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A FAMILY OF POTENTIAL WELLS FOR A WAVE EQUATION

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Abstract. In this paper, a family of potential wells are introduced by means of the modified depths of the potential wells. These potential wells are employed to study the initial-boundary value problem for a wave equation. The expression of the depths of the potential wells is derived. Global existence and finite time blow-up of weak solutions with the subcritical initial energy and the critical initial energy are obtained, respectively. Moreover, some numerical simulations of the depths of the potential wells are carried out.

1. **Introduction.** In this paper, we study the following initial-boundary value problem for a wave equation with single source term

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-1}u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, and the power index p of the source term satisfies

$$1 < p < \begin{cases} \infty & \text{if } N \leq 2, \\ \dfrac{N}{N-2} & \text{if } N > 2. \end{cases}$$

Eq. $(1)_1$ is the classical wave equation that has been widely investigated. We omit further comments. We focus on a family of new potential wells and their applications to Eq. $(1)_1$. The potential well is proposed by Sattinger [15] (see also Payne and Sattinger [14]). In general, making use of the energy functional $J(u)$ and the Nehari functional $I(u)$, we are able to define the potential well

$$W = \{u \in H_0^1(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\},$$

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and its outside set

$$V = \{u \in H_0^1(\Omega) | I(u) < 0, J(u) < d\},$$

where the depth of the potential well

$$d = \inf_{u \in \mathcal{N}} J(u),$$

the Nehari manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega) | I(u) = 0\} \setminus \{0\}.$$

When the initial energy is controlled by the depth of the potential well, the well-posedness of solutions for problem under consideration can be investigated by the potential well theory (see e.g. [2, 3, 4, 6, 13, 16, 18, 22] and the references therein).

Moreover, Liu [8] introduced a family of potential wells and their outside sets

$$W_\delta = \{u \in H_0^1(\Omega) | J_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\},$$

$$V_\delta = \{u \in H_0^1(\Omega) | J_\delta(u) < 0, J(u) < d(\delta)\},$$

where $J_\delta(u)$ is an auxiliary functional originated from $J(u)$, and $d(\delta)$ (namely the depths of the potential wells) needs to be estimated in advance. Subsequently, Liu and Zhao [12] introduced

$$W_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\},$$

$$V_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) < 0, J(u) < d(\delta)\},$$

where

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u),$$

$$\mathcal{N}_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) = 0\} \setminus \{0\},$$

and $I_\delta(u)$ is an auxiliary functional originated from $I(u)$. Through the study on the properties of $d(\delta)$, W_δ and V_δ, many nonlinear evolution equations can be handled (see e.g. [11, 19, 20, 21]). Thus the potential well theory has been enriched and developed greatly.

The main purpose of this paper is to construct a family of new potential wells and their outside sets by modifying the depths of the potential wells inspired by [17]. Thus it is unnecessary to introduce the Nehari functional and the Nehari manifold. The innovation of this paper is that the depths of the potential wells can be computed exactly by an effective approach instead of being estimated so that it is easy to understand the spatial structure of the potential wells. Then the applications of this family of potential wells to the problems concerned can simplify the proofs of the corresponding results. We take problem (1) as an object to illustrate our ideas, which can be applied to a lot of other similar models [1, 5, 9, 10].

This paper is organized as follows. In Section 2, we handle problem (1) by modifying the depths of the potential wells and constructing a family of potential wells. Global existence and finite time blow-up of weak solutions for problem (1) with the subcritical initial energy are obtained. In Section 3, global existence and finite time blow-up of weak solutions for problem (1) with the critical initial energy are obtained. In Section 4, through numerical simulations, we show some intuitive relations between the depths of the potential wells and several parameters.

2. Problem (1) with the subcritical initial energy.

2.1. Introduction of a family of potential wells and computation of their depths.

Throughout the paper, in order to simplify the notations, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)},\ \|\cdot\| := \|\cdot\|_2,\ (u,v) := \int_\Omega uv\,\mathrm{d}x.$$

We define the total energy function related to problem (1)

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1},$$

which satisfies the energy identity $E(t) = E(0)$ for all $t \geq 0$. Moreover, we introduce the auxiliary functional

$$J_\delta(u) = \frac{\delta}{2}\|\nabla u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},\quad 0 < \delta \leq 1.$$

Now we are in a position to define the depths of the potential wells

$$d(\delta) = \max_{y\in[0,\infty)} g_\delta(y),$$

where $y = \|\nabla u\|$,

$$g_\delta(y) = \frac{\delta}{2}y^2 - \frac{C_1^{p+1}}{p+1}y^{p+1},$$

and C_1 is the best Sobolev constant for the embedding from $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$. Obviously, this definition is different from those in the works mentioned before.

Let $g'_\delta(y) = 0$, then

$$y_\delta = \delta^{\frac{1}{p-1}} C_1^{-\frac{p+1}{p-1}}. \tag{2}$$

Hence

$$d(\delta) = g_\delta(y_\delta) = \frac{p-1}{2(p+1)}\delta^{\frac{p+1}{p-1}} C_1^{-\frac{2(p+1)}{p-1}}. \tag{3}$$

By virtue of (2) and (3), we can get

$$y_\delta = \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}.$$

Thus we can define a family of potential wells

$$W_\delta = \left\{ u \in H_0^1(\Omega) \middle| \|\nabla u\| < \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}} \right\},$$

and their outside sets

$$V_\delta = \left\{ u \in H_0^1(\Omega) \middle| \|\nabla u\| > \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}} \right\}.$$

Obviously,

$$\partial W_\delta = \partial V_\delta = \left\{ u \in H_0^1(\Omega) \middle| \|\nabla u\| = \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}} \right\}.$$

Lemma 2.1. *Let $u \in H_0^1(\Omega)$.*

(i) If $u \in W_\delta$ and $\|\nabla u\| \neq 0$, then $\delta\|\nabla u\|^2 > \|u\|_{p+1}^{p+1}$.

(ii) If $u \in \partial W_\delta$, then $\delta\|\nabla u\|^2 \geq \|u\|_{p+1}^{p+1}$.

(iii) If $\delta\|\nabla u\|^2 < \|u\|_{p+1}^{p+1}$, then $u \in V_\delta$.

(iv) If $\delta\|\nabla u\|^2 = \|u\|_{p+1}^{p+1}$ *and* $\|\nabla u\| \neq 0$*, then* $u \in H_0^1(\Omega)\backslash W_\delta = V_\delta \cup \partial V_\delta$.

Proof. (i) Since $u \in W_\delta$, we have

$$\|\nabla u\| < \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}},$$

which, together with (3), gives

$$\|\nabla u\| < \delta^{\frac{1}{p-1}} C_1^{-\frac{p+1}{p-1}}.$$

We further get

$$\delta > C_1^{p+1}\|\nabla u\|^{p-1}.$$

Since $\|\nabla u\| \neq 0$, multiplying the above inequality by $\|\nabla u\|^2$, we obtain

$$\delta\|\nabla u\|^2 > C_1^{p+1}\|\nabla u\|^{p+1},$$

and so

$$\delta\|\nabla u\|^2 > \|u\|_{p+1}^{p+1}.$$

(ii) From $u \in \partial W_\delta$ we have

$$\|\nabla u\| = \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}.$$

By similar arguments in (i), we get $\delta\|\nabla u\|^2 \geq \|u\|_{p+1}^{p+1}$.

(iii) Taking into account $\|\nabla u\| \neq 0$, we obtain

$$\delta\|\nabla u\|^2 < \|u\|_{p+1}^{p+1} \leq C_1^{p+1}\|\nabla u\|^{p+1},$$

i.e.,

$$C_1^{p+1}\|\nabla u\|^{p-1} > \delta.$$

We further get

$$\|\nabla u\| > \delta^{\frac{1}{p-1}} C_1^{-\frac{p+1}{p-1}}.$$

Combining this with (3), we obtain

$$\|\nabla u\| > \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}.$$

Hence $u \in V_\delta$.

(iv) From the proof of (iii) we know that $\delta\|\nabla u\|^2 = \|u\|_{p+1}^{p+1}$ and $\|\nabla u\| \neq 0$ imply

$$\|\nabla u\| \geq \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}.$$

Hence $u \in H_0^1(\Omega)\backslash W_\delta = V_\delta \cup \partial V_\delta$.

□

2.2. **Invariance of the potential wells and their outside sets.** In this subsection, we show that W_δ and V_δ are both invariant under the flow of problem (1) with the subcritical initial energy.

Definition 2.2. A function $u = u(x,t)$ is called a weak solution of problem (1) on $\Omega \times [0,T)$ if $u \in L^\infty(0,T;H_0^1(\Omega))$, $u_t \in L^\infty(0,T;L^2(\Omega))$, $u(0) = u_0$ in $H_0^1(\Omega)$, $u_t(0) = u_1$ in $L^2(\Omega)$, and

$$(u_t(t),v) + \int_0^t (\nabla u(\tau), \nabla v)\,\mathrm{d}\tau = \int_0^t (|u(\tau)|^{p-1}u(\tau),v)\,\mathrm{d}\tau + (u_1,v), \tag{4}$$

for all $v \in H_0^1(\Omega)$ and a.e. $t \in (0,T)$.

(4) implies that

$$\langle u_{tt}(t),v\rangle + (\nabla u(t),\nabla v) = (|u(t)|^{p-1}u(t),v),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$.

Theorem 2.3. *Let u be a solution of problem (1) on $\Omega\times[0,T)$. Assume that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $0 < E(0) < d(\delta)$.*

(i) If $u_0 \in W_\delta$, then $u(t) \in W_\delta$ for all $t \in [0,T)$.
(ii) If $u_0 \in V_\delta$, then $u(t) \in V_\delta$ for all $t \in [0,T)$.

Proof. (i) Suppose that $u(t) \notin W_\delta$ for some $0 < t < T$. Then we see from $u_0 \in W_\delta$ that there exists the first time $0 < t_0 < T$ such that $u(t_0) \in \partial W_\delta$. Thus

$$\|\nabla u(t_0)\| = \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}.$$

Consequently, we deduce from (ii) in Lemma 2.1 that

$$\begin{aligned} J_\delta(u(t_0)) &= \frac{\delta}{2}\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1} \\ &= \delta\left(\frac{1}{2}-\frac{1}{p+1}\right)\|\nabla u(t_0)\|^2 + \frac{1}{p+1}\left(\delta\|\nabla u(t_0)\|^2 - \|u(t_0)\|_{p+1}^{p+1}\right) \\ &\geq \frac{(p-1)\delta}{2(p+1)}\|\nabla u(t_0)\|^2 \\ &= d(\delta). \end{aligned}$$

Clearly, this contradicts

$$E(0) = E(t) = \frac{1}{2}\|u_t(t)\|^2 + J_1(u(t)) < d(\delta), \quad \forall t \in [0,T).$$

Hence $u(t) \in W_\delta$ for all $t \in [0,T)$.

(ii) Again arguing by contradiction, there exists the first time $0 < t_0 < T$ such that $u(t_0) \in \partial V_\delta$. The remainder of proof is the same as that in (i), and so it is omitted here.

□

2.3. **Global existence and blow-up.** In this subsection, we address global existence and finite time blow-up of solutions for problem (1).

Theorem 2.4. *Assume that $u_0 \in W_\delta$, $u_1 \in L^2(\Omega)$ and $0 < E(0) < d(\delta)$. Then problem (1) admits a solution $u(t) \in W_\delta$ for all $t \in [0,\infty)$.*

Proof. Let $\{w_j\}_{j=1}^{\infty}$ be an orthogonal basis of $H_0^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. We construct

$$u_n(t) = \sum_{j=1}^{n} \xi_{jn}(t) w_j, \quad n = 1, 2, \cdots,$$

which satisfies

$$(u_{ntt}(t), w_j) + (\nabla u_n(t), \nabla w_j) = (|u_n(t)|^{p-1} u_n(t), w_j), \quad j = 1, 2, \cdots, n, \tag{5}$$

$$u_n(0) = \sum_{j=1}^{n} \xi_{jn}(0) w_j \to u_0 \ \text{ in } \ H_0^1(\Omega), \tag{6}$$

$$u_{nt}(0) = \sum_{j=1}^{n} \xi'_{jn}(0) w_j \to u_1 \ \text{ in } \ L^2(\Omega). \tag{7}$$

Multiplying (5) by $\xi'_{jn}(t)$ and summing for j, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_{nt}(t)\|^2 + \frac{1}{2}\|\nabla u_n(t)\|^2 - \frac{1}{p+1}\|u_n(t)\|_{p+1}^{p+1}\right) = 0.$$

Integrating this with respect to t from 0 to t, we get

$$E_n(t) = \frac{1}{2}\|u_{nt}(t)\|^2 + \frac{1}{2}\|\nabla u_n(t)\|^2 - \frac{1}{p+1}\|u_n(t)\|_{p+1}^{p+1} = E_n(0), \tag{8}$$

where

$$E_n(0) = \frac{1}{2}\|u_{nt}(0)\|^2 + \frac{1}{2}\|\nabla u_n(0)\|^2 - \frac{1}{p+1}\|u_n(0)\|_{p+1}^{p+1}.$$

It follows from (6) and (7) that $E_n(0) \to E(0)$, $0 < E_n(0) < d(\delta)$ and $u_n(0) \in W_\delta$ for sufficiently large n. By similar arguments in (i) in Theorem 2.3, we have $u_n(t) \in W_\delta$ for all $t \in [0, \infty)$. Consequently,

$$\|\nabla u_n(t)\| < \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}, \quad \forall t \in [0, \infty),$$

and

$$\|u_n(t)\|_{p+1} \le C_1 \|\nabla u_n(t)\| < C_1 \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}, \quad \forall t \in [0, \infty).$$

When $\|\nabla u_n(t)\| \neq 0$, in terms of (i) in Lemma 2.1 and (8), we have

$$\|u_{nt}(t)\|^2 < 2d(\delta), \quad \forall t \in [0, \infty).$$

When $\|\nabla u_n(t)\| = 0$, by means of (8), the above inequality remains valid.

Therefore, there exist u, χ and a subsequence of $\{u_n\}$, always relabeled as the same and we shall not repeat, such that, as $n \to \infty$,

$$u_n \rightharpoonup u \text{ weakly star in } L^\infty(0, \infty; H_0^1(\Omega)), \text{ and } u_n \to u \text{ a.e. in } \Omega \times [0, \infty),$$

$$u_{nt} \rightharpoonup u_t \text{ weakly star in } L^\infty(0, \infty; L^2(\Omega)).$$

$$|u_n|^{p-1} u_n \rightharpoonup \chi \text{ weakly star in } L^\infty(0, \infty; L^r(\Omega)), \quad r = \frac{p+1}{p}.$$

According to [7, Chapter 1, Lemma 1.3], we have $\chi = |u|^{p-1} u$.

For fixed j, integrating (5) with respect to t and taking $n \to \infty$, we get

$$(u_t(t), w_j) + \int_0^t (\nabla u(\tau), \nabla w_j)\,\mathrm{d}\tau = \int_0^t (|u(\tau)|^{p-1} u(\tau), w_j)\,\mathrm{d}\tau + (u_1, w_j).$$

Moreover, it is easy to see from (6) and (7) that $u(0) = u_0$ in $H_0^1(\Omega)$, $u_t(0) = u_1$ in $L^2(\Omega)$. Therefore, u is a solution of problem (1) in the sense of Definition 2.2. In addition, according to (i) in Theorem 2.3, we have $u(t) \in W_\delta$ for all $t \in [0, \infty)$. □

Theorem 2.5. *Assume that $u_0 \in V_\delta$, $u_1 \in L^2(\Omega)$ and $E(0) < d(\delta)$. Then solutions of problem (1) blow up in finite time.*

Proof. Let u be a solution of problem (1). Next, we prove $T < \infty$. If it is not true, then $T = \infty$. We consider the auxiliary function $M(t) = \|u(t)\|^2$, $t \in [0, \infty)$. A direct calculation yields $M'(t) = 2(u(t), u_t(t))$, and

$$\begin{aligned} M''(t) &= 2\|u_t(t)\|^2 + 2\langle u(t), u_{tt}(t)\rangle \\ &= 2\|u_t(t)\|^2 - 2\|\nabla u(t)\|^2 + 2\|u(t)\|_{p+1}^{p+1} \\ &= (p+3)\|u_t(t)\|^2 + (p-1)\|\nabla u(t)\|^2 - 2(p+1)E(0). \end{aligned} \tag{9}$$

When $0 < E(0) < d(\delta)$, by virtue of $u_0 \in V_\delta$ and (ii) in Theorem 2.3, we have $u(t) \in V_\delta$ for all $t \in [0, \infty)$, and so

$$\|\nabla u(t)\|^2 > \frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}.$$

Hence

$$\delta(p-1)\|\nabla u(t)\|^2 > 2(p+1)d(\delta) > 2(p+1)E(0),$$

which, together with (9), gives

$$M''(t) > (p+3)\|u_t(t)\|^2.$$

When $E(0) \le 0$, on account of (9), the above inequality still holds. Therefore, there exists a $t_0 > 0$ such that $M(t_0) > 0$ and $M'(t) \ge M'(t_0) > 0$ for a.e. $t \in [t_0, \infty)$. Then $M(t) \ge M'(t_0)(t - t_0) + M(t_0) > 0$ for a.e. $t \in [t_0, \infty)$.

It follows from the Cauchy-Schwarz inequality that

$$M(t)M''(t) - \frac{p+3}{4}(M'(t))^2 \ge (p+3)(\|u(t)\|^2\|u_t(t)\|^2 - (u(t), u_t(t))^2) \ge 0.$$

Consequently,

$$\left(M^{-\beta}(t)\right)' = -\beta M^{-(1+\beta)}(t)M'(t) < 0,$$

and

$$(M^{-\beta}(t))'' = \frac{-\beta}{M^{\beta+2}(t)}(M(t)M''(t) - (\beta+1){M'}^2(t)) \le 0,$$

for a.e. $t \in [t_0, \infty)$, where $\beta = \dfrac{p-1}{4}$. Then there exists a T_0 such that

$$\lim_{t \to T_0} M(t) = \infty,$$

which contradicts $T = \infty$. Thus the proof of Theorem 2.5 is complete. □

3. **Problem (1) with the critical initial energy.** Next, in the critical case $E(0) = d(\delta)$, we discuss the invariance of the outside sets of the potential wells as well as global existence and finite time blow-up of solutions for problem (1).

Lemma 3.1. *Let $u \in H_0^1(\Omega)$ and $\|\nabla u\| \neq 0$. $J_\delta(\rho u)$ is strictly increasing for $\rho \in (0, \rho_{*,\delta})$, strictly decreasing for $\rho \in (\rho_{*,\delta}, \infty)$, and attains the maximum at $\rho = \rho_{*,\delta}$.*

Proof. It follows from definition of $J_\delta(u)$ that

$$J_\delta(\rho u) = \frac{\delta}{2}\|\nabla(\rho u)\|^2 - \frac{1}{p+1}\|\rho u\|_{p+1}^{p+1}$$
$$= \frac{\rho^2\delta}{2}\|\nabla u\|^2 - \frac{\rho^{p+1}}{p+1}\|u\|_{p+1}^{p+1}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\rho}J_\delta(\rho u) = \rho\delta\|\nabla u\|^2 - \rho^p\|u\|_{p+1}^{p+1}.$$

Clearly, there is a $\rho_{*,\delta} = \rho_{*,\delta}(u) > 0$ such that $\delta\|\nabla u\|^2 = \rho_{*,\delta}^{p-1}\|u\|_{p+1}^{p+1}$, i.e., $\frac{\mathrm{d}}{\mathrm{d}\rho}J_\delta(\rho u)\Big|_{\rho=\rho_{*,\delta}} = 0$. Moreover, $\frac{\mathrm{d}}{\mathrm{d}\rho}J_\delta(\rho u) > 0$ for $\rho \in (0, \rho_{*,\delta})$, $\frac{\mathrm{d}}{\mathrm{d}\rho}J_\delta(\rho u) < 0$ for $\rho \in (\rho_{*,\delta}, \infty)$. □

Theorem 3.2. *Suppose that $u_0 \in W_\delta$, $u_1 \in L^2(\Omega)$ and $E(0) = d(\delta)$. Then problem (1) admits $u(t) \in \overline{W_\delta} = W_\delta \cup \partial W_\delta$ for all $t \in [0, \infty)$.*

Proof. We may perform this proof by considering the following two cases.

(i) $\|\nabla u_0\| \neq 0$.

Let $u_{0m} = \lambda_m u_0$, where $\lambda_m = 1 - \frac{1}{m}$, $m = 2, 3, \cdots$. Next, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(x,0) = u_{0m}(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{10}$$

whose energy is

$$E_m(t) = \frac{1}{2}\|u_{mt}(t)\|^2 + \frac{1}{2}\|\nabla u_m(t)\|^2 - \frac{1}{p+1}\|u_m(t)\|_{p+1}^{p+1}.$$

From $u_0 \in W_\delta$ and Lemma 2.1 it follows that

$$\delta\|\nabla u_0\|^2 > \|u_0\|_{p+1}^{p+1}. \tag{11}$$

Hence

$$\delta\|\nabla u_0\|^2 > \lambda_m^{p-1}\|u_0\|_{p+1}^{p+1},$$

and so

$$\delta\|\nabla u_{0m}\|^2 > \|u_{0m}\|_{p+1}^{p+1}.$$

Consequently,

$$J_\delta(u_{0m}) = \frac{\delta}{2}\|\nabla u_{0m}\|^2 - \frac{1}{p+1}\|u_{0m}\|_{p+1}^{p+1} > 0.$$

It follows from (11) and the proof of Lemma 3.1 that there exists a $\rho_{*,\delta} = \rho_{*,\delta}(u_0) > 1$ such that $J_\delta(\rho u_0)$ attains its maximum. Thus, according to Lemma 3.1, $J_\delta(\rho u_0)$ is strictly increasing on $[\lambda_m, 1]$, and $J_1(\lambda_m u_0) < J_1(u_0)$. As a result,

$$0 < E_m(0) = \frac{1}{2}\|u_1\|^2 + J_1(u_{0m}) < \frac{1}{2}\|u_1\|^2 + J_1(u_0) = E(0) = d(\delta).$$

In terms of Theorem 2.4, for each m, problem (10) admits a solution $u_m(t) \in W_\delta$ for all $t \in [0, \infty)$ satisfying

$$(u_{mt}(t), v) + \int_0^t (\nabla u_m(\tau), \nabla v)\ d\tau = \int_0^t \left(|u_m(\tau)|^{p-1} u_m(\tau), v\right)\ d\tau + (u_1, v), \quad (12)$$

for all $v \in H_0^1(\Omega)$. Consequently,

$$\|\nabla u_m(t)\| < \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}, \quad \forall t \in [0, \infty). \quad (13)$$

When $\|\nabla u_m(t)\| \neq 0$, in terms of (i) in Lemma 2.1 and $E_m(t) = E_m(0) < d(\delta)$, we have

$$\|u_{mt}(t)\|^2 < 2d(\delta), \quad \forall t \in [0, \infty).$$

When $\|\nabla u_m(t)\| = 0$, the above inequality still holds. By the compactness arguments used by the proof of Theorem 2.4, there exists a u such that, as $m \to \infty$ in (12),

$$(u_t(t), v) + \int_0^t (\nabla u(\tau), \nabla v)\ d\tau = \int_0^t \left(|u(\tau)|^{p-1} u(\tau), v\right)\ d\tau + (u_1, v).$$

Moreover, it follows from $(10)_3$ that $u(0) = u_0$ in $H_0^1(\Omega)$, $u_t(0) = u_1$ in $L^2(\Omega)$. Therefore, u is a solution of problem (1). By virtue of (13), we have

$$\|\nabla u(t)\| \le \liminf_{m\to\infty} \|\nabla u_m(t)\| \le \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}, \quad \forall t \in [0, \infty).$$

Hence $u(t) \in \overline{W_\delta}$ for all $t \in [0, \infty)$.

(ii) $\|\nabla u_0\| = 0$.

Let $u_{1m} = \lambda_m u_1$, where $\lambda_m = 1 - \dfrac{1}{m}$, $m = 2, 3, \cdots$. Next, we consider

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-1}u, & (x,t) \in \Omega \times (0, \infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0, \infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_{1m}(x), & x \in \Omega. \end{cases} \quad (14)$$

In this case, due to the fact that $J_1(u_0) = 0$ and $\frac{1}{2}\|u_1\|^2 = E(0)$, we get

$$0 < E_m(0) = \frac{1}{2}\|u_{1m}\|^2 + J_1(u_0) = \frac{1}{2}\|\lambda_m u_1\|^2 < E(0) = d(\delta).$$

Thus it follows from Theorem 2.4 that, for each m, problem (14) admits a solution $u_m(t) \in W_\delta$ for all $t \in [0, \infty)$. The remainder of proof is similar to that in (i).

The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Let u be a solution of problem (1) on $\Omega \times [0, T)$. Assume that $u_0 \in V_\delta$, $u_1 \in L^2(\Omega)$, $E(0) = d(\delta)$ and $(u_0, u_1) \ge 0$. Then $u(t) \in V_\delta$ for all $t \in [0, T)$.*

Proof. Suppose that $u(t) \notin V_\delta$ for some $0 < t < T$. Then we see from $u_0 \in V_\delta$ that there exists the first time $0 < t_0 < T$ such that $u(t_0) \in \partial V_\delta$. Thus,

$$\|\nabla u(t_0)\| = \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}, \quad (15)$$

and

$$\|\nabla u(t)\| > \left(\frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}\right)^{\frac{1}{2}}, \quad \forall t \in [0, t_0). \tag{16}$$

By (15) and (ii) in Lemma 2.1, we have

$$\delta\|\nabla u(t_0)\|^2 \geq \|u(t_0)\|_{p+1}^{p+1}.$$

Hence

$$\begin{aligned} J_\delta(u(t_0)) &= \frac{\delta}{2}\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1} \\ &\geq \frac{(p-1)\delta}{2(p+1)}\|\nabla u(t_0)\|^2 \\ &= d(\delta). \end{aligned} \tag{17}$$

Set $M(t) = \|u(t)\|^2$, $t \in [0, T)$. A direct calculation yields $M'(t) = 2(u(t), u_t(t))$. From $E(0) = d(\delta)$, (16) and (9), it follows that $M''(t) > 0$ for $t \in [0, t_0)$. Hence $M'(t)$ is increasing on $[0, t_0]$. We further get $M'(t_0) > M'(0)$, which, together with $M'(0) = (u_0, u_1) \geq 0$, gives $M'(t_0) = (u(t_0), u_t(t_0)) > 0$. Thus $\|u(t_0)\|\|u_t(t_0)\| \geq (u(t_0), u_t(t_0)) > 0$. Again by

$$E(t_0) = \frac{1}{2}\|u_t(t_0)\|^2 + J_1(u(t_0)) = E(0) = d(\delta),$$

we get $J_1(u(t_0)) < d(\delta)$, which contradicts (17). Therefore, $u(t) \in V_\delta$ for all $t \in [0, T)$. □

Theorem 3.4. *Assume that $u_0 \in V_\delta$, $u_1 \in L^2(\Omega)$, $E(0) = d(\delta)$ and $(u_0, u_1) \geq 0$. Then the solutions of problem (1) blow up in finite time.*

Proof. Let u be a solution of problem (1). Next, we prove $T < \infty$. If it is not true, then $T = \infty$. From $u_0 \in V_\delta$, $(u_0, u_1) \geq 0$ and Theorem 3.3, it follows that $u(t) \in V_\delta$ for all $t \in [0, \infty)$. Hence

$$\|\nabla u(t)\|^2 > \frac{2(p+1)}{p-1}\frac{d(\delta)}{\delta}.$$

Combining this with $E(0) = d(\delta)$, we obtain

$$\delta(p-1)\|\nabla u(t)\|^2 > 2(p+1)d(\delta) = 2(p+1)E(0).$$

Hence, for $M(t)$ introduced in the proof of Theorem 3.3, it follows from (9) that

$$M''(t) > (p+3)\|u_t(t)\|^2,$$

for a.e. $t \in [0, \infty)$. The remainder of proof is the same as that in Theorem 2.5, and so it is omitted here. □

Taking $\delta = 1$, we see that $d = d(1)$ equals to that in [14].

4. **Numerical simulations.**

This section is devoted to some numerical simulations of $d(\delta)$. We see from the definition of $J_\delta(u)$ that the parameter δ is introduced in order to adjust the potential energy. We have Figure 1 that shows the relation between $d(\delta)$ and δ.

Next, we explore the relations among $d(\delta)$, C_1, y_δ and p under $N = 2$.

We first demonstrate the relations: $d(\delta) \sim C_1, p$ when $\delta = 0.5$ by Figure 2; $d(\delta) \sim p, \delta$ when $C_1 = 2$ by Figure 3.

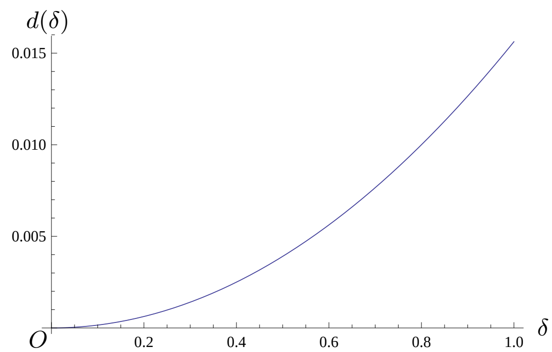

FIGURE 1. $d(\delta) \sim \delta$; $p = 3$, $C_1 = 2$

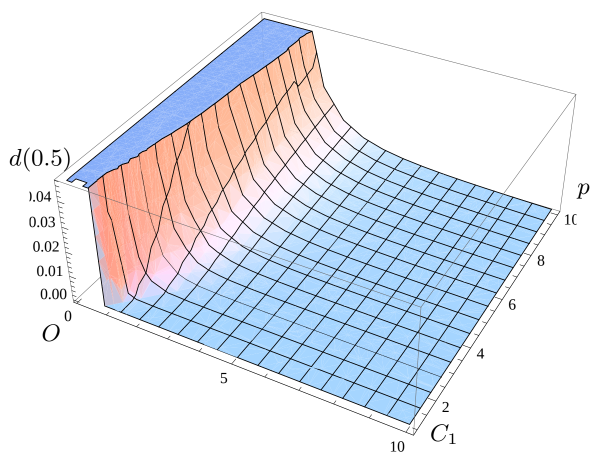

FIGURE 2. $d(0.5) \sim C_1, p$

By fixing $\delta = 1$, $C_1 = 2$, $p = 2, 3, 4, 5$, we obtain the relation: $g_\delta(y) \sim y_\delta$ by Figure 4.

Moreover, we would like to do some simulations about the relations: $y_\delta \sim d(\delta), p$ when $\delta = 1$ by Figure 5; $y_\delta \sim C_1, p$ when $\delta = 0.5$ by Figure 6.

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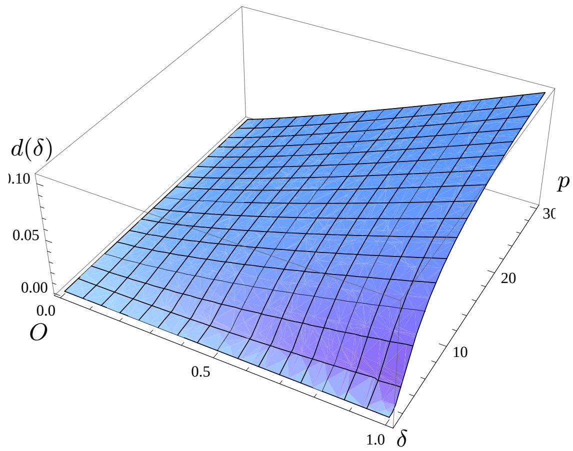

FIGURE 3. $d(\delta) \sim \delta, p$; $C_1 = 2$

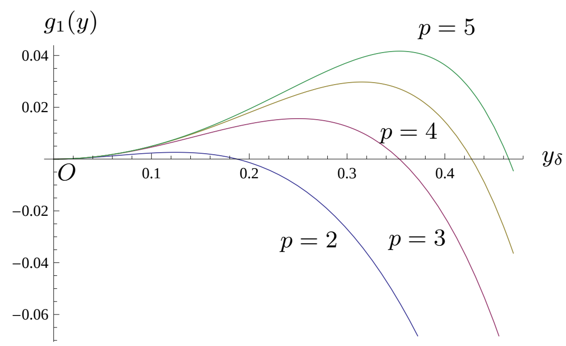

FIGURE 4. $g_1(y) \sim y_\delta$

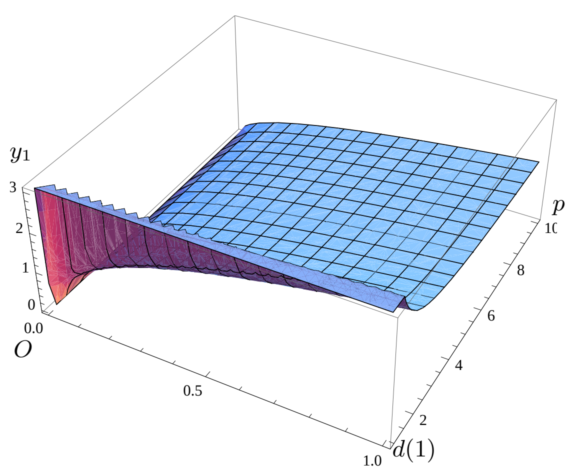

FIGURE 5. $y_1 \sim d(1), p$

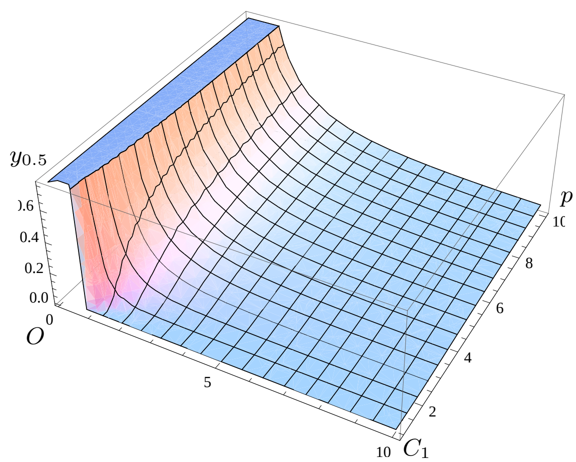

FIGURE 6. $y_{0.5} \sim C_1, p$

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