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H^2 BLOWUP RESULT FOR A SCHRÖDINGER EQUATION WITH NONLINEAR SOURCE TERM

Xuan Liu and Ting Zhang*

School of Mathematical Sciences, Zhejiang University
Hangzhou 310027, China

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Abstract. In this paper, we consider the nonlinear Schrödinger equation on $\mathbb{R}^N, N \geq 1$,

$$\partial_t u = i\Delta u + \lambda |u|^\alpha u,$$

with H^2-subcritical nonlinearities: $\alpha > 0, (N-4)\alpha < 4$ and $\mathrm{Re}\lambda > 0$. For any given compact set $K \subset \mathbb{R}^N$, we construct H^2 solutions that are defined on $(-T, 0)$ for some $T > 0$, and blow up exactly on K at $t = 0$. We generalize the range of the power α in the result of Cazenave, Han and Martel [5]. The proof is based on the energy estimates and compactness arguments.

1. **Introduction.** In this paper, we consider the nonlinear Schrödinger equation with the power nonlinearity

$$\partial_t u = i\Delta u + \lambda |u|^\alpha u \tag{1.1}$$

on $\mathbb{R}^N$, where

$$N \geq 1, \quad \alpha > 0, \quad (N-4)\alpha < 4, \tag{1.2}$$

and $\lambda \in \mathbb{C}$ such that

$$\mathrm{Re}\lambda > \begin{cases} 0, & \text{if } 1 \leq N \leq 3, \\ \frac{\alpha}{2}|\mathrm{Im}\lambda|, & \text{if } N \geq 4. \end{cases} \tag{1.3}$$

Under the assumption (1.2), the equation (1.1) is H^2-subcritical, so that the corresponding Cauchy problem is locally well posed in $H^2(\mathbb{R}^N)$, see [12] and [21]. It is well-known that if $\alpha < \frac{4}{N}$ and the equation (1.1) has a dissipative nonlinearity, i.e. $\mathrm{Re}\lambda < 0$, then all H^1 solutions are global, see [2]. If $\alpha < \frac{2}{N}$ and the nonlinearity is not dissipative, i.e. $\mathrm{Re}\lambda > 0$, it is proved in [2] that the equation (1.1) has no global in time H^1 solution that remains bounded in H^1. The question of the finite-time blow-up is still open. With the restriction $\alpha \geq 2$, it is proved in [6] that under the assumption that $(N-2)\alpha \leq 4$ and $\mathrm{Re}\lambda = 1$, finite time blowup occurs. The construction is based on an appropriate ansatz. This result is extended in [13] to the case $\alpha > 1$ and $(\alpha+2)\mathrm{Re}\lambda \geq \alpha|\lambda|$. Moreover, by refining the initial ansatz (2.7) inductively, the blow-up result is extended to the whole range of H^1 subcritical powers and arbitrary $\mathrm{Re}\lambda > 0$ in [5]. There are some similarly results for the focusing energy subcritical nonlinear wave equation, see [7, 8].

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* Corresponding author: Ting Zhang.

In this paper, we extend the previous blow-up result in [5] to the H^2-subcritical case under the additional technical assumptions (1.3).

Theorem 1.1. *Under the conditions (1.2) and (1.3), for any nonempty compact subset $K \subset \mathbb{R}^N$, there exist $S \in (-1, 0)$ and a solution $u \in C([S, 0), H^2(\mathbb{R}^N)) \cap C^1([S, 0), L^2(\mathbb{R}^N))$ of the equation (1.1) which blows up at time 0 exactly on K in the following sense.*
(1) *If $x_0 \in K$ then for any $r > 0$,*

$$\lim_{t \uparrow 0} \|u(t)\|_{L^2(|x-x_0|<r)} = \infty. \tag{1.4}$$

(2) *If U is a open subset of $\mathbb{R}^N$ such that $K \subset U$, then*

$$\lim_{t\uparrow 0} \|\nabla u(t)\|_{L^2(U)} = \infty, \text{ and } \lim_{t\uparrow 0} \|\Delta u(t)\|_{L^2(U)} = \infty. \tag{1.5}$$

(3) *If Ω is a open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \cap K = \emptyset$, then*

$$\sup_{t\in(S,0]} \|u(t)\|_{H^2(\Omega)} < \infty. \tag{1.6}$$

Remark 1.1. Under the assumptions that $\alpha > 0$, $(N-2)\alpha \leq 4$ and $\mathrm{Re}\lambda > 0$, Cazenave-Han-Martel [5] proved that given any nonempty compact subset K of $\mathbb{R}^N$, there exists a H^1 solution of (1.1) which blows up exactly on K when $t = 0$. We generalize the range of α to the H^2-subcritical case, following the technique developed in [6]. For technical reasons, we require that $\mathrm{Re}\lambda > \frac{\alpha}{2}|\mathrm{Im}\lambda|$ when the dimension $N \geq 4$, which is used in the proof of the estimates of $\|\partial_t \varepsilon_n\|_{L^2}$, see (3.29)-(3.41).

Remark 1.2. It follows from (1.4) and (1.5) that both $\|u(t)\|_2, \|\nabla u(t)\|_2$ and $\|\Delta u(t)\|_2$ blow up when $t \uparrow 0$.

Remark 1.3. The estimate (1.4) can be refined. More precisely, it follows from (4.8) that

$$(-t)^{-\frac{1}{\alpha}+\frac{N}{2k}} \lesssim \|u(t)\|_{L^2(|x-x_0|<r)} \lesssim (-t)^{-\frac{1}{\alpha}}$$

where $k > N\alpha$ is given by (2.2).

We prove Theorem 1.1 by the strategy of [1]. More precisely, we consider the sequence $\{u_n\}_{n\geq 1}$ of solutions of (1.1) with the initial datum $u_n(-\frac{1}{n}) = U_J(-\frac{1}{n})$, where U_J is a refined blowup profile defined in Lemma 2.3. It follows that u_n is defined on $(s_n, -\frac{1}{n})$ for some $s_n < -\frac{1}{n}$. Letting $\varepsilon_n(t) = u_n(t) - U_J(t)$, following the ideas of [5, 15], we show that $\{\varepsilon_n\}_{n\geq 1}$ is uniformly bounded in $L^\infty((S,\tau), H^2) \cap W^{1,\infty}((S,\tau), L^2)$ (S is given by Proposition 3.1) for any $\tau \in (S, 0)$ by the energy arguments. Moreover, by the compactness argument, we find $\varepsilon \in L^\infty((S,0), H^2) \cap W^{1,\infty}((S,0), L^2)$ and a subsequence of $\{\varepsilon_n\}_{n\geq 1}$ weakly converges to ε. Therefore, setting $u(t) = U_J(t) + \varepsilon(t)$, we see that u is a H^2 solution of (1.1). Finally, note that ε is bounded in $H^2(\mathbb{R}^N)$ and U_J blows up at time 0 exactly on K, we deduce that $u(t)$ also blows up at time 0 exactly on K.

The solution u given by Theorem 1.1 blows up at $t = 0$ like the function U_J defined in Lemma 2.3 . Since the function U_0 defined by (2.7) satisfying $U_t = \lambda |U|^\alpha U$, and U_J is a refinement of U_0, we see that the solution u displays an ODE-type blowup. We recall that there are many ODE-type blowup results for several other nonlinear equations, refer to [10, 17, 20] for results in the parabolic context, refer to [1, 18, 23] for the nonlinear wave equations. Recently, there are many well-posedness results for the nonlinear Schrödinger equation, see [9, 14, 24, 25] and references therein.

The rest of the paper is organized as follows. In Section 2, we introduce the blow-up ansatz and the corresponding estimates which are from [5], and recall some useful estimates. Section 3 is devoted to the construction of a sequence of solutions of (1.1) close to the blow-up ansatz and some *a priori* estimates of the approximate solutions. Finally, we complete the proof of Theorem 1.1 in Section 4 by passing to the limit in the approximate solutions.

2. **The blow-up ansatz.** In this section, we introduce the blow-up ansatz constructed in [5].

The first candidate U_0 is defined by (2.7) below, which is a solution of the ordinary differential equation $U_t = \lambda |U_0|^\alpha U_0$. Since the error term $i\Delta U_0$ is not integrable in time near the singularity when α is small, the method used in [1] does not applicable to the case $0 < \alpha \leq 1$. To treat any subcritical α and any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$, Cazenave-Han-Martel [5] refine the blow-up ansatz inductively, using only ODE techniques, see (2.18)-(2.22) for more details. Throughout this section, we choose two integers

$$J = \left[\frac{2}{\alpha} + 4\sigma\right] + 1 \tag{2.1}$$

and

$$k = \max\{2J+4, \frac{16}{3\gamma\sigma}, N\alpha, \frac{1}{(1-\frac{3}{8}\gamma)\sigma}\} \tag{2.2}$$

with

$$\gamma = \min\{\frac{1}{2}, \frac{\alpha}{\alpha+2}, \frac{4}{N}\}, \tag{2.3}$$

$$\sigma = \max\{\frac{4}{\gamma}, (2^{\alpha+1}+4+4K_1)(\alpha+1)|\lambda|M(\alpha\mathrm{Re}\lambda)^{-1}\}, \tag{2.4}$$

where M is given by Lemma 2.4 and $K_1 = |\mathrm{Re}\lambda - \frac{\alpha}{2}|\mathrm{Im}\lambda||^{1-\alpha}(4(\alpha+1)|\lambda|M)^{\alpha-1}$. Let K be any nonempty compact set of $\mathbb{R}^N$ included in the ball of center 0 and radius $R > 0$. It is well-known that there exists a smooth function $Z : \mathbb{R}^N \to [0,\infty)$ which vanishes exactly on K (see Lemma 1.4 in [19]). Define the function $A : \mathbb{R}^N \to [0,\infty)$ by

$$A(x) = (Z(x)\chi(|x|) + (1-\chi(|x|))|x|)^k \tag{2.5}$$

where

$$\chi \in C^\infty(\mathbb{R},\mathbb{R}), \quad \chi(s) = \begin{cases} 1, & 0 \leq s \leq R, \\ 0, & s \geq 2R, \end{cases} \quad \chi'(s) \leq 0 \leq \chi(s) \leq 1, \quad s \geq 0.$$

It follows that the function $A \in C^{k-1}\left(\mathbb{R}^N,\mathbb{R}\right)$, vanishes exactly on K, satisfies

$$\begin{cases} A \geq 0 \text{ and } \left|\partial_x^\beta A\right| \lesssim A^{1-\frac{|\beta|}{k}}, & \text{on } \mathbb{R}^N \text{ for } |\beta| \leq k-1, \\ A(x) = |x|^k, & \text{for } x \in \mathbb{R}^N, |x| \geq 2R. \end{cases} \tag{2.6}$$

Set

$$U_0(t,x) = (\mathrm{Re}\lambda)^{-\frac{1}{\alpha}}(-\alpha t + A(x))^{-\frac{1}{\alpha} - i\frac{\mathrm{Im}\lambda}{\alpha\mathrm{Re}\lambda}}, \quad t<0, x \in \mathbb{R}^N. \tag{2.7}$$

From (1.2), (2.2) and (2.6), we have

$$U_0 \text{ is } C^\infty \text{ in } t<0 \text{ and } C^{k-1} \text{ in } x \in \mathbb{R}^N,$$

$$\partial_t U_0 = \lambda |U_0|^\alpha U_0, \quad t<0, x \in \mathbb{R}^N, \tag{2.8}$$

$$|U_0| = (\mathrm{Re}\lambda)^{-\frac{1}{\alpha}}(-\alpha t + A(x))^{-\frac{1}{\alpha}} \le (\alpha \mathrm{Re}\lambda)^{-\frac{1}{\alpha}}(-t)^{-\frac{1}{\alpha}}, \tag{2.9}$$

and

$$\partial_t |U_0| = \mathrm{Re}\lambda |U_0|^{\alpha+1} \ge 0. \tag{2.10}$$

Next we estimate the profile U_0 given by (2.7). We collect the estimates on U_0 which are from [5] and slight modifications.

Lemma 2.1. *Under the conditions (1.2), (2.2) and (2.6), then we have*

$$\|U_0(t)\|_{L^p} \lesssim (-t)^{-\frac{1}{\alpha}} \tag{2.11}$$

for all $p \ge 1$ and $-1 \le t < 0$. In addition, for every $\rho \in \mathbb{R}, \ell \in \mathbb{N}$ and $|\beta| \le k-1$,

$$\left|\partial_t^\ell \partial_x^\beta U_0\right| \lesssim |U_0|^{1+\ell\alpha+\frac{\alpha}{k}|\beta|} \lesssim (-t)^{-\ell-\frac{|\beta|}{k}} |U_0|, \tag{2.12}$$

$$\left|\partial_x^\beta \left(|U_0|^\rho\right)\right| \lesssim |U_0|^{\rho+\frac{\alpha}{k}|\beta|} \lesssim (-t)^{-\frac{|\beta|}{k}} |U_0|^\rho, \tag{2.13}$$

$$\left|\partial_x^\beta \left(|U_0|^{\rho-1} U_0\right)\right| \lesssim |U_0|^{\rho+\frac{\alpha}{k}|\beta|} \lesssim (-t)^{-\frac{|\beta|}{k}} |U_0|^\rho, \tag{2.14}$$

$$\left|\partial_t \partial_x^\beta |U_0|^\alpha U_0\right| \lesssim (-t)^{-1-\frac{|\beta|}{k}} |U_0|^{\alpha+1}, \tag{2.15}$$

for all $x \in \mathbb{R}^N, t < 0$, and

$$U_0 \in C^\infty\left((-\infty, 0), H^{k-1}\left(\mathbb{R}^N\right)\right). \tag{2.16}$$

Furthermore, for any $x_0 \in \mathbb{R}^N$ such that $A(x_0) = 0$, for any $r > 0, -1 \le t < 0$ and $1 \le p \le \infty$,

$$C_{r,p}(-t)^{-\frac{1}{\alpha}+\frac{N}{pk}} \le \|U_0(t)\|_{L^p(|x-x_0|<r)}, \tag{2.17}$$

where the constant $C_{r,p}$ depends on r and p.

Proof. Estimates (2.11)-(2.14) and the property (2.16) follows by the calculation in [5].

Note that $|U_0|$ is positive for any time $t < 0$, we have

$$\partial_t(|U_0|^\alpha U_0) = \frac{\alpha+2}{2}|U_0|^\alpha \partial_t U_0 + \frac{\alpha}{2}|U_0|^{\alpha-2}U_0^2 \partial_t \overline{U_0}.$$

It follows from Leibnitz's formula, (2.12)-(2.14) that

$$\begin{aligned} |\partial_x^\beta \partial_t(|U_0|^\alpha U_0)| &\lesssim \sum_{\beta_1+\beta_2=\beta} |\partial_x^{\beta_1}|U_0|^\alpha \partial_x^{\beta_2}\partial_t U_0| + \sum_{\beta_1+\beta_2=\beta} |\partial_x^{\beta_1}(|U_0|^{\alpha-2}U_0^2)\partial_x^{\beta_2}\partial_t \overline{U_0}| \\ &\lesssim \sum_{\beta_1+\beta_2=\beta} (-t)^{-\frac{|\beta_1|}{k}}|U_0|^\alpha \cdot (-t)^{-1-\frac{|\beta_2|}{k}}|U_0| \lesssim (-t)^{-1-\frac{|\beta|}{k}}|U_0|^{\alpha+1}, \end{aligned}$$

which proves (2.15).

To prove (2.17), we set $x_0 \in \mathbb{R}^N$ such that $A(x_0) = 0$. For any fixed $x \in \mathbb{R}^N$ satisfying $|x - x_0| < r$, choosing $x_1 \in \mathbb{R}^N$ satisfying $|x_1 - x_0| \le |x - x_0|$ and

$$|A(x_1)| = \max_{|y-x_0|\le|x-x_0|} |A(y)|.$$

From (2.6), we have,

$$\begin{aligned} |A(x_1)| &= |A(x_1) - A(x_0)| = |\nabla A(\eta x_1 + (1-\eta)x_0) \cdot (x_1 - x_0)| \\ &\le C|A(\eta x_1 + (1-\eta)x_0)|^{1-\frac{1}{k}}|x_1 - x_0| \le C|A(x_1)|^{1-\frac{1}{k}}|x_1 - x_0|, \end{aligned}$$

for some $\eta \in [0,1]$, and

$$|A(x_1)| \le C|x_1 - x_0|^k.$$

Then, we have

$$|A(x)| \leq |A(x_1)| \leq C|x_1 - x_0|^k \leq C|x - x_0|^k, \quad \forall\ |x - x_0| < r,$$

and

$$\begin{aligned}\int_{|x-x_0|<r} |U_0|^p dx &\gtrsim \int_{|x-x_0|<r} (-t + |x - x_0|^k)^{-\frac{p}{\alpha}} dx \\ &\gtrsim (-t)^{-\frac{p}{\alpha}+\frac{N}{k}} \int_{|y|<r} (1 + |y|^k)^{-\frac{p}{\alpha}} dy \geq C_{r,p}(-t)^{-\frac{p}{\alpha}+\frac{N}{k}}.\end{aligned}$$

This completes the proof of (2.17). □

Next, we introduce a procedure to reduce the singularity of the error term at any order of $(-t)$ by refining the approximate solution. We consider the linearization of the equation (2.8),

$$\partial_t w = \lambda \frac{\alpha + 2}{2} |U_0|^\alpha w + \lambda \frac{\alpha}{2} |U_0|^{\alpha-2} U_0^2 \overline{w} \tag{2.18}$$

The equation (2.18) has two solutions $w = iU_0$ and $w = \partial_t U_0 = \lambda |U_0|^\alpha U_0$. By means of variation of constants, it is not hard to see that the corresponding nonhomogeneous equation

$$\partial_t w = \lambda(\frac{\alpha + 2}{2}|U_0|^\alpha w + \frac{\alpha}{2}|U_0|^{\alpha-2}U_0^2\overline{w}) + G \tag{2.19}$$

has the solution $w = \mathcal{P}(G)$, where

$$\begin{aligned}\mathcal{P}(G) = \frac{\lambda}{\mathrm{Re}\lambda} |U_0|^\alpha U_0 \int_0^t \left[|U_0|^{-\alpha-2} \mathrm{Re}(\overline{U_0}G)\right](s)ds \\ + i\frac{1}{\mathrm{Re}\lambda} U_0 \int_0^t \left[|U_0|^{-2} \mathrm{Im}(\overline{\lambda U_0}G)\right](s)ds\end{aligned} \tag{2.20}$$

We define $U_j, w_j, \mathcal{E}_j$ by

$$w_0 = iU_0, \quad \mathcal{E}_0 = -\partial_t U_0 + i\Delta U_0 + \lambda|U_0|^\alpha U_0 = i\Delta U_0 \tag{2.21}$$

and then recursively

$$w_j = \mathcal{P}(\mathcal{E}_{j-1}), \quad U_j = U_{j-1} + w_j \quad \mathcal{E}_j = -\partial_t U_j + i\Delta U_j + \lambda|U_j|^\alpha U_j \tag{2.22}$$

for $j \geq 1$, as long as they make sense. We will see that for $j \leq \frac{k-4}{2}, \mathcal{P}(\mathcal{E}_{j-1})$ is well defined at each step, on a sufficiently small time interval. From similar arguments in Lemma 3.2 in [5], by Lemma 2.1 and Faà di Bruno's formula (see Corollary 2.10 in [11]), we have the following estimates. For the convenience of the reader, we briefly sketch the proof.

Lemma 2.2. *Under the conditions (1.2), (2.2) and (2.6), then there exists* $-1 < T < 0$ *such that the following estimates hold for all* $0 \leq j \leq \frac{k-4}{2}$.
(1) *If* $0 \leq |\beta| \leq k - 1 - 2j$, *then*

$$\left|\partial_x^\beta w_j\right| \lesssim (-t)^{j\left(1-\frac{2}{k}\right)-\frac{|\beta|}{k}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N, \tag{2.23}$$

$$\left|\partial_x^\beta (U_j - U_0)\right| \lesssim (-t)^{1-\frac{|\beta|+2}{k}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N, \tag{2.24}$$

$$|\partial_t\partial_x^\beta w_j| \lesssim (-t)^{-1+j(1-\frac{2}{k})-\frac{|\beta|}{k}}|U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N. \tag{2.25}$$

(2) *If* $0 \leq |\beta| \leq k - 3 - 2j$, *then*

$$\left|\partial_x^\beta \mathcal{E}_j\right| \lesssim (-t)^{j\left(1-\frac{2}{k}\right)-\frac{|\beta|+2}{k}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N. \tag{2.26}$$

Moreover

$$\frac{1}{2}|U_0| \le |U_j| \le 2|U_0|, \quad T \le t < 0, x \in \mathbb{R}^N, \tag{2.27}$$

$$U_j \in C^1\left((T,0), H^{k-1-2j}\left(\mathbb{R}^N\right)\right), \tag{2.28}$$

$$|\partial_t U_j| \lesssim (-t)^{-1}|U_0|, \quad T \le t < 0, x \in \mathbb{R}^N, \tag{2.29}$$

$$|\partial_t \mathcal{E}_j| \lesssim (-t)^{-1+j(1-\frac{2}{k})-\frac{2}{k}}|U_0|, \quad T \le t < 0, x \in \mathbb{R}^N. \tag{2.30}$$

Proof. The proof is based on the induction on j. From (2.12), we get that (2.23)-(2.30) hold with $j = 0$.

Assume (2.23)-(2.30) hold with $j \le n$. Then, we only prove (2.25), (2.29) and (2.30) with $j = n+1$, and the other estimates with $j = n+1$, follows from Lemma 3.2 in [5].

In view of (2.20) and (2.22), we see that

$$\partial_t w_{n+1} = \lambda(\frac{\alpha+2}{2}|U_0|^\alpha w_{n+1} + \frac{\alpha}{2}|U_0|^{\alpha-2}U_0^2\overline{w_{n+1}}) + \mathcal{E}_n. \tag{2.31}$$

It follows from Leibnitz's formula, (2.9), (2.13)-(2.14), (2.23) with $j = n+1$ and (2.26) with $j = n$ that

$$|\partial_t \partial_x^\beta w_{n+1}| \lesssim (-t)^{-1+(n+1)(1-\frac{2}{k})-\frac{|\beta|}{k}}|U_0|,$$

which implies (2.25) with $j = n+1$.

Next by (2.22), we see that

$$U_{n+1} = U_n + w_{n+1} = \cdots = w_{n+1} + w_n + \cdots + w_1 + U_0, \tag{2.32}$$

so that $|\partial_t U_{n+1}| \lesssim (-t)^{-1}|U_0|$ by (2.12) and (2.25) with $j \le n+1$. Then (2.29) holds with $j = n+1$.

Finally, we prove (2.30) with $j = n+1$. Since $U_{n+1} - U_n = w_{n+1}$, it follows from (2.19), (2.20) and (2.22) that

$$\begin{aligned}\mathcal{E}_{n+1} - \mathcal{E}_n &= -\partial_t w_{n+1} + i\Delta w_{n+1} + \lambda(|U_{n+1}|^\alpha U_{n+1} - |U_n|^\alpha U_n)\\ &= -\mathcal{E}_n + i\Delta w_{n+1} + \lambda(|U_{n+1}|^\alpha U_{n+1} - |U_n|^\alpha U_n\\ &\quad - \frac{\alpha+2}{2}|U_0|^\alpha w_{n+1} - \frac{\alpha}{2}|U_0|^{\alpha-2}U_0^2\overline{w_{n+1}}).\end{aligned}$$

Writing

$$|U_{n+1}|^\alpha U_{n+1} - |U_n|^\alpha U_n = \int_0^1 \frac{d}{d\theta}[|U_n + \theta w_{n+1}|^\alpha (U_n + \theta w_{n+1})]d\theta$$

$$= \int_0^1 \frac{\alpha+2}{2}|U_n + \theta w_{n+1}|^\alpha w_{n+1} + \frac{\alpha}{2}|U_n + \theta w_{n+1}|^{\alpha-2}(U_n + \theta w_{n+1})^2\overline{w_{n+1}}d\theta,$$

we have

$$\mathcal{E}_{n+1} = i\Delta w_{n+1} + \lambda\int_0^1 \frac{\alpha+2}{2}A_{n+1}(t,\theta)w_{n+1} + \frac{\alpha}{2}B_{n+1}(t,\theta)\overline{w_{n+1}}d\theta, \tag{2.33}$$

where

$$A_{n+1}(t,\theta) = |U_n + \theta w_{n+1}|^\alpha - |U_0|^\alpha,$$
$$B_{n+1}(t,\theta) = |U_n + \theta w_{n+1}|^{\alpha-2}(U_n + \theta w_{n+1})^2 - |U_0|^{\alpha-2}U_0^2.$$

By the directly computation, one can get

$$\begin{aligned} A_{n+1}(t,\theta) &= \int_0^1 \frac{d}{ds}|U_0 + sg_{n+1}(\theta)|^\alpha ds \\ &= \int_0^1 \alpha \mathrm{Re}\left[|U_0 + sg_{n+1}(\theta)|^{\alpha-2}(U_0 + sg_{n+1}(\theta))\overline{g_{n+1}}(\theta)\right] ds \end{aligned} \tag{2.34}$$

where $g_{n+1}(\theta) = U_n + \theta w_{n+1} - U_0$. From (2.12), (2.23) with $j = n+1$, (2.24) with $j = n$, (2.25) with $j = n+1$, (2.32), choosing T satisfying

$$C_0 T^{1-\frac{2}{k}} \le \frac{1}{2}, \tag{2.35}$$

we obtain

$$|g_{n+1}(\theta)| \le C_0(-t)^{1-\frac{2}{k}}|U_0| \le \frac{1}{2}|U_0|, \tag{2.36}$$

$$|\partial_t g_{n+1}(\theta)| \lesssim (-t)^{-\frac{2}{k}}|U_0|, \tag{2.37}$$

$$|\partial_t(U_0 + sg_{n+1}(\theta))| \lesssim (-t)^{-1}|U_0|. \tag{2.38}$$

It follows from (2.34)-(2.38) and Leibnitz's formula that

$$|A_{n+1}(t,\theta)| \lesssim (-t)^{-\frac{2}{k}}, \quad |\partial_t A_{n+1}(t,\theta)| \lesssim (-t)^{-1-\frac{2}{k}}. \tag{2.39}$$

Similarly, using Leibnitz's formula, we see that

$$|B_{n+1}(t,\theta)| \lesssim (-t)^{-\frac{2}{k}}, \quad |\partial_t B_{n+1}(t,\theta)| \lesssim (-t)^{-1-\frac{2}{k}}. \tag{2.40}$$

Now it follows from (2.25) with $j = n+1$, (2.33), (2.39)-(2.40) and Leibnitz's formula that

$$\begin{aligned} |\partial_t \mathcal{E}_{n+1}| &\lesssim |\partial_t \Delta w_{n+1}| + \int_0^1 (|A_{n+1}| + |B_{n+1}|)\,|\partial_t w_{n+1}| d\theta \\ &\quad + \int_0^1 (|\partial_t A_{n+1}| + |\partial_t B_{n+1}|)|w_{n+1}| d\theta \\ &\lesssim (-t)^{-1+(n+1)(1-\frac{2}{k})-\frac{2}{k}}|U_0|, \end{aligned}$$

which implies (2.30) with $j = n+1$. Thus (2.23)-(2.30) hold for all $0 \le j \le \frac{k-4}{2}$ by the induction. □

Then, we get the following lemma immediately.

Lemma 2.3. *Under the conditions in Lemma 2.2, we have*

$$\left|\partial_x^\beta (U_J - U_0)\right| \lesssim (-t)^{1-\frac{|\beta|+2}{k}} |U_0|, \quad 0 \le |\beta| \le k-1-2J, \tag{2.41}$$

$$\left|\partial_x^\beta \mathcal{E}_J\right| \lesssim (-t)^{J\left(1-\frac{2}{k}\right)-\frac{|\beta|+2}{k}} |U_0|, \quad 0 \le |\beta| \le k-3-2J, \tag{2.42}$$

$$\frac{1}{2}|U_0| \le |U_J| \le 2|U_0|, \tag{2.43}$$

$$U_J \in C^1\left((T,0), H^{k-1-2J}\left(\mathbb{R}^N\right)\right), \tag{2.44}$$

$$|\partial_t U_J| \lesssim (-t)^{-1}|U_0|, \tag{2.45}$$

$$|\partial_t \mathcal{E}_J| \lesssim (-t)^{-1+J(1-\frac{2}{k})-\frac{2}{k}}|U_0|, \tag{2.46}$$

$$\mathcal{E}_J = -\partial_t U_J + i\Delta U_J + \lambda |U_J|^\alpha U_J, \tag{2.47}$$

where $T \le t < 0$, $x \in \mathbb{R}^N$, $T \in (-1,0)$.

Finally, we introduce some useful estimates, which will be used in Section 3.

Lemma 2.4. *There exists a constant $M \geq 1$ such that*

$$||u+v|^{\alpha} - |v|^{\alpha}| \leq M(|u|^{\alpha} + 1_{\alpha>1}|u||v|^{\alpha-1}), \tag{2.48}$$

$$||u+v|^{\alpha-2}(u+v)^2 - |v|^{\alpha-2}v^2| \leq M(|u|^{\alpha} + 1_{\alpha>1}|u||v|^{\alpha-1}), \tag{2.49}$$

$$||u|^{\alpha}u - |v|^{\alpha}v| \leq M(|u|^{\alpha} + |v|^{\alpha})|u-v|, \tag{2.50}$$

and if $0 < \alpha \leq 1$,

$$||u+v|^{\alpha} - |u|^{\alpha}| + ||u+v|^{\alpha-2}(u+v)^2 - |u|^{\alpha-2}u^2| \leq M|u|^{\alpha-1}|v|, \tag{2.51}$$

for all $u, v \in \mathbb{C}$, where

$$1_{\alpha>1} = \begin{cases} 0, & \text{if } 0 < \alpha \leq 1, \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Proof. From (2.10) in [4], we can get (2.48) and (2.49), (also see formulas (2.26)-(2.27) in [3]). By the directly computation, one can get (2.50) easily, and omit the details. We prove (2.51) for completeness. Let $z \in \mathbb{C}, |z| \geq \frac{1}{2}$. From $|z|^{\alpha} \leq C|z|$, (2.48) and (2.49) we have

$$||1+z|^{\alpha} - 1| + \left||1+z|^{\alpha-2}(1+z)^2 - 1\right| \leq C|z|^{\alpha} \leq C|z|. \tag{2.52}$$

For $|z| \leq \frac{1}{2}$, writing

$$\begin{aligned} & ||1+z|^{\alpha} - 1| + \left||1+z|^{\alpha-2}(1+z)^2 - 1\right| \\ = & \int_0^1 \frac{d}{d\theta}\left[||1+\theta z|^{\alpha} - 1| + \left||1+\theta z|^{\alpha-2}(1+\theta z)^2 - 1\right|\right] d\theta, \end{aligned} \tag{2.53}$$

we get

$$\begin{aligned} & \left|\frac{d}{d\theta}\left[||1+\theta z|^{\alpha} - 1| + \left||1+\theta z|^{\alpha-2}(1+\theta z)^2 - 1\right|\right]\right| \\ \leq & \ C(\min_{0\leq\theta\leq1}|1+\theta z|)^{\alpha-1}|z| \leq C|z|, \end{aligned} \tag{2.54}$$

which yields (2.52). Now let $u, v \in \mathbb{C}$ with $u \neq 0$, setting $z = v/u$ in (2.52), we obtain that the inequality (2.51) by choosing M larger enough. □

Lemma 2.5. *Assume that $\lambda \in \mathbb{C}, 0 < \alpha, (N-4)\alpha < 4$, $I \subset \mathbb{R}$ is a compact interval and $u \in C(I, H^2(\mathbb{R}^N)) \cap C^1(I, L^2)$ is a strong H^2 solution of the equation*

$$\partial_t u = i\Delta u + \lambda|u|^{\alpha}u,$$

then we have

$$\partial_t(|u|^{\alpha}u) \in \begin{cases} L^2(I, L^{\frac{2N}{N+2}}(\mathbb{R}^N)), & \text{if } 2 \leq (N-2)\alpha, \\ L^2(I, L^2(\mathbb{R}^N)), & \text{if } (N-2)\alpha < 2. \end{cases}$$

Proof. Firstly we recall that u is bounded in $W^{1,q}(I, L^r(\mathbb{R}^N)) \cap L^q(I, H^{2,r}(\mathbb{R}^N))$ for every admissible pair $(q,r) \in \Lambda$ where

$$\Lambda = \{(q,r) : 2 \leq q, r \leq \infty, \frac{2}{q} + \frac{N}{r} = \frac{N}{2}, (q,r,N) \neq (2,\infty,2)\},$$

see [12, 21].

Then if $2 \leq (N-2)\alpha$, we choose two real numbers $r = \frac{2N(\alpha+1)}{N+2(\alpha+1)}, q = \frac{4(\alpha+1)}{(N-2)\alpha-2}$ such that $\frac{N+2}{2N} = \frac{1}{r} + \frac{\alpha}{2\alpha+2}$, and $(q,r) \in \Lambda$. By Hölder's inequality and note that

$2 \le r$, $(N-2)r < 2N$, $q \ge 2$, $H^2 \hookrightarrow L^{2\alpha+2}$, we deduce that

$$\begin{aligned}\|\partial_t(|u|^\alpha u)\|_{L^2(I,L^{\frac{2N}{N+2}}(\mathbb{R}^N))} &\le \|\|\partial_t u\|_{L^r(\mathbb{R}^N)}\|u\|^\alpha_{L^{2\alpha+2}(\mathbb{R}^N)}\|_{L^2(I)} \\ &\le \|u\|^\alpha_{L^\infty(I,H^2(\mathbb{R}^N))}\|\partial_t u\|_{L^2(I,L^r(\mathbb{R}^N))} \\ &\le C(I)\|u\|^\alpha_{L^\infty(I,H^2(\mathbb{R}^N))}\|\partial_t u\|_{L^q(I,L^r(\mathbb{R}^N))} < +\infty.\end{aligned}$$

In the case $(N-2)\alpha < 2$, we may choose $q = \frac{4(\alpha+1)}{N\alpha} > 2$ such that $(q, 2\alpha+2) \in \Lambda$. Thus, by Hölder's inequality and $H^2 \hookrightarrow L^{2\alpha+2}$, we deduce that

$$\begin{aligned}\|\partial_t(|u|^\alpha u)\|_{L^2(I,L^2(\mathbb{R}^N))} &\le \|\|\partial_t u\|_{L^{2\alpha+2}(\mathbb{R}^N)}\|u\|^\alpha_{L^{2\alpha+2}(\mathbb{R}^N)}\|_{L^2(I)} \\ &\le C(I)\|u\|^\alpha_{L^\infty(I,H^2(\mathbb{R}^N))}\|\partial_t u\|_{L^2(I,L^{2\alpha+2}(\mathbb{R}^N))} \\ &\le C(I)\|u\|^\alpha_{L^\infty(I,H^2(\mathbb{R}^N))}\|\partial_t u\|_{L^{q(\alpha)}(I,L^{2\alpha+2}(\mathbb{R}^N))} < +\infty.\end{aligned}$$

□

3. **Construction and estimates of approximate solutions.** In this section, we construct a sequence of solutions u_n of (1.1), close to the ansatz U_J in Lemma 2.3, which will eventually converge to the blowing-up solution of Theorem 1.1. We will estimate $\varepsilon_n = u_n - U_J$ by the energy method. More precisely, we estimate

$$(-t)^{-\sigma}\|\varepsilon_n\|_2 + (-t)^{-(1-\frac{3}{8}\gamma)\sigma}\|\nabla\varepsilon_n\|_2 + (-t)^{-(1-\gamma)\sigma}\|\Delta\varepsilon_n\|_2 + (-t)^{-(1-\frac{\gamma}{2})\sigma}\|\partial_t\varepsilon_n\|_2$$

for some appropriate parameters γ, σ given in (2.3) and (2.4).

Let the ansatz U_J and $T < 0$ be given in Lemma 2.3. From $2J \le k-4$ by (2.2), $U_J\left(-\frac{1}{n}\right) \in H^2\left(\mathbb{R}^N\right)$ by (2.2) and (2.28), we obtain that there exist $s_n < -\frac{1}{n}$ and a unique solution $u_n \in C\left(\left(s_n, -\frac{1}{n}\right], H^2\left(\mathbb{R}^N\right)\right) \cap C^1\left(\left(s_n, -\frac{1}{n}\right], L^2\left(\mathbb{R}^N\right)\right)$ of the following nonlinear Schrödinger equation

$$\begin{cases} \partial_t u_n = i\Delta u_n + \lambda\left|u_n\right|^\alpha u_n, \\ u_n\left(-\frac{1}{n}\right) = U_J\left(-\frac{1}{n}\right), \end{cases} \tag{3.1}$$

defined on the maximal interval $\left(s_n, -\frac{1}{n}\right]$, with the blow-up alternative that if $s_n > -\infty$, then

$$\|u_n(t)\|_{H^2} \underset{t\downarrow s_n}{\longrightarrow} \infty. \tag{3.2}$$

see [12]. Letting $\varepsilon_n \in C(I_n, H^2(\mathbb{R}^N)) \cap C^1(I_n, L^2(\mathbb{R}^N))$ be defined by

$$u_n = U_J + \varepsilon_n, \tag{3.3}$$

with $I_n = (\max\{s_n, T\}, -\frac{1}{n}]$, we have the following estimate.

Proposition 3.1. *There exist $T \le S < 0$ and $n_0 > -\frac{1}{S}$ such that $s_n < S$, for all $n \ge n_0$. Moreover,*

$$\|\varepsilon_n(t)\|_{L^2} \le (-t)^\sigma, \quad \|\nabla\varepsilon_n(t)\|_{L^2} \le (-t)^{(1-\frac{3}{8}\gamma)\sigma}, \tag{3.4}$$

$$\|\Delta\varepsilon_n(t)\|_{L^2} \le (-t)^{(1-\gamma)\sigma}, \quad \|\partial_t\varepsilon_n(t)\|_{L^2} \le (-t)^{(1-\frac{\gamma}{2})\sigma}, \tag{3.5}$$

for all $n \ge n_0$ and $t \in [S, -\frac{1}{n}]$.

Proof. Throughout the proof, we write ε instead of ε_n. Moreover, C denotes a constant that may change from line to line, but is independent of n and t. Unless otherwise specified, all integrals are over $\mathbb{R}^N$. Using (2.22) and (3.3), we have

$$\begin{cases}\partial_t\varepsilon = i\Delta\varepsilon + \lambda(|U_J+\varepsilon|^\alpha(U_J+\varepsilon) - |U_J|^\alpha U_J) + \mathcal{E}_J, \\ \varepsilon(-\frac{1}{n}) = 0.\end{cases} \tag{3.6}$$

Let

$$\tau_n = \inf \quad \Big\{ \quad t \in \left[\max\{T, s_n\}, -\frac{1}{n}\right]; \|\varepsilon(s)\|_{L^2} \le (-s)^{\sigma}, \\ \|\nabla\varepsilon(s)\|_{L^2} \le (-s)^{(1-\frac{3}{8}\gamma)\sigma},\ \|\Delta\varepsilon(s)\|_{L^2} \le (-s)^{(1-\gamma)\sigma}, \\ \|\partial_t\varepsilon(s)\|_{L^2} \le (-s)^{(1-\frac{\gamma}{2})\sigma},\ \text{for all } t < s \le -\frac{1}{n}\Big\}. \tag{3.7}$$

Since $\varepsilon(-\frac{1}{n}) = 0$, we see that $T \le \tau_n < -\frac{1}{n}$. Moreover, it follows from the blow-up alternative (3.2) that $s_n < \tau_n$.

We first estimate $\|\varepsilon(t)\|_{L^2}$. Multiplying (3.6) by $\overline{\varepsilon}$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon(t)\|_{L^2}^2 = \mathrm{Re}\left(\lambda\int[|U_J+\varepsilon|^{\alpha}(U_J+\varepsilon) - |U_J|^{\alpha}U_J]\overline{\varepsilon}\right) + \mathrm{Re}\int \mathcal{E}_J\bar{\varepsilon}.$$

Using Lemma 2.4, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon(t)\|_{L^2}^2 \ge -|\lambda|M\int(|U_J|^{\alpha}+|\varepsilon|^{\alpha})|\varepsilon|^2 - \|\mathcal{E}_J\|_{L^2}\|\varepsilon\|_{L^2}. \tag{3.8}$$

By (2.9) and (2.43), we have

$$\int|U_J|^{\alpha}|\varepsilon|^2 \le 2^{\alpha}(\alpha\mathrm{Re}\lambda)^{-1}(-t)^{-1}\|\varepsilon\|_{L^2}^2. \tag{3.9}$$

In addition, by Gagliardo-Nirenberg's inequality and (3.7), we get

$$\int|\varepsilon|^{\alpha+2} \le C\|\varepsilon\|_2^{\alpha+2-\frac{N}{4}\alpha}\|\Delta\varepsilon\|_2^{\frac{N}{4}\alpha} \le C\|\varepsilon\|_{H^2}^{\alpha+2} \le C(-t)^{(\alpha+2)(1-\gamma)\sigma}. \tag{3.10}$$

Next, by (2.42), we obtain

$$\|\mathcal{E}_J\|_{L^2}\|\varepsilon\|_{L^2} \le C(-t)^{J\left(1-\frac{2}{k}\right)-\frac{2}{k}-\frac{1}{\alpha}+\sigma} = C(-t)^{-1+(J+1)\left(1-\frac{2}{k}\right)-\frac{1}{\alpha}+\sigma}. \tag{3.11}$$

By (2.1), (2.2) and (2.3), we have

$$(J+1)\left(1-\frac{2}{k}\right) - \frac{1}{\alpha} + \sigma \ge \frac{1}{2}(J+1) - \frac{1}{\alpha} + \sigma \ge 3\sigma, \\ (\alpha+2)(1-\gamma)\sigma \ge 2\sigma, \tag{3.12}$$

and

$$|\lambda|M\int|\varepsilon|^{\alpha+2} + \|\mathcal{E}_J\|_{L^2}\|\varepsilon\|_{L^2} \le C(-t)^{2\sigma}, \tag{3.13}$$

where $T \in (-1, 0)$ and $\sigma > 1$ by (2.4). It follows from (3.8), (3.9) and (3.13) that

$$\frac{d}{dt}\|\varepsilon(t)\|_{L^2}^2 \ge -2^{\alpha+1}(\alpha\mathrm{Re}\lambda)^{-1}|\lambda|M(-t)^{-1}\|\varepsilon\|_{L^2}^2 - C(-t)^{2\sigma}$$

and

$$\frac{d}{dt}\left((-t)^{-\sigma}\|\varepsilon(t)\|_{L^2}^2\right) = \sigma(-t)^{-\sigma-1}\|\varepsilon(t)\|_{L^2}^2 + (-t)^{-\sigma}\frac{d}{dt}\|\varepsilon(t)\|_{L^2}^2 \\ \ge \left[\sigma - 2^{\alpha+1}(\alpha\mathrm{Re}\lambda)^{-1}|\lambda|M\right](-t)^{-\sigma-1}\|\varepsilon(t)\|_{L^2}^2 - C(-t)^{\sigma}.$$

Using (2.4), we obtain

$$\frac{d}{dt}\left((-t)^{-\sigma}\|\varepsilon(t)\|_{L^2}^2\right) \ge -C(-t)^{\sigma}.$$

Integrating on $(t, -\frac{1}{n})$ and using $\varepsilon(-\frac{1}{n}) = 0$, we deduce that

$$\|\varepsilon(t)\|_{L^2} \le C_1(-t)^{\sigma+\frac{1}{2}} \tag{3.14}$$

for all $t \in (\tau_n, -\frac{1}{n})$.

Multiplying the equation (3.6) by $-\Delta\bar{\varepsilon}$ and taking the real part, we obtain after integrating by parts

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\varepsilon\|_{L^2}^2 &= \mathrm{Re}\lambda\int \nabla(|U_J+\varepsilon|^\alpha(U_J+\varepsilon) - |U_J|^\alpha U_J - |\varepsilon|^\alpha\varepsilon)\cdot\nabla\bar{\varepsilon} \\ &+\mathrm{Re}\lambda\int \nabla(|\varepsilon|^\alpha\varepsilon)\cdot\nabla\bar{\varepsilon} + \mathrm{Re}\int\nabla\mathcal{E}_J\cdot\nabla\bar{\varepsilon} := N_1+N_2+N_3. \end{aligned} \tag{3.15}$$

By Hölder's and Gagliardo-Nirenberg's inequality, and note that

$$\nabla(|\varepsilon|^\alpha\varepsilon) = \frac{\alpha+2}{2}|\varepsilon|^\alpha\nabla\varepsilon + \frac{\alpha}{2}|\varepsilon|^{\alpha-2}\varepsilon^2\nabla\bar{\varepsilon}$$

we deduce that

$$\begin{aligned} |N_2| &\le C\int|\varepsilon|^\alpha|\nabla\varepsilon|^2 \le C\left(\int|\varepsilon|^{2\alpha+2}\right)^{\frac{\alpha}{2\alpha+2}}\left(\int|\nabla\varepsilon|^{\frac{4(\alpha+1)}{\alpha+2}}\right)^{\frac{\alpha+2}{2\alpha+2}} \\ &\le C\|\varepsilon\|_{H^2}^{\alpha+2} \le C(-t)^{(\alpha+2)(1-\gamma)\sigma} \le C(-t)^{-1+2\sigma}, \end{aligned} \tag{3.16}$$

where $(N-4)(2\alpha+2) < 2N$ and $4(N-2)(\alpha+1)/(\alpha+2) < 2N$ by (1.2), $(\alpha+2)(1-\gamma)\sigma \ge -1+2\sigma$ by (3.12). Next by (2.42) and (3.7), we see that

$$\begin{aligned} |N_3| &\le \|\nabla\mathcal{E}_J\|_{L^2}\|\nabla\varepsilon\|_{L^2} \le C(-t)^{J\left(1-\frac{2}{k}\right)-\frac{3}{k}}\|U_0\|_{L^2}\|\nabla\varepsilon\|_{L^2} \\ &\le C(-t)^{J\left(1-\frac{2}{k}\right)-\frac{3}{k}-\frac{1}{\alpha}+(1-\frac{3}{8}\gamma)\sigma} \le C(-t)^{-1+2\sigma}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} J\left(1-\frac{2}{k}\right)-\frac{3}{k}-\frac{1}{\alpha}+(1-\frac{3}{8}\gamma)\sigma &= -1+(J+1)\left(1-\frac{2}{k}\right)-\frac{1}{k}-\frac{1}{\alpha}+(1-\frac{3}{8}\gamma)\sigma \\ &> -1+\frac{J+1}{2}-\frac{1}{k}-\frac{1}{\alpha}+(1-\frac{3}{8}\gamma)\sigma \\ &> -1+2\sigma-\frac{1}{k}+(1-\frac{3}{8}\gamma)\sigma \ge -1+2\sigma \end{aligned}$$

by (2.1) and (2.2). We now estimate N_1. By the directly computation, we have

$$\begin{aligned} &\nabla(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J-|\varepsilon|^\alpha\varepsilon) \\ &= \frac{\alpha+2}{2}(|U_J+\varepsilon|^\alpha-|\varepsilon|^\alpha)\nabla\varepsilon + \frac{\alpha}{2}(|U_J+\varepsilon|^{\alpha-2}(U_J+\varepsilon)^2-|\varepsilon|^{\alpha-2}\varepsilon^2)\nabla\bar{\varepsilon} \\ &+\frac{\alpha+2}{2}(|U_J+\varepsilon|^\alpha-|U_J|^\alpha)\nabla U_J + \frac{\alpha}{2}(|U_J+\varepsilon|^{\alpha-2}(U_J+\varepsilon)^2-|U_J|^{\alpha-2}U_J^2)\nabla\overline{U_J}, \end{aligned}$$

and

$$|N_1| \le (\alpha+1)|\lambda|(\int B_1|\nabla\varepsilon|^2 + \int B_2|\nabla U_J\nabla\varepsilon|) \tag{3.18}$$

with

$$\begin{aligned} B_1 &= ||U_J+\varepsilon|^\alpha-|\varepsilon|^\alpha| + \left||U_J+\varepsilon|^{\alpha-2}(U_J+\varepsilon)^2-|\varepsilon|^{\alpha-2}\varepsilon^2\right|, \\ B_2 &= ||U_J+\varepsilon|^\alpha-|U_J|^\alpha| + \left|U_J+\varepsilon|^{\alpha-2}(U_J+\varepsilon)^2-|U_J|^{\alpha-2}U_J^2\right|. \end{aligned} \tag{3.19}$$

It follows from Lemma 2.4 and (2.43) that

$$B_1 \le 2^\alpha M|U_0|^\alpha + 2M1_{\alpha>1}|\varepsilon|^{\alpha-1}|U_0|. \tag{3.20}$$

If $\alpha > 1$, then $|\varepsilon|^{\alpha-1}|U_0| \le |\varepsilon|^\alpha + |U_0|^\alpha$, so that

$$B_1 \le (2^\alpha+2)M\,|U_0|^\alpha + C|\varepsilon|^\alpha. \tag{3.21}$$

Then, from (3.20)-(3.21), we obtain

$$\begin{aligned}\int B_1|\nabla\varepsilon|^2 &\leq (2^\alpha+2)M(\alpha\mathrm{Re}\lambda)^{-1}(-t)^{-1}\|\nabla\varepsilon\|_{L^2}^2+C\int|\varepsilon|^\alpha|\nabla\varepsilon|^2\\ &\leq (2^\alpha+2)M(\alpha\mathrm{Re}\lambda)^{-1}(-t)^{-1}\|\nabla\varepsilon\|_{L^2}^2+C(-t)^{-1+2\sigma} \qquad (3.22)\end{aligned}$$

by (2.9) and (3.16).

Next we estimate B_2, separately the cases $\alpha\leq 1$ and $\alpha>1$. When $\alpha\leq 1$, using (2.9), (2.12), (2.41), (3.7) and Lemma 2.4, we deduce that

$$\begin{aligned}\int B_2|\nabla U_J\nabla\varepsilon| &\leq C\int|U_J|^{\alpha-1}|\varepsilon||\nabla U_J||\nabla\varepsilon|\\ &\leq C(-t)^{-1-\frac{1}{k}}\|\varepsilon\|_{L^2}\|\nabla\varepsilon\|_{L^2}\\ &\leq C(-t)^{-1-\frac{1}{k}+(2-\frac{3}{8}\gamma)\sigma}. \qquad (3.23)\end{aligned}$$

When $\alpha>1$, we deduce from Lemma 2.4 and (2.41) that

$$\begin{aligned}&\int B_2|\nabla U_J\nabla\varepsilon|\leq C\int(|U_J|^{\alpha-1}+|\varepsilon|^{\alpha-1})|\varepsilon||\nabla U_J||\nabla\varepsilon|\\ \leq\ & C\|U_J\|_\infty^{\alpha-1}\|\nabla U_J\|_\infty\|\varepsilon\|_2\|\nabla\varepsilon\|_2+C\|\nabla U_J\|_\infty\|\varepsilon\|_{2\alpha}^\alpha\|\nabla\varepsilon\|_2\\ \leq\ & C(-t)^{-1-\frac{1}{k}+(2-\frac{3}{8}\gamma)\sigma}+C(-t)^{-1-\frac{1}{k}+(\alpha-\alpha\frac{N}{2}(\frac{1}{2}-\frac{1}{2\alpha})\gamma)\sigma+(1-\frac{3}{8}\gamma)\sigma}, \qquad (3.24)\end{aligned}$$

where $\|\varepsilon\|_{2\alpha}\leq C\|\varepsilon\|_2^{1-\frac{N}{2}(\frac{1}{2}-\frac{1}{2\alpha})}\|\Delta\varepsilon\|_2^{\frac{N}{2}(\frac{1}{2}-\frac{1}{2\alpha})}$ by Gagliardo-Nirenberg's inequality. Note that

$$\alpha-\alpha\frac{N}{2}(\frac{1}{2}-\frac{1}{2\alpha})\gamma-1=(\alpha-1)(1-\frac{N}{4}\gamma)\geq 0$$

by (2.3) and $\alpha>1$, we deduce that $\int B_2|\nabla U_J\nabla\varepsilon|\leq C(-t)^{-1-\frac{1}{k}+(2-\frac{3}{8}\gamma)\sigma}$. Moreover, we see that $-\frac{1}{k}+(2-\frac{3}{8}\gamma)\sigma\geq 2(1-\frac{3}{8}\gamma)\sigma+\frac{3\gamma\sigma}{16}$ by $k>\frac{16}{3\gamma\sigma}$ in (2.2), hence

$$\int B_2|\nabla U_J\nabla\varepsilon|\leq C(-t)^{-1+2(1-\frac{3}{8}\gamma)\sigma+\frac{3\gamma\sigma}{16}}, \qquad (3.25)$$

so that

$$|N_1|\leq\frac{(\alpha+1)(2^\alpha+2)M|\lambda|}{\alpha\mathrm{Re}\lambda(-t)}\|\nabla\varepsilon\|_{L^2}^2+C(-t)^{-1+2(1-\frac{3}{8}\gamma)\sigma+\frac{3\gamma\sigma}{16}}. \qquad (3.26)$$

Combining (3.15)-(3.17), (3.26) and $-1+2\sigma>-1+2(1-\frac{3}{8}\gamma)\sigma+\frac{3\gamma\sigma}{16}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla\varepsilon(t)\|_2^2\geq &-2(\alpha+1)(2^\alpha+2)M(\alpha\mathrm{Re}\lambda)^{-1}|\lambda|(-t)^{-1}\|\nabla\varepsilon\|_{L^2}^2\\ &-C(-t)^{-1+2(1-\frac{3}{8}\gamma)\sigma+\frac{3\gamma\sigma}{16}}.\end{aligned}$$

Using (2.4), we deduce that

$$\begin{aligned}\frac{d}{dt}\left[(-t)^{-\sigma}\|\nabla\varepsilon(t)\|_2^2\right]&=\sigma(-t)^{-\sigma-1}\|\nabla\varepsilon(t)\|_2^2+(-t)^{-\sigma}\frac{d}{dt}\|\nabla\varepsilon(t)\|_2^2\\ &\geq(\sigma-2(\alpha+1)(2^\alpha+2)|\lambda|M(\alpha\mathrm{Re}\lambda)^{-1})(-t)^{-1-\sigma}\|\nabla\varepsilon\|_2^2\\ &\qquad -C(-t)^{-1+(1-\frac{3}{4}\gamma)\sigma+\frac{3\gamma\sigma}{16}}\\ &\geq -C(-t)^{-1+(1-\frac{3}{4}\gamma)\sigma+\frac{3\gamma\sigma}{16}}.\end{aligned}$$

Integrating on $(t,-\frac{1}{n})$, using $\varepsilon(-\frac{1}{n})=0$, and multiplying by $(-t)^\sigma$, we obtain

$$\|\nabla\varepsilon(t)\|_2\leq C_2(-t)^{(1-\frac{3}{8}\gamma)\sigma+\frac{3\gamma\sigma}{32}} \qquad (3.27)$$

for all $\tau_n<t\leq-\frac{1}{n}$.

Thus, multiplying the equation (3.6) by $\Delta\bar{\varepsilon}$ and taking the imaginary part, we obtain

$$\begin{aligned}\|\Delta\varepsilon\|_2^2 \leq & |\lambda|\int |\nabla(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J-|\varepsilon|^\alpha\varepsilon)||\nabla\bar{\varepsilon}| \\ & +|\lambda|\int|\nabla(|\varepsilon|^\alpha\varepsilon)||\nabla\varepsilon|+\int|\nabla\mathcal{E}_J\nabla\bar{\varepsilon}|+\int|\partial_t\varepsilon\Delta\varepsilon| \\ \leq & C(N_1+N_2+N_3)+\|\partial_t\varepsilon\|_{L^2}\|\Delta\varepsilon\|_{L^2} \\ \leq & C(-t)^{-1+2(1-\frac{3}{8}\gamma)\sigma}+C(-t)^{(2-\gamma-\frac{\gamma}{2})\sigma} \\ \leq & C(-t)^{2(1-\gamma)\sigma+\frac{\gamma\sigma}{2}}\end{aligned}$$

where $-1+2(1-\frac{3}{8}\gamma)\sigma \geq 2(1-\gamma)\sigma+\frac{\gamma\sigma}{2}$ by (2.4), and the (3.16), (3.17), (3.26) for the estimates of N_1, N_2, N_3. So we deduce that

$$\|\Delta\varepsilon\|_2 \leq C_3(-t)^{(1-\gamma)\sigma+\frac{\gamma\sigma}{4}}. \tag{3.28}$$

Finally, we estimate $\|\partial_t\varepsilon\|_{L^2}$, which is similarly to $\|\nabla\varepsilon\|_{L^2}$ and slight modifications. We choose $\rho \in C_0^\infty(\mathbb{R}^N)$ with $\int\rho dx = 1$, and $\rho_\delta(x) = \rho(\frac{x}{\delta})\delta^{-N}(\delta > 0)$. Applying time derivative ∂_t to the equation (3.6), taking convolution with ρ_δ and then multiplying it by $\partial_t\bar{\varepsilon} * \rho_\delta$, taking the real part, we obtain after integrating by parts

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_t\varepsilon*\rho_\delta\|_{L^2}^2 \\ = \ & \mathrm{Re}[\lambda\int(\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J)*\rho_\delta)\cdot(\partial_t\bar{\varepsilon}*\rho_\delta)].\end{aligned} \tag{3.29}$$

Multiplying the equation (3.29) by $(-t)^{-\sigma}$, and then integrating it on the interval $(t, -\frac{1}{n})$, we obtain

$$\begin{aligned}-\frac{1}{2}(-t)^{-\sigma}\|\partial_t\varepsilon*\rho_\delta\|_{L^2}^2 = \int_t^{-\frac{1}{n}}(-s)^{-\sigma}\Big(&\mathrm{Re}[\lambda\int(\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon) \\ &-|U_J|^\alpha U_J+\mathcal{E}_J)*\rho_\delta)\cdot(\partial_t\bar{\varepsilon}*\rho_\delta)]+\frac{\sigma}{2}(-s)^{-1}\|\partial_t\varepsilon*\rho_\delta\|_{L^2}^2\Big)\,ds,\end{aligned} \tag{3.30}$$

where $s_n < t < -\frac{1}{n}$. Now by Lemma 2.5, (2.11), (2.41)-(2.42) and (2.45)-(2.46), we have that $\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J)$ is bounded in $L^2([T,-\frac{1}{n}], L^{\frac{2N}{N+2}}(\mathbb{R}^N))$ if $2 \leq (N-2)\alpha$ or bounded in $L^2([T,-\frac{1}{n}], L^2(\mathbb{R}^N))$ if $(N-2)\alpha < 2$ for any $s_n < T < -\frac{1}{n}$. Note also that $\partial_t\bar{\varepsilon}$ is bounded in $L^2([T,-\frac{1}{n}], L^{\frac{2N}{N-2}}(\mathbb{R}^N))(N \geq 3) \cap L^2([T,-\frac{1}{n}], L^2(\mathbb{R}^N))$ for any $s_n < T < -\frac{1}{n}$. Then, for a.e. $t \in (s_n, -\frac{1}{n})$, we deduce that

$$\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N), \partial_t\bar{\varepsilon} \in L^{\frac{2N}{N-2}}, \text{if } 2 \leq (N-2)\alpha,$$

or

$$\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J) \in L^2(\mathbb{R}^N), \partial_t\bar{\varepsilon} \in L^2, \text{if } (N-2)\alpha < 2.$$

By Young's and Hölder's inequality we deduce that for a.e. $t \in (s_n, -\frac{1}{n})$

$$\begin{aligned}&\mathrm{Re}[\lambda\int(\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J)*\rho_\delta)\cdot(\partial_t\bar{\varepsilon}*\rho_\delta)] \\ &\underset{\delta\to0^+}{\longrightarrow} \mathrm{Re}[\lambda\int(\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J))\cdot\partial_t\bar{\varepsilon}],\end{aligned} \tag{3.31}$$

and the left hand side of (3.31) is dominated by the integrable function

$$|\lambda|\|\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J)\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}\|\partial_t\bar{\varepsilon}\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)},$$

if $2\le (N-2)\alpha$, or dominated by

$$|\lambda|\|\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J+\mathcal{E}_J)\|_{L^2(\mathbb{R}^N)}\|\partial_t\bar{\varepsilon}\|_{L^2(\mathbb{R}^N)},$$

if $(N-2)\alpha<2$. In both cases, the domainated function is integrable on interval $[T,-\frac{1}{n}]$ for any $s_n<T<-\frac{1}{n}$. Thus, we can passing the limit $\delta\to 0$ in (3.30) to get that

$$\begin{aligned}
&-\frac{1}{2}(-t)^{-\sigma}\|\partial_t\varepsilon\|_{L^2}^2\\
=&\int_t^{-\frac{1}{n}}(-s)^{-\sigma}\Big(\mathrm{Re}[\lambda\int(\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J-|\varepsilon|^\alpha\varepsilon))\\
&+\partial_t(|\varepsilon|^\alpha\varepsilon)+\partial_t\mathcal{E}_J]\cdot(\partial_t\bar{\varepsilon})+\frac{\sigma}{2}(-s)^{-1}\|\partial_t\varepsilon\|_{L^2}^2\Big)\,ds.\\
=&\int_t^{-\frac{1}{n}}(-s)^{-\sigma}[M_1+M_2+M_3+\frac{\sigma}{2}(-s)^{-1}\|\partial_t\varepsilon\|_{L^2}^2]ds
\end{aligned}\tag{3.32}$$

We first estimate M_2. If $N\ge 4$, then

$$\begin{aligned}
M_2 &= \mathrm{Re}[\lambda\int\partial_t(|\varepsilon|^\alpha\varepsilon)\partial_t\bar{\varepsilon}]\\
&= (\mathrm{Re}\lambda)\mathrm{Re}\int\partial_t(|\varepsilon|^\alpha\varepsilon)\partial_t\bar{\varepsilon}-\mathrm{Im}\lambda\cdot\mathrm{Im}\int\partial_t(|\varepsilon|^\alpha\varepsilon)\partial_t\bar{\varepsilon}\\
&\ge \mathrm{Re}\lambda\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2-\mathrm{Im}\lambda\frac{\alpha}{2}\mathrm{Im}\int|\varepsilon|^{\alpha-2}\varepsilon^2(\partial_t\bar{\varepsilon})^2\\
&\ge (\mathrm{Re}\lambda-\frac{\alpha}{2}|\mathrm{Im}\lambda|)\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2=\mu\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2.
\end{aligned}\tag{3.33}$$

where $\mu=\mathrm{Re}\lambda-\frac{\alpha}{2}|\mathrm{Im}\lambda|$, and

$$\mathrm{Re}\int\partial_t(|\varepsilon|^\alpha\varepsilon)\partial_t\bar{\varepsilon}=\frac{\alpha+2}{2}\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2+\mathrm{Re}\frac{\alpha}{2}\int|\varepsilon|^{\alpha-2}\varepsilon^2(\partial_t\bar{\varepsilon})^2\ge\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2.$$

When $1\le N\le 3$, we deduce that

$$\begin{aligned}
|M_2| &\le C\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2\le C\|\varepsilon\|_\infty^\alpha\|\partial_t\varepsilon\|_2^2\le C\|\varepsilon\|_{H^2}^\alpha\|\partial_t\varepsilon\|_2^2\\
&\le C(-s)^{((2-\gamma)+(1-\gamma)\alpha)\sigma}\le C(-s)^{-1+2(1-\frac{\gamma}{2})\sigma+\frac{\gamma\sigma}{4}},
\end{aligned}\tag{3.34}$$

by Sobolev's embedding $H^2(\mathbb{R}^N)\hookrightarrow L^\infty(\mathbb{R}^N)$ and (2.3), (3.7).

Next we estimate M_3. By using (2.46), (3.7) and note that $-1+J(1-\frac{2}{k})-\frac{2}{k}+(1-\frac{\gamma}{2})\sigma-\frac{1}{\alpha}\ge -1+2(1-\frac{\gamma}{2})\sigma+\frac{\gamma\sigma}{4}$ by (2.1) and (2.2), we see that

$$\begin{aligned}
|M_3| &= |\mathrm{Re}\int\partial_t\mathcal{E}_J\cdot\partial_t\bar{\varepsilon}|\le C(-s)^{-1+J(1-\frac{2}{k})-\frac{2}{k}+(1-\frac{\gamma}{2})\sigma-\frac{1}{\alpha}}\\
&\le C(-s)^{-1+2(1-\frac{\gamma}{2})\sigma+\frac{\gamma\sigma}{4}}.
\end{aligned}\tag{3.35}$$

We now estimate M_1. By the directly computation, we have

$$\begin{aligned}&\partial_t(|U_J+\varepsilon|^\alpha(U_J+\varepsilon)-|U_J|^\alpha U_J-|\varepsilon|^\alpha\varepsilon)\\&=\frac{\alpha+2}{2}(|U_J+\varepsilon|^\alpha-|\varepsilon|^\alpha)\partial_t\varepsilon+\frac{\alpha}{2}(|U_J+\varepsilon|^{\alpha-2}(U_J+\varepsilon)^2-|\varepsilon|^{\alpha-2}\varepsilon^2)\partial_t\bar{\varepsilon}\\&+\frac{\alpha+2}{2}(|U_J+\varepsilon|^\alpha-|U_J|^\alpha)\partial_tU_J+\frac{\alpha}{2}(|U_J+\varepsilon|^{\alpha-2}(U_J+\varepsilon)^2-|U_J|^{\alpha-2}U_J^2)\partial_t\overline{U_J},\end{aligned}$$

and

$$|M_1|\le(\alpha+1)|\lambda|(\int B_1|\partial_t\varepsilon|^2+\int B_2|\partial_tU_J\partial_t\varepsilon|). \tag{3.36}$$

If $\alpha>1$, then $|\varepsilon|^{\alpha-1}|U_0|\le\mu(4(\alpha+1)|\lambda|M)^{-1}|\varepsilon|^\alpha+K_1|U_0|^\alpha$ by Young's inequality, where $K_1=|\mu|^{1-\alpha}(4(\alpha+1)|\lambda|M)^{\alpha-1}$. By the inequality (2.48), we get

$$\int B_1|\partial_t\varepsilon|^2\le\frac{(2^\alpha+2K_1)M}{\alpha\mathrm{Re}\lambda(-s)}\|\partial_t\varepsilon\|_{L^2}^2+\frac{\mu}{2(\alpha+1)|\lambda|}\int|\varepsilon|^\alpha|\partial_t\varepsilon|^2. \tag{3.37}$$

Moreover, if $1\le N\le3$, we have by (3.34) that

$$\int B_1|\partial_t\varepsilon|^2\le\frac{(2^\alpha+2K_1)M}{\alpha\mathrm{Re}\lambda(-s)}\|\partial_t\varepsilon\|_{L^2}^2+C(-s)^{-1+2(1-\frac{\gamma}{2})\sigma+\frac{\gamma\sigma}{4}}. \tag{3.38}$$

Next we estimate B_2 term, separately the cases $\alpha\le1$ and $\alpha>1$. When $\alpha\le1$, from Lemma 2.4, (2.9), (2.43), (2.45) and (3.7), we get that

$$\int B_2|\partial_tU_J\partial_t\varepsilon|\le C\int|U_J|^{\alpha-1}|\varepsilon||\partial_tU_J||\partial_t\varepsilon|\le C(-s)^{-2+(2-\frac{\gamma}{2})\sigma}. \tag{3.39}$$

When $\alpha>1$, we deduce from Lemma 2.4, (2.9), (2.43), (2.45) and (3.7), that

$$\begin{aligned}\int B_2|\partial_tU_J\partial_t\varepsilon|&\le C\int(|U_J|^{\alpha-1}+|\varepsilon|^{\alpha-1})|\varepsilon||\partial_tU_J||\partial_t\varepsilon|\\&\le C(-s)^{-2}\|\varepsilon\|_{L^2}\|\partial_t\varepsilon\|_{L^2}+C(-s)^{-2}\|\varepsilon\|_{2\alpha}^\alpha\|\partial_t\varepsilon\|_2\\&\le C(-s)^{-2+(2-\frac{\gamma}{2})\sigma}+C(-s)^{-2+(\alpha-\alpha\frac{N}{2}(\frac12-\frac{1}{2\alpha})\gamma)\sigma+(1-\frac{\gamma}{2})\sigma}\\&\le C(-s)^{-2+(2-\frac{\gamma}{2})\sigma},\end{aligned} \tag{3.40}$$

where $-2+(\alpha-\alpha\frac{N}{2}(\frac12-\frac{1}{2\alpha})\gamma)\sigma+(1-\frac{\gamma}{2})\sigma\ge-2+(2-\frac{\gamma}{2})\sigma$ by $\alpha>1$ and (2.3). Combining (3.32)-(3.40), and note that $-2+(2-\frac{\gamma}{2})\sigma\ge-1+2(1-\frac{\gamma}{2})\sigma+\frac{\gamma\sigma}{4}$ by (2.4), we obtain for all $N\ge1$

$$\begin{aligned}&-\frac12(-t)^{-\sigma}\|\partial_t\varepsilon\|_{L^2}^2\\\ge\ &\int_t^{-\frac1n}(-s)^{-\sigma-1}[\frac{\sigma}{2}-(2^\alpha+2K_1)(\alpha+1)|\lambda|M(\alpha\mathrm{Re}\lambda)^{-1}]\cdot\|\partial_t\varepsilon\|_2^2ds\\&-C\int_t^{-\frac1n}(-s)^{-1+(1-\gamma)\sigma+\frac{\gamma\sigma}{4}}ds\\\ge\ &-C\int_t^{-\frac1n}(-s)^{-1+(1-\gamma)\sigma+\frac{\gamma\sigma}{4}}ds\ge-C(-t)^{(1-\gamma)\sigma+\frac{\gamma\sigma}{4}},\end{aligned}$$

which implies that

$$\|\partial_t\varepsilon(t)\|_2\le C_4(-t)^{(1-\frac{\gamma}{2})\sigma+\frac{\gamma\sigma}{8}} \tag{3.41}$$

for all $\tau_n<t\le-\frac1n$.

By (3.14), (3.27), (3.28), and (3.41), there exists $S\in[T,0)$ satisfying

$$C_1(-S)^{\frac12}\le1,\quad C_2(-S)^{\frac{3\gamma\sigma}{32}}\le1,\quad C_3(-S)^{\frac{\gamma\sigma}{4}}\le1,\quad C_4(-S)^{\frac{\gamma\sigma}{8}}\le1, \tag{3.42}$$

such that for n sufficiently large such that $S < -\frac{1}{n}$,

$$\|\varepsilon\|_{L^2} \le (-t)^{\sigma}, \|\nabla\varepsilon\|_{L^2} \le (-t)^{(1-\frac{3}{8}\gamma)\sigma}, \|\Delta\varepsilon\|_{L^2} \le (-t)^{(1-\gamma)\sigma}, \|\partial_t\varepsilon\|_{L^2} \le (-t)^{(1-\frac{\gamma}{2})\sigma}$$

for all $\tau_n < t < -\frac{1}{n}$ such that $t \ge S$. By the definition (3.7) of τ_n, this implies that $\tau_n \le S$. Using the blow-up alternative (3.2), we conclude that $s_n < S$, (3.4) and (3.5) hold. □

4. **Proof of Theorem 1.1.** Using estimate (3.4) and (3.5), we deduce that $\{\varepsilon_n\}_{n\ge\frac{1}{\tau}}$ is bounded in $L^\infty([S,\tau], H^2(\mathbb{R}^N)) \cap W^{1,\infty}([S,\tau], L^2(\mathbb{R}^N))$ for any given $\tau \in (S,0)$. Therefore, there exists $\varepsilon \in L^\infty([S,\tau], H^2(\mathbb{R}^N)) \cap W^{1,\infty}([S,\tau], L^2(\mathbb{R}^N))$ such that (after extracting a subsequence)

$$\begin{aligned} \varepsilon_n &\underset{n\to\infty}{\longrightarrow} \varepsilon, \quad \text{weak}^\star \text{ in } L^\infty\left([S,\tau], H^2\left(\mathbb{R}^N\right)\right), \\ \partial_t\varepsilon_n &\underset{n\to\infty}{\longrightarrow} \partial_t\varepsilon, \quad \text{weak}^\star \text{ in } L^\infty\left([S,\tau], L^2\left(\mathbb{R}^N\right)\right). \end{aligned} \tag{4.1}$$

Moreover, note that for any bounded domain $\Omega \subset \mathbb{R}^N$, we have the embedding relation $H^2(\Omega) \hookrightarrow\hookrightarrow L^{2+\alpha}(\Omega) \hookrightarrow L^2(\Omega)$. Since $\{\varepsilon_n\}_{n\ge\frac{1}{\tau}}$ is uniformly bounded in $L^\infty([S,\tau], H^2(\Omega)) \cap W^{1,\infty}([S,\tau], L^2(\Omega))$, then we have (after extracting a subsequence),

$$\varepsilon_n \underset{n\to\infty}{\longrightarrow} \varepsilon \text{ in } L^\infty([S,\tau], L^{\alpha+2}(\Omega)) \tag{4.2}$$

by Aubin-Lions Theorem, see Simon [22]. Moreover, using $L^\infty(\Omega) \hookrightarrow L^{\alpha+2}(\Omega)$, we see that

$$\varepsilon_n \underset{n\to\infty}{\longrightarrow} \varepsilon \text{ in } L^{\alpha+2}([S,\tau] \times \Omega)). \tag{4.3}$$

By the arbitrariness of τ, a standard argument of diagonal extraction shows that there exists $\varepsilon \in L^\infty_{loc}([S,0), H^2(\mathbb{R}^N)) \cap W^{1,\infty}_{loc}([S,0), L^2(\mathbb{R}^N))$, such that (after extracting a subsequence) (4.1)-(4.3) hold for all $S < \tau < 0$, and

$$\|\varepsilon(t)\|_{L^2} \le (-t)^{\sigma}, \quad \|\nabla\varepsilon(t)\|_{L^2} \le (-t)^{(1-\frac{3}{8}\gamma)\sigma}, \tag{4.4}$$

$$\|\Delta\varepsilon(t)\|_{L^2} \le (-t)^{(1-\gamma)\sigma}, \quad \|\partial_t\varepsilon(t)\|_{L^2} \le (-t)^{(1-\frac{\gamma}{2})\sigma}, \tag{4.5}$$

for all $S \le t < 0$. Moreover, it follows easily from (3.6) and the convergence properties (4.1)-(4.3) that

$$\partial_t\varepsilon = i\Delta\varepsilon + \lambda(|U_J+\varepsilon|^\alpha(U_J+\varepsilon) - |U_J|^\alpha U_J) + \mathcal{E}_J, \quad \text{in} \quad L^\infty_{loc}([S,0), L^2(\mathbb{R}^N)). \tag{4.6}$$

Therefore, setting

$$u(t) = U_J(t) + \varepsilon(t), \quad S \le t < 0, \tag{4.7}$$

we see that $u \in L^\infty_{\text{loc}}\left([S,0), H^2\left(\mathbb{R}^N\right)\right) \cap W^{1,\infty}_{\text{loc}}\left([S,0), L^2\left(\mathbb{R}^N\right)\right)$ and that

$$\partial_t u = i\Delta u + \lambda|u|^\alpha u, \quad \text{in} \quad L^\infty_{loc}([S,0), L^2(\mathbb{R}^N)),$$

by (2.47), (4.6) and (4.7). From the local existence in $H^2(\mathbb{R}^N)$ and the uniqueness in $L^\infty_t H^2_x$, we conclude that $u \in C([S,0), H^2(\mathbb{R}^N)) \cap C^1([S,0), L^2(\mathbb{R}^N))$.

We now prove (1.4)-(1.6) in Theorem 1.1. Let Ω be an open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \cap K = \emptyset$. It follows from (2.5) that $A > 0$ on Ω and $A(x) = |x|^k$ when $|x| > 2R$; and so there exists a constant $c > 0$, such that $A(x) \ge c(1+|x|)^k$ on Ω. Moreover using (2.6), (2.7) and (2.9), we deduce that

$$|U_0| \le C(1+|x|)^{-\frac{k}{\alpha}}, \ |\nabla U_0| \le C(1+|x|)^{-\frac{k}{\alpha}-1} \ \text{ and } \ |\Delta U_0| \le C(1+|x|)^{-\frac{k}{\alpha}-2}, \ \text{ on } \Omega.$$

Since $(1+|x|)^{-\frac{k}{\alpha}} \in L^2(\mathbb{R}^N)$ by (2.2), applying (2.41) and (2.43), we conclude that

$$\limsup_{t\uparrow 0} \|U_J\|_{H^2(\Omega)} < \infty.$$

Then the estimate (1.6) follows from (4.7) and the $L^\infty([S,0),H^2(\mathbb{R}^N))$ boundedness of ε (4.4)-(4.5). Let now $x_0 \in K$ and $r>0$, it follows from (2.11), (2.17) and (2.43) that

$$(-t)^{-\frac{1}{\alpha}+\frac{N}{2k}} \lesssim \|U_J(t)\|_{L^2(|x-x_0|<r)} \lesssim (-t)^{-\frac{1}{\alpha}}. \tag{4.8}$$

Using (4.7) and the embedding $H^2(|x-x_0|<r) \hookrightarrow L^2(|x-x_0|<r)$, we deduce that

$$\begin{aligned}\|u(t)\|_{L^2(|x-x_0|<r)} &\geq \|U_J(t)\|_{L^2(|x-x_0|<r)} - \|\varepsilon(t)\|_{L^2(|x-x_0|<r)}\\ &\gtrsim (-t)^{-\frac{1}{\alpha}+\frac{N}{2k}} - C\|\varepsilon(t)\|_{H^2(\mathbb{R}^N)},\end{aligned}$$

which proves the estimate (1.4) in Theorem 1.1. Next, we prove the estimate (1.5) in Theorem 1.1. Since k satisfies $(2+\frac{4\alpha}{k})(N-2)<2N$ by (2.2), we fix a real number p satisfying

$$p > 2 + \frac{4\alpha}{k} \text{ and } p(N-2) < 2N. \tag{4.9}$$

We apply (2.11), (2.17), (2.43) and Gagliardo-Nirenberg's inequality to obtain

$$\begin{aligned}(-t)^{-\frac{1}{\alpha}+\frac{N}{pk}} &\lesssim \|U_J\|_p \lesssim \|\Delta U_J\|_2^{\frac{N}{2}(\frac{1}{2}-\frac{1}{p})} \|U_J\|_2^{1-\frac{N}{2}(\frac{1}{2}-\frac{1}{p})}\\ &\lesssim \|\Delta U_J\|_2^{\frac{N}{2}(\frac{1}{2}-\frac{1}{p})} (-t)^{-\frac{1}{\alpha}(1-\frac{N}{2}(\frac{1}{2}-\frac{1}{p}))}\end{aligned}$$

and

$$\begin{aligned}(-t)^{-\frac{1}{\alpha}+\frac{N}{pk}} &\lesssim \|U_J\|_p \lesssim \|\nabla U_J\|_2^{N(\frac{1}{2}-\frac{1}{p})} \|U_J\|_2^{1-N(\frac{1}{2}-\frac{1}{p})}\\ &\lesssim \|\nabla U_J\|_2^{N(\frac{1}{2}-\frac{1}{p})} (-t)^{-\frac{1}{\alpha}(1-N(\frac{1}{2}-\frac{1}{p}))},\end{aligned}$$

which implies that

$$(-t)^{\frac{4p}{p-2}(\frac{1}{pk}-\frac{1}{4\alpha}+\frac{1}{2p\alpha})} \lesssim \|\Delta U_J\|_2, \quad (-t)^{\frac{2p}{p-2}(\frac{1}{pk}-\frac{1}{2\alpha}+\frac{1}{p\alpha})} \lesssim \|\nabla U_J\|_2.$$

From (4.9), we have

$$\frac{1}{pk} - \frac{1}{2\alpha} + \frac{1}{p\alpha} < \frac{1}{pk} - \frac{1}{4\alpha} + \frac{1}{2p\alpha} < 0$$

and

$$\lim_{t\uparrow 0} \|\nabla U_J\|_2 = \lim_{t\uparrow 0} \|\Delta U_J\|_2 = \infty.$$

Combining (4.7) and (4.4)-(4.5), we have the estimate (1.5), and finish the proof of Theorem 1.1. □

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E-mail address: lxmath@zju.edu.cn
E-mail address: zhangting79@zju.edu.cn