ELECTRONIC RESEARCH ARCHIVE
Volume **28**, Number **2**, June **2020**
doi:10.3934/era.2020038
pp. **739–776**

CERTAIN $*$-HOMOMORPHISMS ACTING ON UNITAL C^*-PROBABILITY SPACES AND SEMICIRCULAR ELEMENTS INDUCED BY p-ADIC NUMBER FIELDS OVER PRIMES p

Ilwoo Cho

421 Ambrose Hall, Saint Ambrose Univ.
Dept. of Math. & Stat., 518 W. Locust St., Davenport, Iowa, 52803, USA

(Communicated by Hemen Dutta)

Abstract. In this paper, we study the Banach $*$-probability space $(A \otimes_{\mathbb{C}} \mathbb{LS},\ \tau_A^0)$ generated by a fixed unital C^*-probability space $(A,\ \varphi_A)$, and the semicircular elements $\Theta_{p,j}$ induced by p-adic number fields $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, where $\mathcal{P}$ is the set of all primes, and $\mathbb{Z}$ is the set of all integers. In particular, from the order-preserving shifts $g \times h_{\pm}$ on $\mathcal{P} \times \mathbb{Z}$, and $*$-homomorphisms θ on A, we define the corresponding $*$-homomorphisms $\sigma_{(\pm,1)}^{1:\theta}$ on $A \otimes_{\mathbb{C}} \mathbb{LS}$, and consider free-distributional data affected by them.

1. **Introduction.** The main purposes of this paper are (i) to re-consider (*weighted-*)*semicircular elements* in a certain *Banach $*$-probability space* induced by *measurable functions* on *p-adic number fields* $\mathbb{Q}_p$, for *primes* p, and to study free-probabilistic properties of the *Banach $*$-probability space* $\mathbb{LS} = (\mathbb{LS},\ \tau^0)$ generated by those mutually-free, (weighted-)semicircular elements, (ii) to extend the structure $\mathbb{LS}$ to the *tensor product Banach $*$-probability space*,

$$\left(A \otimes_{\mathbb{C}} \mathbb{LS},\ \varphi_A \otimes \tau^0\right)$$

for an arbitrarily fixed *unital C^*-probability space* $(A,\ \varphi_A)$, and investigate (weighted-)semicircular elements of this new Banach $*$-probabilistic structure, (iii) to consider certain $*$-homomorphisms acting on $A \otimes_{\mathbb{C}} \mathbb{LS}$ induced by shifting processes on the Cartesian product set $\mathcal{P} \times \mathbb{Z}$, and to investigate how such $*$-homomorphisms affect the free probability on $A \otimes_{\mathbb{C}} \mathbb{LS}$, and (iv) by extending such $*$-homomorphisms of (iii) to certain $*$-homomorphisms induced by $*$-homomorphisms acting on A, to study how such generalized morphisms distort the free probability on $A \otimes_{\mathbb{C}} \mathbb{LS}$.

The main results of this paper are interesting not only in applied number theory, but also in free-probabilistic operator theory. From number-theoretic objects, primes and corresponding p-adic number fields, the free-probabilistic objects, (weighted-)semicircular elements, are well-constructed; and the operator-theoretic objects, $*$-homomorphisms and corresponding Banach-space operators, are acting on such (weighted-)semicircular elements well; and the structures and properties

2010 *Mathematics Subject Classification.* 05E15, 11G15, 11R47, 11R56, 46L10, 46L54, 47L30, 47L55.

Key words and phrases. Free probability, p-adic number fields $\mathbb{Q}_p$, semicircular elements, shifts on $\mathcal{P} \times \mathbb{Z}$, homomorphism semigroups, pi-shifts, unital C^*-probability spaces, tensor product banach $*$-probability spaces.

of them are characterized and shown. Moreover, under tensor product, operator-algebraic properties of tensor product structures, and free-distributional information are studied operator-algebraically. So, our works provide new connections among number theory, free probability, operator theory, and operator algebra theory. i.e., the main results would be applicable to statistical quantum physics, studying analysis on certain physical structures over the non-Archimedean structures (having "very small" distances, or metrics).

For more about number-theoretic motivations of our proceeding works, see e.g., [16], [17], [18], [19], [31] and [32]. And, for more about statistical analysis, see [1], [2], [3], [4], [5], [6], [15], [21], [22] and [25]. Also, for *free probability theory*, see e.g., [26], [27], [28], [29], [30], [24], [20], [33], [34] and [35].

Relations between primes and operators have been studied in various different approaches. For instance, we studied how primes act on certain operator algebras and dynamical systems, as operators, with help of *p-adic*, and *Adelic analysis* (e.g., [9]).

In [8] and [12], we studied weighted-semicircular elements, and corresponding semicircular elements induced by measurable functions on p-adic number fields $\mathbb{Q}_p$, for $p \in \mathcal{P}$. The main results of these papers show that *p-adic analysis* allows us to have *the (weighted-)semicircular law(s)*, statistically. As applications of [8] and [12], *free stochastic calculus* for our (weighted-)semicircular law(s) was considered in [11]. And we globalize the (weighted-)semicircularity of [8] and [11] to those induced by *Adelic analysis* in [10].

In this paper, we are interested in how the (weighted-)semicircular law(s) on $\left(A \otimes_{\mathbb{C}} \mathbb{LS},\ \varphi_A \otimes \tau^0\right)$ is (are) affected, or distorted by certain $*$-homomorphisms acting on $A \otimes_{\mathbb{C}} \mathbb{LS}$.

2. **Preliminaries.**

In this section, we briefly mention about backgrounds of our proceeding works.

2.1. **Free probability.**

Free probability is the noncommutative operator-algebraic version of classical measure theory and statistics. The classical *independence* is replaced by the *freeness* by replacing measures on sets to linear functionals on noncommutative algebras (e.g., [26], [29], [30], [33] and [35]). It has various applications not only in pure mathematics (e.g., [23], [25], [27], [28], [24] and [20]), but also in related fields (e.g., [3] through [12]). In particular, we here use combinatorial approach of *Speicher* (e.g., [29] and [30]).

In the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed. Also, we deal *free product $*$-probability spaces*, without detailed introduction.

Notation and Assumption. As in the "traditional" free probability theory, the pairs $(B,\ \varphi_B)$ of noncommutative algebras B, and fixed linear functionals φ_B on B are said to be *(noncommutative) free probability spaces*. However, for our purposes, even though a given algebra $\mathcal{A}$ is commutative, we will call a pair $(\mathcal{A},\ \psi)$ of a commutative algebra $\mathcal{A}$ and a linear functional ψ on $\mathcal{A}$, a free probability space, "non-traditionally" (e.g., see [8] through [12]). The freeness on such a non-traditional free probability space $(\mathcal{A},\ \psi)$ is trivial by the commutativity of $\mathcal{A}$, but (traditional) free probability theory well-covers functional-and-statistical analysis on $\mathcal{A}$, for ψ. So, without loss of much generality, we call the pairs (B, φ_B) of (noncommutative, or commutative) algebras B, and linear functionals φ_B on B, *free probability spaces*, below. □

2.2. **Analysis on $\mathbb{Q}_p$.** For more about p-adic analysis, see [31] and [32] (also, see [17] and [22]). Let $\mathbb{Q}_p$ be the *p-adic number fields for $p \in \mathcal{P}$*. Recall that $\mathbb{Q}_p$ are the maximal p-norm-topology closures in the *normed space* $(\mathbb{Q}, |.|_p)$ of all *rational numbers*, where $|.|_p$ are the *non-Archimedean norms*, called *p-norms on* $\mathbb{Q}$, for all $p \in \mathcal{P}$.

For any fixed $p \in \mathcal{P}$, the Banach space $\mathbb{Q}_p$ forms a *field* algebraically under the p-adic addition and the p-adic multiplication in the sense of [32], i.e., $\mathbb{Q}_p$ is a *Banach field*.

Also, such a Banach field $\mathbb{Q}_p$ is understood as a *measure space*

$$\mathbb{Q}_p = \left(\mathbb{Q}_p,\ \sigma\left(\mathbb{Q}_p\right),\ \mu_p\right),$$

equipped with the left-and-right additive invariant *Haar measure* μ_p on the *σ-algebra* $\sigma\left(\mathbb{Q}_p\right)$, satisfying that

$$\mu_p\left(\mathbb{Z}_p\right) = 1,$$

where $\mathbb{Z}_p$ is the *unit disk of* $\mathbb{Q}_p$,

$$\mathbb{Z}_p \overset{def}{=} \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \text{ in } \mathbb{Q}_p,$$

consisting of all p-adic integers of $\mathbb{Q}_p$, for all $p \in \mathcal{P}$ (e.g., [31] and [32]).

As a *topological space*, the p-adic number field $\mathbb{Q}_p$ contains its *basis elements,*

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{1}$$

for all $k \in \mathbb{Z}$. (e.g., [32]).

By understanding $\mathbb{Q}_p$ as a measure space, one can establish a *$*$-algebra $\mathcal{M}_p$ over* $\mathbb{C}$ as a *$*$-algebra*,

$$\mathcal{M}_p = \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}\right]$$

consisting of *μ_p-measurable functions* f,

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S\ \chi_S \quad (t_S \in \mathbb{C}),$$

where the sum $\sum$ is the *finite sum*, and χ_S are the usual *characteristic functions of* S.

On $\mathcal{M}_p$, one can naturally define a *linear functional* φ_p by the *p-adic integral*, i.e.,

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f\ d\mu_p, \forall f \in \mathcal{M}_p. \tag{2}$$

Define now subsets ∂_k of $\mathbb{Q}_p$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{3}$$

We call these μ_p-measurable subsets ∂_k of (3), the *k-th boundaries* (of the basis elements U_k of (1)), for all $k \in \mathbb{Z}$. By the basis property of the subsets U_k of (1), one obtains that

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\sqcup} \partial_k, \tag{4}$$

where $\sqcup$ means the *disjoint union*. Also, by measure-theoretic data, one has

$$\mu_p\left(\partial_k\right) = \mu_p\left(U_k\right) - \mu_p\left(U_{k+1}\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{5}$$

for all $k \in \mathbb{Z}$.

Note that, by (4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}. \tag{6}$$

Thus, by (6), one obtains the following proposition.

Proposition 2.1. *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that*

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S,$$

and (7)

$$\varphi_p(\chi_S) = \int_{\mathbb{Q}_p} \chi_S \, d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

where Λ_S is in the sense of (6).

Proof. The computation (7) is shown by (5). See [8], [9], [10], [11] and [12] for details. □

3. **Free-probabilistic models on $\mathcal{M}_p$.** Throughout this section, fix a prime $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the corresponding p-adic number field, and let $\mathcal{M}_p$ be the p-adic $*$-algebra of $\mathbb{Q}_p$. In this section, let's establish a suitable (non-traditional) free-probabilistic model on $\mathcal{M}_p$.

Let $U_k = p^k \mathbb{Z}_p$ be the basis elements (1), and ∂_k, their boundaries (3) of $\mathbb{Q}_p$, i.e.,

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{8}$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by the *p-adic integral* (2),

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f \, d\mu_p, \text{ for all } f \in \mathcal{M}_p. \tag{9}$$

Definition 3.1. The pairs $(\mathcal{M}_p, \varphi_p)$ are called p-adic (non-traditional free) $*$-probability spaces, for all $p \in \mathcal{P}$.

Then, by (7) and (9), one obtains that

$$\varphi_p\left(\chi_{U_j}\right) = \frac{1}{p^j}, \text{ and } \varphi_p\left(\chi_{\partial_j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

since

$$\Lambda_{U_j} = \{k \in \mathbb{Z} : k \geq j\}, \text{ and } \Lambda_{\partial_j} = \{j\},$$

for all $j \in \mathbb{Z}$.

Proposition 3.1. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Let*

$$\Lambda_{S_1,\ldots,S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} are in the sense of (7), for $l = 1, \ldots, N$. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \forall j \in \Lambda_{S_1,\ldots,S_N},$$

and (10)

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l}\right) = \sum_{j \in \Lambda_{S_1,\ldots,S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Proof. The formula (10) is proven by (7), since

$$\prod_{l=1}^{N} \chi_{S_l} = \chi_{\bigcap_{l=1}^{N} S_l} \text{ in } \mathcal{M}_p.$$

See [8] through [12], for details. □

4. **Representations of** $(\mathcal{M}_p,\ \varphi_p)$**.** Fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p,\ \varphi_p)$ be the p-adic $*$-probability space. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2*-space*,

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p,\ \sigma(\mathbb{Q}_p),\ \mu_p\right) = L^2\left(\mathbb{Q}_p\right), \tag{11}$$

over $\mathbb{C}$, equipped with its *inner product* $<,>_2$,

$$\langle f_1,\ f_2\rangle_2 \overset{def}{=} \int_{\mathbb{Q}_p} f_1\ f_2^*\ d\mu_p, \tag{12}$$

for all $f_1,\ f_2 \in H_p$, inducing the L^2*-norm*,

$$\|f\|_2 \overset{def}{=} \sqrt{\langle f,\ f\rangle_2}, \text{ for all } f \in H_p, \tag{12$'$}$$

where $<,>_2$ is the inner product (12) on H_p.

Definition 4.1. We call the Hilbert space H_p of (11), the p-adic Hilbert space.

By the definition (11) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an algebra-action α^p,

$$\alpha^p(f)\,(h) = fh, \text{ for all } h \in H_p, \tag{13}$$

for all $f \in \mathcal{M}_p$. i.e., by (13), for any $f \in \mathcal{M}_p$, the image $\alpha^p(f)$ is a well-defined *multiplication operator on* H_p *with its symbol* f, satisfying

$$\alpha^p(f_1 f_2) = \alpha^p(f_1)\alpha^p(f_2) \text{ on } H_p, \forall f_1,\ f_2 \in \mathcal{M}_p,$$

and (14)

$$\left(\alpha^p(f)\right)^* = \alpha(f^*) \text{ on } H_p, \forall f \in \mathcal{M}_p.$$

Notation. Denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{M}_p$. Also, for convenience, denote $\alpha_{\chi_S}^p$ simply by α_S^p, for all $S \in \sigma\left(\mathbb{Q}_p\right)$. □

Proposition 4.1. *The pair* $(H_p,\ \alpha^p)$ *is a well-determined Hilbert-space representation of* $\mathcal{M}_p$.

Proof. The proof is done by (14) (e.g., see [8] and [12]). □

Definition 4.2. Let

$$M_p \overset{def}{=} \overline{\alpha^p\left(\mathcal{M}_p\right)}^{\|\cdot\|} = \overline{\mathbb{C}\left[\alpha_f^p\ : f \in \mathcal{M}_p\right]}^{\|\cdot\|} \tag{15}$$

in $B(H_p)$, where $\overline{X}^{\|\cdot\|}$ mean the operator-norm closures of subsets X of $B(H_p)$. This C^*-algebra M_p of (15) is called the p-adic C^*-algebra of $(\mathcal{M}_p,\ \varphi_p)$.

5. **Free-probabilistic models on** M_p**.** Throughout this section, let's fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p,\ \varphi_p)$ be the corresponding p-adic $*$-probability space, and M_p, the p-adic C^*-algebra of (15). Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi_j^p\,(a) \overset{def}{=} \left\langle a(\chi_{\partial_j}),\ \chi_{\partial_j}\right\rangle_2, \forall a \in M_p, \tag{16}$$

for all $j \in \mathbb{Z}$, where $<,>_2$ is the inner product (12) on the p-adic Hilbert space H_p of (11).

Definition 5.1. Let $j \in \mathbb{Z}$, and let φ_j^p be the linear functional (16) on the p-adic C^*-algebra M_p. Then the pair $\left(M_p,\ \varphi_j^p\right)$ is said to be the j-th p-adic (non-traditional) C^*-probability space.

Now, fix $j \in \mathbb{Z}$, and take the j-th p-adic C^*-probability space $\left(M_p,\ \varphi_j^p\right)$. For $S \in \sigma\left(\mathbb{Q}_p\right)$, and an element $\alpha_S^p \in M_p$, one has that

$$\varphi_j^p(\alpha_S^p) = \left\langle \alpha_S^p(\chi_{\partial_j}),\ \chi_{\partial_j} \right\rangle_2 = \mu_p\left(S \cap \partial_j\right)$$

$$= r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{17}$$

by (3.8), for some $0 \leq r_S \leq 1$ in $\mathbb{R}$.

Proposition 5.1. *Let ∂_k be the k-th boundaries (8) of $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then*

$$\varphi_j^p\left(\left(\alpha_{\partial_k}^p\right)^n\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{18}$$

for all $n \in \mathbb{N}$, for $k \in \mathbb{Z}$.

Proof. By (17), one has that

$$\varphi_j^p\left(\alpha_{\partial_k}^p\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \text{ for all } k \in \mathbb{N}. \tag{19}$$

Since $\alpha_{\partial_k}^p$ are projections in M_p, in the sense that:

$$\left(\alpha_{\partial_k}^p\right)^2 = \alpha_{\partial_k}^p = \left(\alpha_{\partial_k}^p\right)^* \text{ in } M_p,$$

the formula (18) holds by (19), for all $k \in \mathbb{Z}$. □

6. **Semigroup C^*-subalgebras $\mathfrak{S}_p$ of M_p.** Let M_p be the p-adic C^*-algebra for $p \in \mathcal{P}$. Take projections

$$P_{p,j} = \alpha_{\partial_j}^p \in M_p, \tag{20}$$

induced by boundaries ∂_j of $\mathbb{Q}_p$, for all $j \in \mathbb{Z}$. We now restrict our interests to these projections $P_{p,j}$ of M_p.

Definition 6.1. Fix $p \in \mathcal{P}$. Let $\mathfrak{S}_p$ be the C^*-subalgebra

$$\mathfrak{S}_p = C^*\left(\{P_{p,j}\}_{j\in\mathbb{Z}}\right) = \overline{\mathbb{C}\left[\{P_{p,j}\}_{j\in\mathbb{Z}}\right]} \textit{ of } M_p, \tag{21}$$

where $P_{p,j}$ are projections (20), for all $j \in \mathbb{Z}$. We call this C^*-subalgebra $\mathfrak{S}_p$, the p-adic boundary (C^*-)subalgebra of M_p.

Every p-adic boundary subalgebra $\mathfrak{S}_p$ satisfies the following structure theorem.

Proposition 6.1. *Let $\mathfrak{S}_p$ be the p-adic boundary subalgebra (21) of the p-adic C^*-algebra M_p. Then*

$$\mathfrak{S}_p \overset{*\text{-}iso}{=} \underset{j\in\mathbb{Z}}{\oplus}\left(\mathbb{C} \cdot P_{p,j}\right) \overset{*\text{-}iso}{=} \mathbb{C}^{\oplus\mathbb{Z}}, \tag{22}$$

in M_p.

Proof. It suffices to show that the generating projections $\{P_{p,j}\}_{j\in\mathbb{Z}}$ of $\mathfrak{S}_p$ are mutually orthogonal from each other. But, one can get that, for any $j_1,\ j_2 \in \mathbb{Z}$,

$$P_{p,j_1}P_{p,j_2} = \alpha^p\left(\chi_{\partial_{j_1}^p \cap \partial_{j_2}^p}\right) = \delta_{j_1,j_2}\alpha_{\partial_{j_1}^p}^p = \delta_{j_1,j_2}P_{p,j_1},$$

in $\mathfrak{S}_p$. Therefore, the structure theorem (22) holds. See [8] for more details. □

7. **Weighted-semicircularity.** Let M_p be the p-adic C^*-algebra, and let $\mathfrak{S}_p$ be the boundary subalgebra (21) of M_p, satisfying the structure theorem (22). Throughout this section, let's fix a prime p. Recall that if $\{P_{p,k}\}_{k\in\mathbb{Z}}$ are the generating projections (20) of $\mathfrak{S}_p$, then

$$\varphi_j^p\left(P_{p,k}\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \forall j,\ k \in \mathbb{Z}, \tag{23}$$

by (18).

Let ϕ be the *Euler totient function*, which is an *arithmetic function*

$$\phi : \mathbb{N} \to \mathbb{C},$$

defined by (24)

$$\phi(n) = \left|\{k \in \mathbb{N} : k \leq n, \ \ \gcd(n, k) = 1\}\right|,$$

for all $n \in \mathbb{N}$, where $|X|$ mean the *cardinalities of sets* X, and gcd means the *greatest common divisor*. Then

$$\phi(q) = q - 1 = q\left(1 - \tfrac{1}{q}\right), \ \forall q \in \mathcal{P}, \tag{25}$$

by (24).

So, one can get that

$$\begin{aligned} \varphi_j^p\left(P_{p,j}\right) &= \tfrac{1}{p^j}\left(1 - \tfrac{1}{p}\right) \\ &= \tfrac{p}{p^{j+1}}\left(1 - \tfrac{1}{p}\right) = \tfrac{\phi(p)}{p^{j+1}}, \end{aligned} \tag{26}$$

by (23) and (25), for $j \in \mathbb{Z}$.

Motivated by (26), define the new linear functionals $\tau_j^p : \mathfrak{S}_p \to \mathbb{C}$, by linear morphisms,

$$\tau_j^p = \tfrac{1}{\phi(p)}\ \varphi_j^p \text{ on } \mathfrak{S}_p, \tag{27}$$

satisfying that:

$$\tau_j^p\left(P_{p,k}\right) = \tfrac{\delta_{j,k}}{\phi(p)}\ \varphi_j^p\left(P_{p,j}\right) = \tfrac{\delta_{j,k}}{p^{j+1}},$$

for all $j,\ k \in \mathbb{Z}$.

Proposition 7.1. *Let $\mathfrak{S}_p(j) = (\mathfrak{S}_p,\ \tau_j^p)$ be a (non-traditional) C^*-probability space, and let $P_{p,k}$ be the generating projections of $\mathfrak{S}_p$, for all $k \in \mathbb{Z}$. Then*

$$\tau_j^p\left(P_{p,k}^n\right) = \tfrac{\delta_{j,k}}{p^{j+1}}, \text{ for all } n \in \mathbb{N}. \tag{28}$$

Proof. The free-moment formula (28) is proven by (27). □

7.1. **Semicircular and weighted-semicircular elements.** Let $(A,\ \varphi)$ be a (traditional, or non-traditional) *topological $*$-probability space* (C^*-probability space, or W^*-probability space, or Banach $*$-probability space, etc.) equipped with a (noncommutative, resp., commutative) topological $*$-algebra A (C^*-algebra, resp., W^*-algebra, resp., Banach $*$-algebra), and a (bounded, or unbounded) linear functional φ on A.

Definition 7.1. Let a be a self-adjoint operator in $(A,\ \varphi)$. This operator a is said to be semicircular in $(A,\ \varphi)$, if

$$\varphi(a^n) = \omega_n c_{\frac{n}{2}}, \textit{ for all } n \in \mathbb{N},$$

where (29)

$$\omega_n = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, where c_k are the k-th Catalan number,

$$c_k = \tfrac{1}{k+1}\begin{pmatrix} 2k \\ k \end{pmatrix} = \tfrac{1}{k+1}\tfrac{(2k)!}{(k!)^2} = \tfrac{(2k)!}{k!(k+1)!},$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is well-known that, if $k_n(...)$ is the *free cumulant on A in terms of φ* (in the sense of [29] and [30]), then a self-adjoint operator a is *semicircular* in $(A,\ \varphi)$, if and only if

$$k_n\left(\underbrace{a,\ a,\ \ldots\ldots,\ a}_{n\text{-times}}\right) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

for all $n \in \mathbb{N}$ (e.g., see [12]). The above characterization (30) of the semicircularity (29) is obtained by the *Möbius inversion of* [29] and [30]. Thus, the semicircular operators a of (A, φ) can be re-defined by the self-adjoint operators satisfying the free-cumulant characterization (30).

Motivated by (30), one can define so-called the *weighted-semicircular elements*.

Definition 7.2. Let $a \in (A, \varphi)$ be a self-adjoint operator. It is said to be weighted-semicircular in (A, φ) with its weight t_0 (in short, t_0-semicircular), if there exists $t_0 \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$, such that

$$k_n\left(\underbrace{a,\ a,\ \ldots.,\ a}_{n\text{-}times}\right) = \begin{cases} t_0 & \textit{if } n = 2 \\ 0 & \textit{otherwise,} \end{cases} \tag{31}$$

for all $n \in \mathbb{N}$, where $k_n(\ldots)$ is the free cumulant on A in terms of φ.

By the definition (31), and by the Möbius inversion of [29] and [30], we obtain the following free-moment characterization (32) of (31): A self-adjoint operator a in a $*$-probability space (A, φ) is t_0-*semicircular*, if and only if there exists $t_0 \in \mathbb{C}^{\times}$, such that

$$\varphi(a^n) = \omega_n\ t_0^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{32}$$

where ω_n are in the sense of (29), for all $n \in \mathbb{N}$.

7.2. **Tensor product banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$.** Let $\mathfrak{S}_p(k) = (\mathfrak{S}_p,\ \tau_k^p)$ be a (non-traditional) C^*-probability space for $p \in \mathcal{P}$, $k \in \mathbb{Z}$. Throughout this section, we fix $p \in \mathcal{P}$, $k \in \mathbb{Z}$, and the corresponding C^*-probability space $\mathfrak{S}_p(k)$.

Define now bounded linear transformations $\mathbf{c}_p$ and $\mathbf{a}_p$ "acting on the C^*-algebra $\mathfrak{S}_p$," by linear morphisms satisfying,

$$\mathbf{c}_p\left(P_{p,j}\right) = P_{p,j+1},$$

and (33)

$$\mathbf{a}_p\left(P_{p,j}\right) = P_{p,j-1},$$

on $\mathfrak{S}_p$, for all $j \in \mathbb{Z}$.

By (33), one can understand $\mathbf{c}_p$ and $\mathbf{a}_p$ as bounded operators contained in the *operator space* $B(\mathfrak{S}_p)$, consisting of all bounded linear operators on $\mathfrak{S}_p$, by regarding $\mathfrak{S}_p$ as a *Banach space* (e.g., [15]). Under this sense, the operators $\mathbf{c}_p$ and $\mathbf{a}_p$ of (33) are understood as well-defined *Banach-space operators on* $\mathfrak{S}_p$.

Definition 7.3. The Banach-space operators $\mathbf{c}_p$ and $\mathbf{a}_p$ on $\mathfrak{S}_p$ in the sense of (33) are called the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$. Define a new Banach-space operator l_p by

$$l_p = \mathbf{c}_p + \mathbf{a}_p \ \textit{on}\ \mathfrak{S}_p. \tag{34}$$

We call this operator l_p of (34), the p-radial operator on $\mathfrak{S}_p$.

Let l_p be the p-radial operator (34) in $B(\mathfrak{S}_p)$. Construct a *Banach algebra* $\mathfrak{L}_p$ by

$$\mathfrak{L}_p = \overline{\mathbb{C}[l_p]} \text{ in } B(\mathfrak{S}_p), \tag{35}$$

equipped with the inherited *operator-norm* $\|.\|$ of $B(\mathfrak{S}_p)$, defined by

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{S}_p} : x \in \mathfrak{S}_p \text{ s.t., } \|x\|_{\mathfrak{S}_p} = 1\},$$

where

$$\|x\|_{\mathfrak{S}_p} = \sup\{\|x(h)\|_2 : h \in H_p \text{ s.t. } \|h\|_2 = 1\},$$

is the C^*-*norm on* $\mathfrak{S}_p$, where $\|.\|_2$ is the L^2-*norm* on the p-adic Hilbert space $H_p = L^2(\mathbb{Q}_p)$.

On the Banach algebra $\mathfrak{L}_p$ of (35), define a unary operation $(*)$ by

$$\sum_{k=0}^{\infty} s_k\, l_p^k \in \mathfrak{L}_p \longmapsto \sum_{k=0}^{\infty} \overline{s_k}\, l_p^k \in \mathfrak{L}_p, \tag{36}$$

where $s_k \in \mathbb{C}$, with their conjugates $\overline{s_k} \in \mathbb{C}$.

Then the operation (36) is a well-defined *adjoint on* $\mathfrak{L}_p$ (e.g., [8] and [12]). So, equipped with the adjoint (36), this Banach algebra $\mathfrak{L}_p$ of (35) forms a *Banach* $*$*-algebra* embedded in the topological vector space $B(\mathfrak{S}_p)$.

Definition 7.4. Let $\mathfrak{L}_p$ be a Banach $*$-algebra (35) for a fixed $p \in \mathcal{P}$. We call $\mathfrak{L}_p$, the p-radial (Banach-$*$-)algebra on $\mathfrak{S}_p$.

Let $\mathfrak{L}_p$ be the p-radial algebra on the boundary subalgebra $\mathfrak{S}_p$. Construct now the tensor product $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ by

$$\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p, \tag{37}$$

where $\otimes_{\mathbb{C}}$ is the *tensor product* of Banach $*$-algebras.

Take now a generating element $l_p^n \otimes P_{p,j}$, for some $n \in \mathbb{N}_0$, and $j \in \mathbb{Z}$, where $P_{p,j}$ are the generating projections (20) of $\mathfrak{S}_p$, with axiomatization:

$$l_p^0 = 1_{\mathfrak{S}_p}, \text{ the identity operator of } \mathfrak{S}_p,$$

in $B(\mathfrak{S}_p)$, for all $j \in \mathbb{Z}$.

Define now a bounded linear morphism $E_p : \mathfrak{L}\mathfrak{S}_p \to \mathfrak{S}_p$ by a linear transformation satisfying that:

$$E_p\left(l_p^k \otimes P_{p,j}\right) = \frac{\left(p^{j+1}\right)^{k+1}}{\left[\frac{k}{2}\right]+1}\, l_p^k(P_{p,j}), \tag{38}$$

for all $k \in \mathbb{N}_0$, $j \in \mathbb{Z}$, where $\left[\frac{k}{2}\right]$ is the *minimal integer greater than or equal to* $\frac{k}{2}$, for all $k \in \mathbb{N}_0$.

By the cyclicity (35) of the tensor factor $\mathfrak{L}_p$ of $\mathfrak{L}\mathfrak{S}_p$, and by the structure theorem (22) of $\mathfrak{S}_p$, the above morphism E_p of (38) is indeed a well-defined linear transformation.

Now, consider how our p-radial operator $l_p = \mathbf{c}_p + \mathbf{a}_p$ acts on $\mathfrak{S}_p$. Observe first that

$$\mathbf{c}_p\mathbf{a}_p\left(P_{p,j}\right) = P_{p,j} = \mathbf{a}_p\mathbf{c}_p\left(P_{p,j}\right),$$

for all $j \in \mathbb{Z}$, $p \in \mathcal{P}$, implying that

$$\mathbf{c}_p\mathbf{a}_p = 1_{\mathfrak{S}_p} = \mathbf{a}_p\mathbf{c}_p \text{ on } \mathfrak{S}_p. \tag{39}$$

Lemma 7.2. *Let* $\mathbf{c}_p$, $\mathbf{a}_p$ *be the p-creation, respectively, the p-annihilation on* $\mathfrak{S}_p$. *Then*

$$\mathbf{c}_p^n\mathbf{a}_p^n = \left(\mathbf{c}_p\mathbf{a}_p\right)^n = 1_{\mathfrak{S}_p} = \left(\mathbf{a}_p\mathbf{c}_p\right)^n = \mathbf{a}_p^n\mathbf{c}_p^n,$$

and (40)

$$\mathbf{c}_p^{n_1}\mathbf{a}_p^{n_2} = \mathbf{a}_p^{n_2}\mathbf{c}_p^{n_1}, \text{ on } \mathfrak{S}_p,$$

for all n, n_1, $n_2 \in \mathbb{N}$.

Proof. The formulas in (40) holds by (39). □

By (40), one can have that

$$l_p^n = \left(\mathbf{c}_p + \mathbf{a}_p\right)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \mathbf{c}_p^k\mathbf{a}_p^{n-k},$$

with identity: (41)

$$\mathbf{c}_p^0 = 1_{\mathfrak{S}_p} = \mathbf{a}_p^0,$$

for all $n \in \mathbb{N}$, where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}, \text{ for all } k \leq n \in \mathbb{N}_0.$$

By (41), one obtains the following proposition.

Proposition 7.3. *Let $l_p \in \mathfrak{L}_p$ be the p-radial operator on $\mathfrak{S}_p$. Then*

(42) $\quad l_p^{2m-1}$ *does not contain $1_{\mathfrak{S}_p}$-term, and*

(43) $\quad l_p^{2m}$ *contains its $1_{\mathfrak{S}_p}$-term,* $\begin{pmatrix} 2m \\ m \end{pmatrix} \cdot 1_{\mathfrak{S}_p}$,

for all $m \in \mathbb{N}$.

Proof. The proofs of (42) and (43) are done by straightforward computations (41), with help of (40). See [8] for details. □

7.3. **Weighted-semicircular elements $Q_{p,j}$ in $\mathfrak{L}\mathfrak{S}_p$.** Fix $p \in \mathcal{P}$, and let $\mathfrak{L}\mathfrak{S}_p$ be the tensor product Banach $*$-algebra (37), and let $E_p : \mathfrak{L}\mathfrak{S}_p \to \mathfrak{S}_p$ be the linear transformation (38). Throughout this section, let

$$Q_{p,j} = l_p \otimes P_{p,j} \in \mathfrak{L}\mathfrak{S}_p, \tag{44}$$

for $j \in \mathbb{Z}$, where $P_{p,j}$ are projections (20) generating $\mathfrak{S}_p$. Observe that

$$\begin{aligned} Q_{p,j}^n &= (l_p \otimes P_{p,j})^n \\ &= l_p^n \otimes P_{p,j}^n = l_p^n \otimes P_{p,j}, \end{aligned} \tag{45}$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$.

By (37) and (45), these operators $Q_{p,j}$ of (44) generate $\mathfrak{L}\mathfrak{S}_p$, for all $j \in \mathbb{Z}$. Consider now that, if $Q_{p,j} \in \mathfrak{L}\mathfrak{S}_p$ is in the sense of (44) for $j \in \mathbb{Z}$, then

$$E_p\left(Q_{p,j}^n\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1}\, l_p^n\left(P_{p,j}\right), \tag{46}$$

by (38) and (45), for all $n \in \mathbb{N}$.

For any fixed $j \in \mathbb{Z}$, define a linear functional $\tau_{p,j}^0$ on $\mathfrak{L}\mathfrak{S}_p$ by

$$\tau_{p,j}^0 = \tau_j^p \circ E_p \text{ on } \mathfrak{L}\mathfrak{S}_p, \tag{47}$$

where τ_j^p is a linear functional (27) on $\mathfrak{S}_p$.

By the linearity of both τ_j^p and E_p, the morphism $\tau_{p,j}^0$ of (47) is a well-defined linear functional on $\mathfrak{L}\mathfrak{S}_p$. So, the pair $\left(\mathfrak{L}\mathfrak{S}_p,\ \tau_{p,j}^0\right)$ forms a (non-traditional) *Banach $*$-probability space.*

By (46) and (47), if $Q_{p,j}$ is in the sense of (44), then

$$\tau_{p,j}^0\left(Q_{p,j}^n\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1}\, \tau_j^p\left(l_p^n(P_{p,j})\right), \tag{48}$$

for all $n \in \mathbb{N}$.

Theorem 7.4. *Let $Q_{p,j} = l_p \otimes P_{p,j} \in \left(\mathfrak{L}\mathfrak{S}_p, \tau_{p,j}^0\right)$, for a fixed $j \in \mathbb{Z}$. Then $Q_{p,j}$ is $p^{2(j+1)}$-semicircular in $\left(\mathfrak{L}\mathfrak{S}_p,\ \tau_{p,j}^0\right)$. More precisely, one obtains that*

$$\tau_{p,j}^0\left(Q_{p,j}^n\right) = \omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{49}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (7.1.5). Equivalently, if $k_n^{0,p,j}(...)$ is the free cumulant on $\mathfrak{LS}_p$ in terms of the linear functional $\tau_{p,j}^0$ of (48), then

$$k_n^{0,p,j}\left(\underbrace{Q_{p,j},\ \ldots\ldots,\ Q_{p,j}}_{n\text{-times}}\right) = \begin{cases} \left(p^{j+1}\right)^2 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{50}$$

for all $n \in \mathbb{N}$.

Proof. The formula (49) is proven by the straightforward computations from (48) with help of (28), (42) and (43). Also, the formula (50) is obtained by the Möbius inversion of [12] from (49). See [8] and [12] for more details. □

8. Semicircularity on $\mathfrak{LS}$.

Let $\mathfrak{LS}_p$ and $\tau_{p,j}^0$ be in the sense of (37), respectively, (47). Then, one has the corresponding non-traditional Banach $*$-probability spaces,

$$\mathfrak{LS}_p(j) = \left(\mathfrak{LS}_p,\ \tau_{p,j}^0\right), \tag{51}$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Let $Q_{p,k} = l_p \otimes P_{p,k}$ be the generating elements (44) of the Banach $*$-probability space $\mathfrak{LS}_p(j)$ of (51), for $p \in \mathcal{P}$, $k \in \mathbb{Z}$. Then the "j-th" generating element $Q_{p,j}$ satisfies the $p^{2(j+1)}$-semicircularity:

$$k_n^{0,p,j}\left(Q_{p,j},\ \ldots,\ Q_{p,j}\right) = \begin{cases} p^{2(j+1)} & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

and (52)

$$\tau_{p,j}^0\left(Q_{p,j}^n\right) = \omega_n\left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, by (49) and (50).

8.1. Free product banach $*$-probability space $\left(\mathfrak{LS},\ \tau^0\right)$.

By (51), we have the family

$$\left\{\mathfrak{LS}_p(j) = \left(\mathfrak{LS}_p,\ \tau_{p,j}^0\right) : p \in \mathcal{P},\ j \in \mathbb{Z}\right\}$$

of (non-traditional) Banach $*$-probability spaces.

From this family, one can define the (traditional) *free product Banach $*$-probability space*,

$$\begin{aligned} \left(\mathfrak{LS},\ \tau^0\right) &\overset{def}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \mathfrak{LS}_p(j), \\ &= \left(\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \mathfrak{LS}_p,\ \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \tau_{p,j}^0\right) \end{aligned} \tag{53}$$

in the sense of [29], [30], [33] and [35].

The structures $\mathfrak{LS}_p(j)$ of (51) are the *free blocks* of this free product $*$-probability space $(\mathfrak{LS},\ \tau^0)$ of (53). Note that the structure (53) is a well-determined (traditional) noncommutative Banach $*$-probability space.

Definition 8.1. The Banach $*$-probability space $\mathfrak{LS} \overset{denote}{=} \left(\mathfrak{LS},\ \tau^0\right)$ of (53) is called the free Adelic filterization.

Let $\mathfrak{LS}$ be the free Adelic filterization. Then, we obtain a subset

$$\mathcal{Q} = \left\{Q_{p,j} = l_p \otimes P_{p,j} \in \mathfrak{LS}_p(j)\right\}_{p\in\mathcal{P},\ j\in\mathbb{Z}} \tag{54}$$

of $\mathfrak{LS}$, consisting of $p^{2(j+1)}$-semicircular elements $Q_{p,j}$ in the free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Remark here that, by the choice of $Q_{p,j}$ in the family $\mathcal{Q}$ of (54), all entries $Q_{p,j}$ are taken from the mutually-distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. It means that all elements $Q_{p,j}$ of $\mathcal{Q}$ are mutually free from each other in the free Adelic filterization $\mathfrak{LS}$.

Theorem 8.1. *Let $Q_{p,j} \in \mathcal{Q}$ in the free Adelic filterization $\mathfrak{LS}$ of (53), where $\mathcal{Q}$ is the family (54), for $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Then the operators*

$$\Theta_{p,j} = \frac{1}{p^{j+1}} Q_{p,j} \in \mathfrak{LS} \tag{55}$$

satisfy

$$\tau^0\left(\Theta_{p,j}^n\right) = \omega_n \ c_{\frac{n}{2}},$$

and (56)

$$k_n^0\left(\underbrace{\Theta_{p,j},\ \Theta_{p,j},\ \ldots,\ \Theta_{p,j}}_{n\text{-times}}\right) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, where $k_n^0(\ldots)$ is the free cumulant on $\mathfrak{LS}$ in terms of τ^0. Equivalently, the operators $\Theta_{p,j}$ of (55) are semicircular in $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Proof. Let $\Theta_{p,j} = \frac{1}{p^{j+1}} Q_{p,j}$ be in the sense of (55), where $Q_{p,j} \in \mathcal{Q}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, in the free Adelic filterization $\mathfrak{LS}$, where $\mathcal{Q}$ is the family (54). Since $Q_{p,j}$ are contained in the mutually distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, the operators $\Theta_{p,j}^n$ are contained in $\mathfrak{LS}_p(j)$ in $\mathfrak{LS}$, for all $n \in \mathbb{N}$, as free reduced words with their length-1. Thus, one has that

$$\tau^0\left(\Theta_{p,j}^n\right) = \tau_{p,j}^0\left(\Theta_{p,j}^n\right) = \tau_{p,j}^0\left(\frac{1}{p^{n(j+1)}} Q_{p,j}^n\right)$$

$$= \left(\frac{1}{p^{j+1}}\right)^n \tau_{p,j}^0\left(Q_{p,j}^n\right) = \left(\frac{1}{p^{j+1}}\right)^n \left(\omega_n p^{n(j+1)} c_{\frac{n}{2}}\right)$$

by the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$ in $\mathfrak{LS}_p(j)$

$$= \omega_n \ c_{\frac{n}{2}}, \tag{57}$$

for all $n \in \mathbb{N}$. Therefore, by (29) and (30), the operators $\Theta_{p,j}$ are semicircular in $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Also, by (31) and (57), one obtains the free cumulant formula in (56) by the Möbius inversion of [29] and [30]. □

The above theorem shows that, from the family $\mathcal{Q}$ of (54) consisting of $p^{2(j+1)}$-semicircular elements $Q_{p,j} \in \mathfrak{LS}_p(j)$, one can construct the corresponding semicircular elements $\Theta_{p,j}$ of (55) in the free Adelic filterization $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, by (57). Let

$$\mathcal{X} = \left\{\Theta_{p,j} \in \mathfrak{LS}_p(j) \,|\, p \in \mathcal{P},\ j \in \mathbb{Z}\right\}. \tag{58}$$

Recall that a subset $S = \{a_t\}_{t \in \Delta}$ of an arbitrary $*$-probability space (B, φ_B) is said to be a *free family*, if all elements $a_t \in S$ are free from each other in (B, φ_B) (e.g., [33] and [35]).

Definition 8.2. Let $S = \{a_t\}_{t \in \Delta}$ be a free family in a $*$-probability space (B, φ_B). This family S is said to be a free semicircular family, if every element a_t of

S is semicircular, for all $t \in \Delta$. Similarly, the family S is called a free weighted-semicircular family, if all elements a_t of S are weighted-semicircular, for all $t \in \Delta$.

So, we obtain the following result.

Theorem 8.2. *Let $\mathfrak{LS}$ be the free Adelic filterization (53).*

(59) *The family $\mathcal{Q}$ of (54) is a free weighted-semicircular family in $\mathfrak{LS}$.*

(60) *The family $\mathcal{X}$ of (58) is a free semicircular family in $\mathfrak{LS}$.*

Proof. The proofs of (59) and (60) are done by (52), (53), (54), (56) and (58). See [8] for details. □

8.2. **Free-semicircular Adelic filterization $\mathbb{LS}$.** Let $\mathfrak{LS}$ be the free Adelic filterization (53), and let $\mathcal{Q}$ be the free weighted-semicircular family (59), and $\mathcal{X}$, the free semicircular family (60) in $\mathfrak{LS}$. We now focus on the *Banach $*$-subalgebra $\mathbb{LS}$ of* $\mathfrak{LS}$ generated by the free family $\mathcal{Q}$,

$$\mathbb{LS} \overset{def}{=} \overline{\mathbb{C}\left[\mathcal{Q}\right]} \subset \mathfrak{LS}, \tag{61}$$

where $\overline{X}$ are the Banach-topology closures of subsets X of $\mathfrak{LS}$.

By (61), we obtain the corresponding Banach $*$-probability space,

$$\mathbb{LS} \overset{denote}{=} \left(\mathbb{LS},\ \tau^0\right), \tag{62}$$

as a free-probabilistic sub-structure of the free Adelic filterization $\mathfrak{LS}$, where τ^0 is the restricted linear functional $\tau^0 \mid_{\mathbb{LS}}$ on $\mathbb{LS}$.

Definition 8.3. Let $\mathbb{LS} = \left(\mathbb{LS},\ \tau^0\right)$ be the Banach $*$-probability space (62) in the free Adelic filterization $\mathfrak{LS}$ of (53). Then it is called the (free-)semicircular Adelic filterization (of $\mathfrak{LS}$, generated by the free semicircular family $\mathcal{X}$ of (59)).

Let $\mathbb{LS}$ be the semicircular Adelic filterization (62). Then it satisfies the following structure theorem.

Theorem 8.3. *Let $\mathbb{LS}$ be the semicircular Adelic filterization (62) of the free Adelic filterization $\mathfrak{LS}$. Then the Banach $*$-algebra $\mathbb{LS}$ satisfies that*

$$\begin{aligned} \mathbb{LS} \quad & \overset{*\text{-}iso}{=} \underset{p \in \mathcal{P},\ j \in \mathbb{Z}}{\star} \left(\overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}\right) \\ & \overset{*\text{-}iso}{=} \overline{\mathbb{C}\left[\underset{p \in \mathcal{P}, j \in \mathbb{Z}}{\star} \{\Theta_{p,j}\}\right]}, \end{aligned} \tag{63}$$

in $\mathfrak{LS}$, where the free product $(\star)$ in the first isomorphic relation of (63) means the free-probability-theoretic free product of [12] *and* [14] *(with respect to the linear functional τ^0 of (62)), and the free product $(\star)$ in the second isomorphic relation of (63) means the pure-algebraic free product inducing "finite" noncommutative free words in the free semicircular family Θ.*

Proof. By the definition (62) of our semicircular Adelic filterization $\mathbb{LS}$, we have

$$\begin{aligned} \mathbb{LS} = \overline{\mathbb{C}\left[\mathcal{X}\right]} &= \overline{\mathbb{C}\left[\{Q_{p,j} \in \mathcal{X} : p \in \mathcal{P},\ j \in \mathbb{Z}\}\right]} \\ & \overset{*\text{-iso}}{=} \overline{\underset{p \in \mathcal{P},\ j \in \mathbb{Z}}{\star} \mathbb{C}\left[\{Q_{p,j}\}\right]} = \underset{p \in \mathcal{P},\ j \in \mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}\}\right]}, \end{aligned} \tag{64}$$

since $\mathcal{X}$ is a free family in $\mathfrak{LS}$, equivalently, since $Q_{p,j}$ are contained in the mutually distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Note that, every $p^{2(j+1)}$-semicircular element $Q_{p,j} \in \mathcal{X}$ of $\mathfrak{LS}$ is identified with

$$Q_{p,j} = p^{j+1}\Theta_{p,j}, \text{ for all } p \in \mathcal{P},\ j \in \mathbb{Z},$$

and hence, the free blocks $\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]}$ of (64) generating the semicircular Adelic filterization $\mathbb{LS}$ are identical to

$$\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} = \overline{\mathbb{C}\left[\{p^{j+1}\Theta_{p,j}\}\right]} = \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}, \tag{64$'$}$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Therefore, by (64), the first $*$-isomorphic relation of (63) holds.

Also, by (64), all elements T of $\mathbb{LS}$ are the limits of linear combinations of noncommutative free reduced words in $\mathcal{X}$, under Banach-topology for $\mathbb{LS}$. Since all noncommutative free words in $\mathcal{X}$ have their unique free-reduced-word forms in $\mathbb{LS}$ (as operators under operator-multiplication on $\mathbb{LS}$), one obtains that

$$\begin{aligned}\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star}\overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]} &\overset{*\text{-iso}}{=} \overline{\mathbb{C}[\{\text{free words in } \mathcal{X}\}]} \\ &= \overline{\mathbb{C}\left[\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star}\{\Theta_{p,j}\}\right]}.\end{aligned} \tag{65}$$

Therefore, by (64), (64)$'$ and (65), the second $*$-isomorphic relation of (63) holds true, too. □

In the middle of the proof of (63), one can get the set-equality,

$$\mathbb{LS} \overset{def}{=} \overline{\mathbb{C}\left[\mathcal{Q}\right]} = \overline{\mathbb{C}\left[\mathcal{X}\right]}, \text{ in } \mathfrak{LS}. \tag{66}$$

9. **Semicircular A-tensor Adelic filterization $\mathbb{LS}_A$.** Let $\mathbb{LS} = (\mathbb{LS},\ \tau^0)$ be the semicircular Adelic filterization generated by the free semicircular family $\mathcal{X}$ of (60). Let $(A,\ \varphi_A)$ be an arbitrary (traditional) unital C^*-probability space satisfying

$$\varphi_A(1_A) = 1,$$

where 1_A is the unit (or the multiplication-identity) of the C^*-algebra A.

Define the tensor product Banach $*$-algebra $\mathbb{LS}_A$ by

$$\mathbb{LS}_A \overset{def}{=} A \otimes_{\mathbb{C}} \mathbb{LS}, \tag{67}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras.

On this new Banach $*$-algebra $\mathbb{LS}_A$ of (67), define a linear functional τ_A by a linear morphism satisfying that

$$\tau_A\left(a \otimes T\right) = \tau^0\left(\varphi_A(a)T\right), \tag{68}$$

for all $a \in (A,\ \varphi_A)$, and $T \in \mathbb{LS}$ (under linearity).

By the definition (68) of the linear functional τ_A,

$$\tau_A\left(a \otimes T\right) = \tau^0(T)\varphi_A(a) = \varphi_A(a)\tau^0(T),$$

for all $a \in (A,\ \varphi_A)$, $T \in \mathbb{LS}$.

Then the Banach $*$-probability space

$$\mathbb{LS}_A \overset{denote}{=} (\mathbb{LS}_A,\ \tau_A) \tag{69}$$

is well-defined, where $\mathbb{LS}_A$ and τ_A are in the sense of (67), respectively, (68).

Definition 9.1. Let $\mathbb{LS}_A = (\mathbb{LS}_A,\ \tau_A)$ be the Banach $*$-probability space (69) induced by a fixed unital C^*-probability space $(A,\ \varphi_A)$ and the semicircular Adelic filterization $\mathbb{LS}$. Then we call $\mathbb{LS}_A$, the semicircular A-tensor(-Adelic) filterization (of $(A,\ \varphi_A)$).

On the semicircular A-tensor filterization $\mathbb{LS}_A$, we obtain the following free distributional data.

Proposition 9.1. *Let $Q_{p,j} \in \mathcal{Q}$, and $\Theta_{p,j} \in \mathcal{X}$ in $\mathbb{LS}$, and $a \in (A,\, \varphi_A)$, inducing*

$$T_{p,j}^{a} = a \otimes Q_{p,j},\ \text{and}\ X_{p,j}^{a} = a \otimes \Theta_{p,j}, \tag{70}$$

in the semicircular A-tensor filterization $\mathbb{LS}_A$ of (69). Then

$$\tau_A\left(\left(T_{p,j}^{a}\right)^{n}\right) = \left(\omega_n p^{n(j+1)} c_{\frac{n}{2}}\right)\varphi_A(a^n),$$

and (71)

$$\tau_A\left(\left(X_{p,j}^{a}\right)^{n}\right) = \left(\omega_n c_{\frac{n}{2}}\right)\varphi_A(a^n),$$

for all $n \in \mathbb{N}$.

Proof. The proof of the free-distributional data (71) are shown by the weighted-semicircularity on the free weighted-semicircular family $\mathcal{Q}$, and the semicircularity on the free semicircular family $\mathcal{X}$ in $\mathbb{LS}$. Indeed, if $T_{p,j}^{a}$ and $X_{p,j}^{a}$ are in the sense of (70), then

$$\tau_A\left(\left(T_{p,j}^{a}\right)^{n}\right) = \tau_A\left(a^n \otimes Q_{p,j}^{n}\right) = \varphi_A(a^n)\tau^0\left(Q_{p,j}^{n}\right),$$

and

$$\tau_A\left(\left(X_{p,j}^{a}\right)^{n}\right) = \tau_A\left(a^n \otimes \Theta_{p,j}^{n}\right) = \varphi_A(a^n)\tau^0\left(\Theta_{p,j}^{n}\right),$$

for all $n \in \mathbb{N}$, by (68). □

By the above proposition, we obtain the following free-probabilistic information on the semicircular A-tensor filterization $\mathbb{LS}_A$.

Theorem 9.2. *Let $\mathbb{LS}_A = (\mathbb{LS}_A,\, \tau_A)$ be the semicircular A-tensor filterization, and let $T_{p,j}^{a}$ and $X_{p,j}^{a}$ be free random variables (70) in $\mathbb{LS}_A$. Suppose a is a self-adjoint operator of $(A,\, \varphi_A)$, satisfying*

$$\varphi_A\left(a^{2n}\right) = \left(\varphi_A(a)\right)^{2n},\ \text{with}\ \varphi_A(a^2) \in \mathbb{C}^{\times}, \tag{72}$$

for all $n \in \mathbb{N}$. Then $T_{p,j}^{a}$ is $\left(p^{(j+1)}\varphi_A(a)\right)^2$-semicircular, and $X_{p,j}^{a}$ is $\varphi_A(a)^2$-semicircular in $\mathbb{LS}_A$.

Proof. Let $a \in (A,\, \varphi_A)$ be a self-adjoint free random variable satisfying (72). Then, by the self-adjointness, the operators $T_{p,j}^{a}$ and $X_{p,j}^{a}$ of (70) are self-adjoint in $\mathbb{LS}_A$, too. Indeed, one has that

$$\left(T_{p,j}^{a}\right)^{*} = a^{*} \otimes Q_{p,j}^{*} = T_{p,j}^{a},$$

and

$$\left(X_{p,j}^{a}\right)^{*} = a^{*} \otimes \Theta_{p,j}^{*} = X_{p,j}^{a},$$

in $\mathbb{LS}_A$.

Also, we have that

$$\begin{aligned}\tau_A\left(\left(T_{p,j}^{a}\right)^{n}\right) &= \left(\omega_n p^{n(j+1)} c_{\frac{n}{2}}\right)\varphi_A(a^n)\\ &= \omega_n p^{n(j+1)}\varphi_A(a)^n c_{\frac{n}{2}}\\ &= \omega_n\left(p^{2(j+1)}\varphi_A(a)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},\end{aligned}$$

and (73)

$$\begin{aligned}\tau_A\left(\left(X_{p,j}^{a}\right)^{n}\right) &= \left(\omega_n c_{\frac{n}{2}}\right)\varphi_A(a^n)\\ &= \omega_n\varphi_A(a)^n c_{\frac{n}{2}}\\ &= \omega_n\left(\varphi_A(a)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},\end{aligned}$$

for all $n \in \mathbb{N}$, by (71) and (72).

Therefore, if a free random variable $a \in (A, \varphi_A)$ satisfies the additional condition (72), then $T_{p,j}^{a}$ is $\left(p^{j+1}\varphi_A(a)\right)^2$-semicircular, and $X_{p,j}^{a}$ is $\varphi_A(a)^2$-semicircular in the semicircular A-tensor filterization $\mathbb{LS}_A$, by (73). $\square$

The following corollary is a direct consequence of the above theorem.

Corollary 9.3. *Let $\mathbb{LS}_A$ be the semicircular A-tensor filterization (69) of (A, φ_A).*

(74) *The operator $T_{p,j}^{1_A}$ in the sense of (70) is $p^{2(j+1)}$-semicircular in $\mathbb{LS}_A$.*

(75) *The operator $X_{p,j}^{1_A}$ in the sense of (70) is semicircular in $\mathbb{LS}_A$.*

(76) *If the linear functional $\varphi_A : A \to \mathbb{C}$ is a state in the sense that*

$$\varphi_A\left(a_1 a_2\right) = \varphi_A(a_1)\varphi_A(a_2), \ \forall a_1, \ a_2 \in A,$$

and if $a \in (A, \varphi_A)$ is a self-adjoint free random variable with $\varphi_A(a) \in \mathbb{C}^{\times}$, then the operator $T_{p,j}^{a}$ of (70) is $\left(p^{j+1}\varphi_A(a)\right)^2$-semicircular, and the operator $X_{p,j}^{a}$ of (70) is $\varphi_A(a)^2$-semicircular in $\mathbb{LS}_A$.

Proof. Let 1_A be the unit of (A, φ_A). Since our fixed C^*-probability space (A, φ_A) is unital in the sense that $\varphi_A(1_A) = 1$, one has

$$\varphi_A\left(1_A^n\right) = \varphi_A(1_A) = 1 = 1^n = \left(\varphi_A(1_A)\right)^n,$$

for all $n \in \mathbb{N}$. Therefore, this self-adjoint free random variable 1_A satisfies the condition (72). Thus, by (73), the operator $T_{p,j}^{1_A}$ is $p^{2(j+1)}$-semicircular, and the operator $X_{p,j}^{1_A}$ is semicircular in $\mathbb{LS}_A$. It proves the statements (74) and (75), respectively.

Assume now that the linear functional φ_A is a state on A, equivalently, assume φ_A is a multiplicative linear functional on A. Then, for any self-adjoint free random variable $a \in (A, \varphi_A)$ with $\varphi_A(a) \in \mathbb{C}^{\times}$,

$$\varphi_A\left(a^n\right) = \varphi_A(a)^n, \text{ for all } n \in \mathbb{N}.$$

So, it satisfies the condition (72). Therefore, the statement (76) holds by (73). $\square$

In the above theorem and corollary, we considered the free-distributional information of the generating operators, on the semicircular A-tensor filterization $\mathbb{LS}_A$.

Theorem 9.4. *Let $\mathbb{LS}_A$ be the semicircular A-tensor filterization (69) of a unital C^*-probability space (A, φ_A). Then*

$$\begin{aligned} \mathbb{LS}_A &\overset{*\text{-}iso}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star}\left(A \otimes_{\mathbb{C}} \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}\right) \\ &\overset{*\text{-}iso}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \overline{A\left[\{\Theta_{p,j}\}\right]}, \end{aligned} \tag{77}$$

where $\overline{Z}$ in the first $*$-isomorphic relation of (77) are the Banach-topology closures of subsets Z of the semicircular Adelic filterization $\mathbb{LS}$, and $\overline{Y}$ in the second $*$-isomorphic relation of (77) are the Banach-topology closures of subsets Y of $\mathbb{LS}_A$, where $A[Y]$ mean the polynomial rings (and hence, algebras, in this case) generated by the subsets Y over A in $\mathbb{LS}_A$.

Proof. By the definition (69) of the semicircular A-tensor filterization $\mathbb{LS}_A$,

$$\mathbb{LS}_A \overset{def}{=} A \otimes_{\mathbb{C}} \mathbb{LS} \overset{*\text{-iso}}{=} A \otimes_{\mathbb{C}} \left(\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}\right)$$

by (63) and (66)

$$\overset{*\text{-iso}}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \left(A \otimes_{\mathbb{C}} \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}\right) \overset{*\text{-iso}}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \overline{A\left[\{\Theta_{p,j}\}\right]}$$

(e.g., see [29], [30], [33] and [35]). Therefore, the free-structure theorem (77) holds. □

As corollary, one obtains the following structure theorems.

Corollary 9.5. *Let $\mathbb{LS}_A$ be the semicircular A-tensor filterization of $(A,\ \varphi_A)$.*

(78) *If A is a direct product C^*-algebra $\underset{k\in\Delta}{\oplus} A_k$ of its C^*-subalgebras $\{A_k\}_{k\in\Delta}$, where $\oplus$ is the direct product of C^*-algebras, and Δ is a countable (finite, or infinite) index set, then*

$$\mathbb{LS}_A \overset{*\text{-}iso}{=} \underset{k\in\Delta}{\oplus_{\mathbb{C}}} \left(\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \overline{A_k\left[\{\Theta_{p,j}\}\right]}\right)$$

$$\overset{*\text{-}iso}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \left(\underset{k\in\Delta}{\oplus_{\mathbb{C}}} \overline{A_k\left[\{\Theta_{p,j}\}\right]}\right),$$

where $\oplus_{\mathbb{C}}$ is the direct product of Banach $*$-algebras.

(79) *If A is a tensor product C^*-algebra $\underset{k\in\Delta}{\otimes} A_k$ of its C^*-subalgebras $\{A_k\}_{k\in\Delta}$, where $\otimes$ is the tensor product of C^*-algebras, then*

$$\mathbb{LS}_A \overset{*\text{-}iso}{=} \underset{k\in\Delta}{\otimes_{\mathbb{C}}} \left(\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \overline{A_k\left[\{\Theta_{p,j}\}\right]}\right)$$

$$\overset{*\text{-}iso}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \left(\underset{k\in\Delta}{\otimes_{\mathbb{C}}} \overline{A_k\left[\{\Theta_{p,j}\}\right]}\right),$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras.

(80) *Let $(A,\ \varphi_A)$ be the fixed unital C^*-probability space. For the linear functional φ_A, assume that the C^*-algebra A is a free product C^*-algebra of its C^*-subalgebras $\{A_k\}_{k\in\Delta}$. Then*

$$\mathbb{LS}_A \overset{*\text{-}iso}{=} \underset{k\in\Delta,\ p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \overline{A_k\left[\{\Theta_{p,j}\}\right]}.$$

Proof. The proofs of the statements (78), (79) and (80) are done by (77). Indeed, one has that: if $A = \underset{k\in\Delta}{\oplus} A_k$, then

$$\overline{A\left[\{\Theta_{p,j}\}\right]} \overset{*\text{-iso}}{=} \overline{\left(\underset{k\in\Delta}{\oplus} A_k\right)\left[\{\Theta_{p,j}\}\right]}$$

$$\overset{*\text{-iso}}{=} \overline{\underset{k\in\Delta}{\oplus_{\mathbb{C}}} \left(A_k\left[\{\Theta_{p,j}\}\right]\right)} = \underset{k\in\Delta}{\oplus_{\mathbb{C}}} \overline{A_k\left[\{\Theta_{p,j}\}\right]},$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Similarly, if $A = \underset{k\in\Delta}{\otimes} A_k$, then

$$\overline{A\left[\{\Theta_{p,j}\}\right]} \overset{*\text{-iso}}{=} \underset{k\in\Delta}{\otimes_{\mathbb{C}}} \overline{A_k\left[\{\Theta_{p,j}\}\right]};$$

and if $A = \underset{k \in \Delta}{\star} A_k$, then

$$\overline{A\,[\{\Theta_{p,j}\}]} \overset{*\text{-iso}}{=} \underset{k \in \Delta}{\star} \overline{A_k\,[\{\Theta_{p,j}\}]},$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. □

Our results of this section illustrate that the free probability on $\mathbb{LS}_A$ is characterized by the both free probability on (A, φ_A), and that on the semicircular Adelic filterization $\mathbb{LS}$. In particular, such a characterization is analyzed by the formula (71), and the structure theorem (77).

10. **Shifts on $\mathcal{P}$ acting on $\mathbb{LS}_A$.** Throughout this section, we fix a unital C^*-probability space (A, φ_A), and the corresponding semicircular A-tensor filterization $\mathbb{LS}_A = (\mathbb{LS}_A, \tau_A)$ of (A, φ_A). Also, let

$$X_{p,j}^{a} = a \otimes \Theta_{p,j} \tag{81}$$

be free random variables (70), generating $\mathbb{LS}_A$, for all $a \in (A, \varphi_A)$, and $\Theta_{p,j} \in \mathcal{X} \subset \mathbb{LS}$, where $\mathbb{LS}$ is the semicircular Adelic filterization and $\mathcal{X}$ is the free semicircular family (60). Indeed, all operators $X_{p,j}^{a}$ formed by (81) generate $\mathbb{LS}_A$, by (63) and (66).

Define a subset $\mathcal{X}_A$ of $\mathbb{LS}_A$ by

$$\mathcal{X}_A \overset{def}{=} \{X_{p,j}^{a} \in \mathbb{LS}_A : X_{p,j}^{a} \text{ is in the sense of (81)}\}. \tag{82}$$

Then, as we discussed above this subset $\mathcal{X}_A$ of (82) generates $\mathbb{LS}_A$, i.e.,

$$\mathbb{LS}_A = \overline{\mathbb{C}\,[\mathcal{X}_A]}, \tag{83}$$

set-theoretically, by (63), (66) and (67).

Suppose a given C^*-algebra A is generated by a subset B of A, i.e., by (83), if

$$A = \overline{\mathbb{C}[B]}^{A},$$

where $\overline{Y}^{A}$ mean the C^*-topology closures of subsets Y of A, then one can re-define the generator set $\mathcal{X}_A$ of (82) by

$$\mathcal{X}_A = \{X_{p,j}^{a} \in \mathbb{LS}_A : a \in B,\, \Theta_{p,j} \in \mathcal{X}\}.$$

However, now, we take a C^*-algebra A arbitrarily. So, in the following text, we understand the generator set $\mathcal{X}_A$ of $\mathbb{LS}_A$ as in the general sense of (82).

In this section, we consider how our free-distributional data on $\mathbb{LS}_A$ are affected (or distorted) by certain shift processes on the set $\mathcal{P}$ of all primes.

10.1. **Shifts on $\mathcal{P}$.** Let $\mathcal{P}$ be the set of all primes in $\mathbb{N}$. Note that the set $\mathcal{P}$ is a *totally ordered set* (or, in short, TO*set*) under the usual inequality ($\leq$). So, one can index $\mathcal{P}$ orderly by

$$\mathcal{P} = \{p_1 \leq p_2 \leq p_3 \leq p_4 \leq \cdots\},$$

with (84)

$$p_1 = 2,\, p_2 = 3,\, p_3 = 5,\, p_4 = 7,\, p_5 = 11,\, ...,\text{ etc..}$$

From below, the set $\mathcal{P}$ is understood as the TOset (84).

Define now an injective functional $g : \mathcal{P} \to \mathcal{P}$ by

$$g(p_k) = p_{k+1}, \text{ for all } k \in \mathbb{N}. \tag{85}$$

For the injection g of (85), we define $g^n : \mathcal{P} \to \mathcal{P}$ by

$$g^n = \underbrace{g \circ g \circ g \circ \cdots \circ g}_{n\text{-times}},$$

with axiomatization: (86)

$$g^0 = id_{\mathcal{P}}, \text{ the identity map on } \mathcal{P},$$

for all $n \in \mathbb{N}_0$, where $(\circ)$ is the usual functional *composition*.

By (86), clearly, $g^1 = g$, in the sense of (85), and

$$g^n(p_k) = p_{k+n} \text{ in } \mathcal{P}, \text{ for all } k \in \mathbb{N},$$

for all $n \in \mathbb{N}_0$. For example,

$$g(2) = 3,\ g^2(3) = 7,\ g^5(5) = 19, \text{ etc..}$$

Definition 10.1. Let g^n be in the sense of (86) for all $n \in \mathbb{N}_0$. Then these functions g^n on $\mathcal{P}$ are said to be n-shifts on $\mathcal{P}$, for all $n \in \mathbb{N}_0$. In particular, the 1-shift $g = g^1$ of (85) is simply called the shift on $\mathcal{P}$.

10.2. **Prime-shift $*$-homomorphisms on $\mathbb{LS}_A$.** Let $\mathbb{LS}_A$ be our semicircular A-tensor filterization, and let g be the shift (85) on the TOset $\mathcal{P}$ of (84), inducing the n-shifts g^n of (86) on $\mathcal{P}$. Define a $*$-*homomorphism* G_A on $\mathbb{LS}_A$ by a bounded "multiplicative" linear transformation satisfying

$$G_A\left(X_{p,j}^a\right) = X_{g(p),j}^a = a \otimes \Theta_{g(p),j}, \tag{87}$$

for all $X_{p,j}^a \in \mathcal{X}_A$, where $\mathcal{X}_A$ is the generator set (82) of $\mathbb{LS}_A$, where $g = g^1$ is the shift (85) on $\mathcal{P}$.

By the multiplicativity, the morphism G_A of (87) satisfies that: if

$$S = \prod_{l=1}^{N}\left(X_{p_l,j_l}^{a_l}\right)^{n_l},$$

in $\mathbb{LS}_A$, for $n_1, ..., n_N \in \mathbb{N}$, as a free reduced words with its length-N (in the sense of (77)) for $N \in \mathbb{N}$, then

$$G_A(S) = G_A\left(\prod_{l=1}^{N}\left(X_{p_l,j_l}^{a_l}\right)^{n_l}\right) = \prod_{l=1}^{N} G_A\left(\left(X_{p_l,j_l}^{a_l}\right)^{n_l}\right)$$

by the multiplicativity of G_A

$$= \prod_{l=1}^{N}\left(G_A\left(X_{p_l,j_l}^{a_l}\right)\right)^{n_l}$$

by the multiplicativity of G_A

$$= \prod_{l=1}^{N}\left(X_{g(p_l),j_l}^{a_l}\right)^{n_l} = \prod_{l=1}^{N}\left(a_l \otimes \Theta_{g(p_l),j_l}\right)^{n_l}$$

by (87)

$$= \prod_{l=1}^{N}\left(a_l^n \otimes \Theta_{g(p_l),j_l}^{n_l}\right),$$

i.e.,

$$G_A(S) = \prod_{l=1}^{N}\left(X_{g(p_l),j_l}^{a_l}\right)^{n_l}, \tag{88}$$

in $\mathbb{LS}_A$.

Also, this morphism G_A of (87) satisfies that

$$G_A\left(S^*\right) = G_A\left(\prod_{l=1}^{N} X_{p_{N-l+1},j_{N-l+1}}^{a_{N-l+1}^*}\right)$$

because

$$\left(X_{p,j}^a\right)^* = \left(a \otimes \Theta_{p,j}\right)^* = a^* \otimes \Theta_{p,j} = X_{p,j}^{a^*}, \tag{89}$$

in $\mathbb{LS}_A$, for all $X_{p,j}^a \in \mathcal{X}_A$, and hence, the above formula goes to

$$= \prod_{l=1}^{N} X_{g(p_{N-l+1}),\ j_{N-l+1}}^{a_{N-l+1}^*} = \left(\prod_{l=1}^{N} X_{g(p_l),j_l}^{a_l}\right)^*$$

by (88)

$$= \left(G_A(S)\right)^*, \tag{90}$$

by (89).

By (88) and (90), one can verify that, for all $T \in \mathbb{LS}_A$,

$$G_A(T^*) = G_A(T)^*, \text{ in } \mathbb{LS}_A. \tag{91}$$

Proposition 10.1. *Let G_A be the multiplicative linear transformation (87) on $\mathbb{LS}_A$. Then it is a $*$-homomorphism on $\mathbb{LS}_A$.*

Proof. The proof is done by (91). i.e., this multiplicative linear transformation G_A preserves adjoints in the sense of (91). Thus, it is a well-defined $*$-homomorphism on $\mathbb{LS}_A$. □

For the $*$-homomorphism G_A of (87), one can have the iterated products (or compositions) G_A^n of (n-copies of) G_A, as $*$-homomorphisms on $\mathbb{LS}_A$, with $G_A^1 = G_A$, for all $n \in \mathbb{N}_0$, with axiomatization:

$$G_A^0 = 1_{\mathbb{LS}_A}, \text{ the identity operator on } \mathbb{LS}_A,$$

satisfying

$$G_A^0\left(X_{p,j}^a\right) = X_{g^0(p),\ j}^a = X_{p,j}^a = 1_{\mathbb{LS}}\left(X_{p,j}^a\right),$$

for all $X_{p,j}^a \in \mathcal{X}_A$ in $\mathbb{LS}_A$, where $\mathcal{X}_A$ is the generator set (82) of $\mathbb{LS}_A$.

Then it is not difficult to check that G_A^n satisfy

$$G_A^n\left(X_{p,j}^a\right) = X_{g^n(p),\ j}^a \text{ in } \mathbb{LS}, \forall n \in \mathbb{N}_0, \tag{92}$$

for all $X_{p,j}^a \in \mathcal{X}_A \subset \mathbb{LS}_A$.

Definition 10.2. The $*$-homomorphism G_A of (87) on the semicircular A-tensor filterization $\mathbb{LS}_A$ is called the prime-shift ($*$-homomorphism) on $\mathbb{LS}_A$. Also, the n-th powers G_A^n of (92) are called the n-prime-shift(-$*$-homomorphism)s on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

Based on our n-prime-shifts (92), we obtain the following free-distributional data.

Theorem 10.2. *Let $X_{p,j}^a \in \mathcal{X}_A$ be a generating operator of $\mathbb{LS}_A$, and let G_A^n be the n-prime-shift on $\mathbb{LS}_A$, for $n \in \mathbb{N}_0$. Then*

$$\begin{aligned} \tau_A\left(\left(G_A^n\left(X_{p,j}^a\right)\right)^k\right) &= \left(\omega_k c_{\frac{k}{2}}\right)\varphi_A(a^k) \\ &= \tau_A\left(\left(X_{p,j}^a\right)^k\right), \end{aligned} \tag{93}$$

for all $k \in \mathbb{N}$.

Proof. Let $X_{p,j}^a \in \mathcal{X}_A$ in $\mathbb{LS}_A$, for $a \in (A, \varphi_A)$, $p \in \mathcal{P}$, and $j \in \mathbb{Z}$. Then

$$G_A^n\left(X_{p,j}^a\right) = X_{g^n(p),j}^a = a \otimes \Theta_{g^n(p),j} \in \mathbb{LS}_A,$$

for any $n \in \mathbb{N}_0$. Thus,

$$\left(G_A^n\left(X_{p,j}^a\right)\right)^k = \left(a \otimes \Theta_{g^n(p),j}\right)^k = a^k \otimes \Theta_{g^n(p),j}^k, \tag{94}$$

for all $k \in \mathbb{N}$.

So, one has that

$$\tau_A\left(\left(G_A^n\left(X_{p,j}^a\right)\right)^k\right) = \tau_A\left(\left(X_{g^n(p),j}^a\right)^k\right)$$

by (94)

$$= \varphi_A(a^k)\tau^0\left(\Theta_{g^n(p),j}^k\right) = \varphi_A(a^k)\left(\omega_k c_{\frac{k}{2}}\right), \tag{95}$$

by (71), for all $k \in \mathbb{N}$.

Therefore, the first equality of (93) holds by (95), and the second equality of (93) holds by (71). □

By the above theorem, one can get the following result.

Corollary 10.3. *Let $a \in (A, \varphi_A)$ be a self-adjoint free random variable, and let $X_{p,j}^a \in \mathcal{X}_A$ be a generating operator (81) of $\mathbb{LS}_A$. Let G_A^n be the n-prime shifts (92) on $\mathbb{LS}_A$, for $n \in \mathbb{N}_0$. Then the free distribution of $X_{p,j}^a$ and the free distributions of $G_A^n\left(X_{p,j}^a\right)$ are identical in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.*

Proof. Let $a \in (A, \varphi_A)$ be given as above. Then, by the self-adjointness of a, the corresponding generating operator $X_{p,j}^a$ is self-adjoint in $\mathbb{LS}_A$, too. Indeed,

$$\left(X_{p,j}^a\right)^* = a^* \otimes \Theta_{p,j}^* = X_{p,j}^a \text{ in } \mathbb{LS}_A.$$

Note now that, since $G_A^n\left(X_{p,j}^a\right) = X_{g^n(p),j}^a$,

$$\left(G_A^n\left(X_{p,j}^A\right)\right)^* = a^* \otimes \Theta_{g^n(p),j}^* = X_{g^n(p),j}^a = G_A^n\left(X_{p,j}^A\right),$$

in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. Therefore, the images $G_A^n\left(X_{p,j}^a\right)$ of our n-prime shifts G_A^n preserve the self-adjointness of $X_{p,j}^a$ in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

Recall that the free distributions of self-adjoint operators are characterized by the free-moment sequence. So, the free distribution of $X_{p,j}^a$ is characterized by

$$\left(\tau_A\left(\left(X_{p,j}^a\right)^k\right)\right)_{k=1}^\infty = \left(\omega_k c_{\frac{k}{2}} \varphi_A(a^k)\right)_{k=1}^\infty,$$

by (71).

Also, the free distributions of $G_A^n\left(X_{p,j}^a\right)$ are characterized by

$$\left(\tau_A\left(\left(G_A^n(X_{p,j}^a)\right)^k\right)\right)_{k=1}^\infty = \left(\tau_A\left(\left(X_{p,j}^a\right)^k\right)\right)_{k=1}^\infty,$$

by the self-adjointness of them, and by (93), for all $n \in \mathbb{N}_0$.

It shows that the free distributions of $G_A^n\left(X_{p,j}^a\right)$ are all identically characterized by the free-moment sequence,

$$\left(0,\ c_1\varphi_A(a^2),\ 0,\ c_2\varphi_A(a^4),\ 0,\ c_3\varphi_A(a^6),\ ....\right),$$

for all $n \in \mathbb{N}_0$. □

Let's generalize the above corollary. In fact, the free-distributional formula (93) guarantees that the free distributions of the generators $X_{p,j}^a \in \mathcal{X}_A$ are preserved by the n-prime shifts G_A^n on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$, since, even though a is not self-adjoint in A, one can have

$$\tau_A\left(\left(G_A^n\left(X_{p,j}^a\right)\right)^k\right) = \tau_A\left(\left(X_{p,j}^a\right)^k\right),$$

and (96)

$$\begin{aligned} \tau_A\left(\left(\left(G_A^n\left(X_{p,j}^a\right)\right)^*\right)^k\right) &= \tau_A\left(\left(G_A^n\left(X_{p,j}^{a^*}\right)\right)^k\right) \\ &= \tau_A\left(\left(X_{p,j}^{a^*}\right)^k\right) = \tau_A\left(\left(\left(X_{p,j}^a\right)^*\right)^k\right), \end{aligned}$$

for all $k \in \mathbb{N}$, because

$$\left(X_{p,j}^a\right)^* = a^* \otimes \Theta_{p,j} = X_{p,j}^{a^*} \text{ in } \mathcal{X}_A,$$

in $\mathbb{LS}_A$.

Therefore, one can verify that

$$\tau_A\left(\left(G_A^n(X_{p,j}^{a^{r_1}})\right)\left(G_A^n(X_{p,j}^{a^{r_2}})\right)...\left(G_A^n(X_{p,j}^{a^{r_k}})\right)\right)$$

$$= \tau_A\left(\left(X_{g^n(p),j}^{a^{r_1}}\right)\left(X_{g^n(p),j}^{a^{r_2}}\right)...\left(X_{g^n(p),j}^{a^{r_k}}\right)\right)$$

$$= \tau_A\left((a^{r_1}a^{r_2}...a^{r_k})\otimes\Theta_{g^n(p),j}^k\right)$$

$$= \varphi_A\left(a^{r_1}a^{r_2}...a^{r_k}\right)\tau^0\left(\Theta_{g^n(p),j}^k\right) = \left(\omega_k c_{\frac{k}{2}}\right)\varphi_A\left(a^{r_1}a^{r_2}...a^{r_k}\right)$$

by (71)

$$= \varphi_A(a^{r_1}a^{r_2}...a^{r_k})\tau^0\left(\Theta_{p,j}^k\right) = \tau_A\left(X_{p,j}^{a^{r_1}}X_{p,j}^{a^{r_2}}...X_{p,j}^{a^{r_k}}\right), \quad (97)$$

by (71), for all $(r_1, ..., r_k) \in \{1, *\}^k$, for all $k \in \mathbb{N}$.

Therefore, one obtains the following theorem.

Theorem 10.4. *Let $X_{p,j}^a \in \mathcal{X}_A$ be a generating operator of the semicircular A-tensor filterization $\mathbb{LS}_A$, where $a \in (A, \varphi_A)$ is arbitrarily given, and let G_A^n be the n-prime shifts on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. Then the free distribution of $X_{p,j}^a$ and the free distributions of $G_A^n\left(X_{p,j}^a\right)$ are identically same in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. i.e.,*

(98) *the free distribution of $G_A^n\left(X_{p,j}^a\right)$ = the free distribution of $X_{p,j}^a$,*

in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

Proof. Let $a \in (A, \varphi_A)$ be self-adjoint, and hence, $X_{p,j}^a \in \mathcal{X}_A$, a self-adjoint generating operator of $\mathbb{LS}_A$. Then, by the above corollary, the free distribution of $X_{p,j}^a$ and those of $G_A^n\left(X_{p,j}^a\right)$ are identical in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

Assume now that a is not self-adjoint in A. Then the corresponding operator $X_{p,j}^a$ is not self-adjoint too, since

$$\left(X_{p,j}^a\right)^* = X_{p,j}^{a^*} \neq X_{p,j}^a \text{ in } \mathbb{LS}_A.$$

It also shows that

$$G_A^n\left(X_{p,j}\right)^* = \left(X_{g^n(p),j}^a\right)^* = X_{g^n(p),j}^{a^*} \neq X_{g^n(p),j}^a = G_A^n\left(X_{p,j}\right),$$

in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

So, the free distribution of $X \overset{denote}{=} X_{p,j}^a$ is characterized by the "joint" free moments,

$$\left\{\tau_A\left(X^{r_1}X^{r_2}...X^{r_k}\right) \,\middle|\, \begin{array}{c}(r_1, ..., r_k) \in \{1, *\}^k, \\ \text{for all } k \in \mathbb{N}\end{array}\right\},$$

and similarly, the free distributions of $X_{(n)} \overset{denote}{=} G_A^n\left(X_{p,j}^a\right)$ are characterized by the joint free moments,

$$\left\{\tau_A\left(X_{(n)}^{r_1}X_{(n)}^{t_2}...X_{(n)}^{r_k}\right) \,\middle|\, \begin{array}{c}(r_1, ..., r_k) \in \{1, *\}^k, \\ \text{for all } k \in \mathbb{N}\end{array}\right\},$$

since $X_{(n)}$ are not self-adjoint in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

However, by (96) and (97), one has that

$$\tau_A \left(X_{(n)}^{r_1} X_{(n)}^{r_2} ... X_{(n)}^{r_k} \right) = \tau_A \left(X^{r_1} X^{r_2} ... X^{r_k} \right),$$

for all $(r_1, ..., r_k) \in \{1, *\}$, for all $k \in \mathbb{N}$, for any $n \in \mathbb{N}_0$.

Therefore, the free distributions of $X_{(n)}$ are identically same with the free distribution of X in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. □

The above theorem shows that the n-prime shifts G_A^n preserve the free probability on the semicircular A-tensor filterization $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

Corollary 10.5. *The $*$-homomorphisms, the n-prime shifts, G_A^n preserve the free probability on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.*

Proof. Note that all elements T of $\mathbb{LS}_A$ are the limits of linear combinations of free reduced words in the generator set $\mathcal{X}_A$ of $\mathbb{LS}_A$, by (77). And, by (98), the free distributions for $G_A^n(\mathcal{X}_A)$ are identical to those for $\mathcal{X}_A$ in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. Therefore, the free distributions for

$$G_A^n \left(\text{free words in } \mathcal{X}_A\right)$$

are identical to those for free words in $\mathcal{X}_A$, by (88) and (90), in $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.

It guarantees that free distribution of every element T is identical to the free distribution of $G_A^n(T)$, for all $T \in \mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. Equivalently, the free probability on $\mathbb{LS}_A$ is preserved by the actions of n-prime shifts $\{G_A^n\}_{n \in \mathbb{N}_0}$. □

10.3. **Free-homomorphisms on $\mathbb{LS}_A$.** In this section, motivated by the main results (93) and (98) of Section 10.2, we consider free-homomorphic relations on our semicircular A-tensor filterization $\mathbb{LS}_A$ under n-prime shifts G_A^n, for $n \in \mathbb{N}_0$.

Definition 10.3. Let (B_1, φ_1), and (B_2, φ_2) be topological $*$-probability spaces. Suppose there exists a bounded $*$-homomorphism $\Phi : B_1 \to B_2$, and assume that

$$\varphi_2 \left(\Phi(b)\right) = \varphi_1(b), \textit{ for all } b \in B_1. \tag{99}$$

Then the topological $*$-probability space (B_1, φ_1) is said to be free-homomorphic to (B_2, φ_2). In particular, a $*$-homomorphism Φ is called a free-($*$-)homomorphism from (B_1, φ_1) to (B_2, φ_2).

If Φ is a $*$-isomorphism satisfying (99), then (B_1, φ_1) is said to be free-isomorphic to (B_2, φ_2). In such a case, this $*$-isomorphism Φ is called a free-($*$-)isomorphism.

By the above free-homomorphic relation (99), one can get the following result.

Theorem 10.6. *Let $\mathbb{LS}_A$ be the semicircular A-tensor filterization of (A, φ_A). Then*

(100) *the n-prime shifts G_A^n are free-homomorphisms on $\mathbb{LS}_A$, $\forall n \in \mathbb{N}_0$.*

Proof. For any arbitrarily fixed $n \in \mathbb{N}_0$, take the n-prime shift G_A^n on $\mathbb{LS}_A$. Then, by (93) and (98), for any free reduced words W of $\mathbb{LS}_A$ in the generator set $\mathcal{X}_A$, the free distributions of $G_A^n(W)$ are identical to the free distribution of W in $\mathbb{LS}_A$. Thus, the $*$-homomorphisms G_A^n preserve the free probability on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$, i.e., the statement (100) holds true. □

11. **Shifts on $\mathbb{Z}$ acting on $\mathbb{LS}_A$.** Throughout this section, fix a unital C^*-probability space $(A,\ \varphi_A)$, and the corresponding semicircular A-tensor filterization $\mathbb{LS}_A = (\mathbb{LS}_A,\ \tau_A)$ of $(A,\ \varphi_A)$. In Section 10, we defined the n-prime shifts G_A^n of (92), which are the $*$-homomorphism on $\mathbb{LS}_A$, and showed that they are free-homomorphisms on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$, by (100).

In this section, we consider certain shifting processes $h_\pm$ on $\mathbb{Z}$, and the corresponding $*$-homomorphisms $\beta_\pm$ on $\mathbb{LS}_A$.

11.1. **Shifts $h_\pm$ on $\mathbb{Z}$.** Let $\mathbb{Z}$ be the set of all integers as usual. Define functions h_+ and h_- on $\mathbb{Z}$ by the bijections on $\mathbb{Z}$,

$$h_+(j) = j+1, \text{ and } h_-(j) = j-1, \tag{101}$$

for all $j \in \mathbb{Z}$. By the definition (101), one can have

$$h_+ \circ h_- = id_{\mathbb{Z}} = h_- \circ h_+, \tag{102}$$

where $id_{\mathbb{Z}}$ is the identity map on $\mathbb{Z}$.

Definition 11.1. Let $h_\pm$ be the bijections (101) satisfying (102). Then we call $h_\pm$, the $(\pm)$-shifts on $\mathbb{Z}$.

Let $h_\pm$ be the $(\pm)$-shifts (101) on $\mathbb{Z}$. Define the functions $h_\pm^n$ on $\mathbb{Z}$ by

$$h_\pm^n = \underbrace{h_\pm \circ h_\pm \circ \cdots \circ\ h_\pm}_{n\text{-times}}, \tag{103}$$

for all $n \in \mathbb{N}_0$, with axiomatization:

$$h_\pm^0 = id_{\mathbb{Z}} \text{ on } \mathbb{Z},$$

satisfying

$$h_+^0(j) = j = h_-^0(j), \text{ for all } j \in \mathbb{Z}.$$

Definition 11.2. Let $h_\pm^n$ be in the sense of (103), for all $n \in \mathbb{N}_0$, where $h_\pm$ are the $(\pm)$-shifts (102) on $\mathbb{Z}$. Then they are called the n-$(\pm)$-shifts on $\mathbb{Z}$, for all $n \in \mathbb{N}_0$.

By (101) and (103), the n-$(\pm)$-shifts $h_\pm^n$ satisfy

$$h_+^n(j) = j+n, \text{ for all } j \in \mathbb{Z},$$

and (104)

$$h_-^n(j) = j-n, \text{ for all } j \in \mathbb{Z},$$

for all $n \in \mathbb{N}_0$. Also, by (102), one has

$$h_+^n \circ h_-^n = id_{\mathbb{Z}} = h_-^n \circ h_+^n,\ \forall n \in \mathbb{N}_0. \tag{105}$$

11.2. **Integer-shift $*$-homomorphisms on $\mathbb{LS}_A$.** Let $h_\pm^n$ be the n-$(\pm)$-shifts (103) on $\mathbb{Z}$, satisfying (104) and (105), for $n \in \mathbb{N}_0$. We now define "multiplicative" linear transformations β_+^n and β_-^n on the semicircular A-tensor filterization $\mathbb{LS}_A$ by the morphisms satisfying

$$\beta_+^n\left(X_{p,j}^a\right) = X_{p,h_+^n(j)}^a = X_{p,j+n}^a,$$

and (106)

$$\beta_-^n\left(X_{p,j}^a\right) = X_{p,h_-^n(j)}^a = X_{p,j-n}^a,$$

with

$$\beta_\pm^0\left(X_{p,j}^a\right) = X_{p,h_\pm^0(j)}^a = X_{p,j}^a = 1_{\mathbb{LS}_A}\left(X_{p,j}^a\right),$$

for all $X_{p,j}^a \in \mathcal{X}_A$, for all $n \in \mathbb{N}_0$, where $\mathcal{X}_A$ is the generator set (82) of $\mathbb{LS}_A$ (by (77)).

By the multiplicativity of the morphisms $\beta_\pm^n$ of (106) on $\mathbb{LS}_A$, if

$$T=\prod_{l=1}^{N}\left(X_{p_l,j_l}^{a_l}\right)^{n_l}\in\mathbb{LS}_A,\text{ for }n_1,\ ...,\ n_N\in\mathbb{N},$$

is a free reduced word with its length-N (in the sense of (77)), for $X_{p_l,j_l}^{a_l}\in\mathcal{X}_A$, for $l=1,\ ...,\ N$, for $N\in\mathbb{N}$, then

$$\beta_{\pm}^{n}(T)=\beta_{\pm}^{n}\left(\prod_{l=1}^{N}\left(X_{p_l,j_l}^{a_l}\right)^{n_l}\right)$$

$$=\prod_{l=1}^{N}\beta_{\pm}^{n}\left(\left(X_{p_l,j_l}^{a_l}\right)^{n_l}\right)=\prod_{l=1}^{N}\left(\beta_{\pm}^{n}(X_{p_l,j_l}^{a_l})\right)^{n_l}$$

by the multiplicativity of $\beta_{\pm}^{n}$

$$=\prod_{l=1}^{N}\left(X_{p_l,h_{\pm}^{n}(j_l)}^{a_l}\right)^{n_l}=\prod_{l=1}^{N}\left(X_{p_l,j_l\pm n}^{a_l}\right)^{n_l},\tag{107}$$

in $\mathbb{LS}_A$, for all $n\in\mathbb{N}_0$. Also, the morphisms $\beta_{\pm}^{n}$ satisfy

$$\begin{aligned}\beta_{\pm}^{n}\left(\left(X_{p,j}^{a}\right)^{*}\right)&=\beta_{\pm}^{n}\left(X_{p,j}^{a^*}\right)=X_{p,j\pm n}^{a^*}\\&=\left(X_{p,j\pm n}^{a}\right)^{*}=\left(\beta_{\pm}^{n}(X_{p,j}^{a})\right)^{*},\end{aligned}\tag{108}$$

for all $X_{p,j}^{a}\in\mathcal{X}_A$, in $\mathbb{LS}_A$, for all $n\in\mathbb{N}_0$.

So, by (107) and (108), if W is a free reduced word of $\mathbb{LS}_A$ in $\mathcal{X}_A$, then

$$\beta_{\pm}^{n}(W^*)=\left(\beta_{\pm}^{n}(W)\right)^{*},$$

implying that

$$\beta_{\pm}^{n}(T^*)=\left(\beta_{\pm}^{n}(T)\right)^{*},\text{ for all }T\in\mathbb{LS}_A,\tag{109}$$

for all $n\in\mathbb{N}_0$.

Proposition 11.1. *Let $\beta_{\pm}^{n}$ be the n-($\pm$)-integer shifts on $\mathbb{LS}_A$, for $n\in\mathbb{N}_0$. Then they are $*$-isomorphisms on $\mathbb{LS}_A$.*

Proof. Note that the n-($\pm$)-shifts $h_{\pm}^{n}$ are bijections on $\mathbb{Z}$, for $n\in\mathbb{N}_0$. So, the restrictions $\beta_{\pm}^{n}\mid_{\mathcal{X}_A}$ of our n-($\pm$)-integer shifts (106) are bijections on the generator set $\mathcal{X}_A$, for $n\in\mathbb{N}_0$. Therefore, these morphisms $\beta_{\pm}^{n}$ of (106) are bijective on $\mathbb{LS}_A$, because of the generator-preserving property, for all $n\in\mathbb{N}_0$. Moreover, by (107) and (109), these multiplicative linear transformations $\beta_{\pm}^{n}$ are $*$-homomorphisms on $\mathbb{LS}_A$, and hence, they are $*$-isomorphisms on $\mathbb{LS}_A$, for all $n\in\mathbb{N}_0$. ☐

Definition 11.3. We call the $*$-homomorphisms $\beta_{\pm}^{n}$ of (106), the n-($\pm$)-integer-shifts on $\mathbb{LS}_A$, for all $n\in\mathbb{N}_0$. If $n=1$ in $\mathbb{N}_0$, we simply call $\beta_{\pm}=\beta_{\pm}^{1}$, the ($\pm$)-integer-shifts on $\mathbb{LS}$.

The above proposition shows a difference between our prime-shifts, and the integer-shifts on $\mathbb{LS}_A$.

Remark 11.1. Note that our n-prime shifts G_A^n are injective $*$-homomorphisms, but not $*$-isomorphisms in general. In particular, if $n\neq 0$ in $\mathbb{N}_0$, then they are not $*$-isomorphisms on $\mathbb{LS}_A$. It is easily verified because the n-shifts g^n of (86) are injective but not bijective on the TOset $\mathcal{P}$ of (84), whenever $n\neq 0$ in $\mathbb{N}_0$. It also shows that G_A^n are free-homomorphisms, but not free-isomorphisms on $\mathbb{LS}_A$, in (100), for all $n\neq 0$ in $\mathbb{N}_0$.

Now, consider how our n-($\pm$)-integer shifts $\beta_{\pm}^{n}$ affect the free probability on $\mathbb{LS}_A$, for $n\in\mathbb{N}_0$.

Theorem 11.2. *Let $n \in \mathbb{N}_0$, and $\beta_{\pm}^{n}$, the corresponding n-($\pm$)-integer shifts on the semicircular A-tensor filterization $\mathbb{LS}_A$. Then, for any $X_{p,j}^{a} \in \mathcal{X}_A$, we have*

$$\tau_A\left(\left(\beta_{\pm}^{n}(X_{p,j}^{a})\right)^{k}\right) = \omega_k c_{\frac{k}{2}} \varphi_A(a^k) = \tau_A\left(\left(X_{p,j}^{a}\right)^{k}\right), \tag{110}$$

for all $k \in \mathbb{N}$.

Proof. Under hypothesis, consider that

$$\tau_A\left(\left(\beta_{\pm}^{n}(X_{p,j}^{a})\right)^{k}\right) = \tau_A\left(\left(X_{p,j\pm n}^{a}\right)^{k}\right)$$

by (107)

$$= \varphi_A(a^k)\tau^0\left(\Theta_{p,j\pm n}^{k}\right) = \omega_k c_{\frac{k}{2}} \varphi_A(a^k)$$

by (71)

$$= \varphi_A(a^k)\tau^0\left(\Theta_{p,j}^{k}\right) = \tau_A\left(\left(X_{p,j}^{a}\right)^{k}\right),$$

for all $k \in \mathbb{N}$, for all $n \in \mathbb{N}_0$.

Therefore, the free-distributional data (110) is obtained. □

Similar to the proof of (98) and that of (100), we obtain the following theorem by (110).

Theorem 11.3. *Let $\mathbb{LS}_A$ be the semicircular A-tensor filterization, and let $\beta_{\pm}^{n}$ be the n-($\pm$)-integer shifts on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$. Then*

(111) $\beta_{\pm}^{n}$ *are free-isomorphisms on* $\mathbb{LS}_A$.

Proof. By (110), the $*$-isomorphisms $\beta_{\pm}^{n}$ preserves free distributions of generating operators of $\mathbb{LS}_A$, contained in $\mathcal{X}_A$. Therefore, by the similar arguments of the proofs of (98) and (100), the free probability on $\mathbb{LS}_A$ is preserved by the action of $\beta_{\pm}^{n}$, for all $n \in \mathbb{N}_0$. □

12. **Shifts on $\mathcal{P}\times\mathbb{Z}$ and $*$-homomorphisms on $\mathbb{LS}_A$.** In this section, we consider both prime shifts, and integer shifts, which are well-defined free-homomorphisms on the semicircular A-tensor filterization $\mathbb{LS}_A$ of a fixed unital C^*-probability space (A, φ_A). In particular, we showed that the prime shifts are injective free-homomorphisms, and the integer shifts are free-isomorphisms on $\mathbb{LS}_A$, by (100), respectively, by (111).

Now, we consider certain $*$-homomorphisms on $\mathbb{LS}_A$ induced by both prime shifts and integer shifts. From below, for convenience, we let

$$\mathbb{N}_0^{\pm} \overset{denote}{=} \{\pm\} \times \mathbb{N}_0.$$

12.1. **Shifts on $\mathbb{P} = \mathcal{P} \times \mathbb{Z}$.** Now, consider the *Cartesian product set* $\mathbb{P}$,

$$\mathbb{P} \overset{def}{=} \mathcal{P} \times \mathbb{Z}. \tag{112}$$

Let g^n be the n-shifts on $\mathcal{P}$, and let h_e^k be the k-(e)-shifts on $\mathbb{Z}$, for $n \in \mathbb{N}_0$, and $(e, k) \in \mathbb{N}_0^{\pm}$, with axiomatization,

$$g^0 = id_{\mathcal{P}}, \text{ and } h_{\pm}^{0} = id_{\mathbb{Z}}.$$

Define now shifts on the set $\mathbb{P}$ of (112) by

$$s_{(e,n_2)}^{n_1} \overset{def}{=} g^{n_1} \times h_e^{n_2} \overset{denote}{=} \left(g^{n_1},\ h_e^{n_2}\right), \tag{113}$$

for all $n_1 \in \mathbb{N}_0$, and $(e, n_2) \in \mathbb{N}_0^{\pm}$. i.e., for any $(p, j) \in \mathbb{P}$,

$$s_{(e,n_2)}^{n_1}(p, j) = (g^{n_1}(p),\ h_e^{n_2}(j)) = (g^{n_1}(p),\ jen_2)$$

in $\mathbb{P}$, where

$$jen_2 = \begin{cases} j + n_2 & \text{if } e = + \\ j - n_2 & \text{if } e = -. \end{cases}$$

For example,

$$s^2_{(-,5)}(3, \ -1) = \left(g^2(3), \ h^5_-(-1)\right) = (7, \ -6)$$

in $\mathbb{P}$.

Definition 12.1. Let $s^{n_1}_{(e,n_2)}$ be injections (113) on the set $\mathbb{P}$ of (112), for $n_1 \in \mathbb{N}_0$, and $(e, n_2) \in \mathbb{N}^{\pm}_0$, with identity,

$$s^0_{(e,0)} = id_{\mathcal{P}} \times id_{\mathbb{Z}} = id_{\mathbb{P}},$$

where $id_{\mathbb{P}}$ is the identity map on $\mathbb{P}$, satisfying

$$id_{\mathbb{P}}(p, j) = (p, \ j) \textit{ in } \mathbb{P}, \textit{ for all } (p, j) \in \mathbb{P}.$$

Then these injections $s^{n_1}_{(e,n_2)}$ are called the shift(-function)s on $\mathbb{P}$.

12.2. **Prime-integer shifts on $\mathbb{LS}_A$.** Let $\mathbb{P}$ be the Cartesian product set (112), and let $s^{n_1}_{(e,n_2)}$ be shifts (113) on $\mathbb{P}$. Then, for such a shift $s^{n_1}_{(e,n_2)}$, one can construct the corresponding $*$-homomorphism $\sigma^{n_1}_{(e,n_2)}$ on the semicircular A-tensor filterization $\mathbb{LS}_A$, defined by the bounded multiplicative linear transformation on $\mathbb{LS}_A$,

$$\sigma^{n_1}_{(e,n_2)} = G^{n_1}_A \ \beta^{n_2}_e \text{ on } \mathbb{LS}_A, \tag{114}$$

for all $n_1 \in \mathbb{N}_0$, and $(e, n_2) \in \mathbb{N}^{\pm}_0$, where G^{n_1} are the n_1-prime shifts, and $\beta^{n_2}_e$ are n_2-(e)-integer shifts on $\mathbb{LS}_A$.

Notation and Assumption. From below, for convenience, we simply write our n-prime shifts G^n_A simply by G^n, for all $n \in \mathbb{N}_0$. □

Since G^{n_1} are $*$-homomorphisms, and $\beta^{n_2}_e$ are $*$-isomorphisms on $\mathbb{LS}_A$, the morphism $\sigma^{n_1}_{(e,n_2)}$ of (114) are indeed well-defined $*$-homomorphisms on $\mathbb{LS}_A$.

Proposition 12.1. *Let $\sigma^{n_1}_{(e,n_2)}$ be a $*$-homomorphism (114) on $\mathbb{LS}_A$. Then*

$$\sigma^{n_1}_{(e,n_2)} \overset{def}{=} G^{n_1}\beta^{n_2}_e = \beta^{n_2}_e G^{n_1} \textit{ on } \mathbb{LS}_A, \tag{115}$$

for all $n_1 \in \mathbb{N}_0$, $(e, n_2) \in \mathbb{N}^{\pm}_0$.

Proof. By the very definition (114),

$$\begin{aligned}\sigma^{n_1}_{(e,n_2)}\left(X^a_{p,j}\right) &= G^{n_1}\left(\beta^{n_2}_e\left(X^a_{p,j}\right)\right) \\ &= G^{n_1}\left(X^a_{p,jen_2}\right) = X^a_{g^{n_1}(p), \ jen_2} \\ &= \beta^{n_2}_e\left(X^a_{g^{n_1}(p),j}\right) = \beta^{n_2}_e\left(G^{n_1}\left(X^a_{p,j}\right)\right) \\ &= \beta^{n_2}_e G^{n_1}\left(X^a_{p,j}\right),\end{aligned}$$

for all generating operators $X^a_{p,j} \in \mathcal{X}_A$.

Since all elements of $\mathbb{LS}_A$ are the limits of linear combinations of free reduced words in $\mathcal{X}_A$ by (77), we have

$$\sigma^{n_1}_{(e,n_2)} \overset{def}{=} G^{n_1}\beta^{n_2}_e = \beta^{n_2}_e G^{n_1} \text{ on } \mathbb{LS}_A,$$

for all $n_1 \in \mathbb{N}_0$, $(e, \ n_2) \in \mathbb{N}^{\pm}_0$. □

Let $Hom\,(\mathbb{LS}_A)$ be the $(*\text{-})$*homomorphism semigroup* acting on the semicircular A-tensor filterization $\mathbb{LS}_A$, consisting of all $*$-homomorphisms on $\mathbb{LS}_A$. Define now the subset $\sigma(\mathbb{LS}_A)$ of $Hom(\mathbb{LS}_A)$ by

$$\sigma(\mathbb{LS}_A) = \{\sigma^{n_1}_{(e,n_2)} : n_1 \in \mathbb{N}_0,\, (e, n_2) \in \mathbb{N}_0^{\pm}\}, \tag{116}$$

where $\sigma^{n_1}_{(e,n_2)}$ are the $*$-homomorphisms (114) on $\mathbb{LS}_A$.

Definition 12.2. We call the $*$-homomorphisms $\sigma^{n_1}_{(e,n_2)}$ of (114), the prime-integer shift(-$*$-homomorphism)s (in short, pi-shifts) on $\mathbb{LS}_A$.

Now, let's consider the following structure theorem of the system $\sigma(\mathbb{LS}_A)$ of (116) in the homomorphism semigroup $Hom(\mathbb{LS}_A)$.

Theorem 12.2. *Let $\sigma(\mathbb{LS}_A)$ be the system (116) in $Hom(\mathbb{LS}_A)$. Then*

(117) $\sigma(\mathbb{LS}_A)$ *is a commutative sub-monoid of $Hom(\mathbb{LS}_A)$.*

Proof. Let $\sigma(\mathbb{LS}_A)$ be the subset (116) of $Hom(\mathbb{LS}_A)$. Then one can obtain that

$$\sigma^{n_1}_{(e,n_2)}\,\sigma^{k_1}_{(r,k_2)} = \left(G^{n_1}\beta_e^{n_2}\right)\left(G^{k_1}\beta_r^{k_2}\right)$$

by (114)

$$= \left(G^{n_1}G^{k_1}\right)\left(\beta_e^{n_2}\beta_r^{k_2}\right)$$

by (115)

$$= G^{n_1+k_1}\,\beta^{|en_2+rk_2|}_{sgn(en_2+rk_2)} = \sigma^{n_1+k_1}_{sgn(en_2+rk_2)}, \tag{118}$$

where sgn is the sign map on $\mathbb{Z}$, satisfying

$$sgn(j) = \begin{cases} + & \text{if } j \geq 0 \\ - & \text{if } j < 0, \end{cases}$$

for all $j \in \mathbb{Z}$, and $|.|$ means the absolute value on $\mathbb{Z}$, for all $n_1, k_1, n_2, k_2 \in \mathbb{N}_0$, and $e,\, r \in \{\pm\}$.

The formula (118) shows that the product (or composition), inherited from that on $Hom(\mathbb{LS}_A)$, is closed on the set $\sigma(\mathbb{LS}_A)$. Thus, one can consider $\sigma(\mathbb{LS}_A)$ as an algebraic sub-structure $(\sigma(\mathbb{LS}_A),\ \cdot)$ in $Hom(\mathbb{LS}_A)$.

Observe now that

$$\left(\beta^{n_1}_{e_1}\beta^{n_2}_{e_2}\right)\beta^{n_3}_{e_3} = \beta^{|e_1n_1+e_2n_2|}_{sgn(e_1n_1+e_2n_2)}\,\beta^{n_3}_{e_3}$$

$$= \beta^{||e_1n_1+e_2n_2|+e_3n_3|}_{sgn(e_1n_1+e_2n_2+e_3n_3)} = \beta^{|e_1n_1+|e_2n_2+e_3n_3||}_{sgn(e_1n_1+e_2n_2+e_3n_3)}$$

$$= \beta^{n_1}_{e_1}\beta^{|e_2n_2+e_3n_3|}_{sgn(e_2n_2+e_3n_3)} = \beta^{n_1}_{e_1}\left(\beta^{n_2}_{e_2}\beta^{n_3}_{e_3}\right), \tag{119}$$

on $\mathbb{LS}_A$, for $(e_l, n_l) \in \mathbb{N}_0^{\pm}$, for all $l = 1,\, 2,\, 3$; also, one has

$$\begin{aligned}(G^{n_1}G^{n_2})\,G^{n_3} &= G^{n_1+n_2}G^{n_3} \\ &= G^{n_1+n_2+n_3} = G^{n_1}G^{n_2+n_3} \\ &= G^{n_1}(G^{n_2}G^{n_3}),\end{aligned} \tag{120}$$

on $\mathbb{LS}_A$, for all $n_1,\, n_2,\, n_3 \in \mathbb{N}_0$.

So, one obtains that

$$\left(\sigma^{n_1}_{(e_1,k_1)}\sigma^{n_2}_{(e_2,k_2)}\right)\sigma^{n_3}_{(e_3,k_3)}$$

$$= \sigma^{n_1+n_2}_{(sgn(e_1k_1+e_2k_2),\,|e_1k_1+e_2k_2|)}\sigma^{n_3}_{(e_3,k_3)}$$

by (118)

$$= \sigma^{(n_1+n_2)+n_3}_{(sgn(e_1k_1+e_2k_2+e_3k_3),\ ||e_1k_1+e_2k_2|+e_3k_3|)}$$

$$= \sigma^{n_1+(n_2+n_3)}_{(sgn(e_1k_1+e_2k_2+e_3k_3),\ |e_1k_1+|e_2k_2+e_3k_3||)}$$

$$= \sigma^{n_1}_{(e_1,k_1)}\ \sigma^{n_2+n_3}_{(sgn(e_2k_2+e_3k_3),\ |e_2k_2+e_3k_3|)}$$

$$= \sigma^{n_1}_{(e_1,k_1)}\left(\sigma^{n_2}_{(e_2,k_2)}\ \sigma^{n_3}_{(e_3,k_3)}\right), \tag{121}$$

by (119) and (120), for $n_l \in \mathbb{N}_0$, $(e_l,\ k_l) \in \mathbb{N}_0^{\pm}$, for all $l = 1,\ 2,\ 3$.

Thus, the operation $(\cdot)$ on $\sigma(\mathbb{LS}_A)$ is associative by (121), and hence, the algebraic pair $(\sigma(\mathbb{LS}_A),\ \cdot)$ forms a semigroup.

Definitely, one can take an element

$$\sigma^0_{(e,0)} = G^0\beta^0_e = 1_{\mathbb{LS}_A} \cdot 1_{\mathbb{LS}_A} = 1_{\mathbb{LS}_A} \in \sigma(\mathbb{LS}_A),$$

satisfying that (122)

$$\sigma^n_{(e,k)} \cdot 1_{\mathbb{LS}_A} = \sigma^n_{(e,k)} = 1_{\mathbb{LS}_A} \cdot \sigma^n_{(e,k)} \text{ in } \sigma(\mathbb{LS}_A),$$

for all $n \in \mathbb{N}_0$, and $(e,k) \in \mathbb{N}_0^{\pm}$.

So, the semigroup $(\sigma(\mathbb{LS}_A),\ \cdot)$ contains its $(\cdot)$-identity $1_{\mathbb{LS}_A} = \sigma^0_{(e,0)}$ of (122), and hence, it is a well-defined monoid in $Hom(\mathbb{LS}_A)$.

Finally, consider that

$$G^{n_1}G^{n_2} = G^{n_1+n_2} = G^{n_2+n_1} = G^{n_2}G^{n_1},$$

and (123)

$$\beta^{k_1}_{e_1}\beta^{k_2}_{e_2} = \beta^{|e_1k_1e_2k_2|}_{sgn(e_1k_1e_2k_2)} = \beta^{|e_2k_2e_1k_1|}_{sgn(e_2k_2e_1k_1)} = \beta^{k_2}_{e_2}\beta^{k_1}_{e_1},$$

on $\mathbb{LS}_A$, for all $n_1,\ n_2 \in \mathbb{N}_0$, and $(e_1,k_1),\ (e_2,k_2) \in \mathbb{N}_0^{\pm}$.

Therefore,

$$\begin{aligned} \sigma^{n_1}_{(e_1,k_1)}\sigma^{n_2}_{(e_2,k_2)} &= \sigma^{n_1+n_2}_{(sgn(e_1k_1e_2k_2),|e_1k_1e_2k_2|)} \\ &= \sigma^{n_2}_{(e_2,k_2)}\sigma^{n_1}_{(e_1,k_1)}, \end{aligned} \tag{124}$$

on $\mathbb{LS}_A$, for all $n_1,\ n_2 \in \mathbb{N}_0$, and $(e_1,k_1),\ (e_2,k_2) \in \mathbb{N}_0^{\pm}$, by (115) and (123).

So, the monoid $(\sigma(\mathbb{LS}_A),\ \cdot)$ is commutative by (124). Therefore, the system $\sigma(\mathbb{LS}_A)$ of (116) is a commutative sub-monoid of the homomorphism semigroup $Hom(\mathbb{LS}_A)$. □

The above structure theorem (117) characterizes the algebraic structure of $\sigma(\mathbb{LS}_A)$ as a commutative monoid embedded in $Hom(\mathbb{LS})$.

Definition 12.3. Let $\sigma(\mathbb{LS}_A)$ be a commutative sub-monoid (116) embedded in the homomorphism semigroup $Hom(\mathbb{LS}_A)$. Then this monoid $\sigma(\mathbb{LS}_A)$ is called the prime-integer-shift monoid (in short, the pi-shift monoid) on $\mathbb{LS}_A$.

12.3. **Free-distributional data on $\mathbb{LS}_A$ affected by $\sigma(\mathbb{LS}_A)$.** Let $\mathbb{LS}_A$ be the fixed semicircular A-tensor filterization of (A, φ_A), and let $\sigma(\mathbb{LS}_A)$ be the pi-shift monoid (116) on $\mathbb{LS}_A$, which is a commutative sub-monoid of the homomorphism semigroup $Hom(\mathbb{LS}_A)$ by (117). In this section, we consider how pi-shift monoid $\sigma(\mathbb{LS}_A)$ affects the free-distributional data on $\mathbb{LS}_A$.

Recall-and-note that the prime-shifts G^n are injective free-homomorphisms on $\mathbb{LS}_A$, and hence, they preserves the free probability on $\mathbb{LS}_A$ by (100), for all $n \in \mathbb{N}_0$; and the integer-shifts β_e^n are free-isomorphisms on $\mathbb{LS}_A$, and hence, they preserves the free probability on $\mathbb{LS}_A$, by (111), for all $(e, n) \in \mathbb{N}_0^{\pm}$. So, it is not difficult to verify that every pi-shift $\sigma_{(e,k)}^n \in \sigma(\mathbb{LS}_A)$ preserves the free probability on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$ and $(e, k) \in \mathbb{N}_0^{\pm}$.

Lemma 12.3. *Let $\sigma(\mathbb{LS}_A)$ be the pi-shift monoid (116) on the semicircular A-tensor filterization $\mathbb{LS}_A$, and let*

$$\sigma \overset{denote}{=} \sigma_{(e,k)}^n \in \sigma(\mathbb{LS}_A), \text{ for } n \in \mathbb{N}_0,\ (e,k) \in \mathbb{N}_0^{\pm},$$

be a pi-shift on $\mathbb{LS}_A$. Then

$$\tau_A\left(\left(\sigma(X_{p,j}^a)\right)^l\right) = \omega_l c_{\frac{l}{2}} \varphi_A(a^l) = \tau_A\left(\left(X_{p,j}^a\right)^l\right), \tag{125}$$

for all $l \in \mathbb{N}$.

Proof. Let $\sigma = \sigma_{(e,k)}^n \in \sigma(\mathbb{LS}_A)$ be a pi-shift, for $n \in \mathbb{N}_0$, $(e,k) \in \mathbb{N}_0^{\pm}$. Then, for any generating operator $X_{p,j}^a \in \mathcal{X}_A$ of $\mathbb{LS}_A$, one has

$$\sigma\left(X_{p,j}^a\right)^l = \left(X_{g^n(p),jek}^a\right)^l,$$

and hence,

$$\tau_A\left(\left(\sigma(X_{p,j}^a)\right)^l\right) = \tau_A\left(\left(X_{g^n(p),jek}^a\right)^l\right)$$

$$= \varphi_A(a^l)\tau^0\left(\Theta_{g^n(p),jek}^l\right) = \omega_l c_{\frac{l}{2}} \varphi_A(a^l)$$

$$= \varphi_A(a^l)\tau^0\left(\Theta_{p,j}^l\right) = \tau_A\left(\left(X_{p,j}^a\right)^l\right),$$

for all $l \in \mathbb{N}$.

Therefore, the free-distributional data (125) holds. □

By the above lemma, we obtain the following result.

Theorem 12.4. *Let $\sigma(\mathbb{LS}_A)$ be the pi-shift monoid on the semicircular A-tensor filterization $\mathbb{LS}_A$. Then every pi-shift $\sigma \in \sigma(\mathbb{LS}_A)$ is a free-homomorphism on $\mathbb{LS}_A$.*

Proof. By the similar arguments of the proofs for (100) and (111), all pi-shifts of the pi-shift monoid $\sigma(\mathbb{LS}_A)$ are free-homomorphisms on $\mathbb{LS}_A$ by (125). □

Remark that, by the definition (114), a pi-shift $\sigma_{(e,k)}^n$ is not a free-isomorphism on $\mathbb{LS}_A$, in general. In particular, if $n \neq 0$ in $\mathbb{N}_0$, then $\sigma_{(e,k)}^n = G^n \beta_e^k$ is not bijective, since G^n is not bijective on $\mathbb{LS}_A$, and hence, it cannot be a free-isomorphism.

Theorem 12.5. *Let $\sigma_{(e,k)}^n \in \sigma(\mathbb{LS}_A)$ be a pi-shift. Then*

(126) $\sigma_{(e,k)}^n$ *is a free-isomorphism, if and only if $n = 0$ in $\mathbb{N}_0$.*

Proof. ($\Leftarrow$) Suppose $n = 0$ in $\mathbb{N}_0$. Then

$$\sigma^n_{(e,k)} = \sigma^0_{(e,k)} = G^0 \beta^k_e = 1_{\mathbb{LS}_A} \beta^k_e = \beta^k_e,$$

and β^k_e is a free-isomorphism by (111), in $\sigma(\mathbb{LS}_A)$.

($\Rightarrow$) Assume that $n \neq 0$ in $\mathbb{N}_0$. Then, as we discussed in the very above paragraph, $\sigma^n_{(e,k)}$ is not a free-isomorphism on $\mathbb{LS}_A$.

Therefore, the characterization (126) holds. □

The above theorem characterizes the free-isomorphic property in the pi-shift monoid $\sigma(\mathbb{LS}_A)$.

By the above two theorems, a pi shift $\sigma^n_{(e,k)} \in \sigma(\mathbb{LS}_A)$ is either a free-homomorphism (if $n \neq 0$), or a free-isomorphism (if $n = 0$) on the semicircular A-tensor filterization $\mathbb{LS}_A$, i.e., it preserves the free probability on $\mathbb{LS}_A$.

13. **A-tensor pi-shift monoids $\sigma_A(\mathbb{LS}_A)$.** Let (A, φ_A) be a fixed unital C^*-probability space, and $\mathbb{LS}_A = (\mathbb{LS}_A, \tau_A)$, the semicircular A-tensor filterization of (A, φ_A), and let $\sigma(\mathbb{LS}_A)$ be the pi-shift monoid on $\mathbb{LS}_A$. By the main results of Section 12, all elements of $\sigma(\mathbb{LS}_A)$ are free-homomorphisms in the homomorphism semigroup $Hom(\mathbb{LS}_A)$. In this section, we generalize the pi-shift monoid $\sigma(\mathbb{LS}_A)$ by acting the homomorphism semigroup $Hom(A)$ of the C^*-algebra A, and construct a new subset $\sigma_A(\mathbb{LS}_A)$ of $Hom(\mathbb{LS}_A)$. We study how such a subset $\sigma_A(\mathbb{LS}_A)$ acts on (the free probability on) $\mathbb{LS}_A$.

13.1. **The A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$.** Let $Hom(A)$ be the homomorphism semigroup of A, consisting of all $*$-homomorphisms on the C^*-algebra A, where (A, φ_A) is our fixed unital C^*-probability space. Let $\theta \in Hom(A)$, and $\sigma^n_{(e,k)} \in \sigma(\mathbb{LS}_A)$, for $n \in \mathbb{N}_0$, $(e,k) \in \mathbb{N}^{\pm}_0$. Define a $*$-homomorphism $\sigma^{n:\theta}_{(e,k)}$ on $\mathbb{LS}_A$ by the morphism satisfying

$$\begin{aligned} \sigma^{n:\theta}_{(e,k)} \left(X^a_{p,j}\right) &= \sigma^{n:\theta}_{(e,k)} \left(a \otimes \Theta_{p,j}\right) \\ &\overset{def}{=} \sigma^n_{(e,k)} \left(\theta(a) \otimes \Theta_{p,j}\right) \\ &= \sigma^n_{(e,k)} \left(X^{\theta(a)}_{p,j}\right) = X^{\theta(a)}_{g^n(p),jek}, \end{aligned} \tag{127}$$

for all $X^a_{p,j} \in \mathcal{X}_A$ in $\mathbb{LS}_A$, where $\mathcal{X}_A$ is the generator set (82) of $\mathbb{LS}_A$.

Let $X^{a_l}_{p_l,j_l} \in \mathcal{X}_A$ in $\mathbb{LS}_A$, for $l = 1, 2$. Then, for the morphism $\sigma^{n:\theta}_{(e,k)}$ of (127), one obtains that

$$\sigma^{n:\theta}_{(e,k)} \left(X^{a_1}_{p_1,j_1} X^{a_2}_{p_2,j_2}\right) = \sigma^{n:\theta}_{(e,k)} \left(a_1 a_2 \otimes \Theta_{p_1,j_1}\Theta_{p_2,j_2}\right)$$

$$= \begin{cases} \sigma^{n:\theta}_{(e,k)} \left(a_1 a_2 \otimes \Theta_{p_1,j_1}\Theta_{p_2,j_2}\right) & \text{if } (p_1,j_1) \neq (p_2,j_2) \text{ in } \mathbb{P} \\ \sigma^{n:\theta} \left(a_1 a_2 \otimes \Theta^2_{p_1,j_1}\right) & \text{if } (p_1,j_1) = (p_2,j_2) \text{ in } \mathbb{P} \end{cases}$$

$$= \begin{cases} \theta(a_1 a_2) \otimes \Theta_{g^n(p_1),j_1ek}\Theta_{g^n(p_2),j_2ek} & \text{if } (p_1,j_1) \neq (p_2,j_2) \\ \theta(a_1 a_2) \otimes \Theta^2_{g^n(p_1),j_1ek} & \text{if } (p_1,j_1) = (p_2,j_2) \end{cases}$$

by (127)

$$= \begin{cases} \theta(a_1)\theta(a_2) \otimes \Theta_{g^n(p_1),j_1ek}\Theta_{g^n(p_2),j_2ek} & \text{resp.,} \\ \theta(a_1)\theta(a_2) \otimes \Theta_{g^n(p_1),j_1ek}\Theta_{g^n(p_1),j_{1ek}} & \end{cases}$$

since $\theta \in Hom(A)$

$$= \left(\theta(a_1) \otimes \Theta_{g^n(p_1),j_1ek}\right)\left(\theta(a_2) \otimes \Theta_{g^n(p_2),j_2ek}\right)$$

$$= \left(X^{\theta(a_1)}_{g^n(p_1),j_1ek}\right)\left(X^{\theta(a_2)}_{g^n(p_2),j_2ek}\right)$$

$$= \left(\sigma^{n:\theta}_{(e,k)}\left(X^a_{p_1,j_1}\right)\right)\left(\sigma^{n:\theta}_{(e,k)}\left(X^a_{p_2,j_2}\right)\right),$$

implying that

$$\sigma^{n:\theta}_{(e,k)}\left(T_1T_2\right) = \left(\sigma^{n:\theta}_{(e,k)}(T_1)\right)\left(\sigma^{n:\theta}_{(e,k)}(T_2)\right), \tag{128}$$

in $\mathbb{LS}_A$, for all T_1, $T_2 \in \mathbb{LS}_A$.

Also, we have, for any $X^a_{p,j} \in \mathcal{X}_A$,

$$\sigma^{n:\theta}_{(e,k)}\left(\left(X^a_{p,j}\right)^*\right) = \sigma^{n:\theta}_{(e,k)}\left(X^{a^*}_{p,j}\right) = X^{\theta(a^*)}_{g^n(p),jek}$$

by (127)

$$= X^{\theta(a)^*}_{g^n(p),jek} = \left(X^{\theta(a)}_{g^n(p),jek}\right)^*$$

since $a \in Hom(A)$

$$= \left(\sigma^{n:\theta}_{(e,k)}\left(X^a_{p,j}\right)\right)^*,$$

implying that

$$\sigma^{n:\theta}_{(e,k)}\left(T^*\right) = \left(\sigma^{n:\theta}_{(e,k)}(T)\right)^*, \tag{129}$$

for all $T \in \mathbb{LS}_A$.

Therefore, the morphism $\sigma^{n:\theta}_{(e,k)}$ of (127) is indeed a well-defined $*$-homomorphism on $\mathbb{LS}_A$, by (128) and (129), for any $\theta \in Hom(A)$, and $\sigma^n_{(e,k)} \in \sigma(\mathbb{LS}_A)$.

Define now a subset $\sigma_A(\mathbb{LS}_A)$ of the homomorphism semigroup $Hom(\mathbb{LS}_A)$ of $\mathbb{LS}_A$ by

$$\sigma_A(\mathbb{LS}_A) = \left\{ \sigma^{n:\theta}_{(e,k)} \;\middle|\; \begin{array}{c} \sigma^{n:\theta}_{(e,k)} \text{ are in the sense of (127),} \\ \text{for all } \theta \in Hom(A), \text{ and} \\ \sigma^n_{(e,k)} \in \sigma(\mathbb{LS}_A) \end{array} \right\}. \tag{130}$$

Then one can get the following structure theorem.

Theorem 13.1. *Let $\sigma_A(\mathbb{LS}_A)$ be the subset (130) of $Hom(\mathbb{LS}_A)$. Then*

(131) *$\sigma_A(\mathbb{LS}_A)$ is a noncommutative monoid, in general.*

Moreover, $\sigma_A(\mathbb{LS}_A)$ becomes a commutative sub-monoid of $Hom(\mathbb{LS}_A)$, if and only if the C^*-algebra A is commutative.

Proof. Let $\sigma_A(\mathbb{LS}_A)$ be the subset (130) of $Hom(\mathbb{LS}_A)$. Take

$$\sigma_l \overset{denote}{=} \sigma^{n_l:\theta_l}_{(e_l,k_l)} \in \sigma_A(\mathbb{LS}_A), \text{ for } l = 1, 2.$$

Observe that, for any generating operator $X^a_{p,j} \in \mathcal{X}_A$ of $\mathbb{LS}_A$,

$$\sigma_1\sigma_2\left(X_{p,j}^{a}\right)=\sigma_1\left(X_{g^{n_2}(p),je_2k_2}^{\theta_2(a)}\right)=X_{g^{n_1}(g^{n_2}(p)),je_2k_2e_1k_1}^{\theta_1(\theta_2(a))}$$

$$=X_{g^{n_1+n_2}(p),\ j(sgn(je_1k_1e_2k_2)+)|e_1k_1e_2k_2|}^{\theta_1\theta_2(a)} \tag{132}$$

$$=\sigma_{(sgn(e_1k_1e_2k_2)+,\ |e_1k_1e_2k_2|)}^{(n_1+n_2):\theta_1\theta_2}\left(X_{p,j}^{a}\right),$$

by (132). Indeed, note that if $\beta_{e_l}^{k_l}$ are the k_l-(e_l)-integer shifts (106) on $\mathbb{LS}_A$, for $(e_l,k_l)\in\mathbb{N}_0^{\pm}$, for $l=1,\,2$, then

$$\beta_{e_1}^{k_1}\beta_{e_2}^{k_2}=\beta_{sgn(e_1k_1e_2k_2)}^{|e_1k_1e_2k_2|}=\beta_{e_2}^{k_2}\beta_{e_1}^{k_1}\text{ on }\mathbb{LS}_A,$$

because

$$\beta_{e_1}^{k_1}\beta_{e_2}^{k_2}\left(X_{p,j}^{a}\right)=\beta_{e_1}^{k_1}\left(X_{p,je_2k_2}^{a}\right)=X_{p,je_2k_2e_1k_1}^{a}$$

$$=\beta_{sgn(e_2k_2e_1k_1)}^{|e_2k_2e_1k_1|}\left(X_{p,j}^{a}\right)=\beta_{sgn(e_1k_1e_2k_2)}^{|e_1k_1e_2k_2|}\left(X_{p,j}^{a}\right)$$

$$=\beta_{e_2}^{k_2}\beta_{e_1}^{k_1}\left(X_{p,j}^{a}\right), \tag{133}$$

for all $X_{p,j}^{a}\in\mathcal{X}_A$, in $\mathbb{LS}_A$.

So, the formula (132) holds by (133). It shows that

$$\sigma_1\sigma_2=\sigma_{(sgn(e_1k_1e_2k_2),\ |e_1k_1e_2k_2|)}^{(n_1+n_2):\theta_1\theta_2}\in\sigma_A(\mathbb{LS}_A), \tag{134}$$

too.

Therefore, under the inherited product, the algebraic pair $(\sigma_A(\mathbb{LS}_A),\ \cdot)$ is a well-determined algebraic sub-structure of $Hom(\mathbb{LS}_A)$. Now, let σ_1 and σ_2 be given as above in $\sigma_A(\mathbb{LS}_A)$, and let

$$\sigma_3=\sigma_{(e_3,k_3)}^{n_3:\theta_3}\in\sigma_A(\mathbb{LS}_A).$$

Then

$$(\sigma_1\sigma_2)\,\sigma_3=\left(\sigma_{(sgn(e_1k_1e_2k_2),\ |e_1k_1e_2k_2|)}^{(n_1+n_2):\theta_1\theta_2}\right)\sigma_3$$

by (134)

$$=\sigma_{(sgn((e_1k_1e_2k_2)e_3k_3),|(e_1k_1e_2k_2)e_3k_3|)}^{(n_1+n_2)+n_3:(\theta_1\theta_2)\theta_3}$$

by (134)

$$=\sigma_{(sgn(e_1k_1(e_2k_2e_3k_3)),\ |e_1k_1(e_2k_2e_3k_3)|)}^{n_1+(n_2+n_3):\theta_1(\theta_2\theta_3)}$$

$$=\sigma_1\left(\sigma_{(sgn(e_2k_2e_3k_3),\ |e_2k_2e_3k_3|)}^{(n_2+n_3):\theta_2\theta_3}\right)=\sigma_1\left(\sigma_2\sigma_3\right). \tag{135}$$

By (135), the algebraic pair $(\sigma_A(\mathbb{LS}),\ \cdot)$ forms a semigroup in $Hom(\mathbb{LS}_A)$.

Let $\mathbf{1}_A\in Hom(A)$ be the identity map on A, which is a $*$-isomorphism on A. Take $\sigma_{(e,0)}^{0:\mathbf{1}_A}$ in $\sigma_A(\mathbb{LS}_A)$. Then

$$\sigma_{(e,0)}^{0:\mathbf{1}_A}=1_{\mathbb{LS}_A},\text{ the identity map on }\mathbb{LS}_A, \tag{136}$$

which is a $*$-isomorphism in $Hom(\mathbb{LS}_A)$, satisfying that

$$\sigma\cdot 1_{\mathbb{LS}_A}=\sigma=1_{\mathbb{LS}_A}\cdot\sigma,\ \forall\sigma\in\sigma_A(\mathbb{LS}_A).$$

Therefore, by (136), the semigroup $(\sigma_A(\mathbb{LS}_A),\ \cdot)$ contains the $(\cdot)$-identity, $1_{\mathbb{LS}_A}=\sigma_{(e,0)}^{0:\mathbf{1}_A}$, and hence, it is a sub-monoid in $Hom(\mathbb{LS}_A)$. i.e., the statement (131) holds.

Definitely, by (126), an element $\sigma_{(e,k)}^{n:\mathbf{1}_A}$ is not bijective on $\mathbb{LS}_A$, whenever $n \neq 0$ in $\mathbb{N}_0$. So, the monoid $\sigma_A(\mathbb{LS}_A)$ cannot be a group in $Hom(\mathbb{LS}_A)$.

Remark that, homomorphism semigroups are not commutative in general. Since our C^*-algebra A is arbitrarily chosen, it is natural to understand the corresponding homomorphism semigroup $Hom(A)$ is not commutative, in general. Under this sense, even though the pi-shift monoid $\sigma(\mathbb{LS}_A)$ is commutative, the monoid $\sigma_A(\mathbb{LS}_A)$ is not commutative, in general.

However, by (134) and by the commutativity of our pi-shift monoid $\sigma(\mathbb{LS}_A)$, this monoid $\sigma_A(\mathbb{LS}_A)$ can be commutative, if and only if the homomorphism semigroup $Hom(A)$ of A is commutative, if and only if A is a commutative C^*-algebra. □

The above theorem characterizes the algebraic property of the subset $\sigma_A(\mathbb{LS}_A)$ of (130), as a noncommutative sub-monoid of $Hom(\mathbb{LS}_A)$ (in general).

Definition 13.1. Let $\sigma_A(\mathbb{LS}_A)$ be the sub-monoid (130) of $Hom(\mathbb{LS}_A)$. We call it the A-tensor-pi-shift monoid (acting) on $\mathbb{LS}_A$.

13.2. **Free-distribution data on $\mathbb{LS}_A$ affected by $\sigma_A(\mathbb{LS}_A)$.** In Section 12, we showed that all pi-shifts in the pi-shift monoid $\sigma(\mathbb{LS}_A)$ are free-homomorphisms on $\mathbb{LS}_A$, preserving the free probability on the semicircular A-tensor filterization $\mathbb{LS}_A$ of a fixed unital C^*-probability space $(A,\ \varphi_A)$. In Section 13.1, we extended the pi-shift monoid $\sigma(\mathbb{LS}_A)$ to the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$ in the sense of (130); and we showed there that, in general, the algebraic property of $\sigma_A(\mathbb{LS}_A)$ is different from that of $\sigma(\mathbb{LS}_A)$ in the homomorphism semigroup $Hom(\mathbb{LS}_A)$. So, it is natural to consider how the free-distributional data on $\mathbb{LS}_A$ is affected by the action of $\sigma_A(\mathbb{LS}_A)$.

First of all, one can immediately obtain the following corollary of (125) and (126).

Corollary 13.2. *Let* $\sigma = \sigma_{(e,k)}^{n:\mathbf{1}_A} \in \sigma_A(\mathbb{LS}_A)$, *where* $\mathbf{1}_A$ *is the identity* $*$*-isomorphism in* $Hom(A)$. *Then* σ *is a free-homomorphism on* $\mathbb{LS}_A$. *Moreover,* σ *is a free-isomorphism, if and only if* $n = 0$ *in* $\mathbb{N}_0$.

Proof. Let σ be given as above in the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$. Then, by definition,

$$\sigma\left(X_{p,j}^{a}\right) = X_{g^n(p),jek}^{\mathbf{1}_A(a)} = X_{g^n(p),jek}^{a} = \sigma_{(e,k)}^{n}\left(X_{p,j}^{a}\right),$$

for all generating operators $X_{p,j}^{a} \in \mathcal{X}_A$ of $\mathbb{LS}_A$, where $\sigma_{(e,k)}^{n}$ is the pi-shift contained in the pi-shift monoid $\sigma(\mathbb{LS}_A)$ in $Hom(\mathbb{LS}_A)$. Therefore, we have that

$$\sigma = \sigma_{(e,k)}^{n:\mathbf{1}_A} = \sigma_{(e,k)}^{n} \text{ on } \mathbb{LS}_A. \tag{137}$$

Therefore, by (125), σ is a free-homomorphism on $\mathbb{LS}_A$; and, by (126), it is a free-isomorphism, if and only if $n = 0$ in $\mathbb{N}_0$. □

By the above corollary, we have the following result.

Corollary 13.3. *Let* $\sigma(\mathbb{LS}_A)$ *be the pi-shift monoid, and let* $\sigma_A(\mathbb{LS}_A)$ *be the* A*-tensor pi-shift monoid in the homomorphism semigroup* $Hom(\mathbb{LS}_A)$. *Then*

(138) $\sigma(\mathbb{LS}_A)$ *is a commutative sub-monoid of* $\sigma_A(\mathbb{LS}_A)$,

in $Hom(\mathbb{LS}_A)$.

Proof. The proof of (138) is done by (137). □

The above corollaries shows that "some" elements of the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$ preserve the free probability on $\mathbb{LS}_A$, by (137) and (138).

Lemma 13.4. *Let $\sigma_\theta = \sigma_{(e,k)}^{n:\theta} \in \sigma_A(\mathbb{LS}_A)$, for $\theta \in Hom(A)$. Then*

$$\begin{aligned}\tau_A\left(\left(\sigma_\theta(X_{p,j}^a)\right)^l\right) &= \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A\left(\theta(a^l)\right)\\ &= \tau_A\left(\left(X_{p,j}^{\theta(a)}\right)^l\right),\end{aligned} \tag{139}$$

for all $l \in \mathbb{N}$.

Proof. Let σ_θ be given as above in the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$. Then, for any generating operator $X_{p,j}^a \in \mathcal{X}_A$ of $\mathbb{LS}_A$,

$$\begin{aligned}\left(\sigma_\theta\left(X_{p,j}^a\right)\right)^l &= \left(X_{g^n(p),jek}^{\theta(a)}\right)^l = \theta(a)^l \otimes \Theta_{g^n(p),jek}^l\\ &= \theta(a^l) \otimes \Theta_{g^n(p),jek}^l,\end{aligned}$$

in $\mathbb{LS}_A$, for all $l \in \mathbb{N}$, since $\theta \in Hom(A)$.

Thus, one obtains that

$$\tau_A\left(\left(\sigma_\theta(X_{p,j}^a)\right)^l\right) = \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A\left(\theta(a^l)\right),$$

for all $l \in \mathbb{N}$, by (125).

Also, one has that

$$\tau_A\left(\left(X_{p,j}^{\theta(a)}\right)^l\right) = \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A\left(\theta(a^l)\right),$$

for all $l \in \mathbb{N}$, too.

Therefore, the free-distributional data (139) holds. □

More general to (137) and (138), we obtain the following result.

Theorem 13.5. *Let $fHom(A)$ be the sub-semigroup of the homomorphism semigroup $Hom(A)$ of the fixed C^*-algebra A, defined by*

$$fHom(A) \overset{def}{=} \left\{\theta \in Hom(A) \;\middle|\; \begin{array}{c}\theta \text{ is a}\\ \text{free-homomorphism}\\ \text{on } A\end{array}\right\}, \tag{140}$$

Define a subset $\sigma_A^f(\mathbb{LS}_A)$ of the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$ by

$$\sigma_A^f(\mathbb{LS}_A) \overset{def}{=} \left\{\sigma_{(e,k)}^{n:\theta} \in \sigma_A(\mathbb{LS}_A) \,|\theta \in fHom(A)\right\}. \tag{141}$$

(142) *$\sigma_A^f(\mathbb{LS}_A)$ is a sub-monoid of $\sigma_A(\mathbb{LS}_A)$.*

(143) *All elements of $\sigma_A^f(\mathbb{LS}_A)$ are free-homomorphisms on $\mathbb{LS}_A$.*

(144) *$\sigma_{(e,k)}^{n:\theta} \in \sigma_A^f(\mathbb{LS}_A)$ is a free-isomorphism, if and only if*

$n = 0$, and θ is bijective on A.

Proof. Let $\sigma_A^f(\mathbb{LS}_A)$ be a subset (141) of $\sigma_A(\mathbb{LS}_A)$. Then, for any

$$\sigma_l = \sigma_{(e_l,k_l)}^{n_l:\theta_l} \in \sigma_A^f(\mathbb{LS}_A), \text{ for } l = 1, 2,$$

we have

$$\sigma_1\sigma_2 = \sigma_{(sgn(e_1k_1e_2k_2)),\ |e_1k_1e_2k_2|)}^{n_1+n_2:\theta_1\theta_2} \overset{let}{=} \sigma, \tag{145}$$

in $\sigma_A(\mathbb{LS}_A)$, by (134).

Remark that if θ_1 and θ_2 are free-homomorphisms on A, then

$$\varphi_A(\theta_1\theta_2(x)) = \varphi_A(\theta_1(\theta_2(x))) = \varphi_A(\theta_2(x)) = \varphi_A(x),$$

since $\theta_1,\ \theta_2 \in fHom(A)$, for $x \in (A,\ \varphi_A)$.

It shows that if $\theta_1,\ \theta_2 \in fHom(A)$, then $\theta_1\theta_2 \in fHom(A)$, where $fHom(A)$ is the subset (140) of $Hom(A)$. Therefore, the $*$-homomorphism σ of (145) is also contained in $\sigma_A^f(\mathbb{LS}_A)$, too. i.e., $\sigma_A^f(\mathbb{LS}_A)$ is a sub-semigroup of the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$. It is clear that the identity $*$-isomorphism $\mathbf{1}_A$ is contained in $fHom(A)$, and hence, the identity $\sigma_{(e,0)}^{0:\mathbf{1}_A} = 1_{\mathbb{LS}_A}$ is contained in $\sigma_A^f(\mathbb{LS}_A)$, too. Thus, $\sigma_A^f(\mathbb{LS}_A)$ forms a sub-monoid of $\sigma_A(\mathbb{LS}_A)$. Equivalently, the statement (142) holds.

Now, let $\sigma_\theta = \sigma_{(e,k)}^{n:\theta} \in \sigma_A^f(\mathbb{LS}_A)$, with $\theta \in fHom(A)$. Then

$$\begin{aligned} \tau_A\left(\left(\sigma_\theta(X_{p,j}^a)\right)^l\right) &= \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A\left(\theta(a^l)\right) \\ \text{by (139)} \qquad &= \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A\left(a^l\right) = \tau_A\left(\left(X_{p,j}^a\right)^l\right), \end{aligned} \tag{146}$$

since $\theta \in fHom(A)$, for all $l \in \mathbb{N}$, for all $X_{p,j}^a \in \mathcal{X}_A \subset \mathbb{LS}_A$.

Therefore, every element of the sub-monoid $\sigma_A^f(\mathbb{LS}_A)$ preserves the free probability on $\mathbb{LS}_A$, i.e., it is a free-homomorphism on $\mathbb{LS}_A$, by (146). So, the statement (143) is proven.

Let σ_θ be given as above in the sub-monoid $\sigma_A^f(\mathbb{LS}_A)$ of the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$. If either

$$n \neq 0 \text{ in } \mathbb{N}_0, \text{ or } \theta \text{ is not bijective on } A,$$

then σ_θ cannot be a $*$-isomorphism. i.e., it is a free-homomorphism, but not a free-isomorphism on $\mathbb{LS}_A$.

If $n = 0$, and θ is bijective on A, then θ is a free-isomorphism on A, and hence, σ_θ is bijective on $\mathbb{LS}_A$; and since

$$\sigma_\theta\left(X_{p,j}^a\right) = \sigma_{(e,k)}^{0:\theta}\left(X_{p,j}^a\right) = X_{p,jek}^{\theta(a)},$$

we have

$$\begin{aligned} \tau_A\left(\left(\sigma_\theta\left(X_{p,j}^a\right)\right)^l\right) &= \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A\left(\theta(a^l)\right) \\ &= \left(\omega_l c_{\frac{l}{2}}\right)\varphi_A(a^l) = \tau_A\left(\left(X_{p,j}^a\right)^l\right), \end{aligned}$$

for all $l \in \mathbb{N}$, for all $X_{p,j}^a \in \mathcal{X}_A \subset \mathbb{LS}_A$, and hence, it is a free-homomorphism on $\mathbb{LS}_A$. i.e., if $n = 0$ in $\mathbb{N}_0$, and $\theta \in fHom(A)$ is bijective on A, then σ_θ is a bijective free-homomorphism, a free-isomorphism, on $\mathbb{LS}_A$.

Therefore, the characterization (144) holds true. □

The above theorem generalizes the free-homomorphic properties (137) and (138) in the A-tensor pi-shift monoid $\sigma_A(\mathbb{LS}_A)$. i.e., there exists the maximal sub-monoid $\sigma_A^f(\mathbb{LS}_A)$ of $\sigma_A(\mathbb{LS}_A)$, consisting of free-homomorphisms on $\mathbb{LS}_A$ (containing the pi-shift monoid $\sigma(\mathbb{LS}_A)$), by (142) and (143). Moreover, we characterize free-isomorphic property of $\sigma_A^f(\mathbb{LS}_A)$ by (144).

Definitely, if one takes an element σ in $\sigma_A(\mathbb{LS}_A) \setminus \sigma_A^f(\mathbb{LS}_A)$, then it is a $*$-homomorphism in $Hom(\mathbb{LS}_A)$, but not a free-homomorphism on the semicircular A-tensor filterization $\mathbb{LS}_A$. In other words, such a $*$-homomorphism σ distorts the free probability on $\mathbb{LS}_A$.

REFERENCES

[1] S. Albeverio, P. E. T. Jorgensen and A. M. Paolucci, On fractional Brownian motion and wavelets, *Compl. Anal. Oper. Theo.*, **6** (2012), 33–63.

[2] S. Albeverio, P. E. T. Jorgensen and A. M. Paolucci, Multiresolution wavelet analysis of integer scale Bessel functions, *J. Math. Phy.*, **48** (2007), 073516, 24 pp.

[3] D. Alpay, P. E. T. Jorgensen and D. Levanony, On the equivalence of probability spaces, *J. Theo. Prob.*, **30** (2017), 813–841.

[4] D. Alpay, P. E. T. Jorgensen and D. P. Kimsey, Moment problems in an infinite number of variables, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **18** (2015), 1550024, 14 pp.

[5] D. Alpay and P. E. T. Jorgensen, Spectral theory for Gaussian processes: Reproducing kernels, *Random Functions and Operator Theory*, **83** (2015), 211–229.

[6] D. Alpay and P. Jorgensen, Spectral theory for Gaussian processes: Reproducing kernels, boundaries, & L_2 -wavelet generators with fractional scales, *Numb. Funct. Anal, Optim.*, **36** (2015), 1239–1285.

[7] D. Alpay, P. Jorgensen and G. Salomon, On free stochastic processes and their derivatives, *Stochastic Process. Appl.*, **124** (2014), 3392–3411.

[8] I. Cho, Free semicircular families in free product Banach $*$-algebras induced by p-adic number fields over primes p, *Compl. Anal. Oper. Theo.*, **11** (2017), 507–565.

[9] I. Cho, Adelic analysis and functional analysis on the finite adele ring, *Opuscula Math.*, **38** (2017), 139–185.

[10] I. Cho, Semicircular-like and semicircular laws on Banach $*$-probability spaces induced by dynamical systems of the finite adele ring, *Adv. Oper. Theo.*, **4** (2019), 24–70.

[11] I. Cho, p-adic free stochastic integrals for p-adic weighted-semicircular motions determined by primes p, *Libertas Math. (New S.)*, **36** (2016), 65–110.

[12] I. Cho and P. E. T. Jorgensen, Semicircular elements induced by p-adic number fields, *Opuscula Math.*, **37** (2017), 665–703.

[13] A. Connes, *Noncommutative Geometry*, Academic Press, Inc., San Diego, CA, 1994.

[14] J.-B. Bost and A. Connes, Hecke algebras, type III-factors, and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math. (N.S.)*, **1** (1995), 411–457.

[15] A. Connes, Trace formula in noncommutative Geometry and the zeroes of the Riemann zeta functions, Available at: `http://www.alainconnes.org/en/download.php`.

[16] B. Dragovich, Ya. Radyno and A. Khennikov, Distributions on adéles, *J. Math. Sci.*, **142** (2007), 2105–2112.

[17] B. Dragovich, A. Khennikov and D. Mihajiović, Linear fractional p-adic and adelic dynamical systems, *Rep. Math. Phy.*, **60** (2007), 55–68.

[18] T. L. Gillespie, *Superposition of Zeroes of Automorphic L-Functions and Functoriality*, Thesis (Ph.D.)–The University of Iowa, 2011, 75 pp.

[19] T. Gillespie and G. H. Ji, Prime number theorems for Rankin-Selberg L-functions over number fields, *Sci. China Math.*, **54** (2011), 35–46.

[20] U. Haagerup and F. Larsen, Brown's spectrial distribution measure for R-diagonal elements in finite von Neumann algebras, *J. Funct. Anal.*, **176** (2000), 331–367.

[21] P. E. T. Jorgensen and A. M. Paolucci, Markov measures and extended zeta functions, *J. Appl. Math. Comput.*, **38** (2012), 305–323.

[22] P. E. T. Jorgensen and A. M. Paolucci, States on the Cuntz algebras and p-adic random walks, *J. Aust. Math. Soc.*, **90** (2011), 197–211.

[23] I. Kaygorodov, I. Shestakov and U. Umirbaev, Free generic Poisson fields and algebras, *Comm. Alg.*, **46** (2018), 1799–1812.

[24] T. Kemp and R. Speicher, Strong Haagerup inequalities for free R-diagonal elements, *J. Funct. Anal.*, **251** (2007), 141–173.

[25] L. Makar-Limanov and I. Shestakov, Polynomials and Poisson dependence in free Poisson algebras and free Poisson fields, *J. Alg.*, **349** (2012), 372–379.

[26] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, London Mathematical Society Lecture Note Series, 335. Cambridge University Press, Cambridge, 2006.

[27] F. Rădulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group of nonsingular index, *Invent. Math.*, **115** (1994), 347–389.
[28] F. Radulescu, Free group factors and Hecke operators, *The Varied Landscape of Operator Theory, Theta Ser. Adv. Math., Theta, Bucharest*, **17** (2014), 241–257.
[29] R. Speicher, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, *Mem. Amer. Math. Soc.*, **132** (1998).
[30] R. Speicher, A conceptual proof of a basic result in the combinatorial approach to freeness, *Infinit. Dimention. Anal. Quant. Prob. Relat. Topics*, **3** (2000), 213–222.
[31] V. S. Vladimirov and I. V. Volovich, p-adic quantum mechanics, *Comm. Math. Phy.*, **123** (1989), 659–676.
[32] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, Series on Soviet and East European Mathematics, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
[33] D.-V. Voiculescu, Aspects of free analysis, *Jpn. J. Math.*, **3** (2008), 163–183.
[34] D. Voiculescu, Free probability and the von Neumann algebras of free groups, *Rep. Math. Phy.*, **55** (2005), 127–133.
[35] D. V. Voiculescu, K. J. Dykemma and A. Nica, *Free Random Variables: A Noncommutative Probability Approach to Free Products with Applications to Random Matrices, Operator Algebras and Harmonic Analysis on Free Groups*, CRM Monograph Series, 1. American Mathematical Society, Providence, RI, 1992.

Received February 2020; revised March 2020.

E-mail address: choilwoo@sau.edu