred to as (3.10) and (3.11), respectively, as the primal and dual variational problems for the elastostatic system (1.4), or equivalently (3.3)-(3.4). Then we have the following variational principles; see [27] and [28] .

Theorem 3.1. *There holds the primal variational principle that the problem (3.10) is equivalent to the elastic problem (1.4) in the following sense. The infimum*

$$\inf\{\mathbf{I}(\tilde{\mathbf{v}}, \tilde{\mathbf{w}});\ \mathcal{L}_{\lambda_A,\mu_A}\tilde{\mathbf{v}} - \mathcal{L}_{\lambda,\mu}\tilde{\mathbf{w}} = \mathbf{f}\}$$

is attainable at a pair $(\mathbf{v}, \mathbf{w}) \in \mathcal{S}\times\mathcal{S}$*. The minimizing pair* $(\mathbf{v}, \mathbf{w})$ *verifies that the function* $\mathbf{u} := \mathbf{v} + \mathrm{i}\delta^{-1}\mathbf{w}$ *is the unique solution to the elastic problem (1.4) and moreover one has*

$$\mathbf{E}(\mathbf{u}) = \mathbf{I}(\mathbf{v}, \mathbf{w}). \tag{3.12}$$

Similarly, there holds the dual variational principle that the problem (3.11) is equivalent to the elastic problem (1.4) in the following sense. The supremum

$$\sup\{\mathbf{J}(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\psi}}); \mathcal{L}_{\lambda_A,\mu_A}\tilde{\boldsymbol{\psi}} + \delta\mathcal{L}_{\lambda,\mu}\tilde{\mathbf{v}} = \mathbf{0}\}$$

is attainable at a pair $(\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{S}\times\mathcal{S}$*. The maximizing pair* $(\mathbf{v}, \boldsymbol{\psi})$ *verifies that the function* $\mathbf{u} := \mathbf{v} + \mathrm{i}\boldsymbol{\psi}$ *is the unique solution to the elastic problem (1.4), and moreover one has*

$$\mathbf{E}(\mathbf{u}) = \mathbf{J}(\mathbf{v}, \boldsymbol{\psi}). \tag{3.13}$$

After establishing the variational principle, then one can apply the dual variational principle to show that ALR and primal variational principle to show none resonance result. The essential issue for applying the variational principle is to find the perfect plasmon waves, namely the nontrivial solution of a non-elliptic PDE as aforementioned. Indeed, the non-elliptic PDE has the following form:

$$\begin{cases} \mathcal{L}_{\lambda_A,\mu_A}\boldsymbol{\psi} = 0, \\ \boldsymbol{\psi}|_- = \boldsymbol{\psi}|_+, \quad \partial_{\boldsymbol{\nu}_{\lambda_A,\mu_A}}\boldsymbol{\psi}|_- = \partial_{\boldsymbol{\nu}_{\lambda_A,\mu_A}}\boldsymbol{\psi}|_+ \quad \text{on} \ \ \partial B_R, \\ \boldsymbol{\psi}(x) = \mathcal{O}(\|x\|^{-1}) \quad \text{as} \quad \|x\| \to \infty, \end{cases} \tag{3.14}$$

where the function $\boldsymbol{\psi} \in H^1_{\text{loc}}(\mathbb{R}^N)^N : \mathbb{R}^N \to \mathbb{R}^N$ and (λ_A, μ_A) is given of the form (3.5) with

$$A(x) = \begin{cases} c, & \|x\| \le R, \\ +1, & \|x\| > R. \end{cases} \tag{3.15}$$

In [27], the perfect plasmon waves in two dimensions are obtained. If

$$c := -\frac{\lambda+\mu}{\lambda+3\mu}, \tag{3.16}$$

then the perfect plasmon waves $\boldsymbol{\psi} = \widehat{\boldsymbol{\psi}}_k$, $k = 1, 2, \dots$ have the following form

$$\widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k\cos(k\theta) \\ -r^k\sin(k\theta) \end{bmatrix}, & r \le R, \\ R^{2k}\begin{bmatrix} \frac{k\alpha(r^2-R^2)}{r^{k+2}}\cos((k+2)\theta) + \frac{1}{r^k}\cos(k\theta) \\ \frac{k\alpha(r^2-R^2)}{r^{k+2}}\sin((k+2)\theta) - \frac{1}{r^k}\sin(k\theta)+ \end{bmatrix}, & r > R; \end{cases} \tag{3.17}$$

or

$$\widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k\sin(k\theta) \\ r^k\cos(k\theta) \end{bmatrix}, & r \le R, \\ R^{2k}\begin{bmatrix} \frac{1}{r^k}\sin(k\theta) + \frac{k\alpha(r^2-R^2)}{r^{k+2}}\sin((k+2)\theta) \\ \frac{1}{r^k}\cos(k\theta) - \frac{k\alpha(r^2-R^2)}{r^{k+2}}\cos((k+2)\theta) \end{bmatrix}, & r > R; \end{cases} \tag{3.18}$$

where

$$\alpha = -c. \tag{3.19}$$

If

$$c = -\frac{\lambda+3\mu}{\lambda+\mu}, \tag{3.20}$$

then the perfect plasmon waves $\boldsymbol{\psi} = \widehat{\boldsymbol{\psi}}_k$, $k = 2, 3, \dots$ can be written as

$$\widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} r^k\cos(k\theta) - k\alpha(r^2-R^2)r^{k-2}\cos((k-2)\theta) \\ r^k\sin(k\theta) + k\alpha(r^2-R^2)r^{k-2}\sin((k-2)\theta) \end{bmatrix} & r \le R, \\ R^{2k}\begin{bmatrix} r^{-k}\cos(k\theta) \\ r^{-k}\sin(k\theta) \end{bmatrix} & r > R; \end{cases} \tag{3.21}$$

or

$$\widehat{\boldsymbol{\psi}}_k(x) := \begin{cases} \begin{bmatrix} -r^k \sin(k\theta) + k\alpha(r^2 - R^2) r^{k-2} \cos((k-2)\theta) \\ r^k \cos(k\theta) + k\alpha(r^2 - R^2) r^{k-2} \sin((k-2)\theta) \end{bmatrix}, & r \le R, \\ R^{2k} \begin{bmatrix} -r^{-k} \sin(k\theta) \\ r^{-k} \cos(k\theta) \end{bmatrix}, & r > R; \end{cases} \tag{3.22}$$

where α is also given of the form in (3.19).

In three dimensions, the paper [28] presents the perfect plasmon waves. The same as the eigensystem of the N-P operator in three dimensions, the perfect plasmon waves in three dimensions are very complicated. The parameter c in (3.15) is not a constant any more and should be chosen depending on the order n as follows:

$$\begin{aligned} c_1 &= -1 - \frac{3}{n-1}, \\ c_2 &= -\frac{(2n+2)((n-1)\lambda + (3n-2)\mu)}{(2n^2+1)\lambda + (2+2n(n-1))\mu}, \\ c_3 &= -\frac{(2n^2+4n+3)\lambda + (2n^2+6n+6)\mu}{2n((n+2)\lambda + (3n+5)\mu)}. \end{aligned} \tag{3.23}$$

The corresponding perfect plasmon waves are very complicated and we choose not to present them here. Please refer to [28].

As mentioned before, finding the perfect plasmon waves of the corresponding non-elliptic PDE is equivalent to investigate the spectral system of the N-P operator. Next, we elaborate the relationship between the perfect plasmon waves and the spectral system of the N-P operator. Let us consider the non-elliptic PDE in (3.14) again. With the help of the potential theory, the solution, namely the perfect plasmon waves can be written as

$$\boldsymbol{\psi} = \mathbf{S}^0_{\partial\Omega}[\boldsymbol{\varphi}](\mathbf{x}) = \int_{\partial\Omega} \boldsymbol{\Gamma}^0(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^N, \tag{3.24}$$

where $\boldsymbol{\Gamma}^0(\mathbf{x})$ is the fundamental solution defined in (2.3), and $\boldsymbol{\varphi} \in H^{-1/2}(\partial B_R)^N$. With the help of the jump formula given in (2.6) and using the transmission condition across ∂B_R for $\boldsymbol{\psi}$, one can show that

$$(\mathbf{K}^0_{\partial\Omega})^*[\boldsymbol{\varphi}] = \frac{c+1}{2(c-1)}\boldsymbol{\varphi}. \tag{3.25}$$

Clearly, if we can choose the parameter c such that $(c+1)/(2(c-1))$ is an eigenvalue of the Neumann-Poincaré operator $(\mathbf{K}^0_{\partial\Omega})^*$, then the function $\boldsymbol{\psi}$ defined in (3.24) is a solution of the non-elliptic PDE in (3.14), with $\boldsymbol{\varphi}$ being the corresponding eigenvector. Conversely, if we can appropriately choose the constant c such that the non-elliptic PDE in (3.14) has an nontrivial solution $\boldsymbol{\psi}$, then $(c+1)/(2(c-1))$ is an eigenvalue of the Neumann-Poincaré operator $(\mathbf{K}^0_{\partial\Omega})^*$. Indeed, this has provided a way to investigate the spectrum of the N-P operator.

For the variational method, the CALR results can be summarized as follows. Consider the configuration $(\mathbf{C}_0, \mathbf{f})$, where $\mathbf{C}_0$ is given in (3.1) and the source $\mathbf{f}$ is supported on a circle in $\mathbb{R}^N\backslash\overline{\Omega}$. Then if the parameters $(\breve{\lambda}, \breve{\mu})$ in the core D and the parameters $(\hat{\lambda}, \hat{\mu})$ in the shell $\Omega\backslash\overline{D}$ are chosen appropriately, then both the configuration and source are invisible provided the source $\mathbf{f}$ is located inside the critical radius r_*. If the source $\mathbf{f}$ is located outside the critical radius r_*, the ALR will not occur. We would like to mention that the parameters $(\breve{\lambda}, \breve{\mu})$ in the core D could be a arbitrary bounded function and the shape of the core D could be arbitrary, which is different from the case for the spectral method. However, the limitation the CALR can occur for the source $\mathbf{f}$ supported on a line.

4. **Some open problems.** The paper [19] is the only research investigating the CALR for the system (1.4) within finite frequency beyond the quasi-static approximation. However, the authors only consider the radial geometry. Thus how to extend the phenomenon of CALR for the system (1.4) to the general geometry is still open. For the variational method, the papers [27] and [28] only establish the variational principle for the elastostatic system, namely the frequency $\omega = 0$. Thus how to build the variational principle for the system (1.4) within finite frequency beyond the quasi-static approximation is worthy investigating. The system (1.4) is the linear elastic system. However, many physical problem are nonlinear elastic system. Thus considering the ALR for the nonlinear elastic system is also very important.

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E-mail address: hjli@math.cuhk.edu.hk