ON THE MOD P STEENROD ALGEBRA AND THE LEIBNIZ-HOPF ALGEBRA

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Abstract. Let p be a fixed odd prime. The Bockstein free part of the mod p Steenrod algebra, $\mathcal{A}_p$, can be defined as the quotient of the mod p reduction of the Leibniz Hopf algebra, $\mathcal{F}_p$. We study the Hopf algebra epimorphism $\pi\colon \mathcal{F}_p \to \mathcal{A}_p$ to investigate the canonical Hopf algebra conjugation in $\mathcal{A}_p$ together with the conjugation operation in $\mathcal{F}_p$. We also give a result about conjugation invariants in the mod 2 dual Leibniz Hopf algebra using its multiplicative algebra structure.

1. **Introduction.** From a topological view, the mod p Steenrod algebra, $\mathcal{A}$, for any prime p, is the algebra of stable cohomology operations for mod p cohomology. Being a Hopf algebra, it also has a unique Hopf algebra conjugation map, χ. Let p be an odd fixed prime and let $\mathcal{A}_p$ denote the subalgebra of $\mathcal{A}$ at odd primes generated by the *Steenrod reduced pth powers* $\mathcal{P}^i$, $i \geq 1$ [28] (i.e., the Bockstein-free part of $\mathcal{A}$). This is also a Hopf algebra with a unique conjugation map.

The Leibniz-Hopf algebra $\mathcal{F}$ is the free associative algebra on generators S^1, S^2, ..., where S^i has degree i. $\mathcal{F}$ is connected. We refer to [18–22] for more detailed information about this algebra. By the mod p reduction of this algebra, $\mathcal{F}_p = \mathcal{F} \otimes \mathbf{Z}/p$, we mean the free associative $\mathbf{Z}/p$ algebra on generators $S^1, S^2, \ldots$. We may make $\mathcal{F}_p$ a Hopf algebra by defining a comultiplication Δ by

$$\Delta(S^n) = \sum_{i_m + j_k = n} S^{i_m} \otimes S^{j_k}.$$

We can give $\mathcal{F}_p$ a new grading by S^i has degree $2i(p-1)$, then the algebra $\mathcal{A}_p$ is naturally defined as a quotient of $\mathcal{F}_p$ by the Adem relations [34]. It follows that we have a Hopf algebra epimorphism $\pi\colon \mathcal{F}_p \to \mathcal{A}_p$. Recently, this homomorphism and its dual $\pi^*\colon \mathcal{A}_p^* \to \mathcal{F}_p^*$ which is a graded Hopf algebra inclusion play an important role concerning the Adem relations in the Steenrod algebra and the conjugation invariant problem in $\mathcal{A}_p$, $\mathcal{F}_p$ and the dual of these algebras. Let us (briefly) give information about it. When $p = 2$, in an earlier paper [39, Section 3], the homomorphism π is considered for determining a better formula for the conjugation operation in the Steenrod algebra. In the same paper, the conjugation invariant problem in $\mathcal{A}_2$ is also investigated by using the conjugation invariants in the Leibniz-Hopf algebra [10]. In [40, 41] the author used the π^* to introduce an alternative view of the Adem relations for any prime number. In [40], motivated by

the work of Crossley and Whitehouse [11], the author attempts to solve conjugation invariant problem in the mod 2 dual Steenrod algebra by using the homomorphism π^*. Conjugation invariants in the mod p dual Steenrod algebra, $\mathcal{A}_p^*$, are determined in [12]. In [38, Section 3] the author used π^* to introduce a different approach on the conjugation invariant problem $\mathcal{A}_p^*$. In [43] Turgay and Kaji used π^* to give generalisations of some classical results concerning the Steenrod algebra in the literature. There are also a wide variety of Hopf algebras with different rich algebraic structures. For some of these algebras we refer to [4, 7, 14, 17, 26].

The conjugation is a useful tool for studying many problems in the Steenrod algebra. Conjugation map in $\mathcal{A}$ was first studied by Thom [37] in 1954. Afterwards Milnor gave a conjugation formula [28, Theorem 5] for Steenrod powers [34], $\mathcal{P}^i$. Many researchers have used it since it has links with topology and algebra. In 1974, Davis [13] computed certain Steenrod powers, $\mathcal{P}^{p^{n-1}+\cdots+p+1}$. Silverman [33], Straffin [36], Barratt and Miller [3], and Karaca and I.Y. Karaca [24] have obtained many relations through this formula. Walker and Wood [45, 46] have used this formula to give an answer for the nilpotent question in the Steenrod algebra.

There are many descriptions of bases for the Steenrod algebra in literature. There are bases developed by Milnor [28], Wall [47], D. Arnon [2], R. Wood [48], in the Steenrod algebra. One of the traditional ones is the admissible basis. In [32], Serre showed that the set of admissible monomials forms a vector space basis for the Steenrod algebra. After that many researches have investigated relationships between the admissible basis and the other bases. Milnor [28, Lemma 8] showed that the admissible basis is related to the Milnor basis. Monks [30, Section 3] expressed an admissible monomial in the Milnor basis using the Milnor product formula. In 1998, Carlisle [6] et al proved a conjecture of Monks [30] on the relation between the admissible basis and the Milnor basis of the mod 2 Steenrod algebra. In the same article the results are also generalised to odd prime cases. In [2], Arnon expressed admissible monomials in $\mathcal{A}_2$ in a different form. In [16,25], Arnon's results are generalized to odd primes. In [44] Turgay and Karaca give relations using the Arnon Bases.

The Steenrod algebra has many relations among its elements. The complexity in the structure of the Steenrod algebra makes calculations without the aid computer programs time consuming. Examples of such computer-based aid are Monk's and Kaji's Maple packages [23, 31] for $\mathcal{A}_2$. Sage [35] includes useful and efficent codes for calculations in $\mathcal{A}_p$ for all primes. Calculations in table 1 in this present paper agree with what Sage produces.

Now we give the organization of this work with the motivations. The problem of explicitly computing conjugates of monomials in terms of the admissible monomials in the Steenrod algebra is an open problem. Motivated by this in section 3, we investigate if we can have a better understanding of conjugation operation in $\mathcal{A}_p$ using the homomorphism π together with a conjugation formula on $\mathcal{F}_p$. Our method enables us to write the conjugates of the Steenrod squares in terms of the admissible basis elements in $\mathcal{A}_p$ (see Examples 3.1, 3.2 and the property (7)). Our approach cannot solve our problem, but it gives the fact that the Adem relations with better understanding enable us to calculate the conjugate of some Steenrod powers.

In section 4, we consider the decomposition of the conjugation operation in the mod 2 dual Leibniz Hopf algebra, $\mathcal{F}_2^*$. We reprove an important identity concerning the coarsening operation (see Theorem 4.1). Our argument in the proof is new in

that it is purely combinatorial. Conjugation invariant problem in $\mathcal{A}_2^*$ is an important problem in algebraic topology because it has links with commutativity of ring spectra (see [1,11,12] for more details). In [11] Crossley and Whitehouse introduced a partial answer to this problem. In [9] Crossley and Turgay give another approach to this problem and determined a vector space basis for the conjugation invariants in the mod dual Leibniz-Hopf algebra, $\mathcal{F}_2^*$. In this section we give a result for the conjugation invariants in $\mathcal{F}_2^*$ using the multiplicative structure of this algebra (see Theorem 4.3).

2. **The mod p Steenrod algebra.** Steenrod operations, $\mathcal{P}^i$, which are also called Steenrod powers at odd primes , are cohomology operations acting on ordinary mod p cohomology of the form

$$\mathcal{P}^i : H^q(X; \mathbb{Z}_p) \longrightarrow H^{q+2i(p-1)}(X; \mathbb{Z}_p)$$

for all integers $i \geq 0$ and $q \geq 0$. These operations satisfy some certain properties and one of them is the Adem Relations:

$$\mathcal{P}^a \mathcal{P}^b = \sum_{j=0}^{[\frac{a}{p}]} (-1)^{a+j} \left(\begin{array}{c} (p-1)(b-j)-1 \\ a-pj \end{array} \right) \mathcal{P}^{a+b-j} \mathcal{P}^j$$

if $a < pb$, and

$$\mathcal{P}^a \beta \mathcal{P}^b = \sum_{j=0}^{[\frac{a}{p}]} (-1)^{a+j} \left(\begin{array}{c} (p-1)(b-j) \\ a-pj \end{array} \right) \beta \mathcal{P}^{a+b-j} \mathcal{P}^j$$

$$+ \sum_{j=0}^{[\frac{a-1}{p}]} (-1)^{a+j-1} \left(\begin{array}{c} (p-1)(b-j)-1 \\ a-pj-1 \end{array} \right) \mathcal{P}^{a+b-j} \beta \mathcal{P}^j$$

if $a \leq pb$, where β is the Bockstein homomorphism [34, Chapter 6].

Remark 2.1. $[\frac{a}{p}]$ denotes the greatest integer $\leq \frac{a}{p}$ and the binomial coefficients are taken modulo p.

Definition 2.1. The mod p Steenrod algebra is the associative algebra over $\mathbb{F}_p$ generated by $\beta, \mathcal{P}^1, \mathcal{P}^2, \ldots$ subject to $\beta^2 = 0$, the Adem Relations and to $\mathcal{P}^0 = 1$. This algebra is graded where $\mathcal{P}^i$ is degree of $2i(p-1)$ and β is degree of 1.

In $\mathcal{A}_p$ a monomial can be written in the form

$$\beta^{\varepsilon_0} \mathcal{P}^{r_1} \beta^{\varepsilon_1} \cdots \mathcal{P}^{r_k} \beta^{\varepsilon_k}$$

where $\varepsilon_i = 0, 1$ and $r_i = 1, 2, \ldots$. We denote this monomial by $\mathcal{P}^I$, where

$$I = (\varepsilon_0, r_1, \varepsilon_1, r_2, \ldots, r_k, \varepsilon_k, 0, 0, \ldots).$$

$\mathcal{P}^I$ is said to be an admissible monomial if $r_i \geq p r_{i+1} + \varepsilon_i$ for all $i \geq 1$. Let the degree of I be the degree of $\mathcal{P}^I$, which is denoted by $d(I)$, then we have the following formula:

$$d(\mathcal{P}^I) = d(I) = \sum_{i=0}^{k} \varepsilon_i + 2(p-1) \sum_{i=1}^{k} r_i.$$

Remark 2.2. If I is an admissible sequence, then

$$d(I) > 1 + p + p^2 + \cdots + p^k + \varepsilon_0 + \cdots + \varepsilon_k = \frac{p^{k+1}-1}{p-1} + \sum_{i=0}^{k} \varepsilon_i,$$

so that finding an admissible basis for a certain degree is a finite problem which gives rise to a computer algorithm.

2.1. **Conjugation in the Steenrod algebra.** In [28], Milnor has showed that $\mathcal{A}$ is a graded connected Hopf algebra with a coproduct defined by the formula:

$$\psi(\mathcal{P}^i) = \sum_{k=0}^{i} \mathcal{P}^k \otimes \mathcal{P}^{i-k}, \quad \text{and} \quad \psi(\beta) = \beta \otimes 1 + 1 \otimes \beta.$$

As $\mathcal{A}_p$ is a connected Hopf algebra [29], it has a unique Hopf algebra conjugation which is also an anti-automorphism. By Thom's identity, a conjugation formula can be defined [28, Section 7] recursively by

$$\chi(\mathcal{P}^0) = 1, \quad \text{and} \quad \sum_{i=0}^{r} \mathcal{P}^i \chi(\mathcal{P}^{r-i}) = 0 \quad r > 0.$$

Example 2.1. If $p = 3$, to calculate the image of $\mathcal{P}^1$ under χ we need to solve the equation below

$$\mathcal{P}^0 \chi(\mathcal{P}^1) + \mathcal{P}^1 \chi(\mathcal{P}^0) = 0. \tag{1}$$

By Eq. (1) it is easily seen that:

$$\chi(\mathcal{P}^1) = 2\mathcal{P}^1.$$

Moreover we can generalize the above equality to all primes as follow.

Proposition 2.3. $\chi(\mathcal{P}^1) = (p-1)\mathcal{P}^1.$

We also have Davis's useful conjugation formula as follows.

Theorem 2.4 ([13, Theorem 1]).

$$\chi(\mathcal{P}^{p^{n-1}+\cdots+p+1}) = (-1)^n \mathcal{P}^{p^{n-1}} \cdots \mathcal{P}^p \mathcal{P}^1. \tag{2}$$

Lastly we give χ-images of $\mathcal{P}^i$ at prime $p = 3, 5, 7, 11$ is given as follows.

Table 1

Calculations under conjugation operation				
$\chi(\mathcal{P}^i)$	$\mathcal{A}_3$	$\mathcal{A}_5$	$\mathcal{A}_7$	$\mathcal{A}_{11}$
$\chi(\mathcal{P}^1)$	$2\mathcal{P}^1$	$4\mathcal{P}^1$	$6\mathcal{P}^1$	$10\mathcal{P}^1$
$\chi(\mathcal{P}^2)$	$\mathcal{P}^2$	$\mathcal{P}^2$	$\mathcal{P}^2$	$\mathcal{P}^2$
$\chi(\mathcal{P}^3)$	$2\mathcal{P}^3$	$4\mathcal{P}^3$	$6\mathcal{P}^3$	$10\mathcal{P}^3$
$\chi(\mathcal{P}^4)$	$\mathcal{P}^{3,1}$	$\mathcal{P}^4$	$\mathcal{P}^4$	$\mathcal{P}^4$
$\chi(\mathcal{P}^5)$	$\mathcal{P}^5 + 2\mathcal{P}^{4,1}$	$4\mathcal{P}^5$	$6\mathcal{P}^5$	$10\mathcal{P}^5$
$\chi(\mathcal{P}^6)$	$\mathcal{P}^6 + \mathcal{P}^{5,1}$	$\mathcal{P}^{5,1}$	$\mathcal{P}^6$	$\mathcal{P}^6$
$\chi(\mathcal{P}^7)$	$2\mathcal{P}^{6,1}$	$\mathcal{P}^7 + 4\mathcal{P}^{6,1}$	$6\mathcal{P}^7$	$10\mathcal{P}^7$
$\chi(\mathcal{P}^8)$	$\mathcal{P}^{6,2}$	$3\mathcal{P}^8 + \mathcal{P}^{7,1}$	$\mathcal{P}^{7,1}$	$\mathcal{P}^8$
$\chi(\mathcal{P}^9)$	$2\mathcal{P}^9 + \mathcal{P}^{8,1} + 2\mathcal{P}^{7,2}$	$3\mathcal{P}^9 + 4\mathcal{P}^{8,1}$	$\mathcal{P}^9 + 6\mathcal{P}^{8,1}$	$10\mathcal{P}^9$
$\chi(\mathcal{P}^{10})$	$\mathcal{P}^{9,1} + \mathcal{P}^{8,2}$	$\mathcal{P}^{10} + \mathcal{P}^{9,1}$	$5\mathcal{P}^{10} + \mathcal{P}^{9,1}$	$\mathcal{P}^{10}$

3. **Computations in the odd primary Steenrod algebra using the conjugation via π.** The algebra $\mathcal{F}_p$ has a basis of words $S^{i_1} S^{i_2} \cdots S^{i_k}$ in the letters $S^{i_1}, S^{i_2}, \ldots$, which we will abbreviate to $S^{j_1, j_2, \ldots, j_r}$. A formula for the conjugation operation on $\mathcal{F}$ was introduced in [8, 15, 27]. We know that the cojugation is anti-multiplicative. Hence, we can express it as follows

$$\chi'(S^{i_1, \ldots, i_k}) = \sum (-1)^m S^{b_1, \ldots, b_m} \tag{3}$$

where the summation is over all refinements $b_1, \ldots, b_m$ of the *reversed* word $i_k, \ldots, i_1$. For instance,

$$\chi'(S_{2,3}) = S_{3,2} - S_{3,1,1} - S_{1,2,2} + S_{1,2,1,1} - S_{2,1,2} + S_{2,1,1,1} + S_{1,1,1,2} - S_{1,1,1,1,1}.$$

We now interested in the graded Hopf algebra homomorphism $\pi \colon \mathcal{F}_p \to \mathcal{A}$, where $\pi(S^n) = \mathcal{P}^n$. This homomorphism preserves conjugation operations:

$$\chi \circ \pi = \pi \circ \chi'. \tag{4}$$

Let us work on mod 3 and use the above equation for computing χ in the following two examples.

Example 3.1. Applying (4) with the OLP S^4 gives us that

$$\chi\big(\pi(S^4)\big) = \pi\big(\chi'(S^4)\big). \tag{5}$$

Since

$$\chi'(S^4) = -S^4 + S^{3,1} + S^{2,2} - S^{1,2,1} + S^{1,3} - S^{2,1,1} - S^{1,1,2} + S^{1,1,1,1},$$

we have

$$\pi\big(\chi'(S^4)\big) = -\mathcal{P}^4 + \mathcal{P}^{3,1} + \mathcal{P}^{2,2} - \mathcal{P}^{1,2,1} + \mathcal{P}^{1,3} - \mathcal{P}^{2,1,1} - \mathcal{P}^{1,1,2} + \mathcal{P}^{1,1,1,1}.$$

The Adem relations gives us that

$$\mathcal{P}^{2,2} = \mathcal{P}^{1,2,1} = \mathcal{P}^{2,1,1} = \mathcal{P}^{1,1,2} = \mathcal{P}^{1,1,1,1} = 0,$$

and $\mathcal{P}^{1,3} = \mathcal{P}^4$. It follows that (6) turns into $\chi(\mathcal{P}^4) = \mathcal{P}^{3,1}$.

Example 3.2. Applying (4) with the HLP $S^{3,1}$ gives us that

$$\chi\big(\pi(S^{3,1})\big) = \pi\big(\chi'(S^{3,1})\big). \tag{6}$$

Since

$$\chi'(S^{3,1}) = S^{1,3} - S^{1,2,1} - S^{1,1,2} + S^{1,1,1,1},$$

it follows that

$$\pi\big(\chi'(S^{3,1})\big) = \mathcal{P}^{1,3} - \mathcal{P}^{1,2,1} + \mathcal{P}^{1,1,1,1}.$$

By the Adem relations, we have: $\mathcal{P}^{1,2,1} = \mathcal{P}^{1,1,1,1} = 0$ and $\mathcal{P}^{1,3} = \mathcal{P}^4$. Hence, $\chi(\mathcal{P}^{3,1}) = \mathcal{P}^4$.

Using (4) we now reprove a property of the conjugation operation in $\mathcal{A}_p$, which plays an important role in the proof of relationship between the X- and Z bases [44, Proposition 3.2]. By the same argument we used in the above examples, we may compute the $\chi(\mathcal{P}^{p^n})$. Now fix a prime number p and an integer $m \geq$. Now consider the following

$$\chi(\mathcal{P}^{p^n}) = \pi\bigg(\sum (-1)^m S^{j_1, \ldots, j_m}\bigg), \tag{7}$$

where the summation is over all refinements $j_1, \ldots, j_m$ of the word p^n. It follows that the only one-length refinement among these refinements is $\mathcal{P}^{p^n}$ and we know that $\pi(\mathcal{P}^{p^n}) = \mathcal{P}^{p^n}$. Recall from [34] that P^{p^n} is indecomposable. Hence by linearity of π, (7) leads to

$$\chi(\mathcal{P}^{p^n}) = -\mathcal{P}^{p^n} + K, \tag{8}$$

where K is a polynomial which is a sum of products of $\mathcal{P}^r$'s where $r < p^n$ for each r.

4. **Results concerning conjugation in the mod 2 dual Leibniz-Hopf algebra.** Recall from [42, Section 2] that, conjugation operation in $\mathcal{F}_2^*$ has a decomposition $\chi_{\mathcal{F}_2^*} = \mathbf{C} \circ R$, where $\mathbf{C}$ is a coarsening operation. Given a basis element $S_{b_1,\ldots,b_p}$, its image under the coarsening operation $\mathbf{C}$ is given by

$$\mathbf{C}(S_{b_1,\ldots,b_p}) = \sum S_{l_1,\ldots,l_n} \tag{9}$$

summed over all coarsening $l_1,\ldots,l_n$ of $b_1,\ldots,b_p$.

We now reprove the following result using combinatorics

Theorem 4.1. $\mathbf{C}^2 = 1$.

The above equality is already given in [42, Section 2]. We give a combinatoric proof for this identity. For this we first give the following.

Proposition 4.2. *Let $S_{b_1,\ldots,b_p} \in \mathcal{F}_2^*$, then*

$$\sum C(S_{r_1,\ldots,r_m}) = S_{b_1,\ldots,b_p}, \tag{10}$$

where the summation is over all coarsenings $r_1,\ldots,r_m$ of $b_1,\ldots,b_p$.

Proof. Let $S_{c_1,\ldots,c_n}$ be any word. If $S_{c_1,\ldots,c_n} = S_{b_1,\ldots,b_p}$, and is in the sum in (10), then we shall show that it occurs with coefficient one. On the other hand, if $S_{c_1,\ldots,c_n} \neq S_{b_1,\ldots,b_p}$, then we shall show that it occurs with coefficient zero. In other words, we need to show that there is an even number of coarsenings $r_1,\ldots,r_m$ of $b_1,\ldots,b_p$ for which $S_{c_1,\ldots,c_n}$ is a summand of $C(S_{r_1,\ldots,r_m})$.

i. Let $S_{c_1,\ldots,c_n} = S_{b_1,\ldots,b_p}$, and be in the sum in (10), then by (9), $S_{c_1,\ldots,c_n}$ is a summand of $C(S_{b_1,\ldots,b_p})$. Moreover, there are no proper coarsening $r_1,\ldots,r_m$ of $b_1,\ldots,b_p$ for which $S_{c_1,\ldots,c_n}$ is a summand of $C(S_{r_1,\ldots,r_m})$. Because, if $S_{c_1,\ldots,c_n}$ is a summand of $C(S_{r_1,\ldots,r_m})$, where $S_{r_1,\ldots,r_m}$ is a proper coarsening of $S_{b_1,\ldots,b_p}$, then by (9), $S_{c_1,\ldots,c_n}$ is a coarsening of $S_{r_1,\ldots,r_m}$, and hence is a proper coarsening of $S_{b_1,\ldots,b_p}$. But $S_{c_1,\ldots,c_n} = S_{b_1,\ldots,b_p}$, so $S_{c_1,\ldots,c_n}$ cannot be a proper coarsening of $S_{b_1,\ldots,b_p}$. Therefore, there is exactly one coarsening $r_1,\ldots,r_m$ for which $S_{c_1,\ldots,c_n}$ is a summand of $C(S_{r_1,\ldots,r_m})$ which is the improper one, where $m = p$ and $r_1 = b_1, r_2 = b_2,\ldots,r_m = b_p$. So $S_{c_1,\ldots,c_n}$ occurs with a coefficient one in the sum.

ii. Let $S_{c_1,\ldots,c_n} \neq S_{b_1,\ldots,b_p}$. If $S_{c_1,\ldots,c_n}$ is to occur in the sum at all, it must be a summand of $C(S_{r_1,\ldots,r_m})$ for some coarsening $r_1,\ldots,r_m$ of $b_1,\ldots,b_p$. Then by (9), $S_{c_1,\ldots,c_n}$ is a coarsening of $S_{r_1,\ldots,r_m}$. We know $S_{r_1,\ldots,r_m}$ is a coarsening of $S_{b_1,\ldots,b_p}$, then $S_{c_1,\ldots,c_n}$ is also a coarsening of $S_{b_1,\ldots,b_p}$.

On the other hand, each coarsening is obtained by turning some of the $p - 1$ commas of $b_1,\ldots,b_p$ into pluses. Let j ($j \geq 1$), be the number of commas in $b_1,\ldots,b_p$ which are turned into pluses when we form $c_1,\ldots,c_n$. Since $b_1,\ldots,b_p \neq c_1,\ldots,c_n$. Then, $r_1,\ldots,r_m$ corresponds to choosing a subset of these j commas. There are 2^j such subsets and, therefore, there are 2^j coarsenings $r_1,\ldots,r_m$ for which $S_{c_1,\ldots,c_n}$ is a coarsening of $S_{r_1,\ldots,r_m}$. This number will be even since $j > 0$.

Hence, in case *i*, we showed that $S_{c_1,\ldots,c_n}$ occurs with a coefficient one in (10), whereas in case *ii*, there is an even number of coarsenings $r_1,\ldots,r_m$ of $b_1,\ldots,b_p$ for which $S_{c_1,\ldots,c_n}$ is a summand of $C(S_{r_1,\ldots,r_m})$. Therefore, since we work modulo 2, each such $S_{c_1,\ldots,c_n}$ occurs with coefficient 0 in (10). Therefore the only summand which is not canceled in the sum is $S_{b_1,\ldots,b_p}$.

□

Proof of Theorem 4.1. Applying the function C both sides of (10) completes the proof. $\qquad\square$

In recent literature the dual Leiniz-Hopf algebra is also called the overlapping shuffle algebra [18]. Let $\mid x \mid$ denote the degree of the element x. In [9] it is showed that as a vector space $Ker(\chi_{\mathcal{F}_2^*} - 1) = Im(\chi_{\mathcal{F}_2^*} - 1)$ in even degrees. Here $Ker(\chi_{\mathcal{F}_2^*} - 1)$ represents a subspace of $\mathcal{F}_2^*$ which is formed by conjugation invariants under $\chi_{\mathcal{F}_{2^*}}$. We use this results and give the following results using the multiplicative structure of $\mathcal{F}_2^*$.

Theorem 4.3. *Let* $x, y \in Ker(\chi_{\mathcal{F}_2^*} - 1)$ *then* $xy \in Im(\chi_{\mathcal{F}_2^*} - 1)$.

Proof. We know $Ker(\chi_{\mathcal{F}_2^*} - 1)$ is a sub algebra of $\mathcal{F}_2^*$ with overlapping shuffle product. To prove, suppose $x, y \in Ker(\chi_{\mathcal{F}_2^*} - 1)$ we will consider x, y according to their degrees in the following cases:

   i. If $\mid x \mid, \mid y \mid$ are both odd then $\mid xy \mid$ is even, then $xy \in Im(\chi_{\mathcal{F}_2^*} - 1)$.

   ii. If $\mid x \mid, \mid y \mid$ are both even then $\mid xy \mid$ is even then $xy \in Im(\chi_{\mathcal{F}_2^*} - 1)$.

   iii. If $\mid x \mid$ is odd and $\mid y \mid$ is even, then by [9, Theorem 2.7] $y \in Im(\chi_{\mathcal{F}_2^*} - 1)$. Thus there is a $z \in \mathcal{F}_2^*$ such that $y = (\chi_{\mathcal{F}_2^*} - 1)(z)$. So $(\chi_{\mathcal{F}_2^*} - 1)(xz) = \chi_{\mathcal{F}_2^*}(xz) - xz = \chi_{\mathcal{F}_2^*}(z)\chi_{\mathcal{F}_2^*}(x) - xz$. Since $x \in Ker(\chi_{\mathcal{F}_2^*} - 1)$ then $\chi_{\mathcal{F}_2^*}(x) = x$. Therefore $\chi_{\mathcal{F}_2^*}(z)\chi_{\mathcal{F}_2^*}(x) - xz = \chi_{\mathcal{F}_2^*}(z)x - xz = x(\chi_{\mathcal{F}_2^*}(z) - z) = xy$. Thus $(\chi_{\mathcal{F}_2^*} - 1)(xz) = xy$ which means $xy \in Im(\chi_{\mathcal{F}_2^*} - 1)$.

   iv. If $\mid x \mid$ is even and $\mid y \mid$ is odd then $xy \in Im(\chi_{\mathcal{F}_2^*} - 1))$ because of the same argument with *iii*, but just the degrees of x, y are different.

By i,ii,iii and iv $x, y \in Ker(\chi_{\mathcal{F}_{2^*}} - 1)$ then $x.y \in Im(\chi_{\mathcal{F}_2^*} - 1)$. $\qquad\square$

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