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# THE LONGTIME BEHAVIOR OF THE MODEL WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES IN ONLINE SOCIAL NETWORKS

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Abstract. In this paper we consider a free boundary problem with nonlocal diffusion describing information diffusion in online social networks. This model can be viewed as a nonlocal version of the free boundary problem studied by Ren et al. (Spreading-vanishing dichotomy in information diffusion in online social networks with intervention, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019) 1843–1865). We first show that this problem has a unique solution for all $t > 0$, and then we show that its longtime behaviour is determined by a spreading-vanishing dichotomy. We also obtain sharp criteria for spreading and vanishing, and show that the spreading always happen if the diffusion rate of any one of the information is small, which is very different from the local diffusion model.

1. **Introduction.** Popular social networks play an essential role in our daily lives. In recent years, many rumors are spreading on social networks such that our lives are seriously affected. In order to control rumor propagation in social networks, we should understand the process of information propagation. Hence, some mathematical models were proposed to characterize and predict the process of information propagation in online social networks, such as, [26, 27, 17]. In [17], Wang et al. proposed the following diffusive logistic model:

$$\begin{cases} u_t = du_{xx} + r(t)u(1-u/K), & t>1,\ l<x<L, \\ u(1,x)=u_0(x), & l\le x\le L, \\ u_x(t,l)=0,\ u_x(t,L)=0, & t>1, \end{cases}$$

where $r$, $K$ and $d$ represent the intrinsic growth rate, the carrying capacity, and the diffusion rate, respectively. $l$ and $L$ stand for the upper and lower bounds of the distances between the source $s$ and other social networks users.

In above system, $l$ and $L$ are fixed boundary and so information only spreads in this fixed area. But in reality, the spreading area of information is changing with time. This can be addressed by considering this over the varying domain. In 2013,

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Lei et al. [7] introduced the free boundary to study single information diffusion in online social networks,

$$\begin{cases} u_t = du_{xx} + r(t)u(1-u/K), & t>0,\ 0<x<h(t), \\ u_x(t,0)=0,\ u(t,h(t))=0, & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,x)=u_0(x), & 0\le x\le h_0. \end{cases} \tag{1}$$

They presented some sharp criteria for information spreading and vanishing. Furthermore, if the information spreading happens, they gave the asymptotic spreading speed which is determined by a corresponding elliptic equation.

The deduction of free boundary condition in (1) can be found in [2]. In 2010, this condition was introduced by Du and Lin [5] to describe the spreading of the invasive species, and a spreading-vanishing dichotomy was first established. After the work of [5] for a logistic type local diffusion model, free boundary approaches to local diffusion problems similar to problem (1) have been studied by many researchers recently. Among the many further extensions, we only mention the extension to certain Lotka-Volterra two-species systems [20, 21, 22, 23] and the references therein.

The works of [17] and [7] all discussed the spreading of the single information. However, in many practical situations, considering multiple information diffusion process in online social networks is more realistic. In 2013, Peng et al. [14] studied information diffusion initiated from multiple sources in online social networks by numerical simulation. But there are many challenging problems in modeling and analysing multiple information diffusion process. In particular, a simple case was considered by Ren et al. [15]. They assumed that there are three pieces of information A, B and C sent from different sources to compete for influence on online users, where the official information C is viewed as an intervention from the media or government to control the spread of the ordinary information A and B. For simplicity, they further assumed that A and B has no influence on C, A and B compete for influence on each other. Following the approach of [7], they proposed the following model

$$\begin{cases} u_t = d_1u_{xx} + u(a_1 - b_1u - c_1v - r_1w), & t>0,\ 0<x<h(t), \\ v_t = d_2v_{xx} + v(a_2 - b_2v - c_2u - r_2w), & t>0,\ 0<x<h(t), \\ w_t = d_3w_{xx} + w(a_3 - b_3w), & t>0,\ 0<x<h(t), \\ u_x(t,0)=v_x(t,0)=w_x(t,0)=0, & t\ge 0, \\ u(t,h(t))=v(t,h(t))=w(t,h(t))=0, & t\ge 0, \\ h'(t) = -\mu[\rho_1u_x(t,h(t)) + \rho_2v_x(t,h(t)) + w_x(t,h(t))], & t>0, \\ h(0)=h_0, & \\ u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ w(0,x)=w_0(x), & 0<x<h_0, \end{cases} \tag{2}$$

where $u(t,x), v(t,x), w(t,x)$ represent the density of influenced users of information A, B, C at time $t$ and location $x$ respectively, $h(t)$ is the spreading front of the news, $d_i$ $(i=1,2,3)$ is the diffusion rates, $a_i$ $(i=1,2,3)$ is the intrinsic growth rates, $1/b_i$ $(i=1,2,3)$ is the carrying capacities, $c_i$ $(i=1,2)$ and $r_i$ $(i=1,2)$ are the intervention rates, $\mu$ stands for the expanding capacity of information. In [15], they first gave the long time behavior of the information: all information spread; one ordinary information and official information spread, while the other ordinary information vanishes; two pieces of ordinary information vanish and official information spreads. And then they established the criteria for spreading and vanishing.

Furthermore, they provided some estimates of asymptotic spreading speed when spreading happens. Finally, by some numerical simulations, they illustrated the results and all cases of the asymptotic behavior of the solution.

Note that in (2), the dispersal of the information is assumed to follow the rules of random diffusion, which is not realistic in general. This kind of dispersal may be better described by a nonlocal diffusion operator of the form

$$d\int_{\mathbb{R}} J(x-y)u(t,y)dy - du(t,x),$$

which can capture short-range as well as long-range factors in the dispersal by choosing the kernel function $J$ properly [1, 10, 11, 12, 13, 16, 24, 25].

Recently, Cao et al. [3] proposed a nonlocal version of the logistic model of [5], and successfully extended many basic results of [5] to the nonlocal model. Motivated by the work [3], some related models with nonlocal diffusion and free boundaries have been considered in several recent works (see, for example, [6, 8, 9, 18, 19]). In this paper, following the approach of [3], we propose and examine a nonlocal version of (2), which has the form

$$\begin{cases} u_t = d_1\displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(t,y)dy - d_1u & \\ \qquad\qquad +u(a_1-b_1u-c_1v-r_1w), & t>0,\ g(t)<x<h(t), \\ v_t = d_2\displaystyle\int_{g(t)}^{h(t)} J_2(x-y)v(t,y)dy - d_2v & \\ \qquad\qquad +v(a_2-b_2v-c_2u-r_2w), & t>0,\ g(t)<x<h(t), \\ w_t = d_3\displaystyle\int_{g(t)}^{h(t)} J_3(x-y)w(t,y)dy - d_3w & \\ \qquad\qquad +w(a_3-b_3w), & t>0,\ g(t)<x<h(t), \\ u(t,x)=v(t,x)=w(t,x)=0, & t\ge 0,\ x=g(t) \text{ or } h(t), \\ g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} [\rho_1 J_1(x-y)u(t,x) & \\ \qquad +\rho_2J_2(x-y)v(t,x)+J_3(x-y)w(t,x)]dydx, & t>0, \\ h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} [\rho_1 J_1(x-y)u(t,x) & \\ \qquad +\rho_2J_2(x-y)v(t,x)+J_3(x-y)w(t,x)]dydx, & t>0, \\ -g(0)=h(0)=h_0, & \\ u(0,x)=u_0(x), v(0,x)=v_0(x), w(0,x)=w_0(x), & -h_0<x<h_0, \end{cases} \tag{3}$$

where $d_i$ $(i=1,2,3)$, $a_i$ $(i=1,2,3)$, $b_i$ $(i=1,2,3)$, $c_i$ $(i=1,2)$, $r_i$ $(i=1,2)$, $\rho_i$ $(i=1,2)$, $\mu$ and $h_0$ are given positive constants. The initial functions $u_0(x)$, $v_0(x)$ and $w_0(x)$ belong to

$$\mathscr{X}(h_0) := \{u_0 \in C([-h_0,h_0]) \ : \ u_0(\pm h_0)=0,\ u_0>0 \text{ in } (-h_0,h_0)\},$$

where $[-h_0,h_0]$ represents the initial range of the information. Assumed that $u,v,w$ are identically 0 for $x\in\mathbb{R}\backslash[g(t),h(t)]$, and the kernel function $J_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$ is continuous and nonnegative, and have the properties

**(J)** $J(0)>0$, $\displaystyle\int_{\mathbb{R}} J(x)dx=1$, $J$ is symmetric, $\sup_{\mathbb{R}} J<\infty$.

The main results of this paper are the following theorems:

**Theorem 1.1** (Global existence and uniqueness). *Suppose that $J_i (i = 1, 2, 3)$ satisfies* **(J)**. *Then for any given $h_0 > 0$ and $u_0(x), v_0(x), w_0(x)$ belonging to $\mathscr{X}(h_0)$, problem (3) admits a unique solution $(u(t, x), v(t, x), w(t, x), g(t), h(t))$ defined for all $t > 0$.*

**Theorem 1.2** (Spreading-vanishing dichotomy). *Let the conditions of Theorem 1.1 hold and $(u, v, w, g, h)$ be the unique solution of (3). Assume further that $J_1(x) > 0$, $J_2(x) > 0$ in $\mathbb{R}$, then one of the following alternatives must happen:*

(i) Spreading: $\lim_{t\to\infty} [h(t) - g(t)] = \infty$.

(ii) Vanishing: $\lim_{t\to\infty} (g(t), h(t)) = (g_\infty, h_\infty)$ *is a finite interval,*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t, x) = 0, \ \lim_{t\to\infty} \max_{g(t)\le x\le h(t)} v(t, x) = 0$$

*and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} w(t, x) = 0.$$

**Theorem 1.3** (Spreading-vanishing criteria). *Assume that $J_i$ $(i = 1, 2, 3)$ satisfies* **(J)**, *and $J_1(x) > 0$, $J_2(x) > 0$ in $\mathbb{R}$. Then the dichotomy in Theorem 1.2 can be determined as follows:*

(i) *If $a_1 \ge d_1$ or $a_2 \ge d_2$ or $a_3 \ge d_3$, then necessarily $h_\infty - g_\infty = \infty$.*

(ii) *If $a_i < d_i$ for $i = 1, 2, 3$, then*
  (a) *If $h_\infty - g_\infty < \infty$, then $h_\infty - g_\infty \le l_*$.*
  (b) *If $h_0 \ge l_*/2$, then $h_\infty - g_\infty = \infty$.*
  (c) *If $h_0 < l_*/2$, then there exist two positive numbers $\mu^* \ge \mu_* > 0$ such that $h_\infty - g_\infty < \infty$ when $0 < \mu \le \mu_*$ and $\mu = \mu^*$, and $h_\infty - g_\infty = \infty$ when $\mu > \mu^*$,*

  *where $l_*$ is given by (13).*

**Theorem 1.4** (Asymptotic limit). *Let $(u, v, w, g, h)$ be the unique solution of (3) and suppose $\lim_{t\to\infty} [h(t) - g(t)] = \infty$. The following conclusions hold:*

(i) *If $a_1 > c_1 \frac{a_2}{b_2} + r_1 \frac{a_3}{b_3}$ and $a_2 > c_2 \frac{a_1}{b_1} + r_2 \frac{a_3}{b_3}$, then*

$$\lim_{t\to\infty} (u(t, x), v(t, x), w(t, x))$$
$$= \left( \frac{b_2(a_1b_3 - r_1a_3) - c_1(a_2b_3 - r_2a_3)}{b_3(b_1b_2 - c_1c_2)}, \frac{-c_2(a_1b_3 - r_1a_3) + b_1(a_2b_3 - r_2a_3)}{b_3(b_1b_2 - c_1c_2)}, \frac{a_3}{b_3} \right)$$

*locally uniformly for $x \in \mathbb{R}$.*

(ii) *If $a_1 + \frac{c_1}{b_2} \left( c_2 \frac{a_1}{b_1} + r_2 \frac{a_3}{b_3} \right) \le c_1 \frac{a_2}{b_2} + r_1 \frac{a_3}{b_3}$ and $a_2 > c_2 \frac{a_1}{b_1} + r_2 \frac{a_3}{b_3}$, then*

$$\lim_{t\to\infty} (u(t, x), v(t, x), w(t, x)) = \left( 0, \frac{a_2b_3 - r_2a_3}{b_2b_3}, \frac{a_3}{b_3} \right) \quad \textit{locally uniformly for } x \in \mathbb{R}.$$

(iii) *If $a_1 > c_1 \frac{a_2}{b_2} + r_1 \frac{a_3}{b_3}$ and $a_2 + \frac{c_2}{b_1} \left( c_1 \frac{a_2}{b_2} + r_1 \frac{a_3}{b_3} \right) \le c_2 \frac{a_1}{b_1} + r_2 \frac{a_3}{b_3}$, then*

$$\lim_{t\to\infty} (u(t, x), v(t, x), w(t, x)) = \left( \frac{a_1b_3 - r_1a_3}{b_1b_3}, 0, \frac{a_3}{b_3} \right) \quad \textit{locally uniformly for } x \in \mathbb{R}.$$

(iv) *If $a_1 \le r_1 \frac{a_3}{b_3}$ and $a_2 \le r_2 \frac{a_3}{b_3}$, then*

$$\lim_{t\to\infty} (u(t, x), v(t, x), w(t, x)) = \left( 0, 0, \frac{a_3}{b_3} \right) \quad \textit{locally uniformly for } x \in \mathbb{R}.$$

**Remark 1.** Note that for the corresponding local diffusion model in [15], no matter how small the diffusion coefficient $d_i$ is, vanishing can always happen if $h_0$ and $\mu$ are both sufficiently small. However, for (3), Theorem 1.3 indicates that when $d_1 \le a_1$ or $d_2 \le a_2$ or $d_3 \le a_3$, spreading always happens no mater how small $h_0$ and $\mu$ are. This is different from the local diffusion model in [15].

The rest of this paper is organised as follows. In Section 2 we prove Theorem 1.1, namely, problem (3) has a unique solution defined for all $t > 0$. The long-time dynamical behaviour of (3) is investigated in Section 3, where Theorems 1.2, 1.3 and 1.4 are proved. Finally, we conclude this paper with a brief discussion in Section 4.

2. **Global existence and uniqueness.** For convenience, we first introduce some notations. For given $T > 0$, define

$$\begin{aligned} H_T &:= \left\{h \in C^1([0,T]) \ : \ h(0) = h_0, \ h(t) \text{ is strictly increasing}\right\}, \\ G_T &:= \left\{g \in C^1([0,T]) \ : -g \in H_T\right\}, \\ D_T &= D_T^{g,h} := \{(t,x) \in \mathbb{R}^2 : 0 < t < T, \ g(t) < x < h(t)\}. \end{aligned}$$

*The proof of Theorem 1.1.* The existence and uniqueness of solution to the problem (3) can be done in a similar fashion as in [3, 6]. We only list the main steps in the proof.

Noting that $J_3$ satisfies **(J)**, $f_3(w) := w(a_3 - b_3 w)$ satisfies **(f1)** and **(f2)** in [3], and $w_0(x)$ belongs to $\mathscr{X}(h_0)$ for any $h_0 > 0$. For any given $T > 0$ and $(g,h) \in G_T \times H_T$, it follows from [3, Lemma 2.3] that the following problem

$$\begin{cases} w_t = d_3 \displaystyle\int_{g(t)}^{h(t)} J_3(x-y) w(t,y)dy - d_3 w + w(a_3 - b_3 w), & 0 < t < T, g(t) < x < h(t), \\ w(t,g(t)) = w(t,h(t)) = 0, & 0 < t < T, \\ w(0,x) = w_0(x), & -h_0 \le x \le h_0 \end{cases}$$

admits a unique solution $w(t,x)$, and

$$0 < w(t,x) \le \max\{\|w_0\|_\infty, a_3/b_3\} =: A_3 \text{ in } D_T.$$

For such $w(t,x)$, it is easy to check that $f_1(t,x,u,v) := u(a_1 - b_1 u - c_1 v + r_1 w(t,x))$ and $f_2(t,x,u,v) := v(a_2 - b_2 v - c_2 u + r_2 w(t,x))$ satisfy **(f)**, **(f1)** and **(f2)** in [6]. For $(g,h)$ given above, it follows from [6, Lemma 2.3] that the following problem

$$\begin{cases} u_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y) u(t,y)dy - d_1 u & \\ \qquad\qquad + u(a_1 - b_1 u - c_1 v - r_1 w(t,x)), & 0 < t < T, \ g(t) < x < h(t), \\ v_t = d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y) v(t,y)dy - d_2 v & \\ \qquad\qquad + v(a_2 - b_2 v - c_2 u - r_2 w(t,x)), & 0 < t < T, \ g(t) < x < h(t), \\ u(t,x) = v(t,x) = 0, & 0 < t < T, \ x = g(t) \text{ or } h(t), \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), & -h_0 \le x \le h_0 \end{cases}$$

has a unique solution $(u,v)$ and

$$0 < u \le \max\left\{\|u_0\|_\infty, a_1/b_1\right\} =: A_1 \text{ in } D_T,$$

$$0 < v \le \max\left\{\|v_0\|_\infty, a_2/b_2\right\} =: A_2 \text{ in } D_T.$$

For $(u,v,w,g,h)$ above, we define $(\widetilde{g},\widetilde{h})$ for $t\in[0,T]$ by

$$\begin{cases}\widetilde{g}(t):=-h_0-\mu\displaystyle\int_0^t\int_{g(\tau)}^{h(\tau)}\int_{-\infty}^{g(\tau)}[\rho_1J_1(x-y)u(\tau,x)+\rho_2J_2(x-y)v(\tau,x)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+J_3(x-y)w(\tau,x)]dydxd\tau,\\ \widetilde{h}(t):=h_0+\mu\displaystyle\int_0^t\int_{g(\tau)}^{h(\tau)}\int_{h(\tau)}^{+\infty}[\rho_1J_1(x-y)u(\tau,x)+\rho_2J_2(x-y)v(\tau,x)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+J_3(x-y)w(\tau,x)]dydxd\tau.\end{cases}$$

Since $J_i$ $(i=1,2,3)$ satisfies **(J)**, there exist constants $\epsilon_0\in(0,h_0/4)$ and $\delta_0$ such that

$$J_i(x)\geq\delta_0 \text{ if } |x|\leq\epsilon_0,\ i=1,2,3.$$

Let

$$L:=(b_1+c_2)A_1+(c_1+b_2)A_2+(r_1+r_2+b_3)A_3,$$

then

$$\begin{aligned}&f_1(u,v,w)=u(a_1-b_1u-c_1v-r_1w)\geq-(b_1A_1+c_1A_2+r_1A_3)u\geq-Lu,\\&f_2(u,v,w)=v(a_2-b_2v-c_2u-r_2w)\geq-(b_2A_2+c_2A_1+r_2A_3)v\geq-Lv,\\&f_3(w)=w(a_3-b_3w)\geq-b_3A_3w\geq-Lw.\end{aligned}$$

Using this we can follow the corresponding arguments of [6] to show that, for some sufficiently small $T_0=T_0(\mu,A_1,A_2,A_3,h_0,\epsilon_0,\rho_1,\rho_2,J)>0$ and any $T\in(0,T_0]$,

$$\sup_{0\leq t_1<t_2\leq T}\frac{\widetilde{g}(t_2)-\widetilde{g}(t_1)}{t_2-t_1}\leq-\widetilde{\sigma}_0,\quad \inf_{0\leq t_1<t_2\leq T}\frac{\widetilde{h}(t_2)-\widetilde{h}(t_1)}{t_2-t_1}\geq\sigma_0,$$

$$\widetilde{h}(t)-\widetilde{g}(t)\leq 2h_0+\frac{\epsilon_0}{4}\text{ for } t\in[0,T],$$

where

$$\widetilde{\sigma}_0=\frac{1}{4}\epsilon_0\delta_0\mu e^{-(d_1+d_2+d_3+L)T_0}\int_{-h_0}^{-h_0+\frac{\epsilon_0}{4}}[\rho_1u_0(x)+\rho_2v_0(x)+w_0(x)]dx,$$

$$\sigma_0=\frac{1}{4}\epsilon_0\delta_0\mu e^{-(d_1+d_2+d_3+L)T_0}\int_{h_0-\frac{\epsilon_0}{4}}^{h_0}[\rho_1u_0(x)+\rho_2v_0(x)+w_0(x)]dx.$$

Let

$$\Sigma_T:=\Big\{(g,h)\in G_T\times H_T\ :\ \sup_{0\leq t_1<t_2\leq T}\frac{g(t_2)-g(t_1)}{t_2-t_1}\leq-\widetilde{\sigma}_0,$$
$$\inf_{0\leq t_1<t_2\leq T}\frac{h(t_2)-h(t_1)}{t_2-t_1}\geq\sigma_0,\ h(t)-g(t)\leq 2h_0+\frac{\epsilon_0}{4}\text{ for } t\in[0,T]\Big\},$$

and define the mapping

$$\mathcal{F}(g,h)=(\widetilde{g},\widetilde{h}).$$

Then the above analysis indicates that

$$\mathcal{F}(\Sigma_T)\subset\Sigma_T\text{ for } T\in(0,T_0].$$

Next, we will show that $\mathcal{F}$ is a contraction mapping on $\Sigma_T$ for sufficiently small $T\in(0,T_0]$. For any given $(g_i,h_i)\in\Sigma_T$ $(i=1,2)$, denote

$$(\widetilde{g}_i,\widetilde{h}_i)=\mathcal{F}(g_i,h_i).$$

Let

$$U(t,x) := u_1(t,x) - u_2(t,x),\ V(t,x) := v_1(t,x) - v_2(t,x),$$
$$W(t,x) := w_1(t,x) - w_2(t,x).$$

Then we can follow the approach of Step 2 in the proof of [3, Theorem 2.1] to show

$$\begin{aligned} &|\widetilde{g}_1(t) - \widetilde{g}_2(t)| + |\widetilde{h}_1(t) - \widetilde{h}_2(t)| \\ &\le 6h_0\mu\rho_1 T\|U\|_{C([0,T]\times\mathbb{R})} + 6h_0\mu\rho_2 T\|V\|_{C([0,T]\times\mathbb{R})} + 6h_0\mu T\|W\|_{C([0,T]\times\mathbb{R})} \\ &\quad + 3T\mu(\rho_1 A_1 + \rho_2 A_2 + A_3)[\|g_1 - g_2\|_{C([0,T])} + \|h_1 - h_2\|_{C([0,T])}]. \end{aligned}$$

We can apply the same argument as the step 2 in the proof of [6, Theorem 2.1] to obtain that there exist $C_1$ and $T_1 \in (0, T_0)$ such that, for $T \le T_1$,

$$\max\{\|U\|_{C([0,T]\times\mathbb{R})},\ \|V\|_{C([0,T]\times\mathbb{R})}\} \le C_1[\|g_1 - g_2\|_{C([0,T])} + \|h_1 - h_2\|_{C([0,T])}].$$

Meanwhile, by using the same argument as the step 2 in the proof of [3, Theorem 2.1] to obtain that there exist $C_2$ and $T_2 \in (0, T_1)$ such that, for $T \le T_2$,

$$\|W\|_{C([0,T]\times\mathbb{R})} \le C_2[\|g_1 - g_2\|_{C([0,T])} + \|h_1 - h_2\|_{C([0,T])}].$$

Then we have, for $T \le T_2$,

$$|\widetilde{g}_1(t) - \widetilde{g}_2(t)| + |\widetilde{h}_1(t) - \widetilde{h}_2(t)| \le C_3 T[\|g_1 - g_2\|_{C([0,T])} + \|h_1 - h_2\|_{C([0,T])}],$$

where

$$C_3 = 6h_0\mu\rho_1 C_1 + 6h_0\mu\rho_2 C_1 + 6h_0\mu C_2 + 3\mu(\rho_1 A_1 + \rho_2 A_2 + A_3).$$

This shows that if we choose $\widetilde{T}$ such that

$$0 < \widetilde{T} \le \min\left\{T_2, \frac{1}{2C_3}\right\},$$

then, for $T \in (0, \widetilde{T}]$,

$$|\widetilde{g}_1(t) - \widetilde{g}_2(t)| + |\widetilde{h}_1(t) - \widetilde{h}_2(t)| \le \frac{1}{2}[\|g_1 - g_2\|_{C([0,T])} + \|h_1 - h_2\|_{C([0,T])}]$$

and so $\mathcal{F}$ is a contraction mapping on $\Sigma_T$. Hence $\mathcal{F}$ has a unique fixed point $(g, h)$ in $\Sigma_T$, which gives a nonnegative solution $(u, v, w, g, h)$ of (3) for $t \in (0, T]$. Similar to Steps 3 and 4 in the proof of [6, Theorem 2.1], we can show that this is the unique solution of (3) and it can be extended uniquely to all $t > 0$. □

3. **Spreading and vanishing.** Since $u, v$ and $w$ are positive in $D_T$, we have $h'(t) > 0$ and $g'(t) < 0$ for $t > 0$. Thus we can define

$$\lim_{t\to\infty} g(t) = g_\infty \in [-\infty, -h_0), \quad \lim_{t\to\infty} h(t) = h_\infty \in (h_0, \infty].$$

Clearly we have either

$$\text{(i) } h_\infty - g_\infty < \infty, \text{ or (ii) } h_\infty - g_\infty = \infty.$$

We will call (i) the vanishing case, and call (ii) the spreading case. The main purpose of this section is to determine when (i) or (ii) can occur, and to determine the long-time profile of $(u, v, w)$ if (i) or (ii) happens.

3.1. **Criteria for vanishing and spreading.** Before analysing the vanishing phenomenon, we first give some lemmas.

**Lemma 3.1.** *Let the condition* **(J)** *hold for the kernel functions* $J_i$ $(i = 1, 2, 3)$, *and* $\beta_1, \beta_2, \beta_3 > 0$ *be constants. Suppose that* $g$, $h \in C^1([0, \infty))$, $g(0) < h(0)$, $g'(t) \le 0$, $h'(t) \ge 0$ *and* $w_i$, $w_{it} \in C(D_\infty) \cap L^\infty(D_\infty)$ *for* $i = 1, 2, 3$, *where* $D_\infty = \{t > 0,\ g(t) < x < h(t)\}$. *If* $(w_1, w_2, w_3, g, h)$ *satisfies*

$$\begin{cases} h'(t) = \sum\limits_{i=1}^{3} \beta_i \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x-y) w_i(t,x) dy dx, & t \ge 0, \\ g'(t) = -\sum\limits_{i=1}^{3} \beta_i \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_i(x-y) w_i(t,x) dy dx, & t \ge 0, \end{cases} \tag{4}$$

*and*

$$\lim_{t\to\infty} h(t) - \lim_{t\to\infty} g(t) < \infty, \tag{5}$$

*then*

$$\lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0.$$

This lemma can be proven by using the same arguments in [6, Lemma 3.1]. Next we recall another lemma which will be used later.

**Lemma 3.2.** ([6, Lemma 3.2]) *Let* $J$ *satisfy the condition* **(J)** *and* $J(x) > 0$ *in* $\mathbb{R}$. *Suppose that* $g, h \in C^1([0, \infty))$, $g(0) < h(0)$, $g'(t) \le 0$, $h'(t) \ge 0$, *and* (5) *holds. If* $(w, g, h)$ *satisfies, for some positive constants* $\beta$ *and* $M$,

$$0 \le w \le M \ \text{in}\ D_\infty, \quad w(t, g(t)) = w(t, h(t)) = 0, \quad \forall\ t \ge 0,$$

$$h'(t) \ge \beta \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y) w(t,x) dy dx, \quad \forall\ t > 0,$$

*and* $\lim\limits_{t\to\infty} h'(t) = 0$, *then*

$$\lim_{t\to\infty} \int_{g(t)}^{h(t)} w(t,x) dx = 0, \qquad \int_0^\infty \int_{g(t)}^{h(t)} w(t,x) dx dt < \infty.$$

We define the operator $\mathcal{L}_\Omega^{d_i} + \beta : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$(\mathcal{L}_\Omega^{d_i} + \beta)[\phi](x) := d_i \int_\Omega J_i(x-y)\phi(y) dy - d_i \phi(x) + \beta(x)\phi(x),$$

where $\Omega$ is an open bounded interval in $\mathbb{R}$, and $\beta \in C(\overline{\Omega})$. The generalized principal eigenvalue of $\mathcal{L}_\Omega^{d_i} + \beta$ is given by

$$\lambda_p(\mathcal{L}_\Omega^{d_i} + \beta) := \inf \left\{ \lambda \in \mathbb{R} : (\mathcal{L}_\Omega^{d_i} + \beta)[\phi] \le \lambda\phi \text{ in } \Omega \text{ for some } \phi \in C(\overline{\Omega}),\ \phi > 0 \right\}.$$

Then we will use the techniques in [6, Theorem 3.3] to give the vanishing result.

**Lemma 3.3.** *Assume that* $J_i$ $(i = 1, 2, 3)$ *satisfies* **(J)**, $J_i(x) > 0$ $(i = 1, 2)$ *in* $\mathbb{R}$. *Let* $(u, v, w, g, h)$ *be the unique solution of* (3). *If* $h_\infty - g_\infty < \infty$, *then*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = \lim_{t\to\infty} \max_{g(t)\le x\le h(t)} v(t,x) = \lim_{t\to\infty} \max_{g(t)\le x\le h(t)} w(t,x) = 0, \tag{6}$$

*moreover,*

$$\lambda_p\left(\mathcal{L}_{(g_\infty, h_\infty)}^{d_i} + a_i\right) \le 0, \ i = 1, 2, 3. \tag{7}$$

*Proof.* By the similar arguments in [3, Theorem 3.7], we can have

$$\lim_{t\to\infty}\max_{g(t)\le x\le h(t)} w(t,x)=0 \text{ and } \lambda_p\left(\mathcal{L}^{d_3}_{(g_\infty,h_\infty)}+a_3\right)\le 0.$$

In the following, we only prove

$$\lim_{t\to\infty}\max_{g(t)\le x\le h(t)} u(t,x)=0 \text{ and } \lambda_p\left(\mathcal{L}^{d_1}_{(g_\infty,h_\infty)}+a_1\right)\le 0. \tag{8}$$

The conclusion for $v$ can be obtained similarly, so we omit here.

By the same arguments in [6], we can have

$$\lim_{t\to\infty} u(t,x)=\lim_{t\to\infty} v(t,x)=\lim_{t\to\infty} w(t,x)=0 \text{ for almost every } x\in[g(0),h(0)]. \tag{9}$$

Define

$$M(t):=\max_{x\in[g(t),h(t)]} u(t,x)$$

and

$$X(t):=\{x\in(g(t),h(t)): u(t,x)=M(t)\}.$$

Then $X_i(t)$ is a compact set for each $t>0$. Therefore, there exist $\underline{\xi}_i(t)$, $\overline{\xi}_i(t)\in X_i(t)$ such that

$$u_t(t,\overline{\xi}(t))=\max_{x\in X(t)} u_t(t,x),\ \ u_t(t,\underline{\xi}(t))=\min_{x\in X(t)} u_t(t,x).$$

By the arguments in [6], the following claim holds

$$M'(t+0):=\lim_{s>t,s\to t}\frac{M(s)-M(t)}{s-t}=u_t(t,\overline{\xi}(t)),$$
$$M'(t-0):=\lim_{s<t,s\to t}\frac{M(s)-M(t)}{s-t}=u_t(t,\underline{\xi}(t)).$$

If $M(t)$ has a local maximum at $t=t_0$, then $M'(t_0)$ exists and $M'(t_0)=0$. Moreover, if $M(t)$ is monotone nondecreasing for all large $t$ and $\lim\limits_{t\to\infty} M(t)=\sigma>0$, then $M'(t-0)\to 0$ as $t\to\infty$; if $M(t)$ is monotone nonincreasing for all large $t$ and $\lim\limits_{t\to\infty} M(t)=\sigma>0$, then $M'(t+0)\to 0$ as $t\to\infty$.

Now we are ready to show that $\lim\limits_{t\to\infty} M(t)=0$. This can be done by the similar argument in Theorem 3.3 of [6]. Arguing indirectly we assume that this claim does not hold. Then

$$\sigma^*:=\limsup_{t\to\infty} M(t)\in(0,\infty). \tag{10}$$

By the above stated properties of $M(t)$, there exists a sequence $t_n>0$ increasing to $\infty$ as $n\to\infty$, and $\xi_n\in\{\overline{\xi}(t_n),\underline{\xi}(t_n)\}$ such that

$$\lim_{n\to\infty} u(t_n,\xi_n)=\sigma^*,\ \ \lim_{n\to\infty} u_t(t_n,\xi_n)=0.$$

By passing to a subsequence of $(t_n,\xi_n)$ if necessary, we may assume, without loss of generality,

$$\lim_{n\to\infty} v(t_n,\xi_n)=\rho\in[0,\infty).$$

By Lemma 3.1, we have $\lim\limits_{t\to\infty} h'(t)=0$. It follows from this fact and Lemma 3.2 that

$$\lim_{t\to\infty}\int_{g(t)}^{h(t)} u(t,y)dy=0.$$

Since $\sup_{x\in\mathbb{R}} J_1(x) < \infty$ by **(J)**, we have

$$\lim_{t\to\infty} \int_{g(t)}^{h(t)} J_1(x-y)u(t,y)dy = 0 \text{ uniformly for } x \in \mathbb{R}.$$

We now make use of the identity

$$u_t = d_1 \int_{g(t)}^{h(t)} J_1(x-y)u(t,y)dy - d_1 u + u(a_1 - b_1 u - c_1 v - r_1 w)$$

with $(t,x) = (t_n, \xi_n)$. Letting $n \to \infty$, we obtain

$$0 \le -d_1\sigma^* + \sigma^*(a_1 - b_1\sigma^* - c_1\rho) < \sigma^*(a_1 - d_1).$$

It follows that $a_1 > d_1$. We show next that this leads to a contradiction.

Indeed, by (9), there exists $x_0 \in (g(0), h(0))$ such that

$$\lim_{t\to\infty} u(t,x_0) = \lim_{t\to\infty} v(t,x_0) = \lim_{t\to\infty} w(t,x_0) = 0.$$

Therefore we can find $T > 0$ large so that

$$-d_1 + a_1 - b_1 u(t,x_0) - c_1 v(t,x_0) - r_1 w(t,x_0) > (a_1 - d_1)/2 > 0 \text{ for } t \ge T.$$

It then follows from the equation satisfied by $u$ that

$$u_t(t,x_0) \ge \frac{a_1 - d_1}{2} u(t,x_0) \quad \text{for } t \ge T,$$

which implies $u(t,x_0) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of $u$. This completes the proof of $\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0$. Similarly, $\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} v(t,x) = 0$.

In the following we prove the second conclusion of (8). Suppose on the contrary that $\lambda_p(\mathcal{L}^{d_1}_{(g_\infty,h_\infty)} + a_1) > 0$. Then there exists small $\epsilon_1 \in (0, \frac{2a_1}{c_1+r_1})$ such that

$$\lambda_p(\mathcal{L}^{d_1}_{(g_\infty+\epsilon,h_\infty-\epsilon)} + a_1 - \frac{\epsilon}{2}(c_1 + r_1)) > 0 \text{ for } \epsilon \in (0, \epsilon_1).$$

Moreover, for such $\epsilon$, it follows from

$$h_\infty - g_\infty < \infty \text{ and } \lim_{t\to\infty} v(t,x) = \lim_{t\to\infty} w(t,x) = 0 \text{ for } x \in \mathbb{R}$$

that there exists $T_\epsilon$ such that

$$g(t) < g_\infty + \epsilon,\ h(t) > h_\infty - \epsilon \text{ for } t > T_\epsilon,$$

and

$$v(t,x) \le \frac{\epsilon}{2},\ w(t,x) \le \frac{\epsilon}{2} \text{ for } t > T_\epsilon \text{ and } x \in \mathbb{R}.$$

Then

$$\begin{aligned} u_t \ge\ & d_1 \int_{g_\infty+\epsilon}^{h_\infty-\epsilon} J_1(x-y)u(t,y)dy - d_1 u \\ & + u[a_1 - b_1 u - \frac{\epsilon}{2}(c_1 + r_1)], \quad t > T_\epsilon,\ x \in [g_\infty + \epsilon, h_\infty - \epsilon]. \end{aligned}$$

Let $\phi(x)$ be the corresponding normalized eigenfunction of $\lambda_p(\mathcal{L}^{d_1}_{(g_\infty+\epsilon,h_\infty-\epsilon)} + a_1 - \frac{\epsilon}{2}(c_1 + r_1))$, namely, $\|\phi\|_\infty = 1$ and

$$d_1 \int_{g_\infty+\epsilon}^{h_\infty-\epsilon} J_1(x-y)\phi(y)dy - d_1\phi(x) + (a_1 - \frac{\epsilon}{2}(c_1 + r_1))\phi(x) = \lambda_p \phi(x).$$

Then, for any $\delta > 0$,

$$d_1 \int_{g_\infty+\epsilon}^{h_\infty-\epsilon} J_1(x-y)\delta\phi(y)dy - d_1\delta\phi(x) + (a_1 - \frac{\epsilon}{2}(c_1+r_1))\delta\phi(x) = \lambda_p \delta\phi(x) > 0.$$

If we choose $\delta$ small enough such that $\delta\phi(x) \leq u(T_\epsilon, x)$ for $x \in [g_\infty + \epsilon, h_\infty - \epsilon]$, then we can use [3, Lemma 3.3] and a simple comparison argument to obtain

$$u(t,x) \geq \delta\phi(x) > 0 \text{ for } t > T_\epsilon \text{ and } x \in [g_\infty + \epsilon, h_\infty - \epsilon].$$

This is a contradiction to $\lim\limits_{t\to\infty} \max\limits_{g(t)\leq x \leq h(t)} u(t,x) = 0$. Thus $\lambda_p \left(\mathcal{L}^{d_1}_{(g_\infty,h_\infty)} + a_1\right) \leq 0$. □

Then Theorem 1.2 can be obtained by Lemma 3.3 directly.

**Corollary 1.** *Suppose that $J_1, J_2$ and $J_3$ satisfy the conditions in Lemma 3.3, and $(u,v,w,g,h)$ is the unique solution of (3). If $a_1 \geq d_1$ or $a_2 \geq d_2$ or $a_3 \geq d_3$, then necessarily $h_\infty - g_\infty = \infty$.*

*Proof.* Arguing indirectly we assume that $h_\infty - g_\infty < \infty$ and $a_i \geq d_i$ for some $i \in \{1,2,3\}$. Thanks to [3, Proposition 3.4],

$$\lambda_p \left(\mathcal{L}^{d_i}_{(g_\infty,h_\infty)} + a_i\right) > 0.$$

This is a contradiction to Lemma 3.3. □

Hence, Theorem 1.3 (i) has been proved.

We next consider the case that

$$a_i < d_i \text{ for } i = 1,2,3. \tag{11}$$

In this case, it follows from [3, Proposition 3.4] that there exists $l_i$ $(i = 1,2,3)$ such that

$$\begin{cases} \lambda_p(\mathcal{L}^{d_i}_{(0,l_i)} + a_i) = 0, & \text{if } l = l_i, \\ (l - l_i)\lambda_p(\mathcal{L}^{d_i}_{(0,l_i)} + a_i) > 0, & \text{if } l - l_i \in (0, +\infty)\backslash\{l_i\}. \end{cases} \tag{12}$$

Define

$$l_* = \min\{l_1, l_2, l_3\}. \tag{13}$$

It is easily seen that conclusions (a) and (b) of Theorem 1.3 follow directly from the definition of $l_*$, (12) and Lemma 3.3. In the following, we prove Theorem 1.3 (c) by several lemmas.

**Lemma 3.4.** *Under the assumptions of Theorem 1.3, if $h_0 < l_*/2$, then there exists a positive number $\mu_0$ such that $h_\infty - g_\infty < \infty$ for any $\mu \in (0, \mu_0]$.*

We need some comparison results to prove this lemma. The proof of the following Lemma 3.5 can be carried out by the same arguments in the proof of [3, Theorem 3.1]. Since the adaptation is rather straightforward, we omit the details here.

**Lemma 3.5.** *For $T \in (0,+\infty)$, suppose that $\overline{g}, \overline{h} \in C([0,T])$, $\overline{u}, \overline{v} \in C(\overline{D_T^{\overline{g},\overline{h}}})$. If $(\overline{u}, \overline{v}, \overline{w}, \overline{g}, \overline{h})$ satisfies*

$$\begin{cases} \overline{u}_t \geq d_1 \int_{\overline{g}(t)}^{\overline{h}(t)} J_1(x-y)\overline{u}(t,y)dy - d_1\overline{u} + \overline{u}(a_1 - b_1\overline{u}), & t>0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{v}_t \geq d_2 \int_{\overline{g}(t)}^{\overline{h}(t)} J_2(x-y)\overline{v}(t,y)dy - d_2\overline{v} + \overline{v}(a_2 - b_2\overline{v}), & t>0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{w}_t \geq d_3 \int_{\overline{g}(t)}^{\overline{h}(t)} J_3(x-y)\overline{w}(t,y)dy - d_3\overline{w} + \overline{w}(a_3 - b_3\overline{w}), & t>0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u}(t,x) \geq 0,\ \overline{v}(t,x) \geq 0,\ \overline{w}(t,x) \geq 0, & t \geq 0,\ x = \overline{g}(t)\ or\, \overline{h}(t), \\ \overline{g}'(t) \leq -\mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{-\infty}^{\overline{g}(t)} [\rho_1 J_1(x-y)\overline{u}(t,x) & \\ \qquad + \rho_2 J_2(x-y)\overline{v}(t,x) + J_3(x-y)\overline{w}(t,x)]dydx, & t>0, \\ \overline{h}'(t) \geq \mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{\overline{h}(t)}^{+\infty} [\rho_1 J_1(x-y)\overline{u}(t,x) & \\ \qquad + \rho_2 J_2(x-y)\overline{v}(t,x) + J_3(x-y)\overline{w}(t,x)]dydx, & t>0, \\ \overline{g}(0) \leq -h_0,\ \overline{h}(0) \geq h_0, & \\ \overline{u}(0,x) \geq u_0(x),\ \overline{v}(0,x) \geq v_0(x),\ \overline{w}(0,x) \geq w_0(x), & -h_0 < x < h_0. \end{cases}$$

*then the unique solution $(u,v,w,g,h)$ of (3) satisfies*

$$\begin{aligned} &u(t,x) \leq \overline{u}(t,x),\ v(t,x) \leq \overline{v}(t,x),\ w(t,x) \leq \overline{w}(t,x), \\ &g(t) \geq \overline{g}(t),\ h(t) \leq \overline{h}(t) \text{ for } 0 < t \leq T,\ g(t) \leq x \leq h(t). \end{aligned}$$

*The proof of Lemma 3.4.* Since $2h_0 < l_*$, we have $\lambda_p(\mathcal{L}^{d_i}_{(-h_0,h_0)} + a_i) < 0$ $(i = 1,2,3)$. There exists some small $\varepsilon > 0$ such that $h^* := h_0(1+\varepsilon)$ satisfies

$$\lambda_p^i := \lambda_p(\mathcal{L}^{d_i}_{(-h^*,h^*)} + a_i) < 0.$$

Let $\phi_i$ $(i = 1,2,3)$ be the positive normalized eigenfunction corresponding to $\lambda_p^i$, namely, $\|\phi_i\|_\infty = 1$ and

$$d_i \int_{-h^*}^{h^*} J_i(x-y)\phi_i(y)dy - d_i\phi_i(x) + a_i\phi_i = \lambda_p^i\phi_i,\ x \in [-h^*, h^*]. \tag{14}$$

Choose positive constants $K_i$ $(i = 1,2,3)$ large enough such that

$$K_1\phi_1(x) \geq u_0(x),\ K_2\phi_2(x) \geq v_0(x) \text{ and } K_3\phi_3(x) \geq w_0(x) \text{ for } x \in [-h_0, h_0].$$

Define

$$\begin{aligned} &\overline{h}(t) = h_0\left[1 + \varepsilon\left(1 - e^{-\delta t}\right)\right],\ \overline{g}(t) = -\overline{h}(t),\ t \geq 0, \\ &z_i(t,x) = K_i e^{-\delta t}\phi_i(x),\ t \geq 0,\ x \in [\overline{g}(t), \overline{h}(t)],\ i = 1,2,3, \end{aligned}$$

where $\delta > 0$ will be determined later. Clearly $h_0 \leq \overline{h}(t) \leq h^*$.

For $t > 0$ and $x \in (\overline{g}(t), \overline{h}(t))$,

$$z_{it} - d_i \int_{\overline{g}(t)}^{\overline{h}(t)} J_i(x-y)z_i(t,y)dy + d_iz_i - z_i(a_i - b_iz_i)$$

$$\geq K_i e^{-\delta t}\Big( -\delta\phi_i(x) - d_i \int_{\overline{g}(t)}^{\overline{h}(t)} J_i(x-y)\phi_i(y)dy + d_i\phi_i - a_i\phi_i\Big)$$
$$\geq K_i e^{-\delta t}\left(-\delta - \lambda_p^i\right)\phi_i(x) \geq 0,\ i=1,2,3,$$

if we can choose $\delta$ small enough such that

$$\delta \leq \min\{-\lambda_p^1, -\lambda_p^2, -\lambda_p^3\}.$$

Moreover, $\overline{h}'(t) = h_0\varepsilon\delta e^{-\delta t}$ and

$$\mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{\overline{h}(t)}^{+\infty} [\rho_1 J_1(x-y)z_1(t,x) + \rho_2 J_2(x-y)z_2(t,x) + J_3(x-y)z_3(t,x)]dydx$$
$$\leq 2\mu(\rho_1 K_1 + \rho_2 K_2 + K_3)e^{-\delta t}h^*.$$

If

$$\mu \leq \frac{h_0\varepsilon\delta}{2(\rho_1 K_1 + \rho_2 K_2 + K_3)h^*} := \mu_0,$$

then we have

$$\overline{h}'(t) \geq \mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{\overline{h}(t)}^{+\infty} [\rho_1 J_1(x-y)z_1(t,x)+\rho_2 J_2(x-y)z_2(t,x)+J_3(x-y)z_3(t,x)]dydx.$$

Similarly, we can derive

$$\overline{g}'(t) \leq -\mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{-\infty}^{\overline{g}(t)} [\rho_1 J_1(x-y)z_1(t,x)+\rho_2 J_2(x-y)z_2(t,x)+J_3(x-y)z_3(t,x)]dydx.$$

We may now apply Lemma 3.5 to obtain

$$u(t,x) \leq z_1(t,x),\ v(t,x) \leq z_2(t,x),\ w(t,x) \leq z_3(t,x),$$
$$g(t) \geq \overline{g}(t),\ h(t) \leq \overline{h}(t) \text{ for } t>0 \text{ and } x \in [g(t),h(t)].$$

It follows that $\lim_{t\to\infty}(h(t)-g(t)) \leq \lim_{t\to\infty}(\overline{h}(t)-\overline{g}(t)) \leq 2h^* < \infty$. □

**Lemma 3.6.** *Under the assumptions of Theorem 1.3, if $h_0 < l_*/2$, then there exists a positive number $\mu^0$ such that $h_\infty - g_\infty = \infty$ for any $\mu > \mu^0$.*

*Proof.* Consider the following problem

$$\begin{cases} \underline{w}_t = d_3 \int_{\underline{g}(t)}^{\underline{h}(t)} J_3(x-y)\underline{w}(t,y)dy - d_3\underline{w} + \underline{w}(a_3 - b_3\underline{w}), & t>0,\ \underline{g}(t)<x<\underline{h}(t),\\ \underline{w}(t,\underline{g}(t)) = \underline{w}(t,\underline{h}(t)) = 0, & t \geq 0,\\ \underline{g}'(t) = -\mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{-\infty}^{\underline{g}(t)} J_3(x-y)\underline{w}(t,x)dydx, & t>0,\\ \underline{h}'(t) = \mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J_3(x-y)\underline{w}(t,x)dydx, & t>0,\\ -\underline{g}(0) = \underline{h}(0) = h_0, & \\ \underline{w}(0,x) = w_0(x), & -h_0<x<h_0. \end{cases}$$

By [3, Theorem 3.1], we have

$$w(t,x) \geq \underline{w}(t,x),\ g(t) \leq \underline{g}(t),\ h(t) \geq \underline{h}(t),\ \text{for } t>0,\ x \in (\underline{g}(t),\underline{h}(t)).$$

It follows from [3, Theorem 3.13] that there exists some $\mu^0$ such that $\lim_{t\to\infty}[\underline{h}(t) - \underline{g}(t)] = \infty$ for any $\mu > \mu^0$, and so $h_\infty - g_\infty = \infty$. □

Then Theorem 1.3 (c) can follow from Lemmas 3.4 and 3.6 by argument in [23, Theorem 5.2]. Next we give the details below for completeness.

*The proof of Theorem 1.3 (c).* Define $\Sigma^* := \{\mu > 0 : h_\infty - g_\infty \le l_*\}$. By Lemma 3.4, we have $(0, \mu_0] \subset \Sigma^*$. It follows from Lemma 3.6 that $\Sigma^* \cap [\mu^0, \infty) = \emptyset$. Therefore, $\mu^* := \sup \Sigma^* \in [\mu_0, \mu^0]$. By this definition and Theorem 1.3 (a), we find that $h_\infty - g_\infty = \infty$ when $\mu > \mu^*$.

We claim that $\mu^* \in \Sigma^*$. Otherwise $h_\infty - g_\infty = \infty$ for $\mu = \mu^*$. Hence, we can find $T > 0$ such that $h(T) - g(T) > l_*$. To stress the dependence of the solution $(u, v, w, g, h)$ of (3) on $\mu$, we write $(u_\mu, v_\mu, w_\mu, g_\mu, h_\mu)$ instead of $(u, v, w, g, h)$. So we have $h_{\mu^*}(T) - g_{\mu^*}(T) > l_*$. By the continuous dependence of $(u_\mu, v_\mu, w_\mu, g_\mu, h_\mu)$ on $\mu$, we can find $\varepsilon > 0$ small so that $h_\mu(T) - g_\mu(T) > l_*$ for $\mu \in [\mu^* - \varepsilon, \mu^* + \varepsilon]$. It follows that for all such $\mu$,

$$\lim_{t\to\infty}[h_\mu(t) - g_\mu(t)] > [h_\mu(T) - g_\mu(T)] > l_*.$$

This implies that $[\mu^* - \varepsilon, \mu^* + \varepsilon] \cap \Sigma^* = \emptyset$, and $\sup \Sigma^* \le \mu^* - \varepsilon$, contradicting to the definition of $\mu^*$. This proves our claim.

Define $\Sigma_* := \{\nu > 0 : \nu \ge \mu_0$ such that $h_\infty - g_\infty \le l_*$ for all $0 < \mu < \nu\}$, then $\mu_* := \sup \Sigma_* \le \mu^*$ and $(0, \mu_*) \subset \Sigma_*$. Similarly to the above, we can prove that $\mu_* \in \Sigma_*$. The proof is completed. □

3.2. **Long-time behaviour in the case of spreading.** Finally, we will examine the long-time behaviour of the solution to (3) when $h_\infty - g_\infty = \infty$. Before proving Theorem 1.4, we first give the following lemma:

**Lemma 3.7.** *$h_\infty = +\infty$ if and only if $g_\infty = -\infty$.*

*Proof.* This follows the idea in the proof of [3, Lemma 3.8]. For example, if $g_\infty = -\infty$ but $h_\infty < +\infty$, then we may argue as in the proof of [3, Theorem 3.7] to obtain $h'(t) \ge \xi_0 > 0$ for all large $t$, which yields a contradiction. □

*The proof of Theorem 1.4.* By [3, Theorem 3.9], we have

$$\lim_{t\to\infty} w(t,x) = \frac{a_3}{b_3} =: C \text{ locally uniformly for } x \in \mathbb{R}. \tag{15}$$

(i) We will prove it by the following steps.

**Step 1.** Let $q(t)$ be the solution of

$$\begin{cases} q'(t) = q(a_1 - b_1 q), \quad t > 0, \\ q(0) = \sup\limits_{x\in\mathbb{R}} u_0(x). \end{cases}$$

Then $\lim\limits_{t\to\infty} q(t) = a_1/b_1$. By the comparison principle ([3, Lemma 2.2]), we have $u(t,x) \le q(t)$ for $t > 0$ and $x \in [g(t), h(t)]$. In view of $u(t,x) = 0$ for $t > 0$ and $x \in \mathbb{R}\backslash(g(t), h(t))$, we have $u(t,x) \le q(t)$ for $t > 0$ and $x \in \mathbb{R}$. Hence,

$$\limsup_{t\to\infty} u(t,x) \le \frac{a_1}{b_1} =: \bar{A}_1 \ \text{ locally uniformly in } \mathbb{R}. \tag{16}$$

**Step 2.** By (15) and (16), we have

$$\limsup_{t\to\infty}[c_2 u(t,x) + r_2 w(t,x)] \le c_2 \bar{A}_1 + r_2 C \ \text{ locally uniformly in } \mathbb{R}.$$

By the condition $a_2 > c_2\frac{a_1}{b_1} + r_2\frac{a_3}{b_3}$, we have

$$a_2 - c_2\bar{A}_1 - r_2 C = a_2 - c_2\frac{a_1}{b_1} - r_2\frac{a_3}{b_3} > 0.$$

It follows from above two facts, and [6, Lemma 3.14] that

$$\liminf_{t\to\infty} v(t,x) \geq \frac{a_2 - c_2\bar{A}_1 - r_2 C}{b_2} =: \underline{B}_1 \text{ locally uniformly in } \mathbb{R}. \tag{17}$$

**Step 3.** By (15) and (17), we have

$$\limsup_{t\to\infty}[c_1 v(t,x) + r_1 w(t,x)] \geq c_1\underline{B}_1 + r_1 C \ \text{ locally uniformly in } \mathbb{R}.$$

The condition $a_1 > c_1\frac{a_2}{b_2} + r_1\frac{a_3}{b_3}$ implies

$$a_1 - c_1\underline{B}_1 - r_1 C = a_1 - c_1\frac{a_2 - c_2\bar{A}_1 - r_2 C}{b_2} - r_1 C \geq a_1 - c_1\frac{a_2}{b_2} - r_1\frac{a_3}{b_3} > 0.$$

These two facts and [6, Lemma 3.14] allow us to derive

$$\limsup_{t\to\infty} u(t,x) \leq \frac{a_1 - c_1\underline{B}_1 - r_1 C}{b_1} =: \bar{A}_2 \ \text{ locally uniformly in } \mathbb{R}. \tag{18}$$

**Step 4.** By (15) and (18), we have

$$\limsup_{t\to\infty}[c_2 u(t,x) + r_2 w(t,x)] \leq c_2\bar{A}_2 + r_2 C \ \text{ locally uniformly in } \mathbb{R}.$$

Furthermore, the condition $a_2 > c_2\frac{a_1}{b_1} + r_2\frac{a_3}{b_3}$ implies

$$a_2 - c_2\bar{A}_2 - r_2 C = a_2 - c_2\frac{a_1 - c_1\underline{B}_1 - r_1 C}{b_1} - r_2 C \geq a_2 - c_2\frac{a_1}{b_1} - r_2\frac{a_3}{b_3} > 0.$$

Similar to the above,

$$\liminf_{t\to\infty} v(t,x) \geq (a_2 - c_2\bar{A}_2 - r_2 C)/b_2 =: \underline{B}_2 \ \text{ locally uniformly in } \mathbb{R}.$$

**Step 5.** Repeating the above procedure, we can find two sequences $\bar{A}_i$ and $\underline{B}_i$ such that

$$\limsup_{t\to\infty} u(t,x) \leq \bar{A}_i, \quad \liminf_{t\to\infty} v(t,x) \geq \underline{B}_i \ \text{ locally uniformly in } \mathbb{R},$$

and

$$\bar{A}_{i+1} = (a_1 - c_1\underline{B}_i - r_1 C)/b_1, \quad \underline{B}_i = (a_2 - c_2\bar{A}_i - r_2 C)/b_2, \quad i = 1, 2, \cdots.$$

Let

$$p := \frac{a_1 - r_1 C}{b_1} - \frac{c_1(a_2 - r_2 C)}{b_1 b_2}, \qquad q := \frac{c_1 c_2}{b_1 b_2}.$$

Then $p > 0$ by $a_1 - c_1\frac{a_2}{b_2} - r_1\frac{a_3}{b_3} > 0$, $0 < q < 1$ by $a_1 > c_1\frac{a_2}{b_2}$ and $a_2 > c_2\frac{a_1}{b_1}$. By direct calculation,

$$\bar{A}_{i+1} = p + q\bar{A}_i, \quad i = 1, 2, \cdots.$$

From $\bar{A}_2 < \bar{A}_1$ and the above iteration formula, we immediately obtain

$$0 < \bar{A}_{i+1} < \bar{A}_i, \quad i = 1, 2, \cdots,$$

from which it easily follows that

$$\lim_{i\to\infty}\bar{A}_i = \frac{b_2(a_1 b_3 - r_1 a_3) - c_1(a_2 b_3 - r_2 a_3)}{b_3(b_1 b_2 - c_1 c_2)},$$

$$\lim_{i\to\infty}\underline{B}_i = \frac{-c_2(a_1 b_3 - r_1 a_3) + b_1(a_2 b_3 - r_2 a_3)}{b_3(b_1 b_2 - c_1 c_2)}.$$

Thus we have

$$\limsup_{t\to\infty} u(t,x) \le \frac{b_2(a_1b_3 - r_1a_3) - c_1(a_2b_3 - r_2a_3)}{b_3(b_1b_2 - c_1c_2)} \text{ locally uniformly in } \mathbb{R}.$$

$$\liminf_{t\to\infty} v(t,x) \ge \frac{-c_2(a_1b_3 - r_1a_3) + b_1(a_2b_3 - r_2a_3)}{b_3(b_1b_2 - c_1c_2)} \text{ locally uniformly in } \mathbb{R}.$$

Similarly, we can show

$$\liminf_{t\to\infty} u(t,x) \ge \frac{b_2(a_1b_3 - r_1a_3) - c_1(a_2b_3 - r_2a_3)}{b_3(b_1b_2 - c_1c_2)} \text{ locally uniformly in } \mathbb{R}.$$

$$\limsup_{t\to\infty} v(t,x) \le \frac{-c_2(a_1b_3 - r_1a_3) + b_1(a_2b_3 - r_2a_3)}{b_3(b_1b_2 - c_1c_2)} \text{ locally uniformly in } \mathbb{R}.$$

Thus, (i) is proved.

(ii) By Steps 1 and 2 in (i), we also have

$$\liminf_{t\to\infty} v(t,x) \ge \frac{a_2 - c_2\bar{A}_1 - r_2C}{b_2} =: \underline{B}_1 \text{ locally uniformly in } \mathbb{R}.$$

It follows from this fact and (15) that

$$\limsup_{t\to\infty}[c_1v(t,x) + r_1w(t,x)] \ge c_1\underline{B}_1 + r_1C \quad \text{locally uniformly in } \mathbb{R}.$$

The condition $a_1 + \frac{c_1}{b_2}\left(c_2\frac{a_1}{b_1} + r_2\frac{a_3}{b_3}\right) \le c_1\frac{a_2}{b_2} + r_1\frac{a_3}{b_3}$ implies

$$\begin{aligned} a_1 - c_1\underline{B}_1 - r_1C &= a_1 - c_1\frac{a_2 - c_2\bar{A}_1 - r_2C}{b_2} - r_1C \\ &= a_1 + \frac{c_1}{b_2}\left(c_2\frac{a_1}{b_1} + r_2\frac{a_3}{b_3}\right) - c_1\frac{a_2}{b_2} - r_1\frac{a_3}{b_3} \le 0. \end{aligned}$$

These two facts and [6, Lemma 3.14] allow us to derive

$$\limsup_{t\to\infty} u(t,x) \le 0 \text{ locally uniformly in } \mathbb{R}.$$

Since

$$\liminf_{t\to\infty} u(t,x) \ge 0 \text{ locally uniformly in } \mathbb{R},$$

we have

$$\lim_{t\to\infty} u(t,x) = 0 \text{ locally uniformly in } \mathbb{R}.$$

It follows from this, (15) and [6, Lemma 3.14] that

$$\frac{a_2b_3 - r_2a_3}{b_2b_3} \le \liminf_{t\to\infty} v(t,x) \le \limsup_{t\to\infty} v(t,x) \le \frac{a_2b_3 - r_2a_3}{b_2b_3} \text{ locally uniformly in } \mathbb{R},$$

and so

$$\lim_{t\to\infty} v(t,x) = \frac{a_2b_3 - r_2a_3}{b_2b_3} \text{ locally uniformly in } \mathbb{R}.$$

We have proved (ii).

(iii) This conclusion can be proved by the same arguments in (ii).

(iv) By (15), we have

$$\limsup_{t\to\infty}[c_1v(t,x) + r_1w(t,x)] \ge r_1C \text{ locally uniformly in } \mathbb{R}.$$

The condition $a_1 \le r_1\frac{a_3}{b_3}$ implies

$$a_1 - r_1C = a_1 - r_1\frac{a_3}{b_3} \le 0.$$

These two facts and [6, Lemma 3.14] allow us to derive

$$\limsup_{t\to\infty} u(t,x) \le 0 \text{ locally uniformly in } \mathbb{R},$$

and so

$$\lim_{t\to\infty} u(t,x) = 0 \text{ locally uniformly in } \mathbb{R}.$$

Similarly, we have

$$\lim_{t\to\infty} v(t,x) = 0 \text{ locally uniformly in } \mathbb{R}.$$

Then (iv) has been proved. □

4. **Discussion.** In this paper, we study a free boundary problem with nonlocal diffusion describing information diffusion in online social networks. This system consists of three equations representing three pieces of information propagating via the internet and competing for influence among users. We obtain the criteria for information spreading and vanishing. If the diffusion rate of any piece of information is small, i.e., $d_1 \le a_1$ or $d_2 \le a_2$ or $d_3 \le a_3$, information will always spread. But when the diffusion rates of three pieces of information are all large, i.e., $d_i > a_i$ $(i = 1, 2, 3)$, whether information spread or vanish depends on the initial data. If the initial spreading area $[-h_0, h_0]$ is within the critical size, i.e., $h_0 < l^*/2$, information spread or vanish depending on the size of the expanding capacity $\mu$, namely, vanishing happens with small expanding capability and spreading happens with large expanding capability. Regardless of the expanding capability, spreading always occurs if the initial spreading area is beyond the critical size. We find that the result of the nonlocal diffusion model (3) is different from the local diffusion model in [15].

When spreading happens, the longtime behavior of the solution is obtained in Theorem 1.4, which is similar to the result of local diffusion model studied in [15]. According to Theorem 1.4, we can choose suitable official information to control rumor propagation in social networks, namely, we can change the value of $a_3$ and $b_3$ by choosing suitable official information.

For local diffusion model (2), the result in [15] showed the spreading has a finite speed when spreading happens. However, what will happen for the nonlocal diffusion model (3)? Very recently, Du, Li and Zhou [4] investigated the spreading speed of the nonlocal model in [3] and proved that the spreading may or may not have a finite speed, depending on whether a certain condition is satisfied by the kernel function $J$ in the nonlocal diffusion term. This contrasts sharply to the local model of [5], where the spreading has finite speed whenever spreading happens. Since (3) consists of three equations, we expect a more complex result for (3), which will be considered in a future work.

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