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SUPERCLOSE ANALYSIS OF A TWO-GRID FINITE ELEMENT SCHEME FOR SEMILINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

Changling Xu and Tianliang Hou*

School of Mathematics and Statistics, Beihua University
Jilin 132013, Jilin, China

Abstract. In this paper, a two-grid finite element scheme for semilinear parabolic integro-differential equations is proposed. In the two-grid scheme, continuous linear element is used for spatial discretization, while Crank-Nicolson scheme and Leap-Frog scheme are ultilized for temporal discretization. Based on the combination of the interpolation and Ritz projection technique, some superclose estimates between the interpolation and the numerical solution in the H^1-norm are derived. Notice that we only need to solve nonlinear problem once in the two-grid scheme, namely, the first time step on the coarse-grid space. A numerical example is presented to verify the effectiveness of the proposed two-grid scheme.

1. **Introduction.** In this paper, we consider the following semilinear parabolic integro-differential equations:

$$u_t - \triangle u + \int_0^t \triangle u(s)ds = f(u),\ X \in \Omega,\ t \in J, \tag{1}$$

$$u(X,t) = 0,\ X \in \partial\Omega,\ t \in J, \tag{2}$$

$$u(X,0) = u_0(X),\ X \in \Omega, \tag{3}$$

where $\Omega \subset \mathbf{R}^2$ is a rectangle with boundary $\partial\Omega$, $X = (x,y)$, $J = (0,T]$, $f(\cdot)$ is twice continuously differentiable and u_0 is a given function. We assume that

$$|f'(y)| + |f''(y)| \le M,\ y \in \mathbf{R}.$$

There exists a lot of numerical methods for solving nonlinear partial differential equations in the literature. For example, Cannon and Lin [1] derived a priori error estimates of semidiscrete and Crank-Nicolson finite element approximations to the solution of the nonlinear diffusion equations with memory. Eriksson and Johnson [3] used adaptive finite element method to solve nonlinear parabolic problems. Moore [9] considered a posteriori error estimates for semi- and fully discrete finite element methods using a degree polynomial basis for solving nonlinear parabolic equations. Garcia [4] discussed a priori error estimates of fully discrete Raviart-Thomas mixed finite element scheme for nonlinear parabolic equations.

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* Corresponding author.

Two-grid method was first proposed by Xu [11, 12] as an efficient discretization technique for solving the nonlinear and the nonsymmetric problems. Liu et al. exploited the two kinds of two-grid algorithms for finite difference solutions of semilinear parabolic equations in [8]. Chen et al. [2] presented a two-grid scheme of mixed finite element method for fully nonlinear reaction-diffusion equations. Hou et al. [5] presented a two-grid method of P_0^2-P_1 mixed finite element method combined with Crank-Nicolson scheme for a class of nonlinear parabolic equations. Shi and Mu [10] discussed some superclose results of a two-grid finite element method for semilinear parabolic equations. Yang and Xing [13] discussed the convergence of two-grid discontinuous Galerkin scheme for a kind of nonlinear parabolic problems.

This paper is motivated by the ideas of the works [10, 11], we present a two-grid scheme for semilinear parabolic integro-differential equations discretized by finite element method combined with Crank-Nicolson scheme. We mainly discuss the superclose estimates between the numerical solution and the interpolation.

The plan of this paper is as follows. In Section 2, we give the Crank-Nicolson scheme and deduce the superclose result of order $\mathcal{O}(h^2+(\triangle t)^2)$ in the H^1-norm. In Section 3, we present the two-grid method and derive the superclose estimates of order $\mathcal{O}(H^2+(\triangle t)^2)$ and order $\mathcal{O}(h^2+H^4+(\triangle t)^2)$, respectively. In Section 4, we present a numerical example to demonstrate the effectiveness of our method.

2. **Superclose analysis of the Crank-Nicolson scheme.** We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p=\sum\limits_{|\alpha|\le m}\|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p=\sum\limits_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega): v|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$.

We denote by $L^s(J;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))}=\left(\int_0^T\|v\|_{W^{m,p}(\Omega)}^s dt\right)^{\frac{1}{s}}$ for $s\in[1,\infty)$, and the standard modification for $s=\infty$. For simplicity of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^1(J;W^{m,p}(\Omega))$. In addition C denotes a general positive constant independent of h and $\triangle t$, where h is the spatial mesh-size, and $\triangle t$ is the time step.

Let $\mathcal{T}_h$ be a uniform rectangular partition of Ω with mesh size h. V_h be the bilinear finite element space with vanishes on $\partial\Omega$. Let I_h and R_h be the associated interpolation and Ritz projection operators on V_h, respectively(see[10]).

Then, for $u\in H_0^1(\Omega)\cap H^3(\Omega)$, from [10] we know that

$$\|u-R_hu\|+h\|\nabla(u-R_hu)\|\le Ch^2\|u\|_2, \tag{4}$$

$$\|I_hu-R_hu\|_1\le Ch^2\|u\|_3, \tag{5}$$

$$(\nabla(u-I_hu),\nabla v_h)\le Ch^2\|u\|_3\|\nabla v_h\|, \tag{6}$$

$$(\nabla(u-R_hu),\nabla v_h)=0,\ \forall\ v_h\in V_h. \tag{7}$$

The weak formulation of (1) is to find $u: J\to H_0^1(\Omega)$, such that

$$(u_t,v)+(\nabla u,\nabla v)-\int_0^t(\nabla u(s),\nabla v)ds=(f(u),v),\ \forall\ v\in H_0^1(\Omega). \tag{8}$$

Let $\{t_n|t_n=n\triangle t; 0\le n\le N\}$ be a uniform partition in time with time step $\triangle t$, $u^n=u(X,t_n)$ and $t_{n-1/2}=(t_{n-1}+t_n)/2$. For a sequence of functions $\{\phi^n\}_{n=0}^N$, we denote $d_t\phi^n = (\phi^n - \phi^{n-1})/\triangle t$, then the Crank-Nicolson scheme of (1) is to find $u_h^n \in V_h$ for $n = 1, 2, ..., N$, such that

$$(d_t u_h^n, v_h) - \left(\sum_{j=0}^{n-2} \frac{\triangle t}{2} \nabla(u_h^j + u_h^{j+1}) + \frac{\triangle t}{8}\left(3\nabla u_h^{n-1} + \nabla u_h^n\right), \nabla v_h\right) = -\left(\frac{\nabla(u_h^n + u_h^{n-1})}{2}, \nabla v_h\right) + \left(\frac{f(u_h^n) + f(u_h^{n-1})}{2}, v_h\right), \ \forall\ v_h \in V_h, \tag{9}$$

$$u_h^0 = R_h u_0(X), \ X \in \Omega. \tag{10}$$

For the proof of existence and uniqueness of the solution for the nonlinear algebraic problem (9)-(10), please refer to [6].

Theorem 2.1. *Let u and u_h^n be the solutions of (8) and (9), respectively. Assume that $u \in L^\infty(H^3)$, $u_t \in L^2(H^2)$, $u_{tt} \in L^2(L^2)$, $u_{ttt} \in L^2(L^2)$, $\nabla u_{tt} \in L^2(H^2) \cap L^\infty(L^2)$ and $\nabla u_{ttt} \in L^2(L^2)$, then, for $n = 1, 2, \ldots, N$, we have*

$$\|u_h^n - I_h u^n\|_1 \le C(h^2 + (\triangle t)^2). \tag{11}$$

Proof. Letting $t = t_{n-1/2}$ and $v = v_h$ in (8), we get

$$(d_t u^n, v_h) + \left(\frac{\nabla(u^n + u^{n-1})}{2}, \nabla v_h\right) - \int_0^{t_{n-1/2}} (\nabla u(s), \nabla v_h)\, ds = (f(u^{n-1/2}), v_h) + (R_1^n, v_h) + (\nabla R_2^n, \nabla v_h), \ \forall\ v_h \in V_h, \tag{12}$$

where $R_1^n = d_t u^n - u_t^{n-1/2}$, $R_2^n = \frac{u^n + u^{n-1}}{2} - u^{n-1/2}$.

Setting $u^n - u_h^n = u^n - R_h u^n + R_h u^n - u_h^n := \eta^n + \xi^n$. Subtracting (9) from (12), with the help of (7), we have

$$\begin{aligned}(d_t\xi^n, v_h) &+ \left(\frac{\nabla(\xi^n + \xi^{n-1})}{2}, \nabla v_h\right) \\ = &- (d_t\eta^n, v_h) + \left(\int_0^{t_{n-1/2}} \nabla u(s) ds - \sum_{j=0}^{n-2} \frac{\triangle t}{2} \nabla(u_h^j + u_h^{j+1})\right. \\ &\left. - \frac{\triangle t}{8}\left(3\nabla u_h^{n-1} + \nabla u_h^n\right), \nabla v_h\right) + (R_1^n, v_h) + (\nabla R_2^n, \nabla v_h) \\ &+ \left(f(u^{n-1/2}) - \frac{f(u_h^n) + f(u_h^{n-1})}{2}, v_h\right), \ \forall\ v_h \in V_h.\end{aligned} \tag{13}$$

Selecting $v_h = d_t\xi^n$ in (13), noting that

$$\left(\frac{\nabla(\xi^n + \xi^{n-1})}{2}, d_t\nabla\xi^n\right) = \frac{1}{2\triangle t}(\|\nabla\xi^n\|^2 - \|\nabla\xi^{n-1}\|^2), \tag{14}$$

then multiplying (13) by $2\triangle t$ and summing from $n = 1, ..., l (1 \le l \le N)$, we conclude that

$$\begin{aligned}2\sum_{n=1}^{l} \|d_t\xi^n\|^2 \triangle t + \|\nabla\xi^l\|^2 = &-2\sum_{n=1}^{l} (d_t\eta^n, d_t\xi^n)\triangle t \\ &+ 2\sum_{n=1}^{l}\left(\int_0^{t_{n-1/2}} \nabla u(s) ds - \sum_{j=0}^{n-2} \frac{\triangle t}{2} \nabla(u^j + u^{j+1})\right.\end{aligned}$$

$$
\begin{aligned}
&-\frac{\triangle t}{4}\nabla(u^{n-1}+u^{n-1/2}),\nabla d_t\xi^n\Bigg)\triangle t+2\sum_{n=1}^{l}\left(\sum_{j=0}^{n-2}\frac{\triangle t}{2}\nabla(\xi^j+\xi^{j+1})\right.\\
&+\frac{\triangle t}{4}\nabla\left(\xi^{n-1}+R_hu^{n-1/2}-\frac{u_h^n+u_h^{n-1}}{2}\right),\nabla d_t\xi^n\Bigg)\triangle t\\
&+2\sum_{n=1}^{l}\left(f(u^{n-1/2})-\frac{f(u_h^n)+f(u_h^{n-1})}{2},d_t\xi^n\right)\triangle t+2\sum_{n=1}^{l}(R_1^n,d_t\xi^n)\triangle t\\
&+2\sum_{n=1}^{l}(\nabla R_2^n,\nabla\xi^n-\nabla\xi^{n-1})=:\sum_{i=1}^{6}I_i,
\end{aligned}
\tag{15}
$$

where we used (7) and $\xi^0=0$.

Now, we estimate the right-hand terms of (15). For I_1, it is easy to check that

$$\|d_t\eta^n\|^2\le\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds,$$

which together with Cauchy inequality, Young's inequality and (4) yields

$$I_1\le Ch^4\|u_t\|^2_{L^2(H^2)}+\frac{1}{8}\sum_{n=1}^{l}\|d_t\xi^n\|^2\triangle t. \tag{16}$$

For I_2, we decompose it as

$$
\begin{aligned}
I_2=&2\sum_{n=1}^{l}\left(\sum_{j=0}^{n-2}\int_{t_j}^{t_{j+1}}\left(\nabla u(s)-\frac{\nabla(u^j+u^{j+1})}{2}\right)ds\right.\\
&\left.+\int_{t_{n-1}}^{t_{n-1/2}}\left(\nabla u(s)-\frac{\nabla(u^{n-1}+u^{n-1/2})}{2}\right)ds,\nabla\xi^n-\nabla\xi^{n-1}\right)\\
=&2\sum_{n=2}^{l}\left(\sum_{j=0}^{n-2}\int_{t_j}^{t_{j+1}}\left(\nabla u(s)-\frac{\nabla(u^j+u^{j+1})}{2}\right)ds,\nabla\xi^n-\nabla\xi^{n-1}\right)\\
&+2\sum_{n=2}^{l}\left(\int_{t_{n-1}}^{t_{n-1/2}}\left(\nabla u(s)-\frac{\nabla(u^{n-1}+u^{n-1/2})}{2}\right)ds,\nabla\xi^n-\nabla\xi^{n-1}\right)\\
&+2\left(\int_0^{t_{1/2}}\left(\nabla u(s)-\frac{\nabla(u^0+u^{1/2})}{2}\right)ds,\nabla\xi^1\right)=:\sum_{i=1}^{3}A_i.
\end{aligned}
\tag{17}
$$

Using Cauchy inequality and Young's inequality, we see that

$$
\begin{aligned}
A_1=&2\left(\sum_{j=0}^{l-2}\int_{t_j}^{t_{j+1}}\left(\nabla u(s)-\frac{\nabla(u^j+u^{j+1})}{2}\right)ds,\nabla\xi^l\right)\\
&-2\left(\int_0^{t_1}\left(\nabla u(s)-\frac{\nabla(u^0+u^1)}{2}\right)ds,\nabla\xi^1\right)\\
&-2\sum_{n=2}^{l-1}\left(\int_{t_{n-1}}^{t_n}\left(\nabla u(s)-\frac{\nabla(u^n+u^{n-1})}{2}\right)ds,\nabla\xi^n\right)\\
\le&C(\triangle t)^4\|\nabla u_{tt}\|^2_{L^2(L^2)}+C\sum_{n=1}^{l}\|\nabla\xi^n\|^2\triangle t+\frac{1}{8}\|\nabla\xi^l\|^2,
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
A_2 =&2\left(\int_{t_{l-1}}^{t_{l-1/2}}\left(\nabla u(s)-\frac{\nabla(u^{l-1}+u^{l-1/2})}{2}\right)ds,\nabla\xi^l\right)\\
&-2\left(\int_{t_1}^{t_{3/2}}\left(\nabla u(s)-\frac{\nabla(u^1+u^{3/2})}{2}\right)ds,\nabla\xi^1\right)\\
&-2\sum_{n=2}^{l-1}\left(\int_{t_n}^{t_{n+1/2}}\left(\nabla u(s)-\frac{\nabla(u^n+u^{n+1/2})}{2}\right)ds\right.\\
&\left.-\int_{t_{n-1}}^{t_{n-1/2}}\left(\nabla u(s)-\frac{\nabla(u^{n-1}+u^{n-1/2})}{2}\right)ds,\nabla\xi^n\right)\\
\le& C(\triangle t)^4\|\nabla u_{tt}\|^2_{L^2(L^2)}+C\sum_{n=1}^{l}\|\nabla\xi^n\|^2\triangle t+\frac{1}{8}\|\nabla\xi^l\|^2,
\end{aligned}
\tag{19}
$$

and

$$
A_3\le C(\triangle t)^4\|\nabla u_{tt}\|^2_{L^2(L^2)}+C\|\nabla\xi^1\|^2\triangle t. \tag{20}
$$

Thus, we get

$$
I_2\le C(\triangle t)^4\|\nabla u_{tt}\|^2_{L^2(L^2)}+C\sum_{n=1}^{l}\|\nabla\xi^n\|^2\triangle t+\frac{1}{4}\|\nabla\xi^l\|^2. \tag{21}
$$

For I_3, by virtue of Cauchy inequality and Young's inequality, we have

$$
\begin{aligned}
I_3 =&2\sum_{n=2}^{l}\left(\sum_{j=0}^{n-2}\frac{\triangle t}{2}\left(\nabla\xi^j+\nabla\xi^{j+1}\right),\nabla\xi^n-\nabla\xi^{n-1}\right)\\
&+\sum_{n=1}^{l}\left(\frac{\triangle t}{2}\left(\nabla\xi^{n-1}+\nabla\left(R_hu^{n-1/2}-\frac{u_h^n+u_h^{n-1}}{2}\right)\right),\nabla\xi^n-\nabla\xi^{n-1}\right)\\
=&2\left(\sum_{j=0}^{l-2}\frac{\triangle t}{2}\left(\nabla\xi^j+\nabla\xi^{j+1}\right),\nabla\xi^l\right)-2\left(\sum_{n=1}^{l-1}\frac{\triangle t}{2}\left(\nabla\xi^{n-1}+\nabla\xi^n\right),\nabla\xi^n\right)\\
&+2\left(\frac{\triangle t}{4}\left(\nabla\xi^{l-1}+\nabla\left(R_hu^{l-1/2}-\frac{u_h^l+u_h^{l-1}}{2}\right)\right),\nabla\xi^l\right)\\
&-2\sum_{n=1}^{l-1}\left(\frac{\triangle t}{4}\left(\nabla\xi^n+\nabla\left(R_hu^{n+1/2}-\frac{u_h^{n+1}+u_h^n}{2}\right)\right)\right.\\
&\left.-\frac{\triangle t}{4}\left(\nabla\xi^{n-1}+\nabla\left(R_hu^{n-1/2}-\frac{u_h^n+u_h^{n-1}}{2}\right)\right),\nabla\xi^n\right)\\
\le& C(\triangle t)^4\left\|\nabla u_{tt}\right\|^2_{L^2(H^3)}+C\sum_{n=1}^{l}\left\|\nabla\xi^n\right\|^2\triangle t,
\end{aligned}
\tag{22}
$$

where we used the following estimate

$$
\begin{aligned}
&\left\|\nabla\left(R_hu^{n-1/2}-\frac{u_h^n+u_h^{n-1}}{2}\right)\right\|\\
=&\left\|\nabla\left(R_hu^{n-1/2}-\frac{R_hu^{n-1}+R_hu^n}{2}+\frac{R_hu^{n-1}+R_hu^n}{2}-\frac{u_h^n+u_h^{n-1}}{2}\right)\right\|
\end{aligned}
$$

$$\begin{aligned}
&=\frac{1}{2}\|\nabla(R_hu^{n-1}-R_hu^{n-1/2})+\nabla(R_hu^n-R_hu^{n-1/2})+\nabla\xi^{n-1}+\nabla\xi^n\|\\
&\le C\|\nabla R_hu_t(\phi^n)(t_n-t_{n-1/2})+\nabla R_hu_t(\psi^n)(t_{n-1}-t_{n-1/2})\|+C\|\nabla\xi^n\|\\
&\quad+C\|\nabla\xi^{n-1}\|\\
&\le C\|\triangle t\nabla R_hu_{tt}(\lambda^n)(\phi^n-\psi^n)\|+C\|\nabla\xi^n\|+C\|\nabla\xi^{n-1}\|\\
&\le C(\triangle t)^2\|\nabla R_hu_{tt}(\lambda^n)\|+C\|\nabla\xi^n\|+C\|\nabla\xi^{n-1}\|\\
&\le C(\triangle t)^2(\|\nabla u_{tt}(\lambda^n)\|+\|\nabla u_{tt}(\lambda^n)-\nabla R_hu_{tt}(\lambda^n)\|)+C\|\nabla\xi^n\|+C\|\nabla\xi^{n-1}\|\\
&\le C(\triangle t)^2\|\nabla u_{tt}(\lambda^n)\|_2+C\|\nabla\xi^n\|+C\|\nabla\xi^{n-1}\|,
\end{aligned}\tag{23}$$

where ϕ^n is located between t_n and $t_{n-1/2}$, ψ^n is located between $t_{n-1/2}$ and t_{n-1}, λ^n is located between ϕ^n and ψ^n, and

$$|\phi^n-\psi^n|\le|t_n-t_{n-1}|=\triangle t.$$

Next, we estimate I_4. By use of mean value theorem and the assumption on f, we conclude that

$$\begin{aligned}
&\left\|f(u^{n-1/2})-\frac{f(u^n)+f(u^{n-1})}{2}\right\|\\
&\le\frac{1}{2}\|f'(\lambda_2^n)(u^{n-1/2}-u^n)+f'(\lambda_1^n)(u^{n-1/2}-u^{n-1})\|\\
&=\frac{\triangle t}{4}\|f'(\lambda_1^n)u_t(\theta_1^n)-f'(\lambda_2^n)u_t(\theta_2^n)\|\\
&=\frac{\triangle t}{4}\|f'(\lambda_1^n)u_t(\theta_1^n)-f'(\lambda_1^n)u_t(\theta_2^n)+f'(\lambda_1^n)u_t(\theta_2^n)-f'(\lambda_2^n)u_t(\theta_2^n)\|\\
&=\frac{\triangle t}{4}\|f'(\lambda_1^n)u_{tt}(\theta_3^n)(\theta_2^n-\theta_1^n)+f''(\lambda_3^n)(\lambda_1^n-\lambda_2^n)u_t(\theta_2^n)\|\\
&\le C(\triangle t)^2\left(\|u_{tt}(\theta_3^n)\|+\|u_t(\theta_2^n)\|\right),
\end{aligned}\tag{24}$$

where we also used

$$\theta_2^n-\theta_1^n\le t_n-t_{n-1}=\triangle t$$

and

$$\begin{aligned}
|\lambda_2^n-\lambda_1^n|&\le|u^n-\lambda_1^n|+|u^{n-1/2}-\lambda_1^n|\\
&\le|u^n-u^{n-1}|+|u^n-u^{n-1/2}|+|u^{n-1/2}-u^{n-1}|\\
&\le 2(|u^n-u^{n-1/2}|+|u^{n-1/2}-u^{n-1}|)\\
&=2(|u_t(\theta_1^n)|+|u_t(\theta_2^n)|)\Delta t,
\end{aligned}$$

where θ_1^n is located between t_{n-1} and $t_{n-1/2}$, θ_2^n is located between $t_{n-1/2}$ and t_n, θ_3^n is located between θ_1^n and θ_2^n, λ_1^n is located between u^{n-1} and $u^{n-1/2}$, λ_2^n is located between $u^{n-1/2}$ and u^n, λ_3^n is located between λ_1^n and λ_2^n.

Using (4), mean value theorem and the assumption on f, we easily get

$$\begin{aligned}
&\left\|\frac{f(u^n)+f(u^{n-1})}{2}-\frac{f(u_h^n)+f(u_h^{n-1})}{2}\right\|\\
&\le C(\|u^n-u_h^n\|+\|u^{n-1}-u_h^{n-1}\|)\\
&\le Ch^2(\|u^n\|_2+\|u^{n-1}\|_2)+C(\|\xi^n\|+\|\xi^{n-1}\|).
\end{aligned}\tag{25}$$

By use of Cauchy inequality, Young's inequality and (24)-(25), we derive

$$
\begin{aligned}
I_4 \leq & C(\triangle t)^4(\|u_{tt}\|^2_{L^2(L^2)} + \|u_t\|^2_{L^2(L^2)}) + Ch^4\|u\|^2_{L^2(H^2)} \\
& + C\sum_{n=1}^{l}\|\nabla\xi^n\|^2\triangle t + \frac{1}{8}\sum_{n=1}^{l}\|d_t\xi^n\|^2\triangle t.
\end{aligned} \tag{26}
$$

For I_5, from Cauchy inequality, Young's inequality and

$$
\|R_1^n\| \leq C(\triangle t)^2\|u_{ttt}(\rho^n)\|,
$$

we have

$$
I_5 \leq C(\triangle t)^4\|u_{ttt}\|^2_{L^2(L^2)} + \frac{1}{8}\sum_{n=1}^{l}\|d_t\xi^n\|^2\triangle t, \tag{27}
$$

where ρ^n is located between t_{n-1} and t_n.

Notice that

$$
\sum_{n=1}^{l}(\nabla R_2^n, \nabla d_t\xi^n)\triangle t = (\nabla R_2^l, \nabla\xi^l) - \sum_{n=1}^{l-1}\left(\nabla R_2^{n+1} - \nabla R_2^n, \nabla\xi^n\right). \tag{28}
$$

Using Taylor expansion, we know that

$$
u^n = u^{n+1/2} - u_t^{n+1/2}\frac{\triangle t}{2} + u_{tt}^{n+1/2}\frac{(\triangle t)^2}{8} - \frac{1}{48}u_{ttt}\left(\beta^n\right)(\triangle t)^3, \tag{29}
$$

$$
u^{n+1} = u^{n+1/2} + u_t^{n+1/2}\frac{\triangle t}{2} + u_{tt}^{n+1/2}\frac{(\triangle t)^2}{8} + \frac{1}{48}u_{ttt}\left(\gamma^n\right)(\triangle t)^3, \tag{30}
$$

where $t_n < \beta^n < t_{n+1/2} < \gamma^n < t_{n+1}$.

Using Cauchy inequality, Young's inequality, (29), (30) and mean value theorem, we get

$$
(\nabla R_2^l, \nabla\xi^l) \leq C(\triangle t)^4\|\nabla u_{tt}(\lambda^l)\|^2 + \frac{1}{2}\|\nabla\xi^l\|^2, \tag{31}
$$

$$
\begin{aligned}
&\sum_{n=1}^{l-1}\left(\nabla R_2^{n+1} - \nabla R_2^n, \nabla\xi^n\right) \\
=&\frac{1}{96}\sum_{n=1}^{l-1}\left(\nabla u_{ttt}\left(\gamma^n\right)(\triangle t)^3 - \nabla u_{ttt}\left(\beta^n\right)(\triangle t)^3, \nabla\xi^n\right) \\
&- \frac{1}{96}\sum_{n=1}^{l-1}\left(\nabla u_{ttt}\left(\gamma^{n-1}\right)(\triangle t)^3 - \nabla u_{ttt}\left(\beta^{n-1}\right)(\triangle t)^3, \nabla\xi^n\right) \\
&+ \sum_{n=1}^{l-1}\left(\nabla\left(u_{tt}^{n+1/2} - u_{tt}^{n-1/2}\right)\frac{(\triangle t)^2}{8}, \nabla\xi^n\right) \\
=&\frac{1}{96}\sum_{n=1}^{l-1}\left(\nabla u_{ttt}\left(\gamma^n\right)(\triangle t)^3 - \nabla u_{ttt}\left(\beta^n\right)(\triangle t)^3, \nabla\xi^n\right) \\
&- \frac{1}{96}\sum_{n=1}^{l-1}\left(\nabla u_{ttt}\left(\gamma^{n-1}\right)(\triangle t)^3 - \nabla u_{ttt}\left(\beta^{n-1}\right)(\triangle t)^3, \nabla\xi^n\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{l-1}\left(\frac{(\triangle t)^2}{8}\nabla u_{ttt}(\delta^n)\triangle t, \nabla\xi^n\right)\\
\le& C(\triangle t)^4\left\|\nabla u_{ttt}\right\|^2_{L^2(L^2)} + C\sum_{n=1}^{l}\left\|\nabla\xi^n\right\|^2\triangle t,
\end{aligned} \tag{32}
$$

where λ^l is located between t_{l-1} and t_l, δ^n is located between $t_{n-1/2}$ and $t_{n+1/2}$.

For I_6, combining Cauchy inequality, Young's inequality, (28) with (31)-(32), we derive

$$
I_6 \le C(\triangle t)^4(\|\nabla u_{ttt}\|^2_{L^2(L^2)} + \|\nabla u_{tt}\|^2_{L^\infty(L^2)}) + C\sum_{n=1}^{l}\|\nabla\xi^n\|^2\triangle t + \frac{1}{2}\|\nabla\xi^l\|^2. \tag{33}
$$

Now, substituting the estimates for I_1-I_6 into (15), then applying discrete Gronwall's lemma, for sufficiently small $\triangle t$, we have

$$
\begin{aligned}
\|\nabla\xi^l\| \le& Ch^2(\|u\|^2_{L^2(H^2)} + \|u_t\|^2_{L^2(H^2)})^{1/2} + C(\triangle t)^2(\|u_t\|^2_{L^2(L^2)} + \|u_{tt}\|^2_{L^2(L^2)}\\
&+ \|u_{ttt}\|^2_{L^2(L^2)} + \|\nabla u_{tt}\|^2_{L^2(H^3)} + \|\nabla u_{tt}\|^2_{L^\infty(L^2)} + \|\nabla u_{ttt}\|^2_{L^2(L^2)})^{1/2},
\end{aligned} \tag{34}
$$

which together with (5), Poincare's inequality and triangle inequality yields (11). We complete the proof of the theorem. □

3. **Superclose analysis of the two-grid scheme.**

In this section, we present the main algorithm of the paper, which has the following two steps:

Step 1. On the coarse grid $\mathcal{T}_H$, compute $u_H^1 \in V_H$ to satisfy the following original nonlinear system:

$$
\begin{aligned}
&(d_t u_H^1, v_H) - \left(\frac{\triangle t}{8}(3\nabla u_H^0 + \nabla u_H^1), \nabla v_H\right)\\
=& -\left(\frac{\nabla(u_H^1 + u_H^0)}{2}, \nabla v_H\right) + \left(\frac{f(u_H^1) + f(u_H^0)}{2}, v_H\right),\ \forall\ v_H \in V_H,
\end{aligned} \tag{35}
$$

$$
u_H^0 = R_H u_0(X),\ \ X \in \Omega. \tag{36}
$$

For $n = 1, ..., N-1$, compute $u_H^{n+1} \in V_H$ to satisfy the following linear system:

$$
\begin{aligned}
&\left(\frac{u_H^{n+1} - u_H^{n-1}}{2\triangle t}, v_H\right) - \left(\sum_{j=1}^{n}\frac{\triangle t}{2}\nabla(u_H^j + u_H^{j-1}), \nabla v_H\right)\\
=& -\left(\frac{\nabla(u_H^{n+1} + u_H^{n-1})}{2}, \nabla v_H\right) + (f(u_H^n), v_H),\ \forall\ v_H \in V_H.
\end{aligned} \tag{37}
$$

Step 2. On the fine grid $\mathcal{T}_h$, for $n = 1, ..., N$, compute $\widetilde{u}_h^n \in V_h$ to satisfy the following linear system:

$$
\begin{aligned}
&(d_t\widetilde{u}_h^n, v_h) - \left(\sum_{j=0}^{n-2}\frac{\triangle t}{2}\nabla(\widetilde{u}_h^j + \widetilde{u}_h^{j+1}) + \frac{\triangle t}{8}(3\nabla\widetilde{u}_h^{n-1} + \nabla\widetilde{u}_h^n), \nabla v_h\right)\\
&+\left(\frac{\nabla(\widetilde{u}_h^n + \widetilde{u}_h^{n-1})}{2}, \nabla v_h\right) = \left(\frac{f(u_H^n) + f'(u_H^n)(\widetilde{u}_h^n - u_H^n)}{2}, v_h\right)\\
&+\left(\frac{f(u_H^{n-1}) + f'(u_H^{n-1})(\widetilde{u}_h^{n-1} - u_H^{n-1})}{2}, v_h\right),\ \forall\ v_h \in V_h,
\end{aligned} \tag{38}
$$

$$
\widetilde{u}_h^0 = R_h u_0(X),\ \ X \in \Omega. \tag{39}
$$

Now, we shall discuss the superclose estimates of the above two-grid algorithm in the following theorem.

Theorem 3.1. *Let u, u_H^n and $\widetilde{u}_h^n$ be the solutions of (8), (35)-(37) and (35)-(39), respectively. Then under the conditions of Theorem 2.1, for $n = 1, 2, \ldots, N$, we have*

$$\|u_H^n - I_H u^n\|_1 \le C(H^2 + (\triangle t)^2), \tag{40}$$

$$\|\widetilde{u}_h^n - I_h u^n\|_1 \le C(h^2 + H^4 + (\triangle t)^2). \tag{41}$$

Proof. Setting $u^n - u_H^n = u^n - R_H u^n + R_H u^n - u_H^n := \omega^n + \varphi^n$. From Theorem 2.1, (40) is obvious for $n = 1$.

For $n = 1, ..., N-1$, letting $t = t_n$ and $v = v_H$ in (8), we get

$$\begin{aligned}\left(\frac{u^{n+1} - u^{n-1}}{2\triangle t}, v_H\right) + \left(\frac{\nabla(u^{n+1} + u^{n-1})}{2}, \nabla v_H\right) - \int_0^{t_n} (\nabla u(s), \nabla v_H)\, ds \\ = (f(u^n), v_H) + (R_3^n, v_H) + (\nabla R_4^n, \nabla v_H),\ \forall\ v_H \in V_H,\end{aligned} \tag{42}$$

where $R_3^n = \frac{u^{n+1}-u^{n-1}}{2\triangle t} - u_t^n$, $R_4^n = \frac{u^{n+1}+u^{n-1}}{2} - u^n$.

Subtracting (37) from (42), with the help of (7), we have

$$\begin{aligned}&\left(\frac{\varphi^{n+1} - \varphi^{n-1}}{2\triangle t}, v_H\right) + \left(\frac{\nabla(\varphi^{n+1} + \varphi^{n-1})}{2}, \nabla v_H\right) \\ =& -\left(\frac{\omega^{n+1} - \omega^{n-1}}{2\triangle t}, v_H\right) + \left(\int_0^{t_n} \nabla u(s) ds - \sum_{j=1}^{n} \frac{\triangle t}{2} \nabla(u_H^j + u_H^{j-1}), \nabla v_H\right) \\ &+ (f(u^n) - f(u_H^n), v_H) + (R_3^n, v_H) + (\nabla R_4^n, \nabla v_H),\ \forall\ v_H \in V_H.\end{aligned} \tag{43}$$

Selecting $v_H = \frac{\varphi^{n+1}-\varphi^{n-1}}{\triangle t}$ in (43), using $\varphi^0 = 0$ and the equality

$$\begin{aligned}&\left(\frac{\nabla(\varphi^{n+1} + \varphi^{n-1})}{2}, \frac{\nabla(\varphi^{n+1} - \varphi^{n-1})}{\triangle t}\right) \\ =&\frac{1}{2\triangle t}(\|\nabla\varphi^{n+1}\|^2 + \|\nabla\varphi^n\|^2 - \|\nabla\varphi^n\|^2 - \|\nabla\varphi^{n-1}\|^2),\end{aligned} \tag{44}$$

then multiplying the resulting equation by $2\triangle t$ and summing from $n = 1, ..., l(1 \le l \le N-1)$, we conclude that

$$\begin{aligned}&\sum_{n=1}^{l} \left\|\frac{\varphi^{n+1} - \varphi^{n-1}}{\triangle t}\right\|^2 \triangle t + \|\nabla\varphi^{l+1}\|^2 + \|\nabla\varphi^l\|^2 \\ =& -\sum_{n=1}^{l} \left(\frac{\omega^{n+1} - \omega^{n-1}}{\triangle t}, \frac{\varphi^{n+1} - \varphi^{n-1}}{\triangle t}\right) \triangle t \\ &+ 2\sum_{n=1}^{l} \left(\int_0^{t_n} \nabla u(s) ds - \sum_{j=1}^{n} \frac{\triangle t}{2} \nabla(u^j + u^{j+1}), \nabla\varphi^{n+1} - \nabla\varphi^{n-1}\right) \\ &+ 2\sum_{n=1}^{l} \left(\sum_{j=1}^{n} \frac{\triangle t}{2} (\nabla\varphi^j + \nabla\varphi^{j-1}), \nabla\varphi^{n+1} - \nabla\varphi^{n-1}\right)\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{n=1}^{l}\left(f(u^n)-f(u_H^n),\frac{\varphi^{n+1}-\varphi^{n-1}}{\triangle t}\right)\triangle t\\
&+2\sum_{n=1}^{l}\left(R_3^n,\frac{\varphi^{n+1}-\varphi^{n-1}}{\triangle t}\right)\triangle t\\
&+2\sum_{n=1}^{l}(\nabla R_4^n,\nabla\varphi^{n+1}-\nabla\varphi^{n-1})+\|\nabla\varphi^1\|^2=:\sum_{i=1}^{7}D_i,
\end{aligned}
\tag{45}
$$

where we used (7) and $\varphi^0=0$.

Similar to the estimates of I_1-I_6, we can estimate D_1-D_6 as

$$D_1\le CH^4\|u_t\|_{L^2(H^2)}^2+\frac{1}{8}\sum_{n=1}^{l}\left\|\frac{\varphi^{n+1}-\varphi^{n-1}}{\triangle t}\right\|^2\triangle t, \tag{46}$$

$$D_2\le C(\triangle t)^4\|\nabla u_{tt}\|_{L^2(L^2)}^2+C\sum_{n=0}^{l}\|\nabla\varphi^{n+1}\|^2\triangle t+\frac{1}{4}\|\nabla\varphi^{l+1}\|^2, \tag{47}$$

$$D_3\le C\sum_{n=0}^{l}\|\nabla\varphi^{n+1}\|^2\triangle t, \tag{48}$$

$$D_4\le CH^4\|u\|_{L^2(H^2)}^2+C\sum_{n=1}^{l}\|\nabla\varphi^{n}\|^2\triangle t+\frac{1}{8}\sum_{n=1}^{l}\left\|\frac{\varphi^{n+1}-\varphi^{n-1}}{\triangle t}\right\|^2\triangle t, \tag{49}$$

$$D_5\le C(\triangle t)^4\|u_{ttt}\|_{L^2(L^2)}^2+\frac{1}{8}\sum_{n=1}^{l}\left\|\frac{\varphi^{n+1}-\varphi^{n-1}}{\triangle t}\right\|^2\triangle t, \tag{50}$$

$$
\begin{aligned}
D_6\le& C(\triangle t)^4(\|\nabla u_{ttt}\|_{L^2(L^2)}^2+\|\nabla u_{tt}\|_{L^\infty(L^2)}^2)\\
&+C\sum_{n=0}^{l}\|\nabla\varphi^{n+1}\|^2\triangle t+\frac{1}{4}\|\nabla\varphi^{l+1}\|^2.
\end{aligned}
\tag{51}
$$

At last, for D_7, using (5), (40) and triangle inequality, we see that

$$D_7\le 2\|\nabla(I_Hu^1-R_Hu^1)\|^2+2\|\nabla(I_Hu^1-u_H^1)\|^2\le C(H^4+(\triangle t)^4). \tag{52}$$

Now, substituting the estimates for D_1-D_7 into (45), then applying discrete Gronwall's lemma, for sufficiently small $\triangle t$, we have

$$
\begin{aligned}
\|\nabla\varphi^{l+1}\|\le& CH^2(\|u\|_{L^2(H^2)}^2+\|u_t\|_{L^2(H^2)}^2)^{1/2}+C(\triangle t)^2(\|u_{ttt}\|_{L^2(L^2)}^2\\
&+\|\nabla u_{tt}\|_{L^2(H^3)}^2+\|\nabla u_{tt}\|_{L^\infty(L^2)}^2+\|\nabla u_{ttt}\|_{L^2(L^2)}^2)^{1/2},
\end{aligned}
\tag{53}
$$

which together with (5), Poincare's inequality and triangle inequality yields

$$
\begin{aligned}
&\|u_H^n-I_Hu^n\|_1\le\|u_H^n-R_Hu^n\|_1+\|R_Hu^n-I_Hu^n\|_1\\
\le& CH^2(\|u\|_{L^\infty(H^3)}^2+\|u_t\|_{L^2(H^2)}^2)^{1/2}+C(\triangle t)^2(\|u_{ttt}\|_{L^2(L^2)}^2\\
&+\|\nabla u_{tt}\|_{L^\infty(L^2)}^2+\|\nabla u_{tt}\|_{L^2(H^3)}^2+\|\nabla u_{ttt}\|_{L^2(L^2)}^2)^{1/2},\ n\ge 2.
\end{aligned}
\tag{54}
$$

Using Taylor expansion, we have

$$f(u^n)=f\,(u_H^n)+f'(u_H^n)(u^n-u_H^n)+\frac{f''(\alpha^n)(u^n-u_H^n)^2}{2}, \tag{55}$$

where α^n is located between u^n and u_H^n.

Setting $R_h u^n - \widetilde{u}_h^n := \widetilde{\xi}^n$. Subtracting (38) from (12), similar to (15), we conclude that

$$\begin{aligned}
&2\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t + \|\nabla\widetilde{\xi}^l\|^2\\
=&-2\sum_{n=1}^{l}(d_t\eta^n, d_t\widetilde{\xi}^n)\triangle t + 2\sum_{n=1}^{l}\Bigg(\int_0^{t_{n-1/2}}\nabla u(s)ds - \sum_{j=0}^{n-2}\frac{\triangle t}{2}\nabla(\widetilde{u}_h^j + \widetilde{u}_h^{j+1})\\
&-\frac{\triangle t}{8}(3\nabla\widetilde{u}_h^{n-1} + \nabla\widetilde{u}_h^n), \nabla\widetilde{\xi}^n - \nabla\widetilde{\xi}^{n-1}\Bigg)\\
&+2\sum_{n=1}^{l}\left(f(u^{n-1/2}) - \frac{f(u^n)+f(u^{n-1})}{2}, d_t\widetilde{\xi}^n\right)\triangle t\\
&+2\sum_{n=1}^{l}\left(\frac{f'(u_H^n)(u^n - \widetilde{u}_h^n)}{2} + \frac{f''(\alpha^n)(u^n - u_H^n)^2}{4}, d_t\widetilde{\xi}^n\right)\triangle t\\
&+2\sum_{n=1}^{l}\left(\frac{f'(u_H^{n-1})\left(u^{n-1} - \widetilde{u}_h^{n-1}\right)}{2} + \frac{f''(\alpha^{n-1})(u^{n-1} - u_H^{n-1})^2}{4}, d_t\widetilde{\xi}^n\right)\triangle t\\
&+2\sum_{n=1}^{l}(R_1^n, d_t\widetilde{\xi}^n)\triangle t + 2\sum_{n=1}^{l}(\nabla R_2^n, \nabla\widetilde{\xi}^n - \nabla\widetilde{\xi}^{n-1}) =: \sum_{i=1}^{7}B_i.
\end{aligned} \tag{56}$$

Now, we estimate B_1-B_7, respectively. For B_3, similar to (24), we know that

$$B_3 \le C(\triangle t)^4\|u_{tt}\|_{L^2(L^2)}^2 + \frac{1}{8}\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t. \tag{57}$$

For B_4, we find from Cauchy inequality and the assumption on f that

$$\begin{aligned}
B_4 &\le \frac{1}{2}\sum_{n=1}^{l}\left(2\|f'(u_H^n)(u^n - \widetilde{u}_h^n)\| + \|f''(\alpha^n)(u^n - u_H^n)^2\|\right)\cdot\|d_t\widetilde{\xi}^n\|\triangle t\\
&\le C\sum_{n=1}^{l}\|u^n - \widetilde{u}_h^n\|\cdot\|d_t\widetilde{\xi}^n\|\triangle t + C\sum_{n=1}^{l}\|(u^n - u_H^n)^2\|\cdot\|d_t\widetilde{\xi}^n\|\triangle t\\
&=: G_1 + G_2.
\end{aligned} \tag{58}$$

Using Young's inequality and (4), we know that

$$G_1 \le Ch^4\|u\|_{L^2(H^2)}^2 + C\sum_{n=1}^{l}\|\widetilde{\xi}^n\|^2\triangle t + \frac{1}{8}\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t. \tag{59}$$

Combining (40), Cauchy inequality, Young's inequality and interpolation theory with $H^1 \hookrightarrow L^4$, we derive

$$\begin{aligned}
G_2 \le& \sum_{n=1}^{l}(\|u^n - I_H u^n\|_{0,4}^2 + \|I_H u^n - u_H^n\|_{0,4}^2)\|d_t\widetilde{\xi}^n\|\triangle t\\
\le& C\sum_{n=1}^{l}(H^4\|u^n\|_{2,4}^2 + \|I_H u^n - u_H^n\|_1^2)\|d_t\widetilde{\xi}^n\|\triangle t
\end{aligned}$$

$$\leq C((\triangle t)^4 + H^8) + \frac{1}{8}\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t. \tag{60}$$

Now, from (59)-(60), we find that

$$B_4 \leq C\sum_{n=1}^{l}\|\widetilde{\xi}^n\|^2\triangle t + C(h^4 + (\triangle t)^4 + H^8) + \frac{1}{4}\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t. \tag{61}$$

Similarly, we can estimate B_5 as

$$B_5 \leq C\sum_{n=1}^{l}\|\widetilde{\xi}^{n-1}\|^2\triangle t + C(h^4 + (\triangle t)^4 + H^8) + \frac{1}{4}\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t. \tag{62}$$

Similar to (16), (21), (27) and (33), we easily have

$$\begin{aligned}|B_1 + B_2 + B_6 + B_7| \leq& Ch^4\|u_t\|^2_{L^2(H^2)} + C\sum_{n=1}^{l}\|\widetilde{\xi}^n\|^2\triangle t + \frac{1}{2}\|\nabla\widetilde{\xi}^n\|^2\\ &+ \frac{1}{8}\sum_{n=1}^{l}\|d_t\widetilde{\xi}^n\|^2\triangle t + C(\triangle t)^4(\|u_{ttt}\|^2_{L^2(L^2)}\\ &+ \|\nabla u_{tt}\|^2_{L^2(L^2)} + \|\nabla u_{ttt}\|^2_{L^2(L^2)}).\end{aligned} \tag{63}$$

It follows from (56)-(57), (61)-(63) and Poincare's inequality that

$$\|\nabla\widetilde{\xi}^l\|^2 \leq C(h^4 + H^8 + (\triangle t)^4) + C\sum_{n=1}^{l}\|\nabla\widetilde{\xi}^n\|^2\triangle t. \tag{64}$$

Thus, for sufficiently small $\triangle t$, using discrete Gronwall's lemma and Poincare's inequality, we arrive at

$$\|\widetilde{\xi}^l\|_1 \leq C(h^2 + H^4 + (\triangle t)^2), \tag{65}$$

which together with (5) and triangle inequality yields

$$\|\widetilde{u}_h^n - I_hu^n\|_1 \leq \|\widetilde{u}_h^n - R_hu^n\|_1 + \|R_hu^n - I_hu^n\|_1 \leq C(h^2 + H^4 + (\triangle t)^2). \tag{66}$$

The proof is complete. □

4. **Numerical experiments.** In this section, we are going to validate the superclose estimates for two-grid discretization method for semilinear parabolic integro-differential equations by a concrete numerical example.

We consider the following semi-linear parabolic integro-differential equations

$$u_t - \triangle u + \int_0^t \triangle u(s)ds = -u^3 + g(X,t),\ X \in \Omega,\ t \in J, \tag{67}$$

$$u(X,t) = 0,\ X \in \partial\Omega,\ t \in J, \tag{68}$$

$$u(X,0) = u_0(X),\ X \in \Omega, \tag{69}$$

where $\Omega = (0,1)^2$ and $J = (0,1]$. We choose $u(X,t) = \sin(\pi t)\sin(\pi x_1)\sin(\pi x_2)$ as the exact solution. Then, the explicit formulation of $g(X,t)$ is

$$g(X,t) = (\pi\cos(\pi t) + 2\pi^2\sin(\pi t) + 2\pi(\cos(\pi t) - 1))\sin(\pi x_1)\sin(\pi x_2) + (u(X,t))^3.$$

We first test the example for the Crank-Nicolson scheme. The error and the convergence order of $\|u_h^n - I_hu^n\|_1$ at $t = 0.125$ with $h = \Delta t$ are presented in Table 1. Obviously, it is the same with the result in Theorem 2.1. Next, the two-grid scheme is tested. The error and the convergence order of $\|u_H^n - I_Hu^n\|_1$ and

$\|\widetilde{u}_h^n - I_h u^n\|_1$ are provided in Table 2 and Table 3. We find from these two tables that the result coincides with that in Theorem 3.1. Finally, we show the efficiency of the two-grid method by comparing the cpu time in Table 4.

h	$\|u_h^n - I_h u^n\|_1$	order
1/32	9.6461e-04	-
1/64	2.4062e-04	2.00
1/128	6.0130e-05	2.00
1/256	1.5037e-05	2.00

Table 1. The error and the convergence order of $\|u_h^n - I_h u^n\|_1$ at $t = 0.125$ with $h = \Delta t$.

H	$\|u_H^n - I_H u^n\|_1$	order
1/16	1.9302e-02	-
1/32	4.8349e-03	2.00
1/64	1.2096e-03	2.00
1/128	3.0244e-04	2.00

Table 2. The error and the convergence order of $\|u_H^n - I_H u^n\|_1$ at $t = 0.0625$ with $H = \Delta t$.

H	$\|\widetilde{u}_h^n - I_h u^n\|_1$	order
1/2	8.7973e-04	-
1/4	7.1420e-05	3.51
1/8	4.6798e-06	3.91
1/16	2.9328e-07	3.99

Table 3. The error and the convergence order of $\|\widetilde{u}_h^n - I_h u^n\|_1$ at $t = 0.001$ with $\Delta t = 0.0001$ and $h = H^2$.

(H, h)	two-grid time (s)	Crank-Nicolson time (s)
$(1/4, 1/16)$	0.0998	0.1164
$(1/8, 1/64)$	0.9118	1.2019
$(1/16, 1/256)$	13.6126	17.9624

Table 4. The cpu time of two-grid scheme and Crank-Nicolson scheme for each time step ($h = \Delta t$).

REFERENCES

[1] J. R. Cannon and Y. P. Lin, A priori L^2 error estimates for finite-element methods for nonlinear diffusion equations with memory, *SIAM J. Numer. Anal.*, **27** (1990), 595–607.

[2] L. P. Chen and Y. P. Chen, Two-grid method for nonlinear reaction-diffusion equations by mixed finite element methods, *J. Sci. Comput.*, **49** (2011), 383–401.

[3] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems IV: Nonlinear problems, *SIAM J. Numer. Anal.*, **32** (1995), 1729–1749.

[4] S. M. F. Garcia, Improved error estimates for mixed finite element approximations for nonlinear parabolic equations: The discrete-time case, *Numer. Methods Partial Differ. Equ.*, **10** (1994), 149–169.

[5] T. L. Hou, W. Z. Jiang, Y. T. Yang and H. T. Leng, Two-grid P_0^2-P_1 mixed finite element methods combined with Crank-Nicolson scheme for a class of nonlinear parabolic equations, *Appl. Numer. Math.*, **137** (2019), 136–150.

[6] M.-N. Le Roux and V. Thomée, Numerical solution of semilinear integro-differential equations of parabolic type with nonsmooth data, *SIAM J. Numer. Anal.*, **26** (1989), 1291–1309.

[7] Q. Lin and J. Lin, *Finite Element Methods: Accuracy and Improvement*, Science Press, Beijing, 2006.

[8] W. Liu, H. Rui and Y. P. Bao, Two kinds of two-grid algorithms for finite difference solutions of semilinear parabolic equations, *J. Sys. Sci. Math. Sci.*, **30** (2010), 181–190.

[9] P. K. Moore, A posterior error estimation with finite element semi- and fully discrete methods for nonlinear parabolic equations in one space dimension, *SIAM J. Numer. Anal.*, **31** (1994), 149–169.

[10] D. Y. Shi and P. C. Mu, Superconvergence analysis of a two-grid method for semilinear parabolic equations, *Appl. Math. Lett.*, **84** (2018), 34–41.

[11] J. C. Xu, A novel two-grid method for semilinear elliptic equations, *SIAM J. Sci. Comput.*, **15** (1994), 231–237.

[12] J. C. Xu, Two-grid discretization techniques for linear and nonlinear PDEs, *SIAM J. Numer. Anal.*, **33** (1996), 1759–1777.

[13] J. M. Yang and X. Q. Xing, A two-grid discontinuous Galerkin method for a kind of nonlinear parabolic problems, *Appl. Math. Comput.*, **346** (2019), 96–108.

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E-mail address: 386270479@qq.com
E-mail address: htlchb@163.com