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A HYBRIDIZED WEAK GALERKIN FINITE ELEMENT SCHEME FOR GENERAL SECOND-ORDER ELLIPTIC PROBLEMS

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Abstract. In this paper, a hybridized weak Galerkin (HWG) finite element scheme is presented for solving the general second-order elliptic problems. The HWG finite element scheme is based on the use of a Lagrange multiplier defined on the element boundaries. The Lagrange multiplier provides a numerical approximation for certain derivatives of the exact solution. It is worth pointing out that a skew symmetric form has been used for handling the convection term to get the stability in the HWG formulation. Optimal order error estimates are derived for the corresponding HWG finite element approximations. A Schur complement formulation of the HWG method is introduced for implementation purpose.

1. **Introduction.** In this paper, we consider the following general second-order elliptic problem

$$
\begin{aligned}
-\nabla\cdot(\mathcal{A}\nabla u)+\nabla\cdot(\boldsymbol{b}u)+cu &= f \text{ in } \Omega, &(1)\\
u &= g \text{ on } \partial\Omega, &(2)
\end{aligned}
$$

where Ω is a polygonal/plolyhedral domain in $\mathbb{R}^d(d=2,3)$, $\mathcal{A}$ is a symmetric matrix, $\boldsymbol{b}=(b_i(\boldsymbol{x}))_{d\times 1}\in[L^\infty(\Omega)]^d$ is a vector-valued function, $c=c(\boldsymbol{x})\in L^\infty(\Omega)$ is a scalar function on Ω, $f\in L^2(\Omega)$ is a source term and $g\in H^{\frac{1}{2}}(\Omega)$ is the boundary

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condition. Assume that the matrix $\mathcal{A}$ satisfies the following property: there exists a constant $\lambda > 0$ such that

$$\xi^t \mathcal{A} \xi \geq \lambda \xi^t \xi, \ \forall \xi \in \mathbb{R}^d,$$

where ξ is understood as a column vector and ξ^t is the transpose of ξ. Taking $\mathcal{A}, \boldsymbol{b}, c, f, g$ to be specific functions in the problem(1)-(2), we can obtain various specific partial differential equations [3, 9, 25, 2, 22, 13, 23, 17]. Many numerical methods have been developed to get the numerical solution to the general second-order elliptic problem(1)-(2), e,g., the finite volume element method [12], the finite element method [3], the mixed finite element method [10], the discontinous Galerkin method [11], the hybridizable discontinuous Galerkin method [6], the discontinuous hp finite elelnent method [1].

Recently, a weak Galerkin (WG) finite element method has been developed to solve the problem (1)-(2) [19, 14]. The weak Galerkin finite element method is a efficient numerical technique in which differential operators are approximated by their weak forms as distributions. In the WG method, the weak function and its derivative can be approximated by piecewise polynomials with various degrees. The efficiency and flexibility have made the WG method become an excellent candidate for solving partial differential equations. Since its contribution, the WG finite element method has been applied successfully to the discretization of several classes of partial differential equations, e.g., the second-order elliptic problem [19, 20, 14], the Biharmonic equation [15, 26], the Stokes equation[21, 24].

In the finite element method, hybridization is a useful technique where a Lagrange multiplier is identified to relax certain constrain such as some continuity requirements. This technique has been used in mixed finite element methods to yield hybridized mixed finite element formulations [4, 5]. It is also employed in discontinuous Galerkin finite element methods to yield hybridized discontinuous Galerkin (HDG) methods [7, 8]. In [24, 16, 18], the WG finite element formulations are hybridized to obtain corresponding hybridized weak Galerkin finite element formulations for involved problems.

The aim of this paper is to propose a hybridized weak Galerkin finite element method for the general second-order elliptic problem (1)-(2) based on the use of a Lagrange multiplier defined on the element boundaries. We shall establish the stability and convergence for the proposed hybridized weak Galerkin finite element method. The hybridized weak Galerkin method is further used to derive a Schur complement formulation which lead to a linear system with significantly less number of unknowns than the original WG or HWG formulation. It is also worth pointing out that a skew symmetric form has been used for handling the convection term to get the stability in the HWG formulation for the problem (1)-(2).

The paper is organized as follows. In the next section, we shall introduce the discrete weak differential operators including the discrete weak gradient operator and the discrete weak convective operator. In Section 3, we shall present a hybridized weak Galerkin formulation for the problem (1)-(2) and further show the relation between WG method and HWG method. The stability condition for the proposed hybridized weak Galerkin formulation shall be proved in Section 4. Some error estimates shall be obtain in Section 5. In Section 6, a Schur complement formulation was established for variable reduction. Finally, conclusions are drawn in Section 7.

2. Discrete weak differential operators.

Let K be any domain in $\mathbb{R}^d$, $d = 2, 3$. We use the standard definition for the Sobolev space $H^s(K)$ and their associated inner products $(\cdot,\cdot)_{s,K}$, norms $\|\cdot\|_{s,K}$, and semi-norms $|\cdot|_{s,K}$ for any $s \geq 0$. For instance, for any integer $s \geq 0$, the semi-norm $|\cdot|_{s,K}$ is defined by

$$|v|_{s,K} = \left(\sum_{|\alpha|=s} \int_K |\partial^\alpha v|^2 dK \right)^{1/2}$$

with the usual notation

$$\alpha = (\alpha_1, \cdots, \alpha_d),\ |\alpha| = \alpha_1 + \cdots + \alpha_d,\ \partial^\alpha = \Pi_{j=1}^d \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{m,K}$ is given by

$$\|v\|_{m,K} = \left(\sum_{j=0}^m |v|_{j,K}^2 \right)^{1/2}.$$

The space $H^0(K)$ coincides with $L^2(K)$, for which norm and inner product are denoted by $\|\cdot\|_K$ and $(\cdot,\cdot)_K$, respectively. If $K = \Omega$, we shall drop the subscript K in the L^2 norm and the L^2 inner product notations.

The space $H(div;K)$ is given by the set of vector-valued functions on K which, together with their divergence, are square integrable; i.e.,

$$H(div;K) = \{\boldsymbol{v} : \boldsymbol{v} \in [L^2(K)]^d, \nabla\cdot\boldsymbol{v} \in L^2(K)\}.$$

The norm in $H(div;K)$ is defined as

$$\|\boldsymbol{v}\|_{H(div;K)} = \left(\|\boldsymbol{v}\|_K^2 + \|\nabla\cdot\boldsymbol{v}\|_K^2\right)^{1/2}.$$

Next, we will introduce the discrete weak gradient operator and the discrete weak convective operator. To this end, we denoted by $v = \{v_0, v_b\}$ a weak function on a polygonal/polyhedral domain T with boundary ∂T. Here, $v_0 \in L^2(T)$ and $v_b \in H^{\frac{1}{2}}(\partial T)$. The first component v_0 can be understood as the value of v in T, and the second component v_b represents v on the boundary of T. Note that v_b may not necessarily be related to the trace of v_0 on ∂T. Denote by $W(T)$ the space of weak functions on T; i.e.,

$$W(T) = \{v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in H^{\frac{1}{2}}(\partial T)\}. \tag{3}$$

Define $(v,w)_T = \int_T vwdx$ and $\langle v,w\rangle_\gamma = \int_\gamma vwds$. Denote by $P_r(K)$ the set of polynomials on T with degree no more than r. In the rest of the section, we shall recall the discrete weak gradient operator and the discrete weak convective operator defined in [14].

Definition 2.1. ([14]) The discrete weak gradient operator, denoted by $\nabla_{w,r,T}$, is defined as the unique polynomial $\nabla_w v \in [P_r(T)]^d$ satisfying the following equation

$$(\nabla_{w,r,T} v, \boldsymbol{q})_T = -(v_0, \nabla\cdot\boldsymbol{q})_T + \langle v_b, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial T},\ \forall \boldsymbol{q} \in [P_r(T)]^d. \tag{4}$$

By applying the usual integration by part to the first term on the right hand side of (4), we can rewrite the equation (4) as

$$(\nabla_{w,r,T} v, \boldsymbol{q})_T = (\nabla v_0, \boldsymbol{q})_K + \langle v_b - v_0, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial T},\ \forall \boldsymbol{q} \in [P_r(T)]^d. \tag{5}$$

Definition 2.2. ([14]) The discrete weak convective operator, denoted by $\boldsymbol{b}\cdot\nabla_{w,r,T}$, is defined as the unique polynomial $\boldsymbol{b}\cdot\nabla_{w,r,T} v \in P_r(T)$ satisfying the following equation

$$(\boldsymbol{b}\cdot\nabla_{w,r,T} v, \phi)_T = -(\boldsymbol{b}\cdot\nabla\phi, v_0)_T - ((\nabla\cdot\boldsymbol{b})\phi, v_0) + \langle \boldsymbol{b}\cdot\boldsymbol{n}, v_b\phi\rangle_{\partial K},\ \forall\phi \in P_r(T). \tag{6}$$

3. **A hybridized weak Galerkin formulation.** The goal of this section is to establish a hybridized weak Galerkin finite element scheme for the problem (1)-(2) and further obtain the relation between WG method and HWG method.

3.1. **Notations.** Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that $\mathcal{T}_h$ is shape regular in the sense as defined in [20]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and mesh size $h = max_{T\in\mathcal{E}_h} h_T$ for $\mathcal{T}_h$.

For each element $T \in \mathcal{T}_h$, denote by $W(T)$ the space of weak function defined by (3). Suppose the trace of $W(T)$ on the boundary ∂T is the usual Sobolev space $L^2(\partial T)$. Define the spaces W and Λ by

$$W = \Pi_{T\in\mathcal{T}_h} W(T),\ \Lambda = \Pi_{T\in\mathcal{T}_h} L^2(\partial T).$$

Note that the values of functions in the space W are not correlated between any adjacent elements T_1 and T_2 which share $e \in \mathcal{E}_h^0$ as a common edge or flat face. For example, on each interior edge $e \in \mathcal{E}_h^0$, $v \in W$ has two copies of v_b: one taken from the left (say T_1) and the other from the right (say T_2). Define the jump of $v \in W$ on $e \in \mathcal{E}_h$ by

$$[\![v]\!]_e = \begin{cases} v_b|_{\partial T_1} - v_b|_{\partial T_2}, & e \in \mathcal{E}_h^0 \\ v_b, & e \in \partial\Omega, \end{cases} \tag{7}$$

where $v_b|_{\partial T_i}$ is the value of v on e as seen from the element T_i. The order of T_1 and T_2 is non-essential in (7) as long as the difference is taken in consistent way in all the formulas. Analogously, for any function $\lambda \in \Lambda$, we define its similarity on $e \in \mathcal{E}_h$ by

$$\langle\!\langle \lambda \rangle\!\rangle_e = \begin{cases} \lambda|_{\partial T_1} + \lambda|_{\partial T_2}, & e \in \mathcal{E}_h^0, \\ 0, & e \in \partial\Omega. \end{cases} \tag{8}$$

Denote by $\langle\!\langle \lambda \rangle\!\rangle$ the similarity of λ in $\mathcal{E}_h$.

For any given integer $k \geq 2$, denote by V_k the discrete weak function space given by

$$V_k(T) = \{v = \{v_0, v_b\} : v_0 \in P_k(T), v_b \in P_{k-1}(T), e \in \partial T\}.$$

Define $\Lambda_k(\partial T)$ by

$$\Lambda_k(\partial T) = \{\lambda : \lambda|_e \in P_{k-1}(e), e \in \partial T\}.$$

By patching $V_k(T)$ and $\Lambda_k(\partial T)$ over all the elements $T \in \mathcal{T}_h$, we obtain two weak Galerkin finite element spaces V_h and Λ_h as follows

$$V_h = \Pi_{T\in\mathcal{T}_h} V_k(T),\ \Lambda_h = \Pi_{T\in\mathcal{T}_h} \Lambda_k(\partial T).$$

Denote by V_h^0 the subspace of V_h with vanishing boundary values on $\partial\Omega$, i.e.,

$$V_h^0 = \{v = \{v_0, v_b\} \in V_h : v_b|_{\partial T\cap\partial\Omega} = 0, \forall T \in \mathcal{T}_h\}.$$

Furthermore, let $\mathcal{V}_h$ be the subspace of V_h consisting of functions without jump on each interior edge or flat face, i.e.,

$$\mathcal{V}_h = \{v = \{v_0, v_b\} \in V_h : [\![v]\!]_e = 0,\ e \in \mathcal{E}_h^0\}.$$

Denote by $\mathcal{V}_h^0$ the subspace of $\mathcal{V}_h$ with vanishing boundary values on $\partial\Omega$, i.e.,

$$\mathcal{V}_h^0 = \{v = \{v_0, v_b\} \in \mathcal{V}_h : v_b|_{\partial T\cap\partial\Omega} = 0, \forall T \in \mathcal{T}_h\}.$$

Let Ξ_h be subspace of Λ_h consisting of functions with similarity zero across each edge or flat face, i.e.,

$$\Xi_h = \{\lambda \in \Lambda_h : \langle\!\langle \lambda \rangle\!\rangle_e = 0, e \in \mathcal{E}_h\}.$$

Note that the functions in the space Ξ_h shall serve as Lagrange multipliers in hybridization methods.

On the finite element space V_h, the discrete weak gradient operator $\nabla_{w,k-1}$ and the discrete weak convective operator $\boldsymbol{b} \cdot \nabla_{k-1}$ are respectively given by

$$(\nabla_{w,k-1} v)|_T = \nabla_{w,k-1,T}(v|_T),\ (\boldsymbol{b} \cdot \nabla_{k-1} v)|_T = \boldsymbol{b} \cdot \nabla_{k-1}(v|_T),\ \forall v \in V_h.$$

With an abuse of notation, from now on we shall drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ and $\boldsymbol{b} \cdot \nabla_{k-1}$.

Let Q_0 be the local L^2 projection on $\mathcal{T}_h$ and Q_b be the local L^2 projection on $\mathcal{E}_h$. Thus, $Q_0|_T$ is the L^2 projection from $L^2(T)$ onto $P_k(T)$ and $Q_b|_e$ is the L^2 projection from $L^2(e)$ onto $P_k(e)$. In addition, denote by $\mathbb{Q}_h$ the local L^2 projection onto $[P_{k-1}(T)]^d$. For any $v \in H^1(\Omega)$, we define the projection operator $Q_h : H^1(\Omega) \to V_h$ such that for each element $T \in \mathcal{T}_h$, we have

$$Q_h v = \{Q_0 v_0, Q_b v_b\},\ \{v_0, v_b\} = i_w(v) \in W(T).$$

Lemma 3.1. *([19]) The L^2-projection Q_h and $\mathbb{Q}_h$ have the following commutative property*

$$\nabla_w(Q_h \phi) = \mathbb{Q}_h(\nabla \phi),\ \forall \phi \in H^1(T). \tag{9}$$

3.2. **Algorithm.** In [14], a new variational form of the problem (1)-(2) is given by

$$(\mathcal{A}\nabla u, \nabla v) + \frac{1}{2}(\boldsymbol{b} \cdot \nabla u, v) - \frac{1}{2}(\boldsymbol{b} \cdot \nabla v, u) + (c_0 u, v) = (f, v),\ v \in H_0^1(\Omega), \tag{10}$$

where $c_0 = \frac{1}{2}(\nabla \cdot \boldsymbol{b}) + c$. And the variational form (10) was further used to establish a WG finite element formulation. In this section, we shall establish a hybridized finite element formulation based on (10) for the problem (1)-(2). To this end, we introduce two forms on V_h as follows

$$\begin{aligned}
a(v, w) &= (\mathcal{A}\nabla_w v, \nabla_w w) + \frac{1}{2}(\boldsymbol{b} \cdot \nabla_w v, w_0) - \frac{1}{2}(\boldsymbol{b} \cdot \nabla_w w, v_0) + (c_0 v_0, w_0),\\
s(v, w) &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial T},\\
b(v, \lambda) &= \sum_{T \in \mathcal{T}_h} \langle v_b, \lambda \rangle,
\end{aligned}$$

where $c_0 = \frac{1}{2}(\nabla \cdot \boldsymbol{b}) + c \geq 0$ for all $x \in \Omega$, and the usual L^2 inner product can be written locally on each element by

$$\begin{aligned}
(\mathcal{A}\nabla_w v, \nabla_w w) &= \sum_{T \in \mathcal{T}_h} (\mathcal{A}\nabla_w v, \nabla_w w)_T,\ (\boldsymbol{b} \cdot \nabla_w v, w_0) = \sum_{T \in \mathcal{T}_h} (\boldsymbol{b} \cdot \nabla_w v, w_0)_T,\\
(\boldsymbol{b} \cdot \nabla_w w, v_0) &= \sum_{T \in \mathcal{T}_h} (\boldsymbol{b} \cdot \nabla_w w, v_0)_T,\ (c_0 u_0, v_0) = \sum_{T \in \mathcal{T}_h} (c_0 u_0, v_0)_T.
\end{aligned}$$

Denote by $a_s(\cdot, \cdot)$ a stabilization of $a(\cdot, \cdot)$ given by

$$a_s(v, w) = a(v, w) + s(v, w). \tag{11}$$

For any $v \in \mathcal{V}_h$, let

$$|\!|\!| v |\!|\!| := \sqrt{a_s(v, v)}. \tag{12}$$

It has been verified in [14] that $|\!|\!| \cdot |\!|\!|$ defines a norm in the space $\mathcal{V}_h^0$.

The following weak Galerkin finite element scheme for the general second-order elliptic problem (1)-(2) was introduced and analyzed in [14].

Algorithm 3.2. ([14]) A numerical approximation for the problem (1)-(2) can be obtained by seeking $\bar{u}_h = \{\bar{u}_0, \bar{u}_b\} \in \mathcal{V}_h$ satisfying both $\bar{u}_b = Q_b g$ on $\partial\Omega$ and the following equation

$$a_s(\bar{u}_h, v) = (f, v_0), \ \forall v = \{v_0, v_b\} \in \mathcal{V}_h^0. \tag{13}$$

The weak Galerkin finite algorithm 3.2 can be hybridized in the finite element space $\mathcal{V}_h$ by using a Lagrange multiplier that shall enforce the continuity of the functions in V_h on interior element boundaries. The corresponding formulation can be described as follows

Algorithm 3.3. A numerical approximation for the problem (1)-(2) can be obtained by seeking $(u_h, \lambda_h) \in V_h \times \Xi_h$ satisfying both $u_b = Q_b g$ on $\partial\Omega$ and the following equations

$$a_s(u_h, v) - b(v, \lambda_h) \ = \ (f, v_0), \tag{14}$$

$$b(u_h, \mu) \ = \ 0, \tag{15}$$

for all $v = \{v_0, v_b\} \in V_h^0, \ \mu \in \Xi_h$.

Since $\lambda \in \Xi_h$ indicates $\lambda_L + \lambda_R = 0$ on each interior edge and $\lambda = 0$ on the boundary edge, then for any $v \in V_h$ and $\lambda \in \Xi_h$, we obtain

$$b(v, \lambda) = \sum_{e \in \mathcal{E}_h^0} \langle [\![u_h]\!]_e, \lambda_L \rangle. \tag{16}$$

3.3. **The relation between WG and HWG.** The aim of this subsection is to show that the HWG scheme (14)-(15) is equivalent to WG scheme (13) in that the solutions u_h from (14)-(15) and $\bar{u}_h$ from (13). But the HWG scheme is expected to be advantageous over WG for some special problems such as interface problems.

Theorem 3.4. *Let $u_h = \{u_0, u_b\} \in V_h$ be the first component of the solution of the HWG scheme (14)-(15). Then, one has $[\![u_b]\!]_e = 0$ for any $e \in \mathcal{E}_h^0$; i.e., $u_h \in \mathcal{V}_h$. Furthermore, $u_b = Q_b g$ on $\partial\Omega$ and u_h satisfies (13). Thus, $u_h = \bar{u}_h$.*

Proof. Let e be an interior edge or flat face shared by two elements T_1 and T_2. By letting $\mu = [\![u_h]\!]$ on $e \in \partial T_1$ (i.e. $\mu = -[\![u_h]\!]$ on $e \in \partial T_2$) and $\mu = 0$ otherwise in (15), we obtain from (16) that

$$0 = b(u_h, \mu) = \sum_{T \in \mathcal{T}_h} \langle u_h, \mu \rangle_{\partial T} = \int_e [\![u_h]\!]_e^2 ds,$$

which implies that $[\![u_h]\!]_e = 0$ for each interior edge or flat face $e \in \mathcal{E}_h^0$.

Now by restricting $v \in \mathcal{V}_h$ in (14) and using the fact $b(v, \lambda_h) = 0$, we obtain

$$a_s(u_h, v) = (f, v_0), \ \forall v \in \mathcal{V}_h^0,$$

which is the same as (13). It follows from the solution uniqueness for (13) that $u_h = \bar{u}_h$, which finishes the proof. □

4. **Stability condition for HWG.** It is easy to check that the following defines a norm in the finite element space Ξ_h

$$\|\lambda\|_{\Xi_h} = \left(\sum_{T\in\mathcal{T}_h} h_e \|\lambda\|_e^2 \right)^{1/2}. \tag{17}$$

As to V_h^0, for any $v \in V_h^0$, let

$$\|v\|_{V_h^0} = \left(|\!|\!| v |\!|\!| + \sum_{T\in\mathcal{T}_h} h_e \|[\![v]\!]_e\|_e^2 \right)^{1/2}. \tag{18}$$

We claim that $\|\cdot\|_{V_h^0}$ defines a norm in V_h^0. In fact, if $\|v\|_{V_h^0} = 0$, then $[\![v]\!]_e = 0$ for $e \in \mathcal{E}_h^0$. Thus, $v \in \mathcal{V}_h$. Hence $v = 0$ since $|\!|\!| \cdot |\!|\!|$ define a norm in $\mathcal{V}_h$. This proves the positivity of $\|\cdot\|_{V_h^0}$. The other properties for a norm can be checked trivially. In the rest of the paper, we always suppose $\mathcal{T}_h$ be a shape regular finite element partition detailed by [20].

Lemma 4.1. *([20])(Trace Inequality) Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, there exists a constant C such that for any $T \in \mathcal{T}_h$ and edge/face $e \in \partial T$, we have*

$$\|\varphi\|_e^2 \le C(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2), \tag{19}$$

where $\varphi \in H^1(T)$ is any function.

Lemma 4.2. *([20])(Inverse Inequality) Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Then, there exists a constant $C(n)$ such that*

$$\|\nabla\varphi\|_T \le C(n) h_T^{-1}\|\varphi\|_T, \ \forall T \in \mathcal{T}_h, \tag{20}$$

for any piecewise polynomial φ of degree n on $\mathcal{T}_h$.

Lemma 4.3. *(Boundedness) There exists a constant $C > 0$ such that*

$$|a_s(w, v)| \le C\|w\|_{V_h^0}\|v\|_{V_h^0}, \ \forall w, v \in V_h^0, \tag{21}$$

$$|b(v, \lambda)| \le C\|v\|_{V_h^0}\|\lambda\|_{\Xi_h}, \ \forall v \in V_h^0, \lambda \in \Xi_h. \tag{22}$$

Proof. To verify (21), we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&|a_s(w,v)| \\
= \ & \Big| (\mathcal{A}\nabla_w w, \nabla_w v) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w w, v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, w_0) + (c_0 w_0, v_0) \\
& + \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b w_0 - v_b, Q_b v_0 - v_b\rangle_{\partial T} \Big| \\
\le \ & \sum_{T\in\mathcal{T}_h} \Big(\|\mathcal{A}\|_\infty \|\nabla_w w\|_T \|\nabla_w v\|_T + \|\boldsymbol{b}\|_\infty \|\nabla_w w\|_T \|v\|_T + \|\boldsymbol{b}\|_\infty \|\nabla_w v\|_T \|w\|_T \\
& + \|c_0\|_\infty \|w\|_T \|v\|_T \Big) + \sum_{T\in\mathcal{T}_h} h^{-1}\|Q_b w_0 - v_b\|_{\partial T}\|Q_b v_0 - v_b\|_{\partial T} \\
\le \ & C\Big(\big(\sum_{T\in\mathcal{T}_h} \|\nabla_w w\|_T^2\big)^{1/2} \big(\sum_{T\in\mathcal{T}_h} \|\nabla_w v\|_T^2\big)^{1/2} + \big(\sum_{T\in\mathcal{T}_h} \|\nabla_w w\|_T^2\big)^{1/2} \big(\sum_{T\in\mathcal{T}_h} \|v\|_T^2\big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{T\in\mathcal{T}_h}\|\nabla_w v\|_T^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|w\|_T^2\Big)^{1/2}+\Big(\sum_{T\in\mathcal{T}_h}\|w\|_T^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|v\|_T^2\Big)^{1/2}\\
&+\Big(\sum_{T\in\mathcal{T}_h}h^{-1}\|Q_bw_0-w_b\|_{\partial T}^2\Big)^{1/2}\Big(h^{-1}\sum_{T\in\mathcal{T}_h}\|Q_bv_0-v_b\|_{\partial T}^2\Big)^{1/2}\Bigg)\\
\le\ & C\|w\|_{V_h^0}\|v\|_{V_h^0}.
\end{aligned}
$$

As to (22), it follows from (16) and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
|b(v,\lambda)| &= \Big|\sum_{T\in\mathcal{T}_h}\langle v_b,\lambda\rangle_{\partial T}\Big| = \Big|\sum_{e\in\mathcal{E}_h^0}\langle [\![v]\!]_e,\lambda\rangle_{\partial T}\Big|\\
&\le \left(h_e^{-1}\sum_{e\in\mathcal{E}_h^0}\|[\![v]\!]_e\|_e^2\right)^{1/2}\left(h_e^{-1}\sum_{e\in\mathcal{E}_h^0}\|\lambda\|_e^2\right)^{1/2}\\
&\le \|v\|_{V_h^0}\|\lambda\|_{\Xi_h}.
\end{aligned}
$$

□

Lemma 4.4. *(Coercivity) For any $v\in\mathcal{V}_h^0$, there exists a constant $C>0$ such that*

$$
a_s(v,v)\ge C\|v\|_{V_h^0}^2. \tag{23}
$$

Proof. For any $v\in\mathcal{V}_h^0$, we have $\|v\|_{V_h^0}^2=|\!|\!|v|\!|\!|$, which implies the estimate (23) holds true with $C=1$. □

Lemma 4.5. *(Inf-sup condition) There exists a constant $C>0$ such that*

$$
\sup_{v\in V_h^0}\frac{b(v,\rho)}{\|v\|_{V_h^0}}\ge C\|\rho\|_{\Xi_h},\ \ \forall\rho\in\Xi_h. \tag{24}
$$

Proof. For any $\rho\in\Xi_h$, we have $\langle\!\langle\rho\rangle\!\rangle_e=0$ or equivalently $\rho^L+\rho^R=0$ on each interior edge $e\in\mathcal{T}_h$ and $\rho=0$ on each boundary edge $e\in\partial\Omega$. By letting $v=\{0,h_e\rho\}\in V_h^0$ in $b(v,\rho)$ and $s(v,v)$, we obtain

$$
\begin{aligned}
b(v,\rho) &= \sum_{T\in\mathcal{T}_h}\langle v_b,\rho\rangle_{\partial T}\\
&= \sum_{e\in\mathcal{E}_h^0}\langle v_b^L,\rho^L\rangle_e+\sum_{e\in\mathcal{E}_h^0}\langle v_b^R,\rho^R\rangle_e\\
&= \sum_{e\in\mathcal{E}_h^0}\langle v_b^L-v_b^R,\rho^L\rangle_e\\
&= 2\sum_{e\in\mathcal{E}_h^0}h_e\|\rho\|_e^2,
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
s(v,v) &= \sum_{T\in\mathcal{T}_h}h_T^{-1}h_e^2\|\rho\|_{\partial T}\\
&= \sum_{e\in\mathcal{E}_h^0}h_T^{-1}h_e^2(\|\rho^L\|^2+\|\rho^R\|^2)\\
&\le 2h_e\sum_{e\in\mathcal{E}_h^0}\|\rho\|_e^2.
\end{aligned} \tag{26}
$$

By (4),the Cauchy-Schwarz inequality, the trace inequality (19) and the inverse inequality (20), we have

$$\begin{aligned}(\nabla_w v, \nabla_w v)_T &= \sum_{e\in\partial T}\langle v_b^*, \nabla_w v\cdot\boldsymbol{n}\rangle_e \le \sum_{e\in\partial T} h_e\|\rho^*\|_e\|\nabla_w v\|_e\\ &\le C\sum_{e\in\partial T} h_e^{\frac{1}{2}}\|\rho^*\|_e\|\nabla_w v\|_T,\end{aligned}$$

which implies

$$\|\nabla_w v\|_T \le C\sum_{e\in\partial T} h_e^{\frac{1}{2}}\|\rho^*\|_e^2,$$

where v_b^* is chosen to be v_b^1 or v_b^2 according to the relative position of v_b and e, and the same to ρ^*. Summing over all elements yields

$$(\nabla_w v, \nabla_w v) \le C\sum_{e\in\mathcal{E}_h^0} h_e\|\rho^*\|_e^2. \tag{27}$$

It follows from (26) and (27) that

$$|\!|\!|v|\!|\!|^2 \le C\sum_{e\in\mathcal{E}_h^0} h_e\|\rho^*\|_e^2. \tag{28}$$

By the fact of $\rho^L+\rho^R=0$, we obtain

$$h_e^{-1}\|[\![v]\!]_e\|_e^2 = h_e^{-1}\|h_e\rho^L - h_e\rho^R\|_e^2 = 2h_e\|\rho\|_e^2. \tag{29}$$

Combining(25), (27), (28) and (29) gives

$$\sup_{v\in V_h^0}\frac{b(v,\rho)}{\|v\|_{V_h^0}} \ge C\frac{\sum_{e\in\mathcal{E}_h^0}\|\rho\|_e^2}{\sum_{e\in\mathcal{E}_h^0}\|\rho\|_e} = C\|\rho\|_{\Xi_h},$$

which finishes the proof. □

5. **Error analysis.** The goal of this section is to establish the error estimate for the HWG finite element solution $\{u_h;\lambda_h\}$ arising from (14)-(15). To this end, we let $(u_h;\lambda_h)\in V_h\times\Xi_h$ be the WG finite element solution arising from the numerical scheme (14)-(15), where $u=\{u_0,u_b\}$. Assume that the exact solution of (1)-(2) is given by u. Let λ be given by

$$\lambda = \frac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{n})(Q_b u + 2u) - \mathcal{A}\nabla u\cdot\boldsymbol{n},\ \text{on } \partial T.$$

Defined error functions by

$$e_h = \{e_0, e_b\} = u_h - Q_h u = \{u_0 - Q_0 u, u_b - Q_b u\},\ \epsilon_h = \lambda_h - Q_b\lambda.$$

Lemma 5.1. *Let $\mathcal{T}_h$ be a finite element partition of Ω that is shape regular. Let u and $(u_h;\lambda_h)\in V_h\times\Xi_h$ be the solutions of (1)-(2) and (14)-(15), respectively. Then, for any $v\in V_h^0$ and $\rho\in\Xi_h$, the error functions e_h and ϵ_h satisfies*

$$\begin{aligned} a_s(e_h, v) + b(v,\epsilon_h) &= \xi_{u,\boldsymbol{b}}(v) + t_{u,\boldsymbol{b}}(v) + \lambda_{u,\boldsymbol{b}}(v) - l_u(v) - S(Q_h u, v), &(30)\\ b(e_h,\rho) &= 0, &(31)\end{aligned}$$

where

$$\begin{aligned}
\xi_{u,\boldsymbol{b}}(v) &= \frac{1}{2}(\nabla\cdot\boldsymbol{b},(Q_0u-u)v_0)+(\boldsymbol{b}\cdot\nabla v_0,Q_0u-u)+c_0(u-Q_0u,v_0),\\
l_u(v) &= \sum_{T\in\mathcal{T}_h}\langle v_0-v_b,(\mathcal{A}\nabla u-\mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n}\rangle_{\partial T},\\
t_{u,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},(u-Q_bu)(v_0-v_b)\rangle_{\partial T},\\
\lambda_{u,\boldsymbol{b}}(v) &= \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},(u-Q_0u)(v_0-v_b)\rangle_{\partial T}.
\end{aligned}$$

Proof. Equation (31) is obvious by the fact $[\![e_h]\!]=0$ from Theorem 3.4. It remains to prove (30). Testing (1) by using v_0 of $v=\{v_0,v_b\}\in V_h^0$, we arrive at

$$\begin{aligned}
(f,v_0) &= (\mathcal{A}\nabla u,\nabla v_0)+\frac{1}{2}((\nabla\cdot\boldsymbol{b})u,v_0)+(\boldsymbol{b}\cdot\nabla u,v_0)+(c_0u,v_0)\\
&\quad-\sum_{T\in\mathcal{T}_h}\langle(\mathcal{A}\nabla u)\cdot\boldsymbol{n},v_0-v_b\rangle_{\partial T}-\sum_{T\in\mathcal{T}_h}\langle(\mathcal{A}\nabla u)\cdot\boldsymbol{n},v_b\rangle_{\partial T},
\end{aligned}\tag{32}$$

where $c_0=\frac{1}{2}(\nabla\cdot\boldsymbol{b})+c$. We first deal with the form $(\mathcal{A}\nabla u,\nabla v_0)$ in (32). In fact, for any $v\in V_h^0$, it follows from Lemma 3.1 and (5) that

$$\begin{aligned}
(\mathcal{A}\nabla_w(Q_hu),\nabla_wv)_T &= (\mathcal{A}\mathbb{Q}_h(\nabla u),\nabla_wv)_T\\
&= (\nabla v_0,\mathcal{A}\mathbb{Q}_h(\nabla u))_T-\langle v_0-v_b,(\mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n}\rangle_{\partial T}\\
&= (\mathcal{A}\nabla u,\nabla v_0)_T-\langle(\mathcal{A}\mathbb{Q}_h(\nabla u))\cdot\boldsymbol{n},v_0-v_b\rangle_{\partial T}
\end{aligned}$$

Thus,

$$(\mathcal{A}\nabla u,v_0) = (\mathcal{A}\nabla_w(Q_hu),\nabla_wv)+\langle(\mathcal{A}\mathbb{Q}_h(\nabla u)\cdot\boldsymbol{n},v_0-v_b\rangle_{\partial T}$$

Then, we handle the term $\frac{1}{2}((\nabla\cdot\boldsymbol{b})u,v_0)+(\boldsymbol{b}\cdot\nabla u,v_0)$ in (32). For $w=\{w_0,w_b\}\in V_h$, according to the definition of discrete weak convective operator (6), we have

$$(\boldsymbol{b}\cdot\nabla_ww,v_0)=-(\boldsymbol{b}\cdot\nabla v_0,w_0)-(\nabla\cdot\boldsymbol{b},w_0v_0)+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},w_bv_0\rangle_{\partial T}\tag{33}$$

and

$$\begin{aligned}
(\boldsymbol{b}\cdot\nabla_wv,w_0) &= -(\boldsymbol{b}\cdot\nabla w_0,v_0)-(\nabla\cdot\boldsymbol{b},w_0v_0)+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},v_bw_0\rangle_{\partial T}\\
&= (\boldsymbol{b}\cdot\nabla v_0,w_0)-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},(v_0-v_b)w_0\rangle_{\partial T}.
\end{aligned}\tag{34}$$

By letting $w=Q_hu$ in (33) and $w_0=Q_0u$ in (34), we obtain the following equations

$$\begin{aligned}
(\boldsymbol{b}\cdot\nabla_w(Q_hu),v_0) &= -(\boldsymbol{b}\cdot\nabla v_0,Q_0u)-(\nabla\cdot\boldsymbol{b},(Q_0u)v_0)\\
&\quad+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},Q_bu(v_0-v_b)\rangle_{\partial T}\\
&\quad+\sum_{T\in\mathcal{T}_h}\langle(\boldsymbol{b}\cdot\boldsymbol{n})Q_bu,v_b\rangle_{\partial T},
\end{aligned}\tag{35}$$

$$(\boldsymbol{b}\cdot\nabla_wv,Q_0u)=(\boldsymbol{b}\cdot\nabla v_0,Q_0u)-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{b}\cdot\boldsymbol{n},Q_0u(v_0-v_b)\rangle_{\partial T}.\tag{36}$$

Thus, by using the integration by parts and (35) and (36) , we obtain

$$
\begin{aligned}
& (\boldsymbol{b}\cdot\nabla u, v_0) + \frac{1}{2}((\nabla\cdot\boldsymbol{b})u, v_0) \\
= \;& -(\boldsymbol{b}\cdot\nabla v_0, u) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, uv_0) + \sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, uv_0\rangle_{\partial T} \\
= \;& -(\boldsymbol{b}\cdot\nabla v_0, Q_0 u) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0 u)v_0) + (\boldsymbol{b}\cdot\nabla v_0, Q_0 u - u) \\
& + \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0 u - u)v_0) + \sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, uv_0\rangle_{\partial T} \\
= \;& -\frac{1}{2}(\boldsymbol{b}\cdot\nabla v_0, Q_0 u) - \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0 u)v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla v_0, Q_0 u) + (\boldsymbol{b}\cdot\nabla v_0, Q_0 u - u) \\
& + \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0 u - u)v_0) + \sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, u(v_0 - v_b)\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, uv_b\rangle_{\partial T} \\
= \;& \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w(Q_h u), v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, Q_0 u) + \frac{1}{2}(\nabla\cdot\boldsymbol{b}, (Q_0 u - u)v_0) \\
& + (\boldsymbol{b}\cdot\nabla v_0, Q_0 u - u) + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, (u - Q_b u)(v_0 - v_b)\rangle_{\partial T} \\
& + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle \boldsymbol{b}\cdot\boldsymbol{n}, (u - Q_0 u)(v_0 - v_b)\rangle_{\partial T} + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\langle (\boldsymbol{b}\cdot\boldsymbol{n})(Q_b u + 2u), v_b\rangle_{\partial T}. \quad (37)
\end{aligned}
$$

By the difinition of $a(u,v)$, we can get

$$
\begin{aligned}
a(Q_h u, v) \;=\;& (\mathcal{A}\nabla_w(Q_h u), \nabla_w v) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w(Q_h u), v_0) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla_w v, Q_0 u) \\
& + (c_0 Q_0 u, v_0). \quad (38)
\end{aligned}
$$

With $\lambda = \frac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{n})(Q_b u + 2u) - \mathcal{A}\nabla u\cdot\boldsymbol{n}$ we have

$$
\begin{aligned}
b(v, Q_b\lambda) \;=\;& \sum_{T\in\mathcal{T}_h}\langle v, Q_b\lambda\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\langle v, \lambda\rangle_{\partial T} \\
=\;& \sum_{T\in\mathcal{T}_h}\langle \frac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{n})(Q_b u + 2u) - \mathcal{A}\nabla u\cdot\boldsymbol{n}, v_b\rangle_{\partial T}
\end{aligned}
$$

According to the (13) and the definition of e_h, we can get

$$
\begin{aligned}
a_s(e_h, v) + b(v, \epsilon_h) \;=\;& a_s(u_h, v) + b(v, \lambda_h) - a_s(Q_h u, v) - b(v, Q_b\lambda) \\
=\;& (f, v_0) - a(Q_h u, v) - s(Q_h u, v) - b(v, Q_b\lambda). \quad (39)
\end{aligned}
$$

Combining the (32), (33) and (37)-(39), we obtain

$$
a_s(e_h, v) + b(v, \epsilon_h) = \xi_{u,\boldsymbol{b}}(v) + t_{u,\boldsymbol{b}}(v) + \lambda_{u,\boldsymbol{b}}(v) - l_u(v) - s(Q_h u, v),
$$

which completes the proof. □

In the rest of the section, we will prove the error estimate for the HWG finite element solution $\{u_h; \lambda_h\}$ arising from (14)-(15). We rewrite the error equations (30)-(31) by

$$
\begin{aligned}
a_s(e_h, v) + b(v, \epsilon_h) \;=\;& \zeta_u(v), \;\; \forall v\in V_h^0, \\
b(e_h, \rho) \;=\;& 0, \;\; \forall \rho\in\Xi_h,
\end{aligned}
$$

where

$$\zeta_u(v) = \xi_{u,\boldsymbol{b}}(v) + t_{u,\boldsymbol{b}}(v) + \lambda_{u,\boldsymbol{b}}(v) - l_u(v) - S(Q_h u, v)$$

is a linear function. The above is a saddle problem for which the Brezzi's theorem [4] can be applied for an analysis on its stability and solvability.

Theorem 5.2. *Suppose $\mathcal{T}_h$ is a finite element partition of Ω that is shape regular. Let u, $\{u_h, \lambda_h\} \in V_h \times \Xi_h$ be solutions of (1)-(2) and (14)-(15), respectively. Then, there exists a constant C such that*

$$\|Q_h u - u_h\|_{V_h^0} + \|Q_b \lambda - \lambda_h\|_{\Xi_h} \le Ch^k \|u\|_{k+1}. \tag{40}$$

Proof. Since all the conditions of Brezzi's theorem [4] have been verified in Section 4, it from the Brezzi's theorem that

$$\|Q_h u - u_h\|_{V_h^0} + \|Q_b \lambda - \lambda_h\|_{\Xi_h} \le C\|\zeta_u\|_{V_h^{0\prime}}. \tag{41}$$

For any $v \in V_h^0$, it has been shown in [14] that

$$|\zeta_u(v)| \le Ch^k \|u\|_{k+1} |||v|||.$$

Thus, we have

$$\|\zeta_u\|_{V_h^{0\prime}} = \sup_{v \in V_h^0} \frac{|\zeta_u(v)|}{\|v\|_{V_h^0}} \le \sup_{v \in V_h^0} \frac{|\zeta_u(v)|}{|||v|||} \le Ch^k \|u\|_{k+1}. \tag{42}$$

Substituting (42) into (41) yields the desired estimate (40), which completes the proof. □

6. **Variable reduction.** The degrees of freedom in the WG scheme (13) are created by the interior variables u_0 and the interface variables u_b. For the HWG scheme (14)-(15), more unknowns are added to the picture from the Lagrange multiplier λ_h. Hence, the size of the discrete system arising from either (13) or (14)-(15) is enormously large. In order to reduce the size of the discrete systems, we present a Schur complement for the WG scheme (13) based on HWG scheme (14)-(15) in the rest of this section. The method shall eliminate all the interior unknowns u_0 and the interface unknow λ_h, and produce a much reduced system involving only the interface variables u_b.

6.1. **Theory of variable reduction.** Define the interface finite element space $\mathbb{B}_h$ as the restriction of the finite element space $\mathcal{V}_h$ on the set of edges $\mathcal{E}_h$; i.e.,

$$\mathbb{B}_h = \{v_b : v_b \in P_{k-1}(e), e \in \mathcal{E}_h\}.$$

Then, $\mathbb{B}_h$ is a Hilbert space equipped with the following inner product

$$\langle w_b, v_b \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \langle w_b, v_b \rangle_e, \forall w_b, v_b \in \mathbb{B}_h.$$

Denote by $\mathcal{E}_h^0$ the subspace of $\mathcal{E}_h$ consisting of functions with vanishing boundary.

Now, we define an operator $G_f : \mathcal{E}_h \mapsto \mathcal{E}_h^0$ such that for any $v_b \in \mathcal{E}_h$, the image $G_f(v_b)$ is obtained by the following three steps.

Step 1. On each element $T \in \mathcal{T}_h$, compute w_0 in terms of w_b by solving the following local equations

$$a_{s,T}(w_h, v) = (f, v_0)_T, \ \forall v = \{v_0, 0\} \in V_h(T). \tag{43}$$

Here $w_h = \{w_0, w_b\} \in V_h(T)$. We denote the solution w_0 by $w_0 = F_f(w_b)$.

Step 2. Compute $\theta_{h,T} \in \Lambda_k(\partial T)$ on each element $T \in \mathcal{T}_h$ such that

$$b_T(v, \theta_{h,T}) = a_{s,T}(w_h, v), \ \forall v = \{0, v_b\} \in V_h(T). \tag{44}$$

Thus, we can obtain a function $\theta_h \in \Lambda_h$. Denote θ_h by $\theta_h = L_f(w_b)$.

Step 3. Set $G_f(w_b)$ as the similarity of θ_h on $\mathcal{E}_h$; i.e.

$$G_f(w_b) = \langle\!\langle \theta_h \rangle\!\rangle.$$

Adding the two equations (43) and (44) gives

$$b_T(v, \theta_{h,T}) = a_{s,T}(w_h, v) - (f, v_0)_T, \ \forall v \in \{v_0, v_b\} \in V_k(T). \tag{45}$$

By the superposition principle one has the following Lemma.

Lemma 6.1. *For any $w_b \in \mathbb{B}_h$, one has*

$$G_f(w_b) = G_0(w_b) + G_f(0), \tag{46}$$

where G_0 is the operator with respect to $f = 0$.

Theorem 6.2. *For any $w_b, v_b \in \mathbb{B}_h^0$, one has*

$$\sum_{e \in \mathcal{E}_h^0} \langle G_0(w_b), G_0(v_b) \rangle_e = a_s(w_h, v_h),$$

where $w_h = \{F_f(w_b), w_b\}$ and $v_h = \{F_f(v_b), v_b\}$. In other words, the linear operator G_0, when restricted to the subspace $\mathcal{E}_h^0$, is symmetric and positive definite.

Proof. For any $w_b, v_b \in \mathcal{E}_h^0$, let

$$\begin{aligned} w_h &= \{F_0(w_b), w_b\}, \ \theta_h = L_0(w_b), \\ v_h &= \{F_0(v_b), v_b\}. \end{aligned}$$

Applying (45) with $f = 0$ we arrive at

$$\begin{aligned} \sum_{e \in \mathcal{E}_h^0} \langle G_0(w_b), v_b \rangle_e &= \sum_{e \in \mathcal{E}_h^0} \left\langle \langle\!\langle \theta_h \rangle\!\rangle_e, v_b \right\rangle_e = \sum_{T \in \mathcal{T}_h} \langle \theta_{h,T}, v_b \rangle \\ &= \sum_{T \in \mathcal{T}_h} b(v_h, \theta_{h,T}) = \sum_{T \in \mathcal{T}_h} a_{s,T}(w_h, v_h), \end{aligned}$$

which finishes the proof. $\square$

Theorem 6.3. *Suppose $\bar{u}_b \in \mathbb{B}_h$ be any function such that $\bar{u}_b = Q_b g$ on $\partial\Omega$. Let $\bar{u}_0 = F_f(\bar{u}_b)$. Then, $\bar{u}_h = \{\bar{u}_0, \bar{u}_b\}$ is the solution of (13) if and only if $\bar{u}_b$ satisfies the following operator equation*

$$G_f(\bar{u}_b) = 0. \tag{47}$$

Proof. The proof will be divided into two steps:

Step 1. We will prove the necessity.

Let $\bar{u}_h = \{\bar{u}_0, \bar{u}_b\}$ be the solution of the WG scheme (13). Then, from Theorem 3.4 there exists $\bar{\lambda}_h \in \Xi_h$ such that $(\bar{u}_h; \bar{\lambda}_h)$ is the solution of HWG scheme (14)-(15). Taking $v = \{v_0, 0\} \in V_k(T)$ on T and zero elsewhere in (14), we have

$$a_{s,T}(\bar{u}_h, v) = (f, v_0), \ \forall v = \{v_0, 0\} \in V_k(T),$$

which implies that $\bar{u}_h$ satisfies the local equation (43).

Next taking $v = \{0, v_b\} \in V_k(T)$ on T and zero elsewhere in (14) gives

$$b_T(\bar{\lambda}_{h,T}, v) = a_{s,T}(\bar{u}_h, v),\ \forall v = \{0, v\} \in V_k(T),$$

where $\bar{\lambda}_{h,T}$ is the restriction of $\bar{\lambda}_h$ on ∂T. This implies that $\bar{\lambda}_h$ satisfies (44). Then from the definition of G_f, we have

$$G_f(\bar{u}_b) = \langle\!\langle \bar{\lambda}_h \rangle\!\rangle.$$

Since $\bar{\lambda}_h \in \Xi_h$ implies $\langle\!\langle \bar{\lambda}_h \rangle\!\rangle = 0$,

$$G_f(\bar{u}_b) = 0,$$

which finishes the proof of the necessity. □

Step 2. We will prove the sufficiency.

Suppose $\bar{u}_b \in \mathbb{B}_h$ satisfies (47) and the boundary condition: $\bar{u}_b = Q_b g$ on $\partial\Omega$. Let $\bar{u}_0 = F_f(\bar{u}_b)$. Then, $\bar{u}_0$ is the solution of the following local equations on each $T \in \mathcal{T}_h$,

$$a_{s,T}(\bar{u}_h, v) = (f, v_0)_T,\ \forall v = \{v_0, 0\} \in V_k(T), \tag{48}$$

where $\bar{u}_h = \{\bar{u}_0, \bar{u}_b\}$.

Now on each element T, we compute $\bar{\lambda}_{h,T} \in \Lambda_k(\partial T)$ by solving the local equation

$$b_T(v, \bar{\lambda}_{h,T}) = a_{s,T}(\bar{u}_h, v),\ \forall v = \{0, v_b\} \in V_k(T). \tag{49}$$

We let $\bar{\lambda}_h \in \Lambda_h$ be the function given by $\bar{\lambda}_h|_{\partial T} = \bar{\lambda}_{h,T}$ with modification $\bar{\lambda}_h|_{\partial\Omega} = 0$. By (47) and the definition of G_f, on each $e \in \mathcal{E}_h^0$ we have

$$\langle\!\langle \bar{\lambda}_h \rangle\!\rangle = G_f(u_b) = 0,$$

which means $\bar{\lambda}_h \in \Xi_h$. Subtracting (49) from (48) gives

$$a_{s,T}(\bar{u}_h, v) - b_T(v, \bar{\lambda}_{h,T}) = (f, v_0),\ \forall v = \{v_0, v_b\} \in V_k(T).$$

Summing up the above equations over all element $T \in \mathcal{T}_h$ yields

$$a_s(\bar{u}_h, v) - b(v, \bar{\lambda}_h) = (f, v_0),\ \forall v = \{v_0, v_b\} \in V_h^0. \tag{50}$$

For any $\sigma \in \Xi_h$, we obtain from (16) that

$$b(\bar{u}_h, \sigma) = \sum_{e \in \mathcal{E}_h^0} \langle [\![\bar{u}_h]\!]_e, \sigma_L \rangle_e = 0 \tag{51}$$

Equations (50) and (51) indicate that $(\bar{u}_h; \bar{\lambda}_h)$ is a solution to HDG scheme (14)-(15). Recalling that $\bar{u}_b = Q_b g$ on $\partial\Omega$, we see from Theorem 3.4 that $\bar{u}_h$ is the WG solution defined by the formulation (13). This completes the proof of the sufficiency.

6.2. **Computational algorithm with reduced variables.** From (46), the equation (47) can rewritten as

$$G_0(\bar{u}_b) = -G_f(0). \tag{52}$$

Let $q_b \in \mathbb{B}_h$ satisfy $q_b = Q_b g$ on $\partial\Omega$ and zero elsewhere. It follows from the linearity of G_0 that

$$G_0(\bar{u}_b) = G_0(\bar{u}_b - q_b) + G_0(q_b).$$

Substituting the above into (52) gives

$$G_0(\bar{u}_b - q_b) = -G_f(0) - G_0(q_b).$$

It is easy to check that the function $p_b = \bar{u}_b - q_b$ has vanishing boundary value. By letting $r_b = -G_f(0) - G_0(q_b)$, we have

$$G_0(p_b) = r_b. \tag{53}$$

The reduced system of linear equation (53) is actually a Schur complement formulation for WG scheme (13). Note that (53) involves only the variables representing the value of the function on $\mathcal{E}_h^0$.

Variable Reduction Algorithm The solution u_h to the WG scheme (13) can be obtained step-by-step as fellows

Step 1. On each element $T \in \mathcal{T}_h$, solve for r_b from the following equation

$$r_b = -G_f(0) - G_0(q_b).$$

Step 2. Solve for $p_b \in \mathbb{B}_h^0$ by the equation (53).

Step 3. Compute $\bar{u}_b = p_b + q_b$ to get the solution on element boundaries. Then, on each element T, compute $\bar{u}_0 = F_f(u_b)$ by solving the local equations (43).

Remark 1. Step 1 requires the inversion of local stiffness matrices and can be accomplished in parallel. The computational complexity is linear with respect to the number of unknowns. Step 2 is only computation extensive part of the implementation.

7. Conclusions.

We have presented a hybridized weak Galerkin finite element scheme for general second-order elliptic problems. The scheme is based on the use of a Lagrange multiplier defined on the element boundaries. We further developed a Schur complement formulation for the hybridized weak Galerkin finite element scheme for implementation purpose. The Schur complement formulation have arrived at a much reduced system involving only the interface variables u_b by elimination the interior unknowns and Lagrange multipliers. It is worth pointing out that a skew symmetric form has been used for handling the convection term to get the stability in the hybridized weak Galerkin finite element formulation. Optimal order error estimates are derived for the corresponding hybridized weak Galerkin finite element approximations.

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