QUASINEUTRAL LIMIT FOR THE COMPRESSIBLE TWO-FLUID EULER–MAXWELL EQUATIONS FOR WELL-PREPARED INITIAL DATA

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Abstract. In this paper, we study the quasi-neutral limit for the compressible two-fluid Euler–Maxwell equations for well-prepared initial data. Precisely, we proved the solution of the three-dimensional compressible two-fluid Euler–Maxwell equations converges locally in time to that of the compressible Euler equation as ε tends to zero. This proof is based on the formal asymptotic expansions, the iteration techniques, the vector analysis formulas and the Sobolev energy estimates.

1. **Introduction.** The purpose of this present paper is to investigate the quasi-neutral limit for the two-fluid Euler–Maxwell equations consisting of a set of nonlinear conservation laws for densities and momentums coupled to the Maxwell equations in time $t > 0$ and space $\mathbb{R}^3$, which describes the transport of electrons of charge

$q_e = -1$ and ions of charge $q_i = 1$ without viscosity in a magnetized plasma [3]

$$\begin{cases} \dfrac{\partial n_\nu}{\partial t} + \text{div}(n_\nu u_\nu) = 0, \quad \nu = e, i, & \text{(1a)} \\[2mm] m_\nu\Big[\dfrac{\partial(n_\nu u_\nu)}{\partial t} + \text{div}(n_\nu u_\nu \otimes u_\nu)\Big] + \nabla P_\nu(n_\nu) = q_\nu n_\nu(E + u_\nu \times \tilde{B}), & \text{(1b)} \\[2mm] \partial_t \tilde{B} + \nabla \times E = 0, \quad \epsilon_0 \partial_t E - \mu_0^{-1} \nabla \times \tilde{B} = -(q_i n_i u_i + q_e n_e u_e), & \text{(1c)} \\[2mm] \text{div}\tilde{B} = 0, \quad \epsilon_0 \text{div} E = q_i n_i + q_e n_e, & \text{(1d)} \end{cases}$$

where n_i, u_i (respectively, n_e, u_e) denote the density and velocity of the ions (respectively, electrons), and $E, \tilde{B}$ are the electric field and the magnetic field. The coefficients ϵ_0, μ_0, c are the vacuum permittivity, vacuum permeability and light speed with $\varepsilon_0 \mu_0 c^2 = 1$, and the parameter $\gamma = 1/(\epsilon_0^{\frac{1}{2}} c)$ is usually chosen to be inversely proportional to the light speed c. As in classical fluid dynamics, the pressure functions $P_\nu(n_\nu) = a_\nu^2 n_\nu^{b_\nu}$ $(\nu = i, e)$ are supposed to be smooth and strictly increasing with $a_\nu > 0, b_\nu \geq 1$. Moreover, $q_\nu n_\nu(E + u_\nu \times \tilde{B})$, $q_i n_i u_i + q_e n_e u_e$, $q_i n_i + q_e n_e$ stand for the Lorentz force, the current density and the free charges for the particle, respectively. We introduce the Debye length $\lambda^2 = \frac{\epsilon_0 K_B T_e}{n_0}$, where the physical parameters are the mean density of the plasma $n_0 > 0$, the Boltzmann constant $K_B > 0$ and the temperature of the electron $T_e > 0$. The scaled Debye length is denoted by $\varepsilon^2 = \epsilon_0$.

With these parameters, system (1) under study can be scaled to the following form

$$\begin{cases} \partial_t n_\nu + \nabla \cdot (n_\nu u_\nu) = 0, & \text{(2a)} \\[1mm] m_\nu(\partial_t u_\nu + u_\nu \cdot \nabla u_\nu) + \nabla h_\nu(n_\nu) = q_\nu(E + \gamma u_\nu \times B), & \text{(2b)} \\[1mm] \gamma \partial_t B + \nabla \times E = 0, \quad \gamma \varepsilon^2 \partial_t E - \nabla \times B = -\gamma(n_i u_i - n_e u_e), & \text{(2c)} \\[1mm] \text{div} B = 0, \quad \varepsilon^2 \text{div} E = n_i - n_e, & \text{(2d)} \end{cases}$$

where $B = \frac{\tilde{B}}{\gamma}$.

Usually, the dimensionless parameters γ, ε are small compared with the size of the other quantities as for the physical situation. In quasi-neutral plasma, the Debye length is small compared with the typical length L of the plasma. By taking the limit $\frac{\lambda}{L} \to 0$ formally, we can derive an equilibrium between the positive and negative charges. In the non-relativistic limit, we regard γ as the singular perturbation parameter and let $\gamma \to 0$. For such scales, the plasma can be considered as the compressible Euler–Poisson system. Furthermore, $\gamma = \varepsilon^2 \to 0$ is the combined non-relativistic and quasi-neutral limit, which leads to incompressible (one-fluid) or compressible (two-fluid) Euler equation.

When taking $n_i, u_i = 0$, system (2) is reduced to the unipolar Euler–Maxwell equations, and there have been many interesting results for the well-posedness and asymptotic analysis [11–13, 20]. To list a few, Peng and Wang [13] studied the convergence of Euler–Maxwell equations in three-dimensional case to the e-MHD equations under well prepared initial data in the quasi-neutral limit. This result was then generalized in the quantum counterpart recently [8] and later in [16] for the general initial data. Moreover, in [11], the authors justified rigorously the convergence of Euler–Maxwell equations to compressible Euler–Poisson equations in time intervals independent of γ by an analysis of asymptotic expansions up to first order for general initial data and up to any order for well-prepared initial data. The combined non-relativistic and quasi-neutral limit can be found in [12]. For the

formation of boundary layers, the interested readers can refer to [1, 2, 15, 17, 19] and the references therein for example.

Recently, the two-fluid Euler/Navier-Stokes equations with electromagnetic field become more and more interesting as well as important in fluid dynamics. The local smooth solution was established in [5] since Euler–Maxwell equations are symmetrizable hyperbolic for n^ν in the sense of Friedrichs. The global existence and large time behavior were obtained in [3, 10]. For the asymptotic limits with small parameters, there have been many mathematical investigations for Euler–Maxwell equations, see [14, 21, 22] for example. With boundary effects, one can see [4, 7] and references therein. However, the rigorous study of the quasi-neutral limit for two-fluid Euler–Maxwell equations is also open [14]. The goal of this paper is to consider this problem. For convenience, we assume $\gamma = 1$, which can be chosen independently of the Debye length.

Different from the unipolar case, the formal quasi-neutral limit for the two-fluid Euler–Maxwell equations is the compressible type. Here, we are going to establish the quasi-neutral limit for the two-fluid system (2) under well-prepared initial data, which means the compatibility conditions (9) are satisfied. Based on the asymptotic expansion and the iteration techniques, we proved rigorously the main result stated in Theorem 2.3. Formally, setting $\varepsilon = 0$, we obtain the compressible limit system (5). Note, too, that the displacement current and the charge separation are neglected in (5), which is essential different from the Euler–Maxwell equations which are symmetrizable and hyperbolic. Moreover, the singularities in the coupling electromagnetic field can also not be cancelled by a symmetrizer of hyperbolic systems [6], which leads the straightforward energy method invalid. We solve these difficulties by introducing the general vorticity and vector analysis formulas (see (51) in Lemma 3.3). In the mean time, long-time existence for smooth solutions of the two-fluid Euler–Maxwell equations as $\varepsilon \to 0$ is also obtained provided that the smooth solution of one-fluid Euler equations exists. Indeed, if the initial data are not well-prepared, we cannot obtain the uniform energy estimates because of $u_i^{(0)} - u_e^{(0)} \neq O(\varepsilon)$. Therefore, extending the result to the general case is not so obvious since we not only need to obtain the initial layer corrections but also to construct a new energy method, and thus is also open.

The paper is organized as follows. In Sect. 2, we perform the formal asymptotic analysis and give the main result stated in Theorem 2.3. In Sect. 3, we justify rigorously the uniform (in ε) energy estimates for the error system (22) by the iteration techniques. Finally, we complete the proof of Theorem 2.3 by taking the limit of the sequences satisfying Cauchy's criterion in Sect. 4.

Before proceeding, let us introduce the notations and lemmas which will be frequently used throughout this paper. We denote by $H^s(\mathbb{R}^3)$ the standard Sobolev's space in the whole space $\mathbb{R}^3$, and denote by $\| \cdot \|_{H^s}$ the norm of the Banach space $H^s(\mathbb{R}^3)$. In addition, we denote α as the multi-index, and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Moreover, C is the general constant independent of the Debye length ε.

In the following, we state the basic Moser-type calculus inequalities which will be used widely in the error estimates.

Lemma 1.1. *Let α be any multi-index with $|\alpha| = k$, $k \geq 1$ and $p \in (1, \infty)$. Then there holds*

$$\|\partial^\alpha(fg)\|_{L^p} \leq C\|f\|_{L^{p_1}} \|\partial^\alpha g\|_{L^{p_2}} + C\|\partial^\alpha f\|_{L^{p_3}} \|g\|_{L^{p_4}},$$

$$\|[\partial^\alpha, f]g\|_{L^p} \leq C\|\partial f\|_{L^{p^1}}\|\partial^{\alpha-1}g\|_{L^{p^2}} + C\|\partial^\alpha f\|_{L^{p^3}}\|g\|_{L^{p^4}}, \tag{3}$$

where $f, g \in \mathbb{S}$, the Schwartz class, and $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p^1} + \frac{1}{p^2} = \frac{1}{p^3} + \frac{1}{p^4}.$$

2. **Formal asymptotic analysis and the main result.** We make the following ansatz for $\nu = i, e$ in terms of the Debye length ε to the initial value problem (2)

$$(n_\nu^\varepsilon, u_\nu^\varepsilon, E^\varepsilon, B^\varepsilon) = \sum_{k \geq 0} \varepsilon^{2k}(n_\nu^{(k)}, u_\nu^{(k)}, E^{(k)}, B^{(k)}). \tag{4}$$

Plugging the formal expansion (4) into system (2), we can obtain the following results.

(i)The leading term $(n_i^{(0)}, u_i^{(0)}, n_e^{(0)}, u_e^{(0)}, E^{(0)}, B^{(0)})$ satisfies

$$\begin{cases} \partial_t n^{(0)} + n^{(0)}\mathrm{div}u_\nu^{(0)} + u_\nu^{(0)} \cdot \nabla n^{(0)} = 0, & \text{(5a)} \\ m_\nu(\partial_t u_\nu^{(0)} + u_\nu^{(0)} \cdot \nabla u_\nu^{(0)}) + \nabla h_\nu(n^{(0)}) = q_\nu(E^{(0)} + u_\nu^{(0)} \times B^{(0)}), & \text{(5b)} \\ \nabla \times B^{(0)} = n^{(0)}(u_i^{(0)} - u_e^{(0)}), & \text{(5c)} \\ \partial_t B^{(0)} + \nabla \times E^{(0)} = 0, \ \mathrm{div}B^{(0)} = 0. & \text{(5d)} \end{cases}$$

The local existence of smooth solutions cannot be obtained directly by the result of [5] since the displacement current and the charge separation are neglected in the limit system (5). In order to overcome the difficulty, we introduce the general vorticity $\omega_\nu^{(0)} = \nabla \times (u_\nu^{(0)} + \frac{q_\nu}{m_\nu}A^{(0)})$, where we have used $\mathrm{div}B^{(0)} = 0$, which implies there exists some magnetic potential $A^{(0)}$ such that $\nabla \times A^{(0)} = B^{(0)}$. Hence, we substitute the following equation for (5b)

$$\partial_t \omega_\nu^{(0)} + \nabla \times (u_\nu^{(0)} \times \omega_\nu^{(0)}) = 0, \tag{6}$$

namely,

$$\partial_t \omega_\nu^{(0)} + \omega_\nu^{(0)}\mathrm{div}u_\nu^{(0)} + u_\nu^{(0)} \cdot \nabla\omega_\nu^{(0)} - \omega_\nu^{(0)} \cdot \nabla u_\nu^{(0)} = 0. \tag{7}$$

Taking the inner product of (7) with $\omega_\nu^{(0)}$, we derive

$$\frac{1}{2}\frac{d}{dt}\|\omega_\nu^{(0)}\|_{L^2}^2 \leq C\|u_\nu^{(0)}\|_{H^3}\|\omega_\nu^{(0)}\|_{L^2}^2. \tag{8}$$

We supplement the above limit system (5) with the initial data

$$n_{i0}^{(0)} = n_{e0}^{(0)} = n_0^{(0)}, \omega_{\nu0}^{(0)} = 0, \nabla \times B_0^{(0)} = n_0^{(0)}(u_{i0}^{(0)} - u_{e0}^{(0)}). \tag{9}$$

In view of (8), we get $\omega_\nu^{(0)} = 0$ ($\nu = i, e$) are naturally preserved for all time. Moreover, it follows from (5c) and $\omega_\nu^{(0)} = \nabla \times u_\nu^{(0)} + \frac{q_\nu}{m_\nu}B^{(0)}$ that

$$0 = \nabla \times (\omega_i^{(0)} - \omega_e^{(0)}) = \nabla \times (\nabla \times (u_i^{(0)} - u_e^{(0)})) + \frac{m_i + m_e}{m_i m_e}n^{(0)}(u_i^{(0)} - u_e^{(0)}). \tag{10}$$

Taking the inner product of (10) with $u_i^{(0)} - u_e^{(0)}$, it holds

$$\|\nabla \times (u_i^{(0)} - u_e^{(0)})\|_{L^2}^2 + \frac{m_i + m_e}{m_i m_e}\|\sqrt{n^{(0)}}(u_i^{(0)} - u_e^{(0)})\|_{L^2}^2 = 0,$$

which implies $u_i^{(0)} = u_e^{(0)}$.

Therefore, for well-prepared initial data (9), solutions $(n_\nu^{(0)}, u_\nu^{(0)}, E^{(0)}, B^{(0)})$ $(\nu = i, e)$ to system (5) are the smooth solutions of the following compressible Euler equation

$$\begin{cases} \partial_t n^{(0)} + n^{(0)} \mathrm{div} u^{(0)} + u^{(0)} \cdot \nabla n^{(0)} = 0, & \text{(11a)} \\ \partial_t u^{(0)} + u^{(0)} \cdot \nabla u^{(0)} + \nabla h_0(n^{(0)}) = 0, & \text{(11b)} \\ E^{(0)} = \nabla \psi, & \text{(11c)} \end{cases}$$

where

$$\psi = \frac{m_i m_e}{m_i + m_e} \Big(\frac{h_i(n^{(0)})}{m_i} - \frac{h_e(n^{(0)})}{m_e} \Big), \tag{12}$$

and

$$h_0(n^{(0)}) = \frac{h_i(n^{(0)}) + h_e(n^{(0)})}{m_i + m_e}. \tag{13}$$

Theorem 2.1. *Let* $\tilde{s} \geq 3$, *and* $(n_0^{(0)}, u_0^{(0)}) \in H^{\tilde{s}}$ *be any given initial data satisfying* $n_0^{(0)} > 0$. *Then there exists some* $0 < T_* \leq +\infty$, *the maximal time of existence, such that the initial value problem* (11) *has a unique solution such that, for any* $T_0 < T_*$,

$$(n^{(0)}, u^{(0)}) \in L^\infty(0, T_0; H^{\tilde{s}+3}), E^{(0)} \in L^\infty(0, T_0; H^{\tilde{s}+2}).$$

(ii)For any $j \geq 1$, provided that we have proved the profiles

$$(n_\nu^{(k)}, u_\nu^{(k)}, E^{(k)}, B^{(k)})_{0 \leq k \leq j-1}$$

are smooth as much as we want in previous steps, we can get the following linear system satisfied by $(n_\nu^{(j)}, u_\nu^{(j)}, E^{(j)}, B^{(j)})$ $(\nu = i, e)$

$$\begin{cases} \partial_t n_\nu^{(j)} + n^{(0)} \mathrm{div} u_\nu^{(j)} + n_\nu^{(j)} \mathrm{div} u^{(0)} + u^{(0)} \cdot \nabla n_\nu^{(j)} + u_\nu^{(j)} \cdot \nabla n^{(0)} = f_\nu^1, & \text{(14a)} \\ m_\nu(\partial_t u_\nu^{(j)} + u^{(0)} \cdot \nabla u_\nu^{(j)} + u_\nu^{(j)} \cdot \nabla u^{(0)}) + \nabla(h_\nu(n^{(0)})n_\nu^{(j)}) \\ \quad = q_\nu(E^{(j)} + u^{(0)} \times B^{(j)}) + f_\nu^2, & \text{(14b)} \\ \nabla \times B^{(j)} = \partial_t E^{(j-1)} + n^{(0)}(u_i^{(j)} - u_e^{(j)}) + u^{(0)}(n_i^{(j)} - n_e^{(j)}) + f^3, & \text{(14c)} \\ \partial_t B^{(j)} + \nabla \times E^{(j)} = 0, & \text{(14d)} \\ n_i^{(j)} - n_e^{(j)} = \mathrm{div} E^{(j-1)}, & \text{(14e)} \end{cases}$$

where

$$\begin{cases} f_\nu^1 = -\sum_{k=1}^{j-1}(n_\nu^{(k)} \mathrm{div} u_\nu^{(j-k)} + u_\nu^{(k)} \cdot \nabla n_\nu^{(j-k)}), & \text{(15a)} \\ f_\nu^2 = -m_\nu \sum_{k=1}^{j-1}(u_\nu^{(k)} \cdot \nabla u_\nu^{(j-k)}) - \nabla(h_\nu^{j-1}(n_\nu^{(k)})_{k \leq j-1}) \\ \quad + q_\nu \sum_{k=1}^{j-1}(u_\nu^{(k)} \times B^{(j-k)}), & \text{(15b)} \\ f^3 = \sum_{k=1}^{j-1} n_\nu^{(k)}(u_i^{(j-k)} - u_e^{(j-k)}). & \text{(15c)} \end{cases}$$

Here, we have used the following relation

$$
\begin{aligned}
h_\nu(n^{(0)} + \sum_{j\geq 1}\varepsilon^{2j}n_\nu^{(j)}) =& h_\nu(n^{(0)}) + h_\nu'(n^{(0)})\sum_{j\geq 1}(\varepsilon^{2j}n_\nu^{(j)}) \\
& + \sum_{j\geq 2}\varepsilon^{2j}h_\nu^{j-1}\big((n_\nu^{(k)})_{k\leq j-1}\big),
\end{aligned}
\tag{16}
$$

where

$$
h_\nu^{j-1}\big((n_\nu^{(k)})_{k\leq j-1}\big) = \frac{1}{j!}\frac{d}{d\varepsilon^{2j}}h_\nu(n^{(0)} + \sum_{j\geq 1}\varepsilon^{2j}n_\nu^{(j)})|_{\varepsilon=0} - h_\nu'(n^{(0)})n_\nu^{(j)}.
\tag{17}
$$

Obviously, the profile (f_ν^1, f_ν^2, f^3) only depends on the known terms by the previous steps.

Letting $(n_-^{(j)}, u_-^{(j)}) = (n_i^{(j)}, u_i^{(j)}) - (n_e^{(j)}, u_e^{(j)})$, we rewrite the induction system (14) as the following form

$$
\begin{cases}
\partial_t\omega_-^{(j)} + \omega_-^{(j)}\mathrm{div}u^{(0)} + u^{(0)}\cdot\nabla\omega_-^{(j)} - \omega_-^{(j)}\cdot\nabla u^{(0)} = \nabla\times(f_i^2 - f_e^2), & \text{(18a)} \\
\nabla\times\omega_-^{(j)} = \nabla\mathrm{div}u_-^{(j)} - \Delta u_-^{(j)} + (\frac{1}{m_i} + \frac{1}{m_e})\big(n^{(0)}u_-^{(j)} + \mathrm{div}E^{(j-1)}u^{(0)} \\
\qquad + \partial_t E^{(j-1)} + f^3\big), & \text{(18b)}
\end{cases}
$$

where $\omega_-^j = \nabla\times u_-^{(j)} + (\frac{1}{m_i} + \frac{1}{m_e})B^{(j)}$. Existence of solutions to system (18) has been derived by [8]. Moreover, we need the following compatibility conditions

$$
n_{i,0}^{(j)} - n_{e,0}^{(j)} = \mathrm{div}E_0^{(j-1)}, \quad \mathrm{div}B_0^{(j)} = 0.
\tag{19}
$$

Letting $(n_+^{(j)}, u_+^{(j)}) = (n_i^{(j)}, u_i^{(j)}) + (n_e^{(j)}, u_e^{(j)})$, we can rewrite the induction system (14) as

$$
\begin{cases}
\partial_t n_+^{(j)} + n^{(0)}\mathrm{div}u_+^{(j)} + n_+^{(j)}\mathrm{div}u^{(0)} + u^{(0)}\cdot\nabla n_+^{(j)} + u_+^{(j)}\cdot\nabla n^{(0)} \\
\quad = f_i^1 + f_e^1, & \text{(20a)} \\
(m_i + m_e)(\partial_t u_+^{(j)} + u^{(0)}\cdot\nabla u_+^{(j)} + u_+^{(j)}\cdot\nabla u^{(0)}) \\
\quad + \nabla\big((h_i(n^{(0)}) + h_e(n^{(0)}))n_+^{(j)}\big) = (m_i - m_e)(\partial_t u_-^{(j)} + u^{(0)}\cdot\nabla u_-^{(j)} \\
\quad + u_-^{(j)}\cdot\nabla u^{(0)}) + \nabla\big((h_i(n^{(0)}) - h_e(n^{(0)}))\mathrm{div}E^{(j-1)}\big) + f_i^2 + f_e^2. & \text{(20b)}
\end{cases}
$$

Then $(n_+^{(j)}, u_+^{(j)})$ $(j\geq 1)$ are the solutions to the linear nonhomogeneous compressible Euler equations [5]. Finally, combining the definition $(n_-^{(j)}, u_-^{(j)}) = (n_i^{(j)}, u_i^{(j)}) - (n_e^{(j)}, u_e^{(j)})$ and the equation (14e), we derive the following result

Theorem 2.2. *Let $\tilde{s}\geq 3$, and $(n_{\nu,0}^{(j)}, u_{\nu,0}^{(j)}, E_0^{(j)}, B_0^{(j)})$ $(\nu = i, e)$ be any given initial data satisfying (19) with $n_{\nu,0}^{(j)} > 0$. Then there exists some $T_0 > 0$, such that the initial value problem (14) has a unique solution that satisfies*

$$
(n_\nu^{(j)}, u_\nu^{(j)}, E^{(j)}, B^{(j)}) \in L^\infty(0, T_0; H^{\tilde{s}+3}\times H^{\tilde{s}+3}\times H^{\tilde{s}+2}\times H^{\tilde{s}+2}).
$$

2.1. **Derivation of the error system.** Take the ansatz for $\nu = i, e$ in terms of the Debye length ε

$$
\begin{cases}
n_\nu^\varepsilon = n^{(0)} + \sum_{k=1}^m \varepsilon^{2k} n_\nu^{(k)} + \varepsilon^{2m}\Psi^\nu = \bar{n}_\nu + \varepsilon^{2m}\Psi^\nu, & \text{(21a)} \\[2mm]
u_\nu^\varepsilon = u^{(0)} + \sum_{k=1}^m \varepsilon^{2k} u_\nu^{(k)} + \varepsilon^{2m}U^\nu = \bar{u}_\nu + \varepsilon^{2m}U^\nu, & \text{(21b)} \\[2mm]
E^\varepsilon = E^{(0)} + \sum_{k=1}^m \varepsilon^{2k} E^{(k)} + \varepsilon^{2m}F = \bar{E} + \varepsilon^{2m}F, & \text{(21c)} \\[2mm]
B^\varepsilon = \sum_{k=1}^m \varepsilon^{2k} B^{(k)} + \varepsilon^{2m}G = \bar{B} + \varepsilon^{2m}G, & \text{(21d)}
\end{cases}
$$

where $(n_i^\varepsilon, u_i^\varepsilon, n_e^\varepsilon, u_e^\varepsilon, E^\varepsilon, B^\varepsilon)$ is the exact solution to system (2), $(n^{(0)}, u^{(0)}, E^{(0)})$ is the solution to the limit system (11), $(n_i^{(j)}, u_i^{(j)}, n_e^{(j)}, u_e^{(j)}, E^{(j)}, B^{(j)})$ is the solution to the linear system (14) and $(\Psi^i, U^i, \Psi^e, U^e, F, G)$ is the remainder term. By careful computation, we derive the following system satisfied by the remainder term

$$
\begin{cases}
\partial_t \Psi^\nu + u_\nu^\varepsilon \cdot \nabla\Psi^\nu + n_\nu^\varepsilon \operatorname{div}U^\nu = -\Psi^\nu \operatorname{div}\bar{u}_\nu - U^\nu \cdot \nabla\bar{n}_\nu - \varepsilon^2 \Re_1^\nu, & \text{(22a)} \\[1mm]
m_\nu(\partial_t U^\nu + u_\nu^\varepsilon \cdot \nabla U^\nu) + h_\nu'(n_\nu^\varepsilon)\nabla\Psi^\nu - q_\nu F - q_\nu(u_\nu^\varepsilon \times G) & \\[1mm]
\quad = -m_\nu U^\nu \cdot \nabla\bar{u}_\nu + q_\nu U^\nu \times \bar{B} - h_\nu''(\bar{n}_\nu)\Psi^\nu\nabla\bar{n}_\nu - \varepsilon^2 \Re_2^\nu, & \text{(22b)} \\[1mm]
\varepsilon^2 \partial_t F - \nabla \times G + n_i^\varepsilon U^i - n_e^\varepsilon U^e + \Psi^i \bar{u}_i - \Psi^e \bar{u}_e = -\varepsilon^2 \Re_3, & \text{(22c)} \\[1mm]
\partial_t G + \nabla \times F = 0, \ \operatorname{div}G = 0, & \text{(22d)} \\[1mm]
\varepsilon^2 \operatorname{div}F - \Psi^i + \Psi^e = 0, & \text{(22e)}
\end{cases}
$$

where the profile $(\Re_1^\nu, \Re_2^\nu, \Re_3)^\top$ is $O(1)$, and depends only on the known and sufficiently smooth functions.

Let $W^\nu = (\Psi^\nu, U^\nu)^\top$. We can rewrite the remainder system (22) in the form

$$
\begin{cases}
\partial_t W^\nu + \sum_{i=1}^3 A_i(n_\nu^\varepsilon, u_\nu^\varepsilon)\partial_{x_i} W^\nu + J = \Re_1^\nu - \varepsilon^2 \Re_0^\nu, & \text{(23a)} \\[2mm]
\varepsilon^2 \partial_t F - \nabla \times G + n_i^\varepsilon U^i - n_e^\varepsilon U^e + \Psi^i \bar{u}_i - \Psi^e \bar{u}_e = -\varepsilon^2 \Re_3, & \text{(23b)} \\[1mm]
\varepsilon^2 \operatorname{div}F - \Psi^i + \Psi^e = 0, & \text{(23c)} \\[1mm]
\partial_t G + \nabla \times F = 0, \ \operatorname{div}G = 0, & \text{(23d)} \\[1mm]
(W^\nu, F, G)|_{t=0} = (W_0^\nu, F_0, G_0), & \text{(23e)}
\end{cases}
$$

where

$$
A_i(n_\nu^\varepsilon, u_\nu^\varepsilon) = \begin{pmatrix} u_\nu^\varepsilon & n_\nu^\varepsilon e_i^\top \\ \frac{h_\nu'(n_\nu^\varepsilon)}{m_\nu} e_i & u_\nu^\varepsilon I \end{pmatrix}, \tag{24}
$$

$$
J = \begin{pmatrix} 0 \\ \frac{-q_\nu F - q_\nu(u_\nu^\varepsilon \times G)}{m_\nu} \end{pmatrix}, \tag{25}
$$

$$
\Re_1^\nu = \begin{pmatrix} -\Psi^\nu \operatorname{div}\bar{u}_\nu - U^\nu \cdot \nabla\bar{n}_\nu \\ -U^\nu \cdot \nabla\bar{u}_\nu + \frac{q_\nu}{m_\nu} U^\nu \times \bar{B} - \frac{1}{m_\nu}\big(h_\nu''(\bar{n}_\nu)\Psi^\nu\nabla\bar{n}_\nu\big) \end{pmatrix}, \tag{26}
$$

and

$$
\Re_0^\nu = (\Re_1^\nu, \Re_2^\nu)^\top.
$$

Moreover, $A_i(n_\nu^\varepsilon, u_\nu^\varepsilon)$ can be symmetrized by the following symmetric and positive matrix

$$A_0(n_\nu^\varepsilon) = \begin{pmatrix} \frac{h_\nu'(n_\nu^\varepsilon)}{m_\nu} & 0^\top \\ 0 & n_\nu^\varepsilon \mathbf{I} \end{pmatrix}, \tag{27}$$

where $\mathbf{I}$ is the 3×3 identity matrix.

Thanks to the symmetrizable structure of system (22), we obtain the standard existence theory of local smooth solutions [5]. Based on this, we will prove the main result stated in the following

2.2. Main results.

Theorem 2.3. *Let $2m$ be any integer with $2m > 4$. Assume that the initial data $(n_{\nu,0}^\varepsilon, u_{\nu,0}^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)$ $(\nu = i, e)$ satisfy the compatibility conditions* (9) *and*

$$\|(n_{\nu,0}^\varepsilon - n_0^{(0)} - \sum_{j=1}^{2m} \varepsilon^j n_{\nu,0}^{(j)}, u_{\nu,0}^\varepsilon - u_0^{(0)} - \sum_{j=1}^{2m} \varepsilon^j u_{\nu,0}^{(j)}, E_0^\varepsilon - E_0^{(0)} - \sum_{j=1}^{2m} \varepsilon^j E_0^{(j)},$$

$$B_0^\varepsilon - \sum_{j=1}^{2m} \varepsilon^j B_0^{(j)})\|_{H^4} \le C\varepsilon^{2m-4},$$

where C is some positive constant independent of ε. Then there exist $\varepsilon_0 > 0$ and solution $(n_\nu^\varepsilon, u_\nu^\varepsilon, E^\varepsilon, B^\varepsilon)$ of system (2) *with initial data $(n_{\nu,0}^\varepsilon, u_{\nu,0}^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)$ on $[0, T^\varepsilon)$ with $\liminf_{\varepsilon \to 0} T^\varepsilon \ge T_*$, the maximal existence time of solutions to the limit system* (11). *In particular, for every $T_0 < T_*$, $0 < \varepsilon < \varepsilon_0$, there holds*

$$\sup_{t \in [0,T_0]} \|(n_\nu^\varepsilon - n^{(0)} - \sum_{j=1}^{2m} \varepsilon^j n_\nu^{(j)}, u_\nu^\varepsilon - u^{(0)} - \sum_{j=1}^{2m} \varepsilon^j u_\nu^{(j)}, E^\varepsilon - E^{(0)} - \sum_{j=1}^{2m} \varepsilon^j E^{(j)},$$

$$B^\varepsilon - \sum_{j=1}^{2m} \varepsilon^j B^{(j)})(t)\|_{H^3} \le C\varepsilon^{2m-4}.$$

3. **Rigorous quasi-neutral limit.** To state the main theorem, we introduce the set S_e^ε, of function in $L^\infty(0, T_0; H^3)$ that satisfies $(W^\nu, F, G)(x, 0) = (W_0^\nu, F_0, G_0)$ and

$$\|(W^\nu, \varepsilon F, G)\|_{H^3} \le \tilde{C}\varepsilon^{-3},$$

where $\tilde{C}$ is a constant independent of ε, which will be determined later. Moreover, we define the weighted norm as

$$\|(W^\nu, \varepsilon F, G)\|_{\varepsilon, s} = \sum_{|\beta|=0}^{s} \varepsilon^{|\beta|} \|\partial^\beta (W^\nu, \varepsilon F, G)\|_{L^2}, \tag{28}$$

for $s \ge 0$. Our next goal is to prove system (23) has a smooth solution $(W^\nu, F, G) \in S_e^\varepsilon$ for appropriate $\tilde{C}$ and ε, which implies the desired estimates stated in Theorem 2.3. As in [9, 18], we consider the nonlinear remainder system coupled with Maxwell equations by the following iteration

$$(W^{\nu,0}, F^0, G^0) = (W_0^\nu, F_0, G_0),$$

where $W^{\nu,0} = (\Psi^{\nu,0}, U^{\nu,0})$, and

$$(W^{\nu,p+1}, F^{p+1}, G^{p+1}) = \Phi(W^{\nu,p}, F^p, G^p),$$

where Φ maps vector $(W^{\nu,p}, F^p, G^p)$ into solution $(W^{\nu,p+1}, F^{p+1}, G^{p+1})$ of the following linear system

$$
\begin{cases}
\partial_t W^{\nu,p+1} + \sum_{i=1}^{3} \tilde{A}_i(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})\partial_{x_i} W^{\nu,p+1} + \tilde{J}^{\nu,p+1} = \tilde{\mathfrak{R}}^{\nu,p+1}, & \text{(29a)} \\[2mm]
\varepsilon^2 \partial_t F^{p+1} - \nabla \times G^{p+1} + n_i^{\varepsilon,p} U^{i,p+1} - n_e^{\varepsilon,p} U^{e,p+1} + \bar{u}_i \Psi^{i,p+1} - \bar{u}_e \Psi^{e,p+1} \\
\quad = -\varepsilon^2 \mathfrak{R}_3, & \text{(29b)} \\
\partial_t G^{p+1} + \nabla \times F^{p+1} = 0, \ \ \text{div} G^{p+1} = 0, & \text{(29c)} \\
\varepsilon^2 \text{div} F^{p+1} - \Psi^{i,p+1} + \Psi^{e,p+1} = 0, & \text{(29d)} \\
(W^{\nu,p+1}, F^{p+1}, G^{p+1})|_{t=0} = (W_0^\nu, F_0, G_0), & \text{(29e)}
\end{cases}
$$

where

$$
(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}) = (\bar{n}_\nu, \bar{u}_\nu) + \varepsilon^{2m}(\Psi^{\nu,p}, U^{\nu,p}), \tag{30}
$$

$$
\tilde{J}^{\nu,p+1} = \begin{pmatrix} 0 \\ \frac{-q_\nu F^{p+1} - q_\nu (u_\nu^{\varepsilon,p} \times G^{p+1})}{m_\nu} \end{pmatrix}, \tag{31}
$$

$$
\tilde{A}_i(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}) = \begin{pmatrix} u_\nu^{\varepsilon,p} & n_\nu^{\varepsilon,p} e_i^\top \\ \frac{h_\nu'(n_\nu^{\varepsilon,p})}{m_\nu} e_i & u_\nu^{\varepsilon,p} \mathbf{I} \end{pmatrix}, \tag{32}
$$

and

$$
\tilde{\mathfrak{R}}^{\nu,p+1} = \begin{pmatrix} -\Psi^{\nu,p+1} \text{div}\bar{u}_\nu - U^{\nu,p+1} \cdot \nabla\bar{n}_\nu \\ -U^{\nu,p+1} \cdot \nabla\bar{u}_\nu + \frac{q_\nu}{m_\nu} U^{\nu,p+1} \times \bar{B} - \frac{1}{m_\nu}\left(h_\nu''(\bar{n}_\nu)\Psi^{\nu,p+1}\nabla\bar{n}_\nu\right) \end{pmatrix} - \varepsilon^2 \mathfrak{R}_0^\nu. \tag{33}
$$

Denote

$$
E_{p,\varepsilon} \triangleq \|(W^{\nu,p}, G^p, \varepsilon F^p)\|_{H^4}. \tag{34}
$$

Proposition 1. *Let $2m$ be any integer with $2m > 4$, and*

$$
(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}, W^{\nu,p+1}, G^{p+1}, F^{p+1}) \ (\nu = i, e)
$$

be the solutions of the iteration equations (29). *Assume that*

$$
\|(W_0^\nu, \varepsilon F_0, G_0)\|_{\varepsilon,4} \leq C,
$$

where C is a generic constant. Then for all $t \in [0, T_0]$, there exists a positive constant ε_0 such that, for all $p \geq 1$,

$$
\|(W^{\nu,p}, \varepsilon F^p, G^p)(t)\|_{\varepsilon,4} \leq C.
$$

Further, by the definitions (28) *and* (34) *that*

$$
E_{p,\varepsilon} \leq C\varepsilon^{-4},
$$

for any $0 < \varepsilon < \varepsilon_0$.

The following section is devoted to the proof of Proposition 1.

Obviously, since

$$
(W^{\nu,0}, F^0, G^0) = (W_0^\nu, F_0, G_0),
$$

we can obtain

$$
E_{0,\varepsilon} \leq C\varepsilon^{-4}. \tag{35}
$$

Now we assume there exists a sufficiently small ε such that

$$
E_{p,\varepsilon} \leq C\varepsilon^{-4}. \tag{36}
$$

Hence, we need to prove that

$$E_{p+1,\varepsilon} \le C\varepsilon^{-4}. \tag{37}$$

Recalling the expansion (21), we immediately obtain that there exists positive constant ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, $n_\nu^{\varepsilon,p} = \bar{n}_\nu + \varepsilon^{2m}\Psi^{\nu,p}$ is bounded from above and below, namely,

$$\frac{n^{(0)}}{2} \le n_\nu^{\varepsilon,p} \le \frac{3n^{(0)}}{2}. \tag{38}$$

Here, we need the condition $2m > 4$. Similar arguments applying to $\|(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})\|_{H^k}$ with $0 \le k \le 4$ yields

Lemma 3.1. *For any $2m > 4$ and sufficiently small ε, it holds that*

$$\|(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})\|_{H^k} \le C(1 + \varepsilon^{2m}E_{p,\varepsilon}). \tag{39}$$

Proof. By the expansion (21) and Sobolev imbedding that

$$\begin{aligned}
\|(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})\|_{H^k} &\le C + C\varepsilon^{2m}\|(\Psi^{\nu,p}, U^{\nu,p})\|_{H^k} \\
&\le C(1 + \varepsilon^{2m}E_{p,\varepsilon}).
\end{aligned} \tag{40}$$

$\square$

Analogously, we get

Lemma 3.2. *Under the same assumptions as in Lemma 3.1 hold. Then we obtain*

$$\|\partial_t n_\nu^{\varepsilon,p}\|_{L^\infty} \le C + C\varepsilon^{4m}E_{p,\varepsilon}^2. \tag{41}$$

In the following, we first give the L^2-estimates.

3.1. L^2-estimates.

Lemma 3.3. *For any $t \in [0, T_0]$, there exists a sufficiently small $\varepsilon > 0$ such that*

$$\begin{aligned}
&\frac{d}{dt}\|(W^{i,p+1}, W^{e,p+1}, \varepsilon F^{p+1}, G^{p+1})\|_{L^2}^2 \\
&\le C(1 + \varepsilon^{4m}E_{p,\varepsilon}^2)\|(W^{i,p+1}, W^{e,p+1}, \varepsilon F^{p+1}, G^{p+1})\|_{L^2}^2 + C\varepsilon^2.
\end{aligned} \tag{42}$$

Proof. Applying the operator $m_\nu \hat{A}_0$ to system (29) and taking the inner product with $(W^{\nu,p+1}, F^{p+1}, G^{p+1})^\top$, we obtain

$$\begin{aligned}
&m_\nu \sum_{\nu=i,e} \langle \mathcal{T}W^{\nu,p+1}, W^{\nu,p+1} \rangle + \frac{d}{dt}\|(\varepsilon F^{p+1}, G^{p+1})\|_{L^2}^2 + \int n_i^{\varepsilon,p} U^{i,p+1} \cdot F^{p+1} \\
&- \int n_e^{\varepsilon,p} U^{e,p+1} \cdot F^{p+1} - \sum_{\nu=i,e} q_\nu \int n_\nu^{\varepsilon,p} F^{p+1} \cdot U^{\nu,p+1} \\
&= -\int \bar{u}_i \Psi^{i,p+1} \cdot F^{p+1} + \int \bar{u}_e \Psi^{e,p+1} \cdot F^{p+1} \\
&\quad + \sum_{\nu=i,e} q_\nu \int n_\nu^{\varepsilon,p} u_\nu^{\varepsilon,p} \times G^{p+1} \cdot U^{\nu,p+1} + \sum_{\nu=i,e} \int \hat{A}_0 \tilde{\mathfrak{R}}^{\nu,p+1} W^{\nu,p+1} \\
&\quad - \varepsilon^2 \int \hat{A}_0 \mathfrak{R}_3 F^{p+1} \\
&\triangleq \sum_{i=1}^5 I_0^i,
\end{aligned} \tag{43}$$

where

$$\hat{A}_0 = \begin{pmatrix} \frac{h'_\nu(n_\nu^{\varepsilon,p})}{m_\nu} & 0^\top \\ 0 & n_\nu^{\varepsilon,p}\mathbf{I} \end{pmatrix}, \tag{44}$$

and the abbreviated operator $\mathcal{T}$ is defined by

$$\langle \mathcal{T}W^{\nu,p+1}, W^{\nu,p+1}\rangle = \frac{1}{2}\frac{d}{dt}\|\hat{A}_0 W^{\nu,p+1}\|_{L^2}^2 - \frac{1}{2}\int(\partial_t, \nabla\cdot)(\hat{A}_0, \vec{A})|W^{\nu,p+1}|^2, \tag{45}$$

where $\vec{A} = \hat{A}_0[(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})]$. Here, we have used the vector analysis formula

$$-\int \nabla \times G^{p+1} \cdot F^{p+1} + \int \nabla \times F^{p+1} \cdot G^{p+1} = \int \mathrm{div}(F^{p+1} \times G^{p+1}) = 0. \tag{46}$$

Thanks to Sobolev embedding $H^2 \hookrightarrow L^\infty$, (36), (38) and (41), we derive

$$\begin{aligned}\|\partial_t \hat{A}_0\|_{L^\infty} &\leq \|(1 + h''_\nu(n_\nu^{\varepsilon,p}))\partial_t n_\nu^{\varepsilon,p}\|_{L^\infty} \\ &\leq C + C\varepsilon^{4m}E_{p,\varepsilon}^2,\end{aligned} \tag{47}$$

and

$$\begin{aligned}\|\mathrm{div}\vec{A}\|_{L^\infty} &\leq C\|(1 + h''_\nu(n_\nu^{\varepsilon,p}))\|(\nabla n_\nu^{\varepsilon,p}, \mathrm{div}u_\nu^{\varepsilon,p})\|_{L^\infty} \\ &\quad + C\|(1 + h''_\nu(n_\nu^{\varepsilon,p}))\|\nabla n_\nu^{\varepsilon,p} \cdot u_\nu^{\varepsilon,p}\|_{L^\infty} \\ &\leq C + C\varepsilon^{4m}\|(\Psi^{\nu,p}, U^{\nu,p})\|_{H^3}^2 \\ &\leq C + C\varepsilon^{4m}E_{p,\varepsilon}^2.\end{aligned} \tag{48}$$

Hence,

$$\frac{1}{2}\int(\partial_t, \nabla\cdot)(\hat{A}_0, \vec{A})|W^{\nu,p+1}|^2 \leq C(1 + \varepsilon^{4m}E_{p,\varepsilon}^2)\|W^{\nu,p+1}\|_{L^2}^2. \tag{49}$$

Inserting this into (43), and using (46) and $q_i = 1, q_e = -1$, we have

$$\frac{d}{dt}\|(\hat{A}_0 W^{\nu,p+1}, \varepsilon F^{p+1}, G^{p+1})\|_{L^2}^2 \leq \sum_{i=1}^5 I_0^i + C(1 + \varepsilon^{4m}E_{p,\varepsilon}^2)\|W^{\nu,p+1}\|_{L^2}^2. \tag{50}$$

From (29d), the first and second term on the right hand side of (43) can be accordingly decomposed into

$$\begin{aligned}I_0^1 + I_0^2 &= -\int(\Psi^{i,p+1} - \Psi^{e,p+1})\bar{u_i} \cdot F^{p+1} - \int(\bar{u_i} - \bar{u_e})\Psi^{e,p+1} \cdot F^{p+1} \\ &= -\varepsilon^2 \int \mathrm{div}F^{p+1}F^{p+1} \cdot \bar{u_i} - \int(\bar{u_i} - \bar{u_e})\Psi^{e,p+1} \cdot F^{p+1}.\end{aligned}$$

By the vector analysis formulation

$$\mathrm{div}ff = \mathrm{div}(f \otimes f) - \frac{1}{2}\nabla(|f|^2) - \nabla \times f \times f, \tag{51}$$

where f is a vector function, $I_0^1 + I_0^2$ can be further decomposed by

$$\begin{aligned}I_0^1 + I_0^2 &= -\varepsilon^2 \int \left(\mathrm{div}(F^{p+1} \otimes F^{p+1}) - \frac{1}{2}\nabla(|F^{p+1}|^2) - \nabla \times F^{p+1} \times F^{p+1}\right) \cdot \bar{u_i} \\ &\quad - \int(\bar{u_i} - \bar{u_e})\Psi^{e,p+1} \cdot F^{p+1} \\ &\leq \varepsilon^2 \int \nabla \times F^{p+1} \times F^{p+1} \cdot \bar{u_i} + C\|(\Psi^{e,p+1}, \varepsilon F^{p+1})\|_{L^2}^2,\end{aligned}$$

thanks to $(\bar{u}_i - \bar{u}_e) \sim O(\varepsilon^2)$ since $u_i^{(0)} = u_e^{(0)}$. To deal with the term $\varepsilon^2 \int \nabla \times F^{p+1} \times F^{p+1} \cdot \bar{u}_i$, we apply (29b)$\times G^{p+1}$ −(29c)$\times \varepsilon^2 F^{p+1}$ to derive

$$\varepsilon^2 \partial_t (F^{p+1} \times G^{p+1}) - \nabla \times G^{p+1} \times G^{p+1} - \varepsilon^2 \nabla \times F^{p+1} \times F^{p+1}$$
$$+ \left(n_i^{\varepsilon,p} U^{i,p+1} - n_e^{\varepsilon,p} U^{e,p+1} \right) \times G^{p+1} + \left(\bar{u}_i \Psi^{i,p+1} - \bar{u}_e \Psi^{e,p+1} \right) \times G^{p+1} \qquad (52)$$
$$+ \varepsilon^2 \Re_3 \times G^{p+1} = 0.$$

Thus, we have

$$\varepsilon^2 \int \nabla \times F^{p+1} \times F^{p+1} \cdot \bar{u}_i$$
$$= \varepsilon^2 \frac{d}{dt} \int (F^{p+1} \times G^{p+1}) \cdot \bar{u}_i - \varepsilon^2 \int (F^{p+1} \times G^{p+1}) \cdot \partial_t \bar{u}_i$$
$$\quad - \int \operatorname{div}(G^{p+1} \otimes G^{p+1}) \cdot \bar{u} + \frac{1}{2} \int \nabla(|G^{p+1}|^2) \cdot \bar{u}$$
$$\quad + \int \left(n_i^{\varepsilon,p} U^{i,p+1} - n_e^{\varepsilon,p} U^{e,p+1} \right) \times G^{p+1} \cdot \bar{u}_i$$
$$\quad + \int \left(\bar{u}_i \Psi^{i,p+1} - \bar{u}_e \Psi^{e,p+1} \right) \times G^{p+1} \cdot \bar{u}_i + \varepsilon^2 \int \Re_3 \times G^{p+1} \cdot \bar{u}_i$$
$$\leq \varepsilon^2 \frac{d}{dt} \int (F^{p+1} \times G^{p+1}) \cdot \bar{u}_i + C\varepsilon^4 + C\|(W^{\nu,p+1}, G^{p+1}, \varepsilon F^{p+1})\|_{L^2}^2,$$

thanks to $\operatorname{div} G^{p+1} = 0$ and (38). Recalling $\hat{A}_0$ is positively definite, we have

$$\sum_{\nu=i,e} \|\hat{A}_0 W^{\nu,p+1}\|_{L^2}^2 \geq C\|W^{\nu,p+1}\|_{L^2}^2.$$

In fact, since ε is sufficiently small, we get

$$\|(\varepsilon F^{p+1}, G^{p+1})\|_{L^2}^2 - \varepsilon^2 \int (F^{p+1} \times G^{p+1}) \cdot \bar{u}_i \geq C\|(\varepsilon F^{p+1}, G^{p+1})\|_{L^2}^2.$$

For the last three terms $I_0^3 \sim I_0^5$, by Young's inequality and Hölder inequality, we get

$$I_0^3 \sim I_0^5 \leq C\varepsilon^2 + C(1 + \varepsilon^{2m} E_{p,\varepsilon})\|(\varepsilon F^{p+1}, G^{p+1}, W^{\nu,p+1})\|_{L^2}^2.$$

Putting the above estimates together, the proof of Lemma 3.3 is then complete. □

3.2. Higher order estimates.

Lemma 3.4. *Let $1 \leq k \leq 4$ be an integer, α be a multi-index with $|\alpha| = k$, then we have*

$$\frac{d}{dt}\|(\partial^\alpha W^{i,p+1}, \partial^\alpha W^{e,p+1})\|_{L^2}^2 - \sum_{\nu=i,e} q_\nu \int n_\nu^{\varepsilon,p} \partial^\alpha F^{p+1} \cdot \partial^\alpha U^{\nu,p+1}$$
$$\leq C\varepsilon^4 + C(1 + \varepsilon^{4m} E_{p,\varepsilon}^2)\|(G^{p+1}, W^{\nu,p+1})\|_{H^k}^2. \qquad (53)$$

Proof. First, applying the operator ∂^α to (29a), we derive

$$\partial_t \partial^\alpha W^{\nu,p+1} + \sum_{i=1}^3 \tilde{A}_i(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}) \partial_{x_i} \partial^\alpha W^{\nu,p+1} + \partial^\alpha \tilde{J}^{\nu,p+1} = \partial^\alpha \tilde{\Re}^{\nu,p+1} + H_1, \quad (54)$$

where

$$H_1 = \sum_{i=1}^{3} [\partial^\alpha, \tilde{A}_i(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})] \partial_{x_i} W^{\nu,p+1}. \tag{55}$$

Then taking inner product of (54) with $m_\nu \hat{A}_0 \partial^\alpha W^{\nu,p+1}$, we derive

$$
\begin{aligned}
&m_\nu \langle \mathcal{T} \partial^\alpha W^{\nu,p+1}, \partial^\alpha W^{\nu,p+1} \rangle - q_\nu \int n_\nu^{\varepsilon,p} \partial^\alpha F^{p+1} \cdot \partial^\alpha U^{\nu,p+1} \\
={}&m_\nu \int \hat{A}_0 \partial^\alpha \tilde{\mathfrak{R}}^{\nu,p+1} \partial^\alpha W^{\nu,p+1} + q_\nu \int n_\nu^{\varepsilon,p} \partial^\alpha (u_\nu^{\varepsilon,p} \times G^{p+1}) \cdot \partial^\alpha U^{\nu,p+1} \\
&+ m_\nu \int \hat{A}_0 H_1 \partial^\alpha W^{\nu,p+1} \\
={}&\sum_{i=1}^{3} I_\varepsilon^i.
\end{aligned}
\tag{56}
$$

Again, the abbreviated operator $\mathcal{T}$ is defined by

$$\langle \mathcal{T} W^{\nu,p+1}, W^{\nu,p+1} \rangle = \frac{1}{2} \frac{d}{dt} \|\hat{A}_0 \partial^\alpha W^{\nu,p+1}\|_{L^2}^2 - \frac{1}{2} \int (\partial_t, \nabla \cdot)(\hat{A}_0, \vec{\tilde{A}}) |\partial^\alpha W^{\nu,p+1}|^2,$$

where $\vec{\tilde{A}} = \hat{A}_0 [(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p})]$. For the first term on the left side hand of (56), we can employ arguments similar to those used in the estimate of (45) to obtain

$$m_\nu \langle \mathcal{T} \partial^\alpha W^{\nu,p+1}, \partial^\alpha W^{\nu,p+1} \rangle \le C(1 + \varepsilon^{4m} E_{p,\varepsilon}^2) \|W^{\nu,p+1}\|_{H^k}^2. \tag{57}$$

On the other hand, using Young's inequality, Hölder inequality, Lemma 1.1, together with (38), we get

$$I_\varepsilon^1 \le C \|W^{\nu,p+1}\|_{H^k}^2 + C\varepsilon^4,$$

and

$$I_\varepsilon^2 \le C(1 + \varepsilon^{2m} E_{p,\varepsilon}) \|(G^{p+1}, W^{\nu,p+1})\|_{H^k}^2.$$

For the commeutator term I_ε^3, by Lemma 1.1, we have,

$$
\begin{aligned}
I_\varepsilon^3 \le{}& C\|\hat{A}_0\|_{L^\infty} \|H_1\|_{L^2} \|\partial^\alpha W^{\nu,p+1}\|_{L^2} \\
\le{}& C(\|\nabla \tilde{A}_i\|_{L^\infty} \|\nabla W^{\nu,p+1}\|_{H^{k-1}} + \|\tilde{A}_i\|_{H^k} \|\nabla W^{\nu,p+1}\|_{L^\infty}) \|\partial^\alpha W^{\nu,p+1}\|_{L^2} \\
\le{}& C(1 + \varepsilon^{2m} E_{p,\varepsilon}) \|W^{\nu,p+1}\|_{H^k}^2.
\end{aligned}
$$

Finally, putting all the above estimates together, we completes the proof of Lemma 3.4, thanks to (38). □

Lemma 3.5. *Under the same conditions in Lemma 3.4, we have the following estimate*

$$
\begin{aligned}
&\frac{d}{dt} \|(\varepsilon \partial^\alpha F^{p+1}, \partial^\alpha G^{p+1})\|_{L^2}^2 + \sum_{\nu=i,e} q_\nu \int n_\nu^{\varepsilon,p} \partial^\alpha U^{\nu,p+1} \cdot \partial^\alpha F^{p+1} \\
&\le C \|(G^{p+1}, W^{\nu,p+1}, \varepsilon F^{p+1})\|_{H^k}^2 + \frac{C}{\varepsilon^2}(1 + \varepsilon^{4m} E_{p,\varepsilon}^2) \|W^{\nu,p+1}\|_{H^{k-1}}^2 + C\varepsilon^2.
\end{aligned}
\tag{58}
$$

Proof. An application of the operator ∂^α to (29b) and (29c) leads to

$$
\begin{cases}
\varepsilon^2 \partial_t \partial^\alpha F^{p+1} - \partial^\alpha \nabla \times G^{p+1} + n_i^{\varepsilon,p} \partial^\alpha U^{i,p+1} - n_e^{\varepsilon,p} \partial^\alpha U^{e,p+1} + \bar{u}_i \partial^\alpha \Psi^{i,p+1} \\
\quad - \bar{u}_e \partial^\alpha \Psi^{e,p+1} = -\varepsilon^2 \partial^\alpha \Re_3 + H_2 + H_3, \tag{59a} \\
\partial_t \partial^\alpha G^{p+1} + \partial^\alpha \nabla \times F^{p+1} = 0, \ \partial^\alpha \mathrm{div} G^{p+1} = 0, \tag{59b}
\end{cases}
$$

where

$$
H_2 = [\partial^\alpha, n_i^{\varepsilon,p}] U^{i,p+1} - [\partial^\alpha, n_e^{\varepsilon,p}] U^{e,p+1}, \tag{60}
$$

and

$$
H_3 = [\partial^\alpha, \bar{u}_i] \Psi^{i,p+1} - [\partial^\alpha, \bar{u}_e] \Psi^{e,p+1}. \tag{61}
$$

Multiplying (59) by $(\partial^\alpha F^{p+1}, \partial^\alpha G^{p+1})$, it holds

$$
\frac{d}{dt} \| (\varepsilon \partial^\alpha F^{p+1}, \partial^\alpha G^{p+1}) \|_{L^2}^2 + \int n_i^{\varepsilon,p} \partial^\alpha U^{i,p+1} \cdot \partial^\alpha F^{p+1}
$$

$$
- \int n_e^{\varepsilon,p} \partial^\alpha U^{e,p+1} \cdot \partial^\alpha F^{p+1}
$$

$$
= - \int \bar{u}_i \partial^\alpha \Psi^{i,p+1} \cdot \partial^\alpha F^{p+1} + \int \bar{u}_e \partial^\alpha \Psi^{e,p+1} \cdot \partial^\alpha F^{p+1} + \int H_2 \partial^\alpha F^{p+1} \tag{62}
$$

$$
+ \int H_3 \partial^\alpha F^{p+1} - \varepsilon^2 \int \partial^\alpha \Re_3 \partial^\alpha F^{p+1}
$$

$$
\triangleq \sum_{i=1}^5 J_\varepsilon^i.
$$

Here, the vector analysis formula

$$
\mathrm{div}(f \times g) = (\nabla \times f) \cdot g - (\nabla \times g) \cdot f \tag{63}
$$

has been used again, for any vector functions f and g.

We proceed to control the five terms on the right hand side of (62). For J_ε^1 and J_ε^2, we can employ arguments similar to those used in the estimates of I_0^1 and I_0^2 to obtain

$$
J_\varepsilon^1 + J_\varepsilon^2 \le \varepsilon^2 \frac{d}{dt} \int (\partial^\alpha F^{p+1} \times \partial^\alpha G^{p+1}) \cdot \bar{u}_i + C \| (G^{p+1}, W^{\nu,p+1}, \varepsilon F^{p+1}) \|_{H^k}^2 + C \varepsilon^4.
$$

The usual estimate (3) on commutator leads to

$$
J_\varepsilon^3 \le C \| \varepsilon F^{p+1} \|_{H^k} \| \frac{1}{\varepsilon} H_2 \|_{L^2}
$$

$$
\le C \| \varepsilon F^{p+1} \|_{H^k} \frac{1}{\varepsilon} (\| \nabla n_\nu^{\varepsilon,p} \|_{L^\infty} \| U^{\nu,p+1} \|_{H^{k-1}} + \| U^{\nu,p+1} \|_{L^\infty} \| n_\nu^{\varepsilon,p} \|_{H^k})
$$

$$
\le C \| \varepsilon F^{p+1} \|_{H^k}^2 + \frac{C}{\varepsilon^2} (1 + \varepsilon^{4m} E_{p,\varepsilon}^2) \| W^{\nu,p+1} \|_{H^{k-1}}^2.
$$

Similarly, the forth term J_ε^4 can be bounded by

$$
J_\varepsilon^4 \le C \| \varepsilon F^{p+1} \|_{H^k}^2 + \frac{C}{\varepsilon^2} \| W^{\nu,p+1} \|_{H^{k-1}}^2.
$$

It follows from Young's inequality and Hölder inequality, the last term J_ε^5 can be estimated as

$$J_\varepsilon^5 \leq C\varepsilon^2 + C\|\varepsilon F^{p+1}\|_{H^k}^2.$$

Adding these above estimates together, the proof of Lemma 3.5 is then complete.   □

3.3. The end of the proof of Proposition 1.

Proof. Combining Lemmas 3.4 and 3.5, we have

$$\frac{d}{dt}\|(\partial^\alpha W^{p+1}, \varepsilon\partial^\alpha F^{p+1}, \partial^\alpha G^{p+1})\|_{L^2}^2$$

$$\leq C\varepsilon^2 + C(1 + \varepsilon^{4m}E_{p,\varepsilon}^2)\|(G^{p+1}, W^{\nu,p+1}, \varepsilon F^{p+1})\|_{H^k}^2 \tag{64}$$

$$+ \frac{C}{\varepsilon^2}(1 + \varepsilon^{4m}E_{p,\varepsilon}^2)\|W^{\nu,p+1}\|_{H^{k-1}}^2,$$

where $1 \leq |\alpha| = k \leq 4$.

Recalling Lemma 3.3 and the weighted energy norm (28), we obtain, for $t \in [0, T_0]$,

$$\|(W^{p+1}, \varepsilon F^{p+1}, G^{p+1})(t)\|_{\varepsilon,4}^2$$

$$\leq C\|(W^{p+1}, \varepsilon F^{p+1}, G^{p+1})(0)\|_{\varepsilon,4}^2 + C\varepsilon^2 \tag{65}$$

$$+ C\int_0^t (1 + \varepsilon^{4m}E_{p,\varepsilon}^2)\|(W^{p+1}, \varepsilon F^{p+1}, G^{p+1})(\tau)\|_{\varepsilon,4}^2.$$

From (36), there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, we have

$$\varepsilon^{2m}E_{p,\varepsilon} < 1, \tag{66}$$

thanks to the assumption $2m > 4$. Using Gronwall inequality, (65) and (66), we infer that

$$\sup_{t\in[0,T_0]}\|(W^{\nu,p+1}, \varepsilon F^{p+1}, G^{p+1})(t)\|_{\varepsilon,4}^2 \leq \tilde{C}, \tag{67}$$

where $\tilde{C} = e^{2T_0}(C_0 + 1)T_0$, and C_0 is a constant dependent on the initial data. A straightforward calculation implies

$$E_{p+1,\varepsilon} \leq C\varepsilon^{-4}\|(W^{\nu,p+1}, \varepsilon F^{p+1}, G^{p+1})(t)\|_{\varepsilon,4} \tag{68}$$

$$\leq \tilde{C}\varepsilon^{-4}.$$

This completes the proof of Proposition 1.   □

4. Proof of Theorem 2.3.

Proof. Set

$$(\mathcal{W}^{\nu,p}, \varepsilon\mathcal{F}^p, \mathcal{G}^p) = (W^{\nu,p+1}, \varepsilon F^{p+1}, G^{p+1}) - (W^{\nu,p}, \varepsilon F^p, G^p),$$

where

$$\mathcal{W}^{\nu,p} = (\ominus^{\nu,p}, \mathcal{U}^{\nu,p}) = (\Psi^{\nu,p+1}, U^{\nu,p+1}) - (\Psi^{\nu,p}, U^{\nu,p}).$$

By careful computation, we obtain the system satisfied by $(\mathcal{W}^{\nu,p}, \varepsilon\mathcal{F}^p, \mathcal{G}^p)$

$$
\begin{cases}
\partial_t \mathcal{W}^{\nu,p} + \sum_{i=1}^{3} \tilde{A}_i(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}) \partial_{x_i} \mathcal{W}^{\nu,p} + \tilde{J}^{\nu,p+1} - \tilde{J}^{\nu,p} - \tilde{\Re}^{\nu,p+1} + \tilde{\Re}^{\nu,p} \\
\quad = -\sum_{i=1}^{3} \left(\tilde{A}_i(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}) - \tilde{A}_i(n_\nu^{\varepsilon,p-1}, u_\nu^{\varepsilon,p-1}) \right) \partial_{x_i} W^{\nu,p}, & \text{(69a)} \\
\varepsilon^2 \partial_t \mathcal{F}^p - \nabla \times \mathcal{G}^p + n_i^{\varepsilon,p}\mathcal{U}^{i,p} - n_e^{\varepsilon,p}\mathcal{U}^{e,p} + \bar{u}_i \ominus^{i,p} - \bar{u}_e \ominus^{e,p} \\
\quad = -(n_i^{\varepsilon,p} - n_i^{\varepsilon,p-1})U^{i,p} + (n_e^{\varepsilon,p} - n_e^{\varepsilon,p-1})U^{e,p}, & \text{(69b)} \\
\partial_t \mathcal{G}^p + \nabla \times \mathcal{F}^p = 0, \;\; \mathrm{div}\mathcal{G}^p = 0, & \text{(69c)} \\
\varepsilon^2 \mathrm{div}\mathcal{F}^p - \ominus^{i,p} + \ominus^{e,p} = 0, & \text{(69d)} \\
(\mathcal{W}^{\nu,p}, \varepsilon\mathcal{F}^p, \mathcal{G}^p)|_{t=0} = 0. & \text{(69e)}
\end{cases}
$$

Based on the similar arguments of Lemmas 3.3-3.5 and (64)-(68), we deduce

$$
\sup_{t \in [0,T_0]} \|(\mathcal{W}^{\nu,p}, \varepsilon\mathcal{F}^p, \mathcal{G}^p)(t)\|_{\varepsilon,3} \le c \|(\mathcal{W}^{\nu,p-1}, \varepsilon\mathcal{F}^{p-1}, \mathcal{G}^{p-1})(t)\|_{\varepsilon,3},
$$

where $0 < c < 1$, which depends on the bound C in Proposition 1, thanks to (69e), $2m > 4$ and $(n_\nu^{\varepsilon,p}, u_\nu^{\varepsilon,p}) = (\bar{n}_\nu, \bar{u}_\nu) + \varepsilon^{2m}(\Psi^{\nu,p}, U^{\nu,p})$. This implies

$$
(\Psi^{i,p}, U^{i,p}, \Psi^{e,p}, U^{e,p}, \varepsilon F^p, G^p)
$$

is a Cauchy sequence, and hence there exists

$$
(\Psi^i, U^i, \Psi^e, U^e, \varepsilon F, G) \in C([0,T_0]; H^3)
$$

such that, as $p \to \infty$, we can obtain the convergence of the whole sequence

$$
(W^{i,p}, W^{e,p}, G^p, \varepsilon F^p)_{p \ge 1}
$$

to $(W^i, W^e, G, \varepsilon F)$, as well as

$$
\sup_{t \in [0,T_0]} \|(W^i, W^e, \varepsilon F, G)(t)\|_{H^3} \le C\varepsilon^{-3},
$$

for any $0 < \varepsilon < \varepsilon_0$. Indeed, in a similar manner to [9], we infer that $(W^i, W^e, \varepsilon F, G) \in C^i([0,T_0]; H^{3-i})$ for $i = 0, 1$. Passing the limit $p \to \infty$ in the system (29), we obtain system (23) admits a classical solution (W^i, W^e, F, G) that satisfies

$$
\sup_{t \in [0,T_0]} \|(W^i, W^e, F, G)(t)\|_{H^3} \le C\varepsilon^{-4}.
$$

With the aid of the expansion (21), $(n_i^\varepsilon, u_i^\varepsilon, n_e^\varepsilon, u_e^\varepsilon, E^\varepsilon, B^\varepsilon)$ converges strongly to $(n^{(0)}, u^{(0)}, n^{(0)}, u^{(0)}, E^{(0)}, 0)$ in $C(0, T_0; H^3)$, for any integer $2m > 4$.

The proof of Theorem 2.3 is then complete. □

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