ON SIMULTANEOUS RECOVERY OF SOURCES/OBSTACLES AND SURROUNDING MEDIUMS BY BOUNDARY MEASUREMENTS

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Abstract. We consider a particular type of inverse problems where an unknown source embedded in an inhomogeneous medium, and one intends to recover the source and/or the medium by knowledge of the wave field (generated by the unknown source) outside the medium. This type of inverse problems arises in many applications of practical importance, including photoacoustic and thermoacoustic tomography, brain imaging and geomagnetic anomaly detections. We survey the recent mathematical developments on this type of inverse problems. We discuss the mathematical tools developed for effectively tackling this type of inverse problems. We also discuss a related inverse problem of recovering an embedded obstacle and its surrounding medium by active measurements.

1. Introduction.

1.1. **Background.** Recently, there has been growing interest on the mathematical study of a particular type of inverse problems where an unknown source embedded in an inhomogeneous medium, and one intends to recover the source and/or the medium by knowledge of the wave field (generated by the unknown source) outside the medium. This is known as the inverse problem with passive measurements, and it is in comparison to the inverse problem with active measurements where in order to determine an unknown object, one sends a known probing wave field to generate the associated wave measurements. The aforementioned inverse problem with passive measurements have received considerable attentions in the literature due to its practical importance in many applications including the thermoacoustic tomography (TAT) and photoacoustic tomography (PAT) [19], electroencephalography (EEG) and magnetoencephalography (MEG) [11], geomagnetic monitoring [7, 8]. For a simple mathematical illustration, we consider this type of inverse problem described by the following scalar system:

$$
\begin{cases}
\frac{1}{c^2}\partial_t^2 u(\mathbf{x},t) + \Delta u(\mathbf{x},t) = F(\mathbf{x},t), & (\mathbf{x},t) \in \mathbb{R}^3 \times \mathbb{R}_+, \\
u(\mathbf{x},0) = f(\mathbf{x}), \quad \partial_t u(\mathbf{x},0) = 0, & \text{for} \quad \mathbf{x} \in \mathbb{R}^3,
\end{cases}
\tag{1.1}
$$

where c denotes the speed of wave. f is the initial state which represents an internal source produced by the tissue volume expansion. Let Ω be the unknown object, then $supp(f) \subset \Omega$ and $supp(1-c) \subset \Omega$. F can be treated as an active source term. In some real applications, there is no active source terms, that is $F = 0$; however, in some cases, the active source exists and is even unknown.

By introducing the following temporal Fourier transform:

$$
\hat{\mathbf{A}}(\omega) := \frac{1}{2\pi} \int_0^\infty \mathbf{A}(t) e^{i\omega t} dt, \quad \omega \in \mathbb{R}_+,
\tag{1.2}
$$

where $\mathbf{A}$ is any function (scalar or three dimensional vector) which belongs to $L^1(\mathbb{R}_+)^d \cap L^2(\mathbb{R}_+)^d$ ($d = 1, 3$). By using the temporal Fourier transform, the system (1.1) can be transformed to the following Helmholtz system:

$$
\Delta \hat{u}(\mathbf{x},\omega) + \frac{\omega^2}{c^2}\hat{u}(\mathbf{x},\omega) = \frac{i\omega}{2\pi c^2}f(\mathbf{x}) + \hat{F}(\mathbf{x},\omega).
\tag{1.3}
$$

In order to ensure the well-posedness of the transformed equation (1.3), one imposes the classical Sommerfeld radiation condition

$$
\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|\Big(\frac{\partial \hat{u}(\mathbf{x})}{\partial|\mathbf{x}|} - i\omega\hat{u}(\mathbf{x})\Big) = 0.
\tag{1.4}
$$

In a lot cases the problem is vector system more than the scaler system (1.3), which is governed by the following time-harmonic system in the frequency domain,

$$
\begin{cases}
\nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{F} & \text{in } \mathbb{R}^3, \\
\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} & \text{in } \mathbb{R}^3, \\
\nabla \cdot (\mu\mathbf{H}) = 0, \quad \nabla \cdot (\varepsilon\mathbf{E}) = \hat{\rho} & \text{in } \mathbb{R}^3, \\
\lim_{\|\mathbf{x}\|\to\infty} \|\mathbf{x}\|\big(\sqrt{\mu_0}\mathbf{H} \times \hat{\mathbf{x}} - \sqrt{\varepsilon_0}\mathbf{E}\big) = 0,
\end{cases}
\tag{1.5}
$$

where the last limit is the Silver-Müller radiation condition and it holds uniformly in all directions $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\| \in \mathbb{S}^2$. In (1.5), $\mathbf{E}$ and $\mathbf{H}$, respectively, denote the total electric and magnetic field and $\mathbf{F}$ is some active source term with $supp(\mathbf{F}) \subset \Omega$. $\hat{\rho}(\mathbf{x})$ stands for the charge density. The functions ε, μ and σ signify the electromagnetic (EM) medium parameters in $\mathbb{R}^3$, and are referred to as the electric permittivity,

the magnetic permeability and the electric conductivity, respectively. Let ε_0 and μ_0 denote, respectively, the permittivity and the permeability of the uniformly homogeneous free space $\mathbb{R}^3 \backslash \overline{\Omega}$. Define $k_0 := \omega \sqrt{\varepsilon_0 \mu_0}$ to be the wavenumber with respect to a frequency $\omega \in \mathbb{R}_+$. We shall first introduce the inverse problems that will be considered. We remark that most the inverse problems to be considered can be formulated in a same manner, although some of the problems should be modified a little bit. We will mention the modification in the related sections. We also mention that most of the notations and definitions shall be the same in different inverse problems which will be reviewed and any modifications will be stressed in the specific sections.

1.2. **Inverse problems.** We are mainly concerned with the passive measurements outside the body Ω, although we will also review on the simultaneous recovery of obstacle and surrounding medium with incident wave in the last section. The inverse problems on simultaneous recovery of source and medium with passive measurements can be formulated as follows:

$$\mathcal{D}(\mathbf{x}, \omega) := \mathcal{M}(\mathbf{x}, \omega)\Big|_{(\mathbf{x}, \omega) \in \Gamma \times \mathbb{R}_+} \longrightarrow (\mathcal{F}(\mathbf{x}), \mathcal{P}(\Omega)), \tag{1.6}$$

where $\mathcal{M}(\mathbf{x}, t) = \hat{u}(\mathbf{x}, \omega), (\mathbf{E}, \mathbf{H})(\mathbf{x}, \omega)$, $\mathcal{F}(\mathbf{x}) = f(\mathbf{x}), \mathbf{F}(\mathbf{x})$ is the source term and $\mathcal{P}(\Omega) = c, (\varepsilon, \mu, \sigma)$ stand for the parameter(s) to be recovered. Γ is some smooth surface outside the body Ω. It is noted that the dimension of measurements and unknown parameters are equal. Thus it is possibly that one can recover the unknown parameters uniquely by just using the measurements. On the other hand, since the data $\hat{u}(\mathbf{x}, \omega)$ $((\mathbf{E}, \mathbf{H})(\mathbf{x}, \omega))$ on Γ is analytic with respect to the frequency ω, by unique continuation theorem, one only needs the data collected on a short span with respect to ω. Here, only the low frequency data, that is $\omega \in (0, \omega_0)$, where ω_0 is small enough, is used. For two configurations $(\mathcal{F}_j, \mathcal{P}_j)$ with respect to two measurements $\mathcal{D}_j$, $j = 1, 2$, the uniqueness of the inverse problem (1.6) can then be expressed as

$$\mathcal{D}_1 = \mathcal{D}_2 \longrightarrow (\mathcal{F}_1, \mathcal{P}_1) = (\mathcal{F}_2, \mathcal{P}_2). \tag{1.7}$$

The uniqueness result in (1.7) is not so easy to obtain for the two configurations in a general space, such as L^2-space. Even for only recovery of one part in the configuration such as the parameter $\mathcal{P}$, it is quite challenging to get the uniqueness without any prior information on $\mathcal{P}$. Usually, one needs some assumptions on the configuration $(\mathcal{F}, \mathcal{P})$, which can be treated as a prior information.

1.3. **Layer potential.** Let $\Phi_\omega(\mathbf{x})$ be the fundamental solution to $-\Delta - \omega^2$, that is

$$\Phi_\omega(\mathbf{x}) := \frac{e^{i\omega\|\mathbf{x}\|}}{4\pi\|\mathbf{x}\|}. \tag{1.8}$$

One can also find the fundamental solution, denoted by $\mathbf{G}_\omega(\mathbf{x})$, which is a 6×6 matrix function, to system (1.5) ([10, 21])

$$\mathbf{G}_\omega(\mathbf{x}) = \begin{pmatrix} (k_0^2 I_3 + D^2)\Phi_{k_0}(\mathbf{x}) & i\omega\mu_0 \nabla \times (\Phi_{k_0}(\mathbf{x})I_3) \\ -i\omega\varepsilon_0 \nabla \times (\Phi_{k_0}(\mathbf{x})I_3) & (k_0^2 I_3 + D^2)\Phi_{k_0}(\mathbf{x}) \end{pmatrix},$$

where I_3 is the 3×3 identity matrix and D^2 means double derivative.

In what follows, we shall introduce the notions of layer potentials. Before this, we introduce some Sobolev spaces (cf. [18, 20]). First we define the space $L^2_{loc}(X)^d$,

$d = 1, 3$ to be

$$L^2_{loc}(X)^d = \{\mathbf{U}|_\Omega \in L^2(\Omega)^d; \ \Omega \ \text{is any bounded subdomain of } X\}.$$

We often use the spaces

$$H_{loc}(\text{curl}; X) = \{\mathbf{U}|_\Omega \in H(\text{curl}; \Omega); \ \Omega \ \text{is any bounded subdomain of } X\}$$

and

$$H(\text{curl}; \Omega) = \{\mathbf{U} \in (L^2(\Omega))^3; \ \nabla \times \mathbf{U} \in (L^2(\Omega))^3\}.$$

We also define the following spaces

$$H_{loc}(\text{div}; X) = \{\mathbf{U}|_\Omega \in H(\text{div}; \Omega); \ \Omega \ \text{is any bounded subdomain of } X\}$$

and

$$H(\text{div}; \Omega) = \{\mathbf{U} \in (L^2(\Omega))^3; \ \nabla \cdot \mathbf{U} \in (L^2(\Omega))\}.$$

We next present some elementary theory on layer potentials. For any bounded domain $B \subset \mathbb{R}^3$, we denote by $\mathcal{V}^\omega_B : L^2_{loc}(B)^d \to H^2_{loc}(B)^d$, $d = 1, 3$, the volume potential operator defined by

$$\mathcal{V}^\omega_B[\Psi](\mathbf{x}) := \int_B \Phi_\omega(\mathbf{x} - \mathbf{y})\Psi(\mathbf{y})d\mathbf{y}. \tag{1.9}$$

We also denote by $\mathcal{S}^\omega_B : H^{-1/2}(\partial B) \to H^1(\mathbb{R}^3 \setminus \partial B)$, the single layer potential operator given by

$$\mathcal{S}^\omega_B[\varphi](\mathbf{x}) := \int_{\partial B} \Phi_\omega(\mathbf{x} - \mathbf{y})\varphi(\mathbf{y})ds_{\mathbf{y}}, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial B, \tag{1.10}$$

and the double layer potential $\mathcal{D}^\omega_B : H^{1/2}(\partial B) \to H^1(\mathbb{R}^3 \setminus \partial B)$ given by

$$\mathcal{D}^\omega_B[\varphi](\mathbf{x}) := \int_{\partial B} \frac{\partial}{\partial \nu_y}\Phi_\omega(\mathbf{x} - \mathbf{y})\varphi(\mathbf{y})ds_{\mathbf{y}}, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial B,$$

and $\mathcal{K}^\omega_B : H^{1/2}(\partial B) \to H^{1/2}(\partial B)$ the Neumann-Poincaré operator

$$\mathcal{K}^\omega_B[\phi](\mathbf{x}) := \text{p.v.} \quad \int_{\partial B} \frac{\partial \Phi_\omega(\mathbf{x} - \mathbf{y})}{\partial \nu_y}\phi(\mathbf{y})ds_{\mathbf{y}}, \tag{1.11}$$

where p.v. stands for the Cauchy principle value. In (1.11) and also in what follows, ν denotes the exterior unit normal vector to the boundary of the domain concerned. It is known that the single layer potential operator $\mathcal{S}^\omega_B$ and the double layer potential operator $\mathcal{D}^\omega_B$ satisfy the trace formulae

$$\frac{\partial}{\partial \nu}\mathcal{S}^\omega_B[\phi]\Big|_\pm = (\mp \frac{1}{2}I + (\mathcal{K}^\omega_B)^*)[\phi] \quad \text{on } \partial B,$$
$$\mathcal{D}^\omega_B[\varphi](\mathbf{x})\Big|_\pm = (\pm \frac{1}{2}I + \mathcal{K}^\omega_B)[\varphi] \quad \text{on } \partial B, \tag{1.12}$$

where $(\mathcal{K}^\omega_B)^*$ is the adjoint operator of $\mathcal{K}^\omega_B$ with respect to the L^2 inner product.

Let $\nabla_{\partial B}\cdot$ denote the surface divergence. Denote by $L^2_T(\partial B) := \{\Psi \in L^2(\partial B)^3, \nu \cdot \Psi = 0\}$. Let $H^s(\partial B)$ be the usual Sobolev space of order $s \in \mathbb{R}$ on ∂B. Set

$$\text{TH}(\text{div}, \partial B) := \Big\{\Psi \in L^2_T(\partial B) : \nabla_{\partial B} \cdot \Psi \in L^2(\partial B)\Big\},$$

$$\text{TH}(\text{curl}, \partial B) := \Big\{\Psi \in L^2_T(\partial B) : \nabla_{\partial B} \cdot (\Psi \times \nu) \in L^2(\partial B)\Big\},$$

endowed with the norms

$$\|\Psi\|_{\mathrm{TH}(\mathrm{div},\partial B)} = \|\Psi\|_{L^2(\partial B)} + \|\nabla_{\partial B} \cdot \Psi\|_{L^2(\partial B)},$$
$$\|\Psi\|_{\mathrm{TH}(\mathrm{curl},\partial B)} = \|\Psi\|_{L^2(\partial B)} + \|\nabla_{\partial B} \cdot (\Psi \times \nu)\|_{L^2(\partial B)},$$

respectively. For a density function $\Psi \in \mathrm{TH}(\mathrm{div},\partial B)$, we define the vectorial single layer potential by

$$\mathcal{A}_B^\omega[\Psi](\mathbf{x}) := \int_{\partial B} \Phi_\omega(\mathbf{x} - \mathbf{y})\Psi(\mathbf{y})ds_{\mathbf{y}}, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial B. \tag{1.13}$$

It is known that $\nabla \times \mathcal{A}_B^\omega$ satisfies the following jump formula

$$\nu \times \nabla \times \mathcal{A}_B^\omega[\Psi]\big|_\pm = \pm \frac{\Psi}{2} + \mathcal{M}_B^\omega[\Psi] \quad \text{on } \partial B, \tag{1.14}$$

where

$$\forall \mathbf{x} \in \partial B, \quad \nu \times \nabla \times \mathcal{A}_B^\omega[\Psi]\big|_\pm(\mathbf{x}) = \lim_{t \to 0^+} \nu \times \nabla \times \mathcal{A}_B^\omega[\Psi](\mathbf{x} \pm t\nu)$$

and

$$\mathcal{M}_B^\omega[\Psi](\mathbf{x}) = \text{p.v.} \quad \nu \times \nabla \times \int_{\partial B} \Phi_\omega(\mathbf{x} - \mathbf{y})\Psi(\mathbf{y})ds_{\mathbf{y}}. \tag{1.15}$$

We also define $\mathrm{TH}(\mathrm{div},\partial B) \to \mathrm{TH}(\mathrm{div},\partial B)$ by

$$\mathcal{L}_B^\omega[\Psi](\mathbf{x}) := \nu_{\mathbf{x}} \times \nabla \times \nabla \times \mathcal{A}_B^\omega[\Psi](\mathbf{x}), \tag{1.16}$$

and $\mathcal{N}_B^\omega : \mathrm{TH}(\mathrm{div},\partial B) \to L^2(\partial B)$ by

$$\mathcal{N}_B^\omega[\Psi](\mathbf{x}) := \nu_{\mathbf{x}} \cdot \left(\nabla \times \mathcal{A}_B^\omega[\Psi](\mathbf{x}) \right). \tag{1.17}$$

It is mentioned that $\frac{I}{2} \pm \mathcal{M}_B^\omega$ is invertible on $\mathrm{TH}(\mathrm{div},\partial B)$ when ω is sufficiently small (see e.g., [1, 22]). In the following, if $\omega = 0$, we formally set Φ_ω introduced in (1.8) to be $\Phi_0 = -1/(4\pi\|\mathbf{x}\|)$, and the other integral operators introduced above can also be formally defined when $\omega = 0$.

1.4. **Lippmann-Schwinger equation.** Solution to the system (1.3) and (1.5) can be transformed to boundary integral equation by using the Lippmann-Schwinger equations, respectively, (see, e.g., [6])

$$\hat{u}(\mathbf{x}) = \omega^2 \mathcal{V}_{\mathbb{R}^3}^\omega \left[(1 - \frac{1}{c^2})\hat{u} \right](\mathbf{x}) + \frac{i\omega}{2\pi} \mathcal{V}_{\mathbb{R}^3}^\omega \left[\frac{f}{c^2} \right](\mathbf{x}) + \mathcal{V}_{\mathbb{R}^3}^\omega[\hat{F}](\mathbf{x}), \tag{1.18}$$

and

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \int_{\mathbb{R}^3} \mathbf{G}_\omega(\cdot - \mathbf{y}) \begin{pmatrix} \tilde{\gamma}(\mathbf{y})\mathbf{E}(\mathbf{y}) \\ \tilde{\mu}(\mathbf{y})\mathbf{H}(\mathbf{y}) \end{pmatrix} d\mathbf{y} + i/(\omega\varepsilon_0) \int_{\mathbb{R}^3} \mathbf{G}_\omega(\cdot - \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ 0 \end{pmatrix} d\mathbf{y}, \tag{1.19}$$

where $\tilde{\gamma}(\mathbf{y}) := \varepsilon(\mathbf{y}) + i\sigma(\mathbf{y})/\omega - \varepsilon_0$ and $\tilde{\mu}(\mathbf{y}) = \mu(\mathbf{y}) - \mu_0$. The integral equations (1.18) and (1.19) are fundamental tools in analysis of the behavior of the solutions ($\hat{u}$, or $(\mathbf{E}, \mathbf{H})$). With some additional information, such as the measurements outside the body Ω or information from the body like obstacle inside, the integral equation need be altered in order to comply with the additional information. We will show some details later in the related topics.

2. **TAT/PAT.** Thermoacoustic tomography (TAT) and photoacoustic tomography (PAT) are two coupled-physics modalities which attempt to combine the ultrasound and the high contrast capabilities of electromagnetic waves. The main mathematical model is (1.3) and (1.4), with $\hat{F} \equiv 0$. Suppose that Ω is the main object with $c(\mathbf{x})$ be the wave speed in Ω and f is the internal source. By using the Lippmann-Schwinger equation (1.18) and low frequency asymptotic expansion, the solution to (1.2) can be written by

$$
\begin{aligned}
\hat{u}(\mathbf{x}) =&\omega^2 \mathcal{V}_{\mathbb{R}^3}^{\omega}\Big[(1-\frac{1}{c^2})\hat{u}\Big](\mathbf{x}) + \frac{i\omega}{2\pi}\mathcal{V}_{\mathbb{R}^3}^{\omega}\Big[\frac{f}{c^2}\Big](\mathbf{x}) \\
=&\frac{i\omega}{2\pi}\mathcal{V}_{\Omega}^{0}\Big[\frac{f}{c^2}\Big](\mathbf{x}) + \mathcal{O}(\omega^2).
\end{aligned}
\tag{2.1}
$$

By using the low frequency asymptotic formula (2.1), one can first recover the term fc^{-2} by using some additional prior knowledge on it.

Theorem 2.1 (Theorem 2.2 in [19])**.** *Let (f_j, c_j), $j = 1, 2$, be two admissible TAT/PAT configurations such that*

$$
\hat{u}_1(\mathbf{x}, \omega)\bigg|_{(\mathbf{x},\omega)\in\Gamma\times\mathbb{R}_+} = \hat{u}_2(\mathbf{x}, \omega)\bigg|_{(\mathbf{x},\omega)\in\Gamma\times\mathbb{R}_+}.
\tag{2.2}
$$

Define $\varphi(\mathbf{x}) := f_1(\mathbf{x})c_1(\mathbf{x})^{-2} - f_2(\mathbf{x})c_2(\mathbf{x})^{-2}$. If one of the following two conditions is satisfied

   i) $\varphi(\mathbf{x})$ is a harmonic function in Ω;
   ii) $\varphi(\mathbf{x})$ is independent of one variable, say $\mathbf{x}_j$, $1 \le j \le 3$.

then one has

$$
f_1(\mathbf{x})c_1(\mathbf{x})^{-2} = f_2(\mathbf{x})c_2(\mathbf{x})^{-2}, \quad for \quad a.e. \quad \mathbf{x} \in \Omega.
\tag{2.3}
$$

Remark 2.1. The proof of (2.2) is based on harmonic analysis and Fourier analysis. A key observation together with the leading order in (2.1) and (2.2), one can derive the following integral identity (cf. Theorem 2.1 in [19]):

$$
\int_{\mathbb{R}^3}(f_1(\mathbf{x})c_1(\mathbf{x})^{-2} - f_2(\mathbf{x})c_2(\mathbf{x})^{-2})h(\mathbf{x})d\mathbf{x} = 0,
\tag{2.4}
$$

holds for any harmonic function h. We remark that only the leading order term in low frequency asymptotic expansions (2.1) is used to derive the key integral identity (2.4). There might be more information lies in the higher order terms, which although is not easy to explore.

The equality (2.3) implies that if one of the parameter, the source $f(\mathbf{x})$ or sound speed $c(\mathbf{x})$ is recovered, then the other parameter is automatically recovered.

Theorem 2.2 (Theorem 2.4 in [19])**.** *Let (f, c) be an admissible TAT/PAT configuration such that*

$$
c^{-2}(\mathbf{x}) = c_b^{-2}(\mathbf{x}) + \upsilon\chi(\Sigma)
\tag{2.5}
$$

where c_b is a positive background sound speed which is supposed to be known in advance, and υ is a positive constant denoting an anomalous inclusion supported in Ω. In (2.5) and also in what follows, χ denotes the characteristic function. Furthermore, we assume that both c_b and f are independent of the variable x_j, $1 \le j \le 3$ and

$$
\int_{\Sigma}\mathcal{V}_{\mathbb{R}^3}^{0}[c^{-2}f](\mathbf{x})h(\mathbf{x})d\mathbf{x} \ne 0,
\tag{2.6}
$$

for a harmonic function h. Then both f and v are uniquely determined by boundary measurement $\hat{u}(\mathbf{x}, \omega)$ for $(\mathbf{x}, \omega) \in \Gamma \times \mathbb{R}_+$.

Theorem 2.2 shows that if the parameter, the sound speed c, satisfies some priori condition (2.5), together with (2.6), then both the sound speed and the source function can be uniquely recovered.

3. **Brain imaging.** The electromagnetic activity of the human brain is governed by the quasi-static theory of Maxwell equations which, to some extent, decouples the electric from the magnetic behavior. The brain is modelled as a conductive medium with certain conductivity and a magnetic permeability which is equal to the permeability of the air. The brain is surrounded first by the cerebrospinal fluid, then by the skull, and finally by the scalp, all having different conductivities but the same magnetic permeability. In most cases though the brain-head system is assumed to be a single homogeneous medium. A primary neuronal current within the conductive brain tissue excites a secondary induction current and both currents generate an electric potential, which is measured on the surface of the head, and a magnetic flux, which is measured outside, but close to, the head. Recordings of the electric potential concern the theory of electroencephalography (EEG) and recordings of the magnetic flux density concern the theory of magnetoencephalography (MEG). The forward problem of EEG or MEG consists of the calculation of the electric potential or the magnetic induction, respectively, from a complete knowledge of the primary neuronal current. The inverse problem of EEG or MEG consists of the identification of the location and intensity of the neuronal current from the electric potential or the surface of the head, or of the magnetic induction outside the head, respectively.

Suppose B is the main object (the brain) and $\mathbf{J}$ is the internal source in B with $supp(\mathbf{J}) \Subset B$ and $\mathbf{J} \in H(\text{div}; B)$, where

$$H(\text{div}; \Omega) = \{\mathbf{U} \in (L^2(\Omega))^3; \ \nabla \cdot \mathbf{U} \in (L^2(\Omega))^3\}.$$

The governing equation for brain imaging, slightly different from (1.5), is given by

$$\begin{cases} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = 0 & \text{in} \quad \mathbb{R}^3, \\ \nabla \times \mathbf{H} + i\omega(\epsilon + i\frac{\sigma}{\omega})\mathbf{E} = \mathbf{J} & \text{in} \quad \mathbb{R}^3, \\ \lim_{\|\mathbf{x}\| \to \infty} \|\mathbf{x}\|(\mathbf{H} \times \hat{\mathbf{x}} - \mathbf{E}) = 0, & \hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|. \end{cases} \tag{3.1}$$

For simplicity we introduce the space $H_0(\text{div}; B)$ the functions in $H(\text{div}; B)$ and has compactly support in B. By using the Lippmann-Schwinger equation (1.19), and the layer potential techniques, one can obtain that

$$\begin{pmatrix} \omega\mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathbf{M}_B^{k_0} \begin{pmatrix} \tilde{\gamma}\omega\mathbf{E} \\ \tilde{\mu}\mathbf{H} \end{pmatrix} + \begin{pmatrix} i/\varepsilon_0(k_0^2 I_3 + D^2)\mathcal{V}_B^{k_0}[\mathbf{J}] \\ \nabla \times \mathcal{V}_B^{k_0}[\mathbf{J}] \end{pmatrix} \quad \text{in} \quad \mathbb{R}^3, \tag{3.2}$$

where the operator $\mathbf{M}_B^{k_0}$ is defined by

$$\mathbf{M}_B^{k_0} := \begin{pmatrix} (k_0^2 I_3 + D^2)\mathcal{V}_B^{k_0} & i\omega^2\mu_0\nabla \times \mathcal{V}_B^{k_0} \\ -i\varepsilon_0\nabla \times \mathcal{V}_B^{k_0} & (k_0^2 I_3 + D^2)\mathcal{V}_B^{k_0} \end{pmatrix}. \tag{3.3}$$

It is remarked that in (3.2), if σ is not identically zero, one clearly shall encounter the blow-up issue of the quantity $\tilde{\gamma}$ as $\omega \to +0$. However, in the asymptotic analysis, $\omega\tilde{\gamma}$ always appears as a whole term and hence the aforementioned blow-up issue can be avoided. With carefully asymptotic analysis with respect to ω, one can find the following leading order term:

Proposition 3.1. *There holds the following asymptotic expansions:*

$$\mathcal{A}_B^{k_0} \begin{pmatrix} \omega\mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} i/\varepsilon_0(k_0^2 I_3 + D^2)\mathcal{V}_B^{k_0}[\mathbf{J}] \\ \nabla \times \mathcal{V}_B^{k_0}[\mathbf{J}] \end{pmatrix} \quad in \quad \mathbb{R}^3, \tag{3.4}$$

where $\mathcal{A}_B^{k_0}$ is defined by

$$\mathcal{A}_B^{k_0} := I_6 - \mathbf{M}_B^{k_0} \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & \tilde{\mu} \end{pmatrix}. \tag{3.5}$$

For $\omega \in \mathbb{R}_+$ sufficiently small, we have the following asymptotic expansions of $\mathcal{A}_B^{k_0}$ with respect to ω,

$$\mathcal{A}_B^{k_0} = I_6 - \mathbf{M}_B^0 \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & \tilde{\mu} \end{pmatrix} + \omega^2 \mathcal{R}_B^{k_0}, \tag{3.6}$$

where $\mathbf{M}_B^0$ is defined by

$$\mathbf{M}_B^0 := \begin{pmatrix} D^2\mathcal{V}_B^0 & 0 \\ -i\varepsilon_0 \nabla \times \mathcal{V}_B^0 & D^2\mathcal{V}_B^0 \end{pmatrix}, \tag{3.7}$$

and $\mathcal{R}_B^{k_0}$ is a remainder term which is a bounded operator on $H_{loc}^2(\mathbb{R}^3)^3 \times H_{loc}^2(\mathbb{R}^3)^3$. I_6 is a six-by-six identity matrix.

The integral identity (3.4) shows the direct connection between the electromagnetic fields $(\mathbf{E}, \mathbf{H})$ and the source function $\mathbf{J}$. If the invertibility of the matrix operator $\mathcal{A}_B^{k_0}$ in (3.6) is solved, then one can build the direct connection between the source function $\mathbf{J}$ and boundary measurements, which is the key to prove the uniqueness on recovery of the source function $\mathbf{J}$. However, the invertibility of the operator $\mathcal{A}_B^{k_0}$ relies on the parameters $\tilde{\gamma}$ and $\tilde{\mu}$, which are unknown parameters and need to be recovered. To ensure the invertibility of $\mathcal{A}_B^{k_0}$, some priori information on the parameters $\tilde{\gamma}$ and $\tilde{\mu}$ should be assumed. In [11], the authors present some piecewise constants assumptions on the permittivity ε and conductivity σ.

Assumption 1. *Assume that (ε, σ) satisfies one of the following two conditions:*

(i) *$\sigma = c\varepsilon$ in B, where $c \geq 0$ is a constant;*
(ii) *ε and σ are piecewise constants in $\mathbb{R}^3$ in the following sense. Set $\Sigma_0 := B$ and assume that Σ_j, $j = 1, 2, \ldots, N$, are Lipschitz subdomains of Σ_0 such that $\Sigma_j \Subset \Sigma_{j-1}$ and $\Sigma_{j-1}\backslash\overline{\Sigma_j}$ is connected for $1 \leq j \leq N$. Set $\varepsilon^{(0)} = \varepsilon_0$, $\sigma^{(0)} = 0$ and let $\varepsilon^{(j)}$ and $\sigma^{(j)}$ be constants for $j = 1, 2, \ldots, N + 1$. The medium parameters are given as follows,*

$$\varepsilon = \varepsilon^{(0)}\chi(\mathbb{R}^3\backslash\overline{\Sigma_0}) + \sum_{j=1}^{N}\varepsilon^{(j)}\chi(\Sigma_{j-1}\backslash\overline{\Sigma_j}) + \varepsilon^{(N+1)}\chi(\Sigma_N),$$

$$\sigma = \sigma^{(0)}\chi(\mathbb{R}^3\backslash\overline{\Sigma_0}) + \sum_{j=1}^{N}\sigma^{(j)}\chi(\Sigma_{j-1}\backslash\overline{\Sigma_j}) + \sigma^{(N+1)}\chi(\Sigma_N). \tag{3.8}$$

It is emphasized that Assumption 1 shall be mainly needed in the invertibility of the $\mathcal{A}_B^{k_0}$. At certain places, the asymptotic expansion results could hold with more general parameters. Furthermore, it is pointed out that in Assumption 1, the form of μ is not mentioned. In fact, in the subsequent uniqueness study, μ shall be taken to be a constant. In essence, the analysis can be extended to cover the case that μ is also piecewise constant in a form as that in (3.8), but should be with much more complicated and technical arguments.

By using the piecewise constants assumption 1, one can prove the invertibility of $\mathcal{A}_B^{k_0}$ by using some ingenious idea in transforming the integral equations back to some classical partial differential equations and, instead by proving the invertibility, one can prove the existence and uniqueness of the derived partial differential equations.

The recovery of the source and surrounding medium regarding brain imaging is theoretically and technically quite challenging in three folds. First, since the governing equation is Maxwell equation, the integral representation of the solutions, together with the asymptotic analysis with the low frequency should be taken much care of. Second, the simultaneous recovery of both internal source and medium is challenging, even the recovery of the medium alone from boundary measurements is a quite difficult problem and usually some priori information on the parameter should be given to reduce the complexity of the problem. Third, the recovery of parameters contain three different parameters, the electric permittivity, the magnetic permeability and the electric conductivity, which are coupled together. One can see from the integral representation that the source function is usually much easier to recover than the parameters. Thus it is normal to first recover the source function and then use some additional information, one can recover the parameters.

Besides the main piecewise constants assumption 1, to recover the medium, one needs the following admissible assumption:

Definition 3.1. Suppose $\mu \neq \mu_0$ is a constant in B and σ, ε are piecewise constants in B which satisfy (3.8) in Assumption 1 with $N = 2$ and $\sigma \neq 0$. Let $(\mathbf{E}, \mathbf{H})$ be the corresponding solutions to (3.1), associated with $(\varepsilon, \mu, \sigma, \mathbf{J})$. Let $\mathbf{E}^{(1)}$ be the solution to

$$
\begin{cases}
\nabla \times \mathbf{E}^{(1)} = i\mu \mathbf{H}^{(0)} & \text{in} \quad B, \\
\nabla \cdot \sigma \mathbf{E}^{(1)} = 0 & \text{in} \quad B, \\
\nu \cdot \sigma \mathbf{E}^{(1)} = -i\epsilon_0 \nu \cdot \mathbf{E}|_+ & \text{on} \quad \partial B.
\end{cases}
\tag{3.9}
$$

We call $(\mathbf{J}, \mu, \sigma)$ an admissible two-layer structure, if there exists $l_1, l_2 \in L_0^2(\partial \Sigma_1)$, such that

$$
\begin{aligned}
\int_{\partial \Sigma_1} l_1 \nu \cdot \mathbf{E}^{(1)}|_+ = C_1 \int_{\partial B} g_{l_1} \nu \cdot \mathbf{E}^{(1)}|_-, \\
\int_{\partial \Sigma_1} l_2 \nu \cdot \mathbf{E}^{(1)}|_+ = C_2 \int_{\partial B} g_{l_2} \nu \cdot \mathbf{E}^{(1)}|_-,
\end{aligned}
\tag{3.10}
$$

where $C_1 \neq C_2$ are two constants and $g_{l_j} = u_j|_{\partial B}$, with u_j, $j = 1, 2$ solutions to

$$
\begin{cases}
\Delta u_j = 0 \quad \text{in} \quad B, \\
u_j = -\mathcal{S}_B^0 (\mathcal{S}_B^0)^{-1} \left[\mathcal{S}_{\Sigma_1}^0 (\mathcal{S}_{\Sigma_1}^0)^{-1} (\sigma_1^{(1)})^{-1} \sigma_1^{(2)} \left(\frac{I}{2} + \mathcal{K}_{\Sigma_1}^0 \right) [l_j] - \mathcal{D}_{\Sigma_1}^0 [l_j] \right] \\
\quad + (\sigma_1^{(1)})^{-1} \sigma_1^{(2)} \left(\frac{I}{2} + \mathcal{K}_{\Sigma_1}^0 \right) [l_j] + \left(\frac{I}{2} - \mathcal{K}_B^0 \right) [l_j] \quad \text{on} \quad \partial \Sigma_1.
\end{cases}
\tag{3.11}
$$

Then one can generate the following uniqueness result:

Theorem 3.1. *Let $(\varepsilon_1, \mu_1, \sigma_1, \mathbf{J}_1)$ and $(\varepsilon_2, \mu_2, \sigma_2, \mathbf{J}_2)$ be two sets of EM configurations. Suppose that $(\varepsilon_j, \mu_j, \sigma_j)$ and $\mathbf{J}_j \in H_0(\mathrm{div}; B)$, $j = 1, 2$. Suppose further that $\mu_1 = \mu_2 = \mu'$ and $\nabla \cdot \mathbf{J}_1 = \nabla \cdot \mathbf{J}_2 = 0$ in B, where μ' is a positive constant, and σ_j, ε_j are piecewise constants in B which satisfy (3.8) in Assumption 1 with*

$\sigma_1 = \sigma_2 \neq 0$. *Let* $(\mathbf{E}_1, \mathbf{H}_1)$ *and* $(\mathbf{E}_2, \mathbf{H}_2)$ *be the corresponding solutions to (3.1) associated with* $(\varepsilon_1, \mu_1, \sigma_1, \mathbf{J}_1)$ *and* $(\varepsilon_2, \mu_2, \sigma_2, \mathbf{J}_2)$, *respectively. Suppose that*

$$\mathbf{E}_1(\mathbf{x}, \omega)\bigg|_{(\mathbf{x}, \omega) \in \Gamma \times (0, \omega_0)} = \mathbf{E}_2(\mathbf{x}, \omega)\bigg|_{(\mathbf{x}, \omega) \in \Gamma \times (0, \omega_0)}, \tag{3.12}$$

where ω_0 *is any given positive constant. Then we have* $\mathbf{J}_1 = \mathbf{J}_2$, *provided there exists* $\xi \in \mathbb{S}^2$ *such that* $\Phi := \xi \cdot (\nabla \times (\mathbf{J}_1 - \mathbf{J}_2))$ *satisfies the one of the following:*

(i) $\Phi(\mathbf{x}) = h(\mathbf{x})$ *for* $\mathbf{x} \in B$, *and* h *is a harmonic function in* $\mathbb{R}^3$;
(ii) *There exists a unit vector* $\mathbf{d} \in \mathbb{S}^2$ *such that* $\mathbf{d} \cdot \nabla \Phi = 0$.

Furthermore, suppose $N = 2$ *in (3.8), let* $\mathbf{E}^{(1)}$ *be the solution to (3.9). If* $\nu \cdot \mathbf{E}|_+ \neq 0$ *as* $\omega \to 0$ *on* ∂B *and* $(\mathbf{J}_1, \mu', \sigma_1)$ *is an admissible two-layer structure, then* $\varepsilon_1 = \varepsilon_2$.

The proof for uniqueness of the two-layer structure is highly technical and in fact can be extended to multi-layer structure. One can also refer to [12] for similar idea in recovering the piecewise constants conductivity with one measurement.

4. **Geomagnetic monitoring.** Another recovery of source and medium problem by using only passive measurements can be found in the field of geomagnetic monitoring. Geomagnetic detections have been successfully applied in science and engineering including military submarine detection, minerals searching and geoexploration. In fact, the operations of the Swarm satellites, a set of three probes sent by the European Space Agency (ESA) in the year 2013, are capable of measuring variations in the Earth's magnetic field. These so-called secular variations can be detected by studying what happens beneath the Earth. In the mathematical formulation of geomagnetic monitoring problem, there exist several different approximation models. The most acceptable model is the magnetohydrodynamic (MHD) model (see, [2, 5, 13] and references therein), where the dynamo mechanism states that the convecting currents of fluid metal in the Earth's outer core, driven by heat flow from the outer core, organized into rolls by the Coriolis force, create circulating electric currents, which generate the magnetic field. In general, there are three basic partial differential equations including the Navier-Stokes equations, the heat equation and the Maxwell equations, and they are coupled together to form the MHD system. In a simple case, if one assumes that the influence of the magnetic field on velocity of the fluid inside the outer core can be neglected, then the governing equation to be considered is just the Maxwell equation (1.5), where the source function $\mathbf{F}$ has the following form:

$$\mathbf{F} = 1/\lambda \mathbf{v} \times \mathbf{H}, \tag{4.1}$$

where $\lambda := 1/(\sigma\mu)$ denotes for the magnetic diffusivity. The source function $\mathbf{F}$ can be treated as the source generated inside the core of the Earth, denoted by Σ_c, thus $supp(\mathbf{F}) \subset \Sigma_c$. Suppose that a collection of magnetized anomalies presented in the shell of the Earth. Let D_l, $l = 1, 2, \ldots, l_0$, denote the magnetized anomalies, where D_l, $1 \leq l \leq l_0$ is a simply-connected Lipschitz domain such that the corresponding material parameters are given by ε_l, μ_l and σ_l. It is assumed that ε_l, μ_l and σ_l are all positive constants with $\mu_l \neq \mu_0$, $1 \leq l \leq l_0$. With the presence of the magnetized anomalies $(D_l; \varepsilon_l, \mu_l, \sigma_l)$, $l = 1, 2, \ldots, l_0$, in the shell of the Earth, the EM medium configuration in the space $\mathbb{R}^3$ is then described by

$$\sigma(\mathbf{x}) = \sigma_c(\mathbf{x})\chi(\Sigma_c) + \sum_{l=1}^{l_0} \sigma_l \chi(D_l),$$

$$\mu(\mathbf{x}) = (\mu_c(\mathbf{x}) - \mu_0)\chi(\Sigma_c) + \sum_{l=1}^{l_0} (\mu_l - \mu_0)\chi(D_l) + \mu_0, \tag{4.2}$$

and

$$\varepsilon(\mathbf{x}) = (\varepsilon_c(\mathbf{x}) - \varepsilon_0)\chi(\Sigma_c) + (\varepsilon_s(\mathbf{x}) - \varepsilon_0)\chi(\Sigma_s \setminus \overline{\bigcup_{l=1}^{l_0} D_l}) + \sum_{l=1}^{l_0} (\varepsilon_l - \varepsilon_0)\chi(D_l) + \varepsilon_0. \tag{4.3}$$

Here, Σ signifies the Earth and $\Sigma_s := \Sigma\setminus\overline{\Sigma_c}$ denotes the shell part of the Earth. The inverse problem can then be recast as

$$\mathbf{H}^\infty(\hat{\mathbf{x}};\omega) - \mathbf{H}_0^\infty(\hat{\mathbf{x}};\omega)\big|_{(\hat{\mathbf{x}},\omega)\in\hat{\Gamma}\times\mathbb{R}_+} \longrightarrow \bigcup_{l=1}^{l_0}(D_l;\varepsilon_l,\mu_l,\sigma_l),$$

$$\hat{\Gamma} := \{\hat{\mathbf{x}} \in \mathbb{S}^2; \hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}, \ \mathbf{x} \in \Gamma\} \subset \mathbb{S}^2, \tag{4.4}$$

where $\mathbf{H}^\infty(\hat{\mathbf{x}};\omega)$ and $\mathbf{H}_0^\infty(\hat{\mathbf{x}};\omega)$ are, respectively, far field pattern of magnetic field before and after the presence of magnetic anomalies D_l, $l = 1, 2, \ldots, l_0$. In geomagnetic monitoring, the size of the anomalies are usually quite small compared with the size of the Earth, and it is important to detect the position of the anomalies. In other words, one can assume that

$$D_l = \delta\Omega + \mathbf{z}_l, \quad l = 1, 2, \ldots, l_0, \tag{4.5}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, and $\delta \in \mathbb{R}_+$ is sufficiently small. $\mathbf{z}_l$, $l = 1, 2, \ldots, l_0$ are positions of the anomalies. Thus the inverse problem (4.4) can be reduced to recovery of position and parameters of the anomalies. We refer that in a recent work in [9], the variation of the magnetic anomalies is considered, which means that the anomalies may change its size at different time. Here, for the sake of simplicity, we only review on the work on the recovery of anomalies which stay the same size after their presence. Let $\mathbf{H}_0^{(0)}$ and $\mathbf{H}^{(0)}$ be the the essential main parts of the magnetic fields before and after the presence of the anomalies, respectively. Then the main part on the difference of the magnetic fields can be found in the following asymptotic expansions

Theorem 4.1 (Theorem 3.2 in [7]). *Suppose D_l, $l = 1, 2, \ldots, l_0$ are defined in (4.5) with $\delta \in \mathbb{R}_+$ sufficiently small. Then for $\mathbf{x} \in \mathbb{R}^3 \setminus \Sigma$, there holds the following asymptotic expansion result*

$$\mathbf{H}^{(0)}(\mathbf{x}) = \mathbf{H}_0^{(0)}(\mathbf{x}) + \delta^3 \sum_{l=1}^{l_0} \nabla\big(\nabla\Phi_0(\mathbf{x} - \mathbf{z}_l)^T \mathbf{P}_l \mathbf{H}_0^{(0)}(\mathbf{z}_l)\big) + \mathcal{O}(\delta^4). \tag{4.6}$$

The matrix $\mathbf{P}_l$, $l = 1, 2, \ldots, l_0$ are denoted by

$$\mathbf{P}_l := \mu_0 \mathbf{M}_l - \varepsilon_0 \mathbf{D}_l - \mathbf{P}_0, \tag{4.7}$$

where $\mathbf{P}_0$ is defined by

$$\mathbf{P}_0 := \int_{\partial\Omega} \tilde{\mathbf{y}}(\lambda_\varepsilon I - (\mathcal{K}_\Omega^0)^*)^{-1}[\nu_l]ds_{\tilde{\mathbf{y}}}, \quad \lambda_\varepsilon := \frac{\varepsilon_s + \varepsilon_0}{2(\varepsilon_s - \varepsilon_0)}. \tag{4.8}$$

The polarization tensors $\mathbf{D}_l$ *and* $\mathbf{M}_l$ *are* 3×3 *matrices defined by*

$$\mathbf{D}_l = \frac{1}{\gamma_l - \varepsilon_s} \frac{\varepsilon_s}{\varepsilon_s - \varepsilon_0} \int_{\partial \Omega} \tilde{\mathbf{y}} (\lambda_{\gamma_l} I + (\mathcal{K}_\Omega^0)^*)^{-1} (\lambda_\varepsilon I - (\mathcal{K}_\Omega^0)^*)^{-1} [\nu_l] ds_{\tilde{\mathbf{y}}}, \qquad (4.9)$$

and

$$\mathbf{M}_l = \frac{1}{\mu_l - \mu_0} \frac{\varepsilon_s}{\varepsilon_s - \varepsilon_0} \int_{\partial \Omega} \tilde{\mathbf{y}} (\lambda_{\mu_l} I - (\mathcal{K}_\Omega^0)^*)^{-1} (\lambda_\varepsilon I - (\mathcal{K}_\Omega^0)^*)^{-1} [\nu_l] ds_{\tilde{\mathbf{y}}}, \qquad (4.10)$$

respectively, $l = 1, 2, \ldots, l_0$. *Here* $\gamma_l := \varepsilon_l + i\sigma_l/\omega$ *and the parameters* λ_{μ_l} *and* λ_{γ_l} *are defined by*

$$\lambda_{\mu_l} := \frac{\mu_l + \mu_0}{2(\mu_l - \mu_0)}, \quad \lambda_{\gamma_l} := \frac{\gamma_l + \varepsilon_s}{2(\gamma_l - \varepsilon_s)}, \quad l = 1, 2, \ldots, l_0. \qquad (4.11)$$

The asymptotic expansion (4.6) is the key formula in deriving the unique recovery results. One can see from the formula that the positions of the anomalies lie in the fundamental solution term $\Phi_0(\mathbf{x} - \mathbf{z}_l)$ and the magnetic field without anomalies $\mathbf{H}_0^{(0)}(\mathbf{z}_l)$. In real applications, the magnetic field $\mathbf{H}_0^{(0)}(\mathbf{x})$ is not easy to obtain. Thus one needs to get rid of the term $\mathbf{H}_0^{(0)}(\mathbf{z}_l)$ in (4.6) in order to recover the position of the anomalies. In fact, one can estimate the whole term $\mathbf{P}_l \mathbf{H}_0^{(0)}(\mathbf{z}_l)$ by using the following harmonic expansion of fundamental solution $\Phi_0(\mathbf{x} - \mathbf{z})$,

Lemma 4.1. *Let* $\mathbf{z} \in B_{R_0}$ *be fixed, where* B_{R_0} *stands for a ball of radius* $R_0 \in \mathbb{R}_+$. *Let* $\mathbf{x} \in \partial B_{R_1}$ *and suppose* $R_0 < R_1$. *There holds the following asymptotic expansion*

$$\nabla \Phi_0(\mathbf{x} - \mathbf{z}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n+1)Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}} - \nabla_S Y_n^m(\hat{\mathbf{x}})}{(2n+1)R_1^{n+2}} \overline{Y_n^m(\hat{\mathbf{z}})} \|\mathbf{z}\|^n, \qquad (4.12)$$

where $\hat{\mathbf{z}} = \mathbf{z}/\|\mathbf{z}\|$ *and* $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|$. Y_n^m *is the spherical harmonics of order* m *and degree* n.

By taking the boundary integral of the difference of the measurement on a surface of a big ball (B_{R_0}), the result is approximately $\delta^3 \mathbf{P}_l \mathbf{H}_0^{(0)}(\mathbf{z}_l)$. Thus one can first recover the position of the anomalies and then the parameters. With the asymptotic expansion (4.12) on hand, one can first derive the following interesting result:

Lemma 4.2. *Let* $(\mathbf{E}_j, \mathbf{H}_j)$, $j = 1, 2$, *be the solutions to* (1.5) *with* $\mathbf{F}$ *be defined in* (4.1) *for two objects* $D_l^{(1)}$ *and* $D_l^{(2)}$, *respectively. Let* $\mathbf{P}_l^{(1)}$ *and* $\mathbf{P}_l^{(2)}$ *the polarization tensors defined in* (4.7) *for* $D_l^{(1)}$ *and* $D_l^{(2)}$, *respectively,* $l = 1, 2, \ldots, l_0$. *If there holds*

$$\nu \cdot \mathbf{H}_1 = \nu \cdot \mathbf{H}_2 \neq 0 \quad on \ \Gamma, \qquad (4.13)$$

then one has

$$\sum_{m=-n}^{n} \mathbf{N}_{n+1}^m(\hat{\mathbf{x}})^T \mathbf{d}_1^{n,m} = \sum_{m=-n}^{n} \mathbf{N}_{n+1}^m(\hat{\mathbf{x}})^T \mathbf{d}_2^{n,m}, \quad \hat{\mathbf{x}} \in \mathbb{S}^2 \qquad (4.14)$$

for any $n \in \mathbb{N} \cup \{0\}$, *where*

$$\mathbf{d}_j^{n,m} = \sum_{l=1}^{l_0} \overline{Y_n^m(\hat{\mathbf{z}}_l^{(j)})} \|\mathbf{z}_l^{(j)}\|^n (\mathbf{P}_l^{(j)})^T \mathbf{H}_0^{(0)}(\mathbf{z}_l^{(j)}), \quad j = 1, 2, \qquad (4.15)$$

and

$$\mathbf{N}_{n+1}^m(\hat{\mathbf{x}}) = (n+1)Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}} - \nabla_S Y_n^m(\hat{\mathbf{x}}). \qquad (4.16)$$

The results in Lemma 4.2 derived an interesting identity (4.14), which puts all the position information, together with the parameter information in simply the vectors $\mathbf{d}_j^{n,m}$ in (4.15) for all $n \in \mathbb{N} \cup \{0\}$. Due to the orthogonality of the vectorial spherical harmonic functions in (4.16) for different degree n and order m, the recovery of position and parameters can be implemented level by level. For example, one can first recover the position of the anomalies by using only the equality (4.14) for $N = 0$. Further information on the parameters of the magnetic anomalies can be recovered using $N = 1$, $N = 2$, etc.

5. **Simultaneously recovery of obstacle and surrounding medium.** In this section, we shall review on some recent simultaneously recovery results with active measurements. In inverse scattering study area, the unknown/inaccessible object is usually referred to as a scatterer and it could be an impenetrable obstacle or a penetrable inhomogeneous medium. Many existing studies tend to consider the recovery of either an obstacle or an inhomogeneous medium. However, in lots of real applications, the object is not simply an impenetrable obstacle or a penetrable inhomogeneous medium, but rather both objects are contained in the environment. Thus one should consider the recovery in a complex scenario where the scatterer consists of an inhomogeneous medium and a possibly embedded impenetrable obstacle. The corresponding study becomes radically more challenging compared to the existing ones, where the recovery is mainly concerned with either one of the obstacle and the medium by assuming the other one is known. We shall give a brief review on the simultaneously recovery of medium and obstacle for Maxwell system [10], which is an extension of the work on Helmholtz system [16]. Since the recovery is based on active measurements, some incident waves are required. Introduce the incident electromagnetic plane waves as follows

$$\mathbf{E}^i(\mathbf{x}, \omega, \mathbf{d}, \mathbf{q}) := \frac{1}{\sqrt{\varepsilon_0}} \mathbf{p} e^{ik_0 \mathbf{x} \cdot \mathbf{d}}, \quad \mathbf{H}^i(\mathbf{x}, \omega, \mathbf{d}, \mathbf{q}) := \frac{1}{\sqrt{\mu_0}} (\mathbf{d} \times \mathbf{p}) e^{ik_0 \mathbf{x} \cdot \mathbf{d}},$$

where $\mathbf{d} \in \mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$ signifies the incident direction and $\mathbf{p} \in \mathbb{R}^3$ with $\mathbf{p} \perp \mathbf{d}$ denotes a polarisation vector. Let Ω be the object and $B \Subset \Omega$ denote a perfectly electric conducting (PEC) obstacle that is embedded inside an inhomogeneous medium in $\Omega \backslash \overline{B}$. For the sake of simplicity, the conductivity of the inhomogeneous medium is supposed to be zero and magnetic permeability is supposed to be μ_0, which is equivalent to that of background. It is also assumed that there is no source contained in the inhomogeneous medium. Because of the existence of incident wave and obstacle, the governing equation is slightly different from (1.5). Instead, the total electric $\mathbf{E}$ and magnetic field $\mathbf{H}$ satisfy the time-harmonic Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E} - i\omega\mu_0\mathbf{H} = 0, & \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ \nabla \times \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0, & \text{in } \mathbb{R}^3 \setminus \overline{B}, \end{cases} \tag{5.1}$$

along with the following PEC boundary condition,

$$\nu \times \mathbf{E} = 0, \quad \text{on } \partial B, \tag{5.2}$$

and the scattered fields $\mathbf{E}^s := \mathbf{E} - \mathbf{E}^i$, $\mathbf{H}^s := \mathbf{H} - \mathbf{H}^i$ satisfy the Silver-Müller radiation condition

$$\lim_{\|\mathbf{x}\| \to \infty} \|\mathbf{x}\| \left(\sqrt{\mu_0} \mathbf{H}^s \times \hat{\mathbf{x}} - \sqrt{\varepsilon_0} \mathbf{E}^s \right) = 0, \tag{5.3}$$

where $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|$ for $\mathbf{x} \in \mathbb{R}^3\backslash\{0\}$. The radiation condition (5.3) characterises the outgoing nature of the EM fields and it also implies the following asymptotic expansion of the scattered electric wave,

$$\mathbf{E}^s(\mathbf{x},\omega,\mathbf{d},\mathbf{q}) = \frac{e^{ik\|\mathbf{x}\|}}{\|\mathbf{x}\|}\left\{\mathbf{E}^\infty(\hat{\mathbf{x}},\omega,\mathbf{d},\mathbf{q}) + \mathcal{O}\left(\frac{1}{\|\mathbf{x}\|}\right)\right\}, \qquad \text{as } \|\mathbf{x}\| \to \infty, \quad (5.4)$$

which holds uniformly in all directions $\hat{\mathbf{x}}$. The term $\mathbf{E}^\infty(\hat{\mathbf{x}},\omega,\mathbf{d},\mathbf{q})$ is referred as electric far-field pattern. The Lippmann-Schwinger integral equation to (5.1)-(5.3) admit the following form

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^i \\ \mathbf{H}^i \end{pmatrix} + \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} + \int_{\mathbb{R}^3\backslash\overline{B}} \mathbf{G}_\omega(\cdot - \mathbf{y}) \begin{pmatrix} \tilde{\varepsilon}(\mathbf{y})\mathbf{E}(\mathbf{y}) \\ 0 \end{pmatrix} d\mathbf{y}, \quad (5.5)$$

where

$$\tilde{\varepsilon} := (\varepsilon - \varepsilon_0)/\varepsilon_0. \quad (5.6)$$

Since the existence of the obstacle inside the inhomogeneous medium, there is on additional term $(\mathbf{E}_0, \mathbf{H}_0)$ in (5.5), besides the incident wave, comparing with (1.19). Furthermore, the additional field $(\mathbf{E}_0, \mathbf{H}_0)$ in (5.5) takes the following form

$$\mathbf{E}_0 = \nabla \times \mathcal{A}_B^{k_0}[\Phi_E], \quad \mathbf{H}_0 = -i/(\omega\mu_0)\nabla \times \nabla \times \mathcal{A}_B^{k_0}[\Phi_E], \quad (5.7)$$

for some $\Phi_E \in \mathrm{TH}(\mathrm{div},\partial B)$. The density function Φ_E can be determined by the PEC condition (5.2). In fact, by using (5.2) and (5.7), one has

$$\nu \times (k_0^2 I_3 + D^2)\mathcal{V}_{\Omega\backslash\overline{B}}^{k_0}\tilde{\varepsilon}[\mathbf{E}] + (\frac{I}{2} + \mathcal{M}_B^{k_0})[\Phi_E] = -\nu \times \mathbf{E}^i \quad \text{on } \partial B. \quad (5.8)$$

Recall that $I/2 + \mathcal{M}_B^{k_0}$ is invertible on $\mathrm{TH}(\mathrm{div},\partial B)$ when k_0 is sufficiently small (see [1, 22]), one then can obtain from the third equation in (5.8) that

$$\Phi_E = -\left(\frac{I}{2} + \mathcal{M}_B^{k_0}\right)^{-1}\nu \times \mathbf{E}^i - \left(\frac{I}{2} + \mathcal{M}_B^{k_0}\right)^{-1}\nu \times (k_0^2 I_3 + D^2)\mathcal{V}_{\Omega\backslash\overline{B}}^{k_0}\tilde{\varepsilon}[\mathbf{E}]. \quad (5.9)$$

With the integral representation of the solution (5.5), one can consider the unique recovery using the following steps (similar to the idea in [10]). First, by considering the low wavenumber asymptotics in terms of ω, we can derive some integral identities, which can serve to decouple the scattering information of B from that of $(\Omega\backslash\overline{B},\varepsilon)$. Then, by using certain harmonic analysis techniques, one can invert the previously obtained integral identities to recover the conductor and the medium. To be more precise, the obstacle is always the first to recover, and after its determination, the recovery of surrounding medium is just an inverse medium problem. The final asymptotic expansions regarding the electric filed $\mathbf{E}$ and magnetic field $\mathbf{H}$ can be found as follows

Lemma 5.1. *Let* $(\mathbf{E},\mathbf{H})$ *be the solution to system (5.1) and (5.3) with* ε *be a constant on* ∂B. *Then there holds the following in* $\Omega \setminus \overline{B}$

$$\mathbf{E} = (I_3 - \mathfrak{K}_1(\varepsilon))^{-1}\mathfrak{C}_1[\mathbf{p}] + i\omega(I_3 - \mathfrak{K}_1(\varepsilon))^{-1}\mathfrak{C}_1[(\mathbf{x}\cdot\mathbf{d})\mathbf{p}] + \mathcal{O}(\omega^2), \quad (5.10)$$

and

$$\mathbf{H} = i\mathfrak{C}_2[(\mathbf{x}\cdot\mathbf{d})\mathbf{p}] + \mathcal{O}(\omega) \quad as \ \varepsilon \to 0, \quad (5.11)$$

where the operators appeared in (5.10) and (5.10) are defined by

$$\mathfrak{K}_1(\varepsilon) = D^2\mathcal{V}_{\Omega\backslash\overline{B}}^0\tilde{\varepsilon} - \nabla \times \mathcal{A}_B^0\left(\frac{I}{2} + \mathcal{M}_B^0\right)^{-1}\nu \times D^2\mathcal{V}_{\Omega\backslash\overline{B}}^0\tilde{\varepsilon}, \quad (5.12)$$

and

$$\mathfrak{C}_1[\mathbf{V}] = -\nabla \times \mathcal{A}_B^0 \left(\frac{I}{2} + \mathcal{M}_B^0\right)^{-1} [\nu \times \mathbf{V}] + \mathbf{V},$$

$$\mathfrak{C}_2[\mathbf{V}] = i\mu_0^{-1} D^2 \mathcal{A}_B^0 \left(\frac{I}{2} + \mathcal{M}_B^0\right)^{-1} [\nu \times \mathbf{V}] - i\mu_0^{-1}\nabla \times \mathbf{V},$$

$$(5.13)$$

for any vector-valued function $\mathbf{V} \in H(curl; \Omega \setminus \overline{B})$. $H(curl; \Omega \setminus \overline{B})$ *is defined as follows*

$$H(curl; \Omega \setminus \overline{B}) := \{\mathbf{U} \in (L^2(\Omega \setminus \overline{B}))^3;\ \nabla \times \mathbf{U} \in (L^2(\Omega \setminus \overline{B}))^3\}.$$

In Lemma 5.1, the permittivity ε is assumed to be constant on the boundary of B. This technical assumption is used to prove the invertibility of the operator $I_3 - \mathfrak{K}_1(\varepsilon)$, where $\mathfrak{K}_1(\varepsilon)$ is defined in (5.12). With Lemma 5.1 on hand, one can prove the uniqueness in recovery of the obstacle B,

Theorem 5.1. *Assume that the electric permittivities* ε_j *are constants on* ∂B_j, $j = 1, 2$. *Let* ω_0 *be a positive number. For any fixed incident direction* $\mathbf{d}$ *and polarization* $\mathbf{q}$, *if*

$$\mathbf{E}_1^\infty(\hat{\mathbf{x}}, \omega) = \mathbf{E}_2^\infty(\hat{\mathbf{x}}, \omega) \qquad (5.14)$$

for all observation directions $\hat{\mathbf{x}} \in \mathbb{S}^2$ *and all frequencies* $\omega \in (0, \omega_0)$, *then we have* $B_1 = B_2$.

Theorem 5.1 shows the uniqueness of the obstacle when the far-field pattern is given for any fixed incident direction $\mathbf{d}$ and polarization $\mathbf{q}$. The result equally holds when the far-field data are replaced by the near-field data, consisting of the tangential components of the electric and magnetic fields measured on any open surface away from the scatterer. Indeed, by the unique continuation principle for the Maxwell system [6], those two sets of data are equivalent to each other. When the obstacle is recovered, the rest of the problem is to recover the medium, which turns to an inverse medium problem again. One can use a similar assumption like in the brain imaging part, that is , assume the electric permittivity ε to be piecewise constant. As a simple case, one can prove the following result

Theorem 5.2. *[Theorem 4.2 in [10]] Suppose that* ε_j, $j = 1, 2$, *are constant in* $\Omega \setminus \overline{B}$. *For any fixed incident direction* $\mathbf{d}$ *and polarization* $\mathbf{q}$, *if*

$$\mathbf{E}_1^\infty(\hat{\mathbf{x}}, \omega) = \mathbf{E}_2^\infty(\hat{\mathbf{x}}, \omega) \qquad (5.15)$$

for all observation directions $\hat{\mathbf{x}} \in \mathbb{S}^2$ *and all frequencies* $\omega \in (0, \omega_0)$, *then* $\varepsilon_1 = \varepsilon_2$.

In Theorem 5.2, the electric permittivity inside the surrounding medium is supposed to be a constant, which is a simplest assumption. As we have mentioned, one can in fact assume that the electric permittivity is piecewise constants inside $\Omega \setminus \overline{B}$. However, the proof requires much more technical ingredients and also some admissible assumptions (mild) should be used (see Theorem 3.1).

6. **Conclusions.** In this paper, we survey some recent mathematical developments on simultaneous recovery of sources/obstacles and surrounding mediums by boundary measurements. Those problems have several features in common. First and fundamentally, they are all concerned with the simultaneous recovery problem with boundary measurements, and most are by passive measurements. Second and most importantly, the measurements are on an interval of the frequency (or an interval of the time). More precisely, one only needs the measurements on a low frequency

part, although it is in some sense equivalent to the measurements on all frequency, and low frequency asymptotic analysis is a main tool in deriving of the uniqueness. Third, although there are a lot of technical tools which should be explored inside the problems, the main methods are all based on layer potential technique, harmonic analysis, Fourier analysis, etc. Finally, there are always some prior assumptions on the source or medium, especially the medium. Thus, future works on the relaxation of the assumption, or using more measurements are worth studying. Finally, we would like to mention in passing some further developments of this type of inverse problems associated with random PDEs [14, 15, 17] and fractional PDEs [3, 4], respectively.

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