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APPROXIMATION BY MULTIVARIATE BASKAKOV–KANTOROVICH OPERATORS IN ORLICZ SPACES

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Abstract. Utilizing some properties of multivariate Baskakov–Kantorovich operators and using K-functional and a decomposition technique, the authors find two equivalent theorems between the K-functional and modulus of smoothness, and obtain a direct theorem in the Orlicz spaces.

1. **Motivations.**

For proceeding smoothly, we recall from [31] some definitions and related results.

A continuous convex function $\Phi(t)$ on $[0,\infty)$ is called a Young function if it satisfies

$$\lim_{t\to0^+}\frac{\Phi(t)}{t}=0 \quad \text{and} \quad \lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty.$$

For a Young function $\Phi(t)$, its complementary Young function is denoted by $\Psi(t)$.

It is clear that the convexity of $\Phi(t)$ can lead to $\Phi(\alpha t)\le\alpha\Phi(t)$ for $\alpha\in[0,1]$. In particular, one has $\Phi(\alpha t)<\alpha\Phi(t)$ for $\alpha\in(0,1)$.

A Young function $\Phi(t)$ is said to satisfy the Δ_2-condition, denoted by $\Phi\in\Delta_2$, if there exist $t_0\ge0$ and $C\ge1$ such that $\Phi(2t)\le C\Phi(t)$ for $t\ge t_0$.

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Throughout the paper, we shall use the following standard notations:

$$\mathbb{N}_0=\{0,1,2,\dots\},\quad \mathbb{N}=\{1,2,3,\dots\},\quad m\in\mathbb{N},$$

$$\boldsymbol{x}=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m,\quad |\boldsymbol{x}|=\sum_{i=1}^m x_i,\quad \boldsymbol{k}=(k_1,k_2,\dots,k_m)\in\mathbb{N}_0^m,$$

$$\boldsymbol{x}^k=x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m},\quad \boldsymbol{k}!=k_1!k_2!\cdots k_m!,\quad |\boldsymbol{k}|=\sum_{i=1}^m k_i,$$

$$\binom{n}{\boldsymbol{k}}=\frac{n!}{\boldsymbol{k}!(n-|\boldsymbol{k}|)!},\quad \sum_{\boldsymbol{k}=0}^{\infty}=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty},$$

$$\mathbb{R}_0^m=\{\boldsymbol{x}=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m:0\le x_i<\infty,1\le i\le m\},$$

$$D^{\boldsymbol{k}}=D_1^{k_1}D_2^{k_2}\cdots D_m^{k_m},\quad D_i^r=\frac{\partial^r}{\partial x_i^r}$$

for $r\in\mathbb{N}$.

Let $\Phi(t)$ be a Young function. We define the Orlicz class $L_\Phi(\mathbb{R}_0^m)$ as the collection of all Lebesgue measurable functions $f(\boldsymbol{x})$ on $\mathbb{R}_0^m$ such that

$$\rho(f,\Phi)=\int_{\mathbb{R}_0^m}\Phi(|f(\boldsymbol{x})|)\mathrm{d}\boldsymbol{x}<\infty.$$

We also define the Orlicz space $L_\Phi^*(\mathbb{R}_0^m)$ as the collection of all Lebesgue measurable functions $f(\boldsymbol{x})$ on $\mathbb{R}_0^m$ such that $\int_{\mathbb{R}_0^m}\Phi(|\alpha f(\boldsymbol{x})|)\mathrm{d}\boldsymbol{x}<\infty$ for some $\alpha>0$. The Orlicz space $L_\Phi^*(\mathbb{R}_0^m)$ is a Banach space under the Luxemburg norm

$$\|f\|_{(\Phi)}=\inf_{\lambda>0}\left\{\lambda:\rho\left(\frac{f}{\lambda},\Phi\right)\le 1\right\}.$$

The Orlicz norm $\|f\|_\Phi$ on $L_\Phi^*(\mathbb{R}_0^m)$, which is equivalent to the Luxemburg norm on $L_\Phi^*(\mathbb{R}_0^m)$, is given by

$$\|f\|_\Phi=\sup_{\rho(g,\Psi)\le 1}\left|\int_{\mathbb{R}_0^m}f(\boldsymbol{x})g(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|$$

and satisfies

$$\|f\|_{(\Phi)}\le\|f\|_\Phi\le 2\|f\|_{(\Phi)}. \tag{1}$$

Throughout this paper, we use C to denote a constant, which may be not necessarily the same in different cases, independent of n and $\boldsymbol{x}$.

For $\boldsymbol{x}\in\mathbb{R}_0^m$, we introduce weight functions $\varphi(x)=\sqrt{x(1+x)}$ for $m=1$ and $\varphi_i(\boldsymbol{x})=\sqrt{x_i(1+|\boldsymbol{x}|)}$ for $m>1$ and $1\le i\le m$. We also define weighted Sobolev space

$$W_\varphi^{r,\Phi}(\mathbb{R}_0^m)=\left\{f\in L_\Phi^*(\mathbb{R}_0^m):D^{\boldsymbol{k}}f\in A.C.loc\left(\overset{\circ}{\mathbb{R}_0^m}\right),\ \varphi_i^rD_i^rf\in L_\Phi^*(\mathbb{R}_0^m)\right\},$$

where $|\boldsymbol{k}|\le r$ and $\overset{\circ}{\mathbb{R}_0^m}$ is the interior of $\mathbb{R}_0^m$.

The Peetre K-functional is defined in [6] by

$$K_{r,\varphi}\big(f,t^r\big)_\Phi=\inf\left\{\|f-g\|_\Phi+t^r\sum_{i=1}^m\|\varphi_i^rD_i^rg\|_\Phi:g\in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)\right\}$$

for $t>0$. Now we define the modified K-functional as

$$\overline{K}_{r,\varphi}(f,t^r)_\Phi = \inf\left\{\|f-g\|_\Phi + t^r\sum_{i=1}^m \|\varphi_i^r D_i^r g\|_\Phi + t^{2r}\sum_{i=1}^m \|D_i^r g\|_\Phi : g\in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)\right\}$$

for $t>0$.

For any vector $\boldsymbol{e}\in\mathbb{R}^m$, we write

$$\Delta_{h\boldsymbol{e}}^r f(\boldsymbol{x}) = \begin{cases} \displaystyle\sum_{i=0}^r \binom{r}{i}(-1)^i f(\boldsymbol{x}+ih\boldsymbol{e}), & \boldsymbol{x}, \boldsymbol{x}+rh\boldsymbol{e}\in\mathbb{R}_0^m \\ 0, & \text{otherwise} \end{cases}$$

for the rth forward difference of a function f in the direction of $\boldsymbol{e}$. We define the modulus of smoothness of $f\in L_\Phi^*(\mathbb{R}_0^m)$ as

$$\omega_{r,\varphi}(f,t)_\Phi = \sup_{0<h\le t}\sum_{i=1}^m \|\Delta_{h\varphi_i\boldsymbol{e}_i}^r f\|_\Phi.$$

Let

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad x\in[0,\infty), \quad n\in\mathbb{N}.$$

The well known Baskakov operators [3] were defined by

$$B_n(f,x) = \sum_{k=0}^\infty p_{n,k}(x) f\left(\frac{k}{n}\right)$$

which can be used to approximate any function f defined on $[0,\infty)$. In order to consider the approximation in $L_p[0,\infty)$, Ditzian and Totik [8] modified the form of the Baskakov operators as

$$V_{n,1}(f,x) = \sum_{k=0}^\infty p_{n,k}(x)n\int_{k/n}^{(k+1)/n} f(u)\mathrm{d}u$$

which are called Baskakov–Kantorovich's operators. There are many approximation results about one variable operator of the Baskakov type in $C[0,\infty)$ or $L_p[0,\infty)$. See [1, 2, 3, 8, 9, 10, 11, 13, 14, 15, 26, 33, 34, 35] and closely related references therein.

The multivariate Baskakov–Kantorovich's operators [5] were defined by

$$V_{n,m}(f,\boldsymbol{x}) = \sum_{\boldsymbol{k}=0}^\infty p_{n,\boldsymbol{k}}(\boldsymbol{x})Q_{n,\boldsymbol{k}}(f),$$

where

$$p_{n,\boldsymbol{k}}(\boldsymbol{x}) = \binom{n+|\boldsymbol{k}|-1}{\boldsymbol{k}}\frac{\boldsymbol{x}^{\boldsymbol{k}}}{(1+|\boldsymbol{x}|)^{n+|\boldsymbol{k}|}}$$

and

$$\begin{aligned} Q_{n,\boldsymbol{k}}(f) &= n^m\int_{k_1/n}^{(k_1+1)/n}\int_{k_2/n}^{(k_2+1)/n}\cdots\int_{k_m/n}^{(k_m+1)/n} f(u_1,u_2,\ldots,u_m)\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_m \\ &\triangleq n^m\int_{\boldsymbol{k}/n}^{(\boldsymbol{k}+1)/n} f(\boldsymbol{u})\mathrm{d}\boldsymbol{u}. \end{aligned}$$

There are few results about multivariate Baskakov type operators. Cao and An introduced in [4] multivariate Baskakov–Durrmeyer operators and obtained a direct inequality in $L_p[0,\infty)$. Cao and Ding [5] established a direct theorem of multivariate Baskakov–Kantorovich operators in $L_p[0,\infty)$ as

$$\|V_{n,m}(f)-f\|_p \le C\left[\omega_{2,\varphi}\left(f,\frac{1}{n^{1/2}}\right)_p+\frac{\|f\|_p}{n}\right].$$

For more information on approximation properties for operators in the Orlicz, Morrey, Baskakov–Durrmeyer–Stancu, or other type spaces, we recommend three groups of references, [21, 22, 23, 24, 25], [16, 17, 18, 19, 20], [12, 27, 28, 29, 30], to interested readers.

In this paper, basing on the above conclusions, utilizing K-functional and a decomposition technique, considering properties of multivariate Baskakov–Kantorovich operators in the form of Lemmas 2.1 to 2.4 in Section 2, we establish two equivalent theorems, Theorems 3.1 and 3.2 in Section 3, between the K-functional and modulus of smoothness, and obtain a direct theorem, Theorem 4.1 in Section 4, in the Orlicz spaces $L_\Phi^*(\mathbb{R}_0^m)$.

2. **Lemmas.** In order to prove the direct theorem, we need several lemmas below.

Lemma 2.1. *Let $f\in L_\Phi^*(\mathbb{R}_0^m)$, $n>m$. Then*

$$\|V_{n,m}(f)\|_\Phi \le C\|f\|_\Phi.$$

Proof. By the decomposition formula

$$\begin{aligned}V_{n,m}(f,\boldsymbol{x})&=\sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\mathrm{d}u_1\sum_{k_2=0}^\infty p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)\\ &\times n\int_{k_2/n}^{(k_2+1)/n}\mathrm{d}u_2\cdots\sum_{k_m=0}^\infty p_{n+k_1+\cdots+k_{m-1},k_m}\left(\frac{x_m}{1+x_1+\cdots+x_{m-1}}\right)\\ &\times n\int_{k_m/n}^{(k_m+1)/n}f(u_1,u_2,\cdots,u_m)\mathrm{d}u_m,\end{aligned}$$

Jensen's inequality, and the double inequality (1), we obtain

$$\begin{aligned}\|V_{n,m}(f)\|_\Phi&\le 2\|V_{n,m}(f)\|_{(\Phi)}\\ &=2\inf_{\lambda>0}\left\{\lambda:\int_{\mathbb{R}_0^m}\Phi\left(\frac{1}{\lambda}\left|\sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\mathrm{d}u_1\right.\right.\right.\\ &\times\sum_{k_2=0}^\infty p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)n\int_{k_2/n}^{(k_2+1)/n}\mathrm{d}u_2\cdots\\ &\times\sum_{k_m=0}^\infty p_{n+k_1+\cdots+k_{m-1},k_m}\left(\frac{x_m}{1+x_1+\cdots+x_{m-1}}\right)\\ &\left.\left.\left.\times n\int_{k_m/n}^{(k_m+1)/n}f(u_1,u_2,\cdots,u_m)\mathrm{d}u_m\right|\right)\mathrm{d}\boldsymbol{x}\le 1\right\}\\ &\le 2\inf_{\lambda>0}\left\{\lambda:\int_{\mathbb{R}_0^m}\sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\mathrm{d}u_1\sum_{k_2=0}^\infty p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&\times n\int_{k_2/n}^{(k_2+1)/n}\mathrm{d}u_2\cdots\sum_{k_m=0}^{\infty}p_{n+k_1+\cdots+k_{m-1},k_m}\Big(\frac{x_m}{1+x_1+\cdots+x_{m-1}}\Big)\\
&\qquad\times n\int_{k_m/n}^{(k_m+1)/n}\Phi\Big(\frac{1}{\lambda}|f(u_1,u_2,\cdots,u_m)|\Big)\mathrm{d}u_m\mathrm{d}\boldsymbol{x}\le 1\Big\}\\
&=2\inf_{\lambda>0}\Big\{\lambda:\sum_{k_1=0}^{\infty}\int_0^{\infty}p_{n,k_1}(x_1)(1+x_1)\mathrm{d}x_1 n\int_{k_1/n}^{(k_1+1)/n}\mathrm{d}u_1\sum_{k_2=0}^{\infty}\\
&\times\int_0^{\infty}p_{n+k_1,k_2}\Big(\frac{x_2}{1+x_1}\Big)(1+x_1+x_2)\mathrm{d}\Big(\frac{x_2}{1+x_1}\Big)n\int_{k_2/n}^{(k_2+1)/n}\mathrm{d}u_2\cdots\\
&\times\sum_{k_{m-1}=0}^{\infty}\int_0^{\infty}p_{n+k_1+\cdots+k_{m-2},k_{m-1}}\Big(\frac{x_{m-1}}{1+x_1+\cdots+x_{m-2}}\Big)(1+x_1+\cdots+x_{m-1})\\
&\qquad\times\mathrm{d}\Big(\frac{x_{m-1}}{1+x_1+\cdots+x_{m-2}}\Big)n\int_{k_{m-1}/n}^{(k_{m-1}+1)/n}\mathrm{d}u_{m-1}\sum_{k_m=0}^{\infty}\\
&\times\int_0^{\infty}p_{n+k_1+\cdots+k_{m-1},k_m}\Big(\frac{x_m}{1+x_1+\cdots+x_{m-1}}\Big)\mathrm{d}\Big(\frac{x_m}{1+x_1+\cdots+x_{m-1}}\Big)\\
&\qquad\times n\int_{k_m/n}^{(k_m+1)/n}\Phi\Big(\frac{1}{\lambda}|f(u_1,u_2,\cdots,u_m)|\Big)\mathrm{d}u_m\le 1\Big\}\\
&=2\inf_{\lambda>0}\Big\{\lambda:\sum_{k_1=0}^{\infty}\frac{n^m}{(n-1)\cdots(n-m)}\int_{k_1/n}^{(k_1+1)/n}\mathrm{d}u_1\\
&\times\sum_{k_2=0}^{\infty}\int_{k_2/n}^{(k_2+1)/n}\mathrm{d}u_2\cdots\sum_{k_m=0}^{\infty}\int_{k_m/n}^{(k_m+1)/n}\Phi\Big(\frac{1}{\lambda}|f(u_1,u_2,\cdots,u_m)|\Big)\mathrm{d}u_m\le 1\Big\}\\
&\le 2\inf_{\lambda>0}\Big\{\lambda:\int_0^{\infty}\mathrm{d}u_1\int_0^{\infty}\mathrm{d}u_2\cdots\int_0^{\infty}\Phi\Big(\frac{C}{\lambda}|f(u_1,u_2,\cdots,u_m)|\Big)\mathrm{d}u_m\le 1\Big\}\\
&\qquad=C\|f\|_{(\Phi)}\le C\|f\|_{\Phi}.
\end{aligned}
$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2 ([24])**.** *For $f\in L_\Phi^*[0,\infty)$ and $\Psi\in\Delta_2$, we have*

$$\|\theta(f)\|_\Phi\le C\|f\|_\Phi,$$

where

$$\theta(f,x)=\sup_{\substack{0\le t<\infty\\ t\ne x}}\Big[\frac{1}{t-x}\int_x^t f(u)\mathrm{d}u\Big]$$

is the Hardy-Littlewood function of $f(x)$.

Lemma 2.3. *Let $f\in L_\Phi^*[0,\infty)$. Then*

$$|B_n(f,x)-f(x)|\le\frac{C}{n}\theta\big(\varphi^2|f''|,x\big).$$

Proof. By Taylor's formula

$$f(t)=f(x)+f'(x)(t-x)$$

$$+\int_0^{t-x}\frac{t-x-\tau}{(x+\tau)(1+x+\tau)}(x+\tau)(1+x+\tau)f''(x+\tau)\mathrm{d}\tau$$

and the inequality

$$\left|\frac{t-x-\tau}{(x+\tau)(1+x+\tau)}\right|\le\begin{cases}\dfrac{4|t-x|}{x(1+x)}, & t\ge\dfrac{x}{2}, x\le 1\\ \dfrac{2|t-x|}{x(1+t)}, & 0\le t<\dfrac{x}{2}, x>1\end{cases}$$

in [32, Eq. (6.1)], one acquires

$$\begin{aligned}|B_n(f,x)-f(x)|&\le CB_n\left(\frac{(t-x)^2}{\varphi^2(x)}\max\left\{1,\frac{1+x}{1+t}\right\},x\right)\theta\big(\varphi^2|f''|,x\big)\\&\le\frac{C}{n}\theta\big(\varphi^2|f''|,x\big),\end{aligned}$$

where we used

$$B_n(1,x)=1,\quad B_n(t-x,x)=0,\quad B_n\big((t-x)^2,x\big)=\frac{\varphi^2(x)}{n},$$

and

$$B_n\left(\frac{(t-x)^2}{\varphi^2(x)}\max\left\{1,\frac{1+x}{1+t}\right\},x\right)\le\frac{C}{n}.$$

The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Let $f\in L_\Phi^*(\mathbb{R}_0^2)$ and $\Psi\in\Delta_2$. Then*

$$\|V_{n,2}(f)-f\|_\Phi\le\frac{C}{n}\left(\|f\|_\Phi+\sum_{i=1}^{2}\|\varphi_i^2D_i^2f\|_\Phi\right).\tag{2}$$

Proof. Let

$$z=\frac{x_2}{1+x_1}\quad\text{and}\quad g_{u_1}(t)=f\big(u_1,(1+u_1)t\big)$$

for $0\le t<\infty$. Utilizing the decomposition formula

$$\begin{aligned}V_{n,2}(f,\boldsymbol{x})=&\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)\\&\times n\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}f\big(u_1,(1+u_1)u_2\big)(1+u_1)\mathrm{d}u_2\mathrm{d}u_1\end{aligned}$$

in [5] concludes

$$\begin{aligned}V_{n,2}(f,\boldsymbol{x})-f(\boldsymbol{x})=&\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}(z)n\\&\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}[g_{u_1}(t)-g_{u_1}(z)](1+u_1)\mathrm{d}t\mathrm{d}u_1\\&+V_{n,1}\left(f\left(u_1,(1+u_1)\frac{x_2}{1+x_1}\right)\right)-f(x_1,x_2)\\&\triangleq J_1+J_2,\quad 0\le u_1<\infty.\end{aligned}\tag{3}$$

Now we start out to estimate J_1. Using Jensen's inequality and the convexity of $\Phi(t)$, it follows

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \Phi\Big(\frac{1}{\lambda}|J_1|\Big)\mathrm{d}x_1\mathrm{d}x_2 = \int_0^\infty\int_0^\infty \Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\\
&\sum_{k_2=0}^\infty p_{n+k_1,k_2}\Big(\frac{x_2}{1+x_1}\Big)n(1+u_1)\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}[g_{u_1}(t)-g_{u_1}(z)]\mathrm{d}t\mathrm{d}u_1\Bigg|\Bigg)\mathrm{d}x_1\mathrm{d}x_2\\
&\le \int_0^\infty\int_0^\infty\sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}\Big(\frac{x_2}{1+x_1}\Big)n(1+u_1)\\
&\qquad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}[g_{u_1}(t)-g_{u_1}(z)]\mathrm{d}t\Bigg|\Bigg)\mathrm{d}u_1\mathrm{d}x_1\mathrm{d}x_2\\
&= \int_0^\infty\int_0^\infty\sum_{k_1=0}^\infty p_{n,k_1}(x_1)(1+x_1)n\int_{k_1/n}^{(k_1+1)/n}\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)n(1+u_1)\\
&\qquad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}[g_{u_1}(t)-g_{u_1}(z)]\mathrm{d}t\Bigg|\Bigg)\mathrm{d}u_1\mathrm{d}x_1\mathrm{d}z\\
&= \sum_{k_1=0}^\infty\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)n(1+u_1)\\
&\qquad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}[g_{u_1}(t)-g_{u_1}(z)]\mathrm{d}t\Bigg|\Bigg)\mathrm{d}z\mathrm{d}u_1\\
&= \sum_{k_1=0}^\infty\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)n(1+u_1)\\
&\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\Bigg[g_{u_1}\Big(\frac{k_2}{n+k_1}\Big)-g_{u_1}(z)+\int_{k_2/(n+k_1)}^t g'_{u_1}(s)\mathrm{d}s\Bigg]\mathrm{d}t\Bigg|\Bigg)\mathrm{d}z\mathrm{d}u_1\\
&\le\frac12\sum_{k_1=0}^\infty\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\Bigg\{\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)2n(1+u_1)\\
&\qquad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\Bigg[g_{u_1}\Big(\frac{k_2}{n+k_1}\Big)-g_{u_1}(z)\Bigg]\mathrm{d}t\Bigg|\Bigg)+\\
&\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)2n(1+u_1)\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\int_{k_2/(n+k_1)}^t g'_{u_1}(s)\mathrm{d}s\mathrm{d}t\Bigg|\Bigg)\Bigg\}\mathrm{d}z\mathrm{d}u_1\\
&=\frac12\sum_{k_1=0}^\infty\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\Bigg\{\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)2\Bigg[g_{u_1}\Big(\frac{k_2}{n+k_1}\Big)\\
&\qquad-g_{u_1}(z)\Bigg]\Bigg|\Bigg)+\Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_2=0}^\infty p_{n+k_1,k_2}(z)2n(1+u_1)\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\\
&\qquad\times\int_{k_2/(n+k_1)}^t g'_{u_1}(s)\mathrm{d}s\mathrm{d}t\Bigg|\Bigg)\Bigg\}\mathrm{d}z\mathrm{d}u_1 \triangleq J_{11}+J_{12}.
\end{aligned}\tag{4}$$

Employing Lemmas 2.2 and 2.3 yields

$$
\begin{aligned}
J_{11} &= \frac{1}{2}\sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty \Phi\biggl(\frac{2}{\lambda}\bigl|B_{n+k_1}(g_{u_1},z)-g_{u_1}(z)\bigr|\biggr)\mathrm{d}z\mathrm{d}u_1\\
&\le \frac{1}{2}\sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty \Phi\biggl(\frac{1}{\lambda}\frac{C}{n+k_1}\theta(\varphi^2|g''_{u_1}|,z)\biggr)\mathrm{d}z\mathrm{d}u_1\\
&\le \frac{1}{2}\sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty \Phi\biggl(\frac{C}{\lambda(n+k_1)}\varphi^2(z)|g''_{u_1}(z)|\biggr)\mathrm{d}z\mathrm{d}u_1.
\end{aligned}
$$

On the other hand, by definition, we can deduce

$$
\varphi^2(t)g''_{u_1}(t)=t(1+t)(1+u_1)^2D_2^2f(u_1,(1+u_1)t)=\bigl(\varphi_2^2D_2^2f\bigr)\bigl(u_1,(1+u_1)t\bigr)
$$

and

$$
\begin{aligned}
J_{11} &\le \frac{1}{2}\sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\\
&\qquad\Phi\biggl(\frac{C}{\lambda(n+k_1)}\bigl|\bigl(\varphi_2^2D_2^2f\bigr)\bigl(u_1,(1+u_1)z\bigr)\bigr|\biggr)\mathrm{d}z\mathrm{d}u_1\\
&\le \sum_{k_1=0}^{\infty}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\frac{1}{1+u_1}\Phi\biggl(\frac{C}{\lambda n}\bigl|\bigl(\varphi_2^2D_2^2f\bigr)\bigl(u_1,(1+u_1)z\bigr)\bigr|\biggr)\mathrm{d}\bigl((1+u_1)z\bigr)\mathrm{d}u_1\\
&\le \int_0^\infty\int_0^\infty\Phi\biggl(\frac{C}{\lambda n}\bigl|\bigl(\varphi_2^2D_2^2f\bigr)(u_1,u_2)\bigr|\biggr)\mathrm{d}u_1\mathrm{d}u_2. \qquad (5)
\end{aligned}
$$

Using Jensen's inequality, we derive

$$
\begin{aligned}
J_{12} &= \frac{1}{2}\sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\Phi\Biggl(\frac{1}{\lambda}\Biggl|\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}(z)n(1+u_1)\\
&\quad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\int_{k_2/(n+k_1)}^{t}2g'_{u_1}(s)\mathrm{d}s\mathrm{d}t\Biggr|\Biggr)\mathrm{d}z\mathrm{d}u_1\\
&\le \sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\int_0^\infty\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}(z)n(1+u_1)\\
&\quad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\Phi\biggl(\frac{1}{\lambda}\biggl|\int_{k_2/(n+k_1)}^{t}2g'_{u_1}(s)\mathrm{d}s\biggr|\biggr)\mathrm{d}t\mathrm{d}z\mathrm{d}u_1\\
&= \sum_{k_1=0}^{\infty}\frac{n}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\sum_{k_2=0}^{\infty}n(1+u_1)\\
&\quad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\Phi\biggl(\frac{1}{\lambda}\biggl|\int_{k_2/(n+k_1)}^{t}2g'_{u_1}(s)\mathrm{d}s\biggr|\biggr)\mathrm{d}t\mathrm{d}u_1\\
&\le \sum_{k_1=0}^{\infty}\frac{n^2}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\sum_{k_2=0}^{\infty}(1+u_1)\\
&\quad\times\int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]}\Phi\biggl(\frac{1}{\lambda}\biggl|\int_{k_2/(n+k_1)}^{(k_2+1)/[n(1+u_1)]}2g'_{u_1}(s)\mathrm{d}s\biggr|\biggr)\mathrm{d}t\mathrm{d}u_1
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k_1=0}^{\infty} \frac{n^2}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} \frac{1}{n} \Phi\left(\frac{1}{\lambda}\left|\int_{k_2/(n+k_1)}^{(k_2+1)/[n(1+u_1)]} 2g'_{u_1}(s)\mathrm{d}s\right|\right)\mathrm{d}u_1 \\
&\le \sum_{k_1=0}^{\infty} \frac{n^2}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} \frac{n+k_1}{n} \\
&\quad \times \int_{k_2/(n+k_1)}^{(k_2+1)/(n+k_1)} \Phi\left(\frac{2|g'_{u_1}(s)|}{\lambda(n+k_1)}\right)\mathrm{d}s\mathrm{d}u_1 \\
&\le \sum_{k_1=0}^{\infty} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} \int_{k_2/(n+k_1)}^{(k_2+1)/(n+k_1)} \Phi\left(\frac{2n}{\lambda(n-1)(n-2)}|g'_{u_1}(s)|\right)\mathrm{d}s\mathrm{d}u_1 \\
&\le \sum_{k_1=0}^{\infty} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} \int_{k_2/(n+k_1)}^{(k_2+1)/(n+k_1)} \Phi\left(\frac{C}{\lambda n}|g'_{u_1}(s)|\right)\mathrm{d}s\mathrm{d}u_1 \\
&\le \int_0^{\infty}\int_0^{\infty} \Phi\left(\frac{C}{\lambda n}|g'_{u_1}(u_2)|\right)\mathrm{d}u_2\mathrm{d}u_1.
\end{aligned}
$$

From Lemma 2.3 in [25], we obtain

$$
\begin{aligned}
\int_0^{\infty} \Phi\left(\frac{1}{\lambda}|g'_{u_1}(u_2)|\right)\mathrm{d}u_2 \le \int_0^{\infty} &\Phi\left(\frac{1}{\lambda}|g_{u_1}(u_2)|\right)\mathrm{d}u_2 \\
&+ \int_0^{\infty} \Phi\left(\frac{1}{\lambda}\varphi^2(u_2)|g''_{u_1}(u_2)|\right)\mathrm{d}u_2.
\end{aligned}
$$

By the above inequalities and

$$
\varphi^2(s)g''_{u_1}(s) = s(1+s)(1+u_1)^2 D_2^2 f\big(u_1,(1+u_1)s\big) = \big(\varphi^2 D_2^2 f\big)\big(u_1,(1+u_1)s\big),
$$

we obtain

$$
\begin{aligned}
J_{12} &\le \int_0^{\infty}\int_0^{\infty}\left[\Phi\left(\frac{C}{\lambda n}|g_{u_1}(u_2)|\right) + \Phi\left(\frac{C}{\lambda n}\varphi^2(u_2)|g''_{u_1}(u_2)|\right)\right]\mathrm{d}u_2\mathrm{d}u_1 \\
&\le \int_0^{\infty}\int_0^{\infty} \Phi\left(\frac{C}{\lambda n}|f(u_1,u_2)|\right)\mathrm{d}u_1\mathrm{d}u_2 \\
&\quad + \int_0^{\infty}\int_0^{\infty} \Phi\left(\frac{C}{\lambda n}\big|\varphi_2^2(u_1,u_2)D_2^2 f(u_1,u_2)\big|\right)\mathrm{d}u_1\mathrm{d}u_2.
\end{aligned}
$$

Combining the above inequality with (4) and (5) acquires

$$
\begin{aligned}
\int_0^{\infty}\int_0^{\infty} \Phi\left(\frac{1}{\lambda}|J_1|\right)\mathrm{d}x_1\mathrm{d}x_2 &\le \int_0^{\infty}\int_0^{\infty} \Phi\left(\frac{C}{\lambda n}|f(u_1,u_2)|\right)\mathrm{d}u_1\mathrm{d}u_2 \\
&+2\int_0^{\infty}\int_0^{\infty} \Phi\left(\frac{C}{\lambda n}\big|\varphi_2^2(u_1,u_2)D_2^2 f(u_1,u_2)\big|\right)\mathrm{d}u_1\mathrm{d}u_2.
\end{aligned} \tag{6}
$$

Now we can start off to estimate J_2. Let

$$
h(u_1) = h(u_1,\boldsymbol{x}) = f\left(u_1,(1+u_1)\frac{x_2}{1+x_1}\right), \quad 0 \le u_1 < \infty.
$$

From [5, 8], we obtain

$$
|V_{n,1}(f,x) - f(x)| \le \frac{C}{n}\big(|f'(x)| + \varphi^2(x)|f''(x)|\big). \tag{7}
$$

Using inequalities (3) and (7), Lemma 2.2, and the convexity of $\Phi(t)$ arrives at

$$\begin{aligned}\int_{\mathbb{R}_0^2}\Phi\left(\frac{1}{\lambda}|J_2|\right)\mathrm{d}\boldsymbol{x}&=\int_0^\infty\int_0^\infty\Phi\left(\frac{1}{\lambda}\big|V_{n,1}(h(\cdot),x_1)-h(x_1)\big|\right)\mathrm{d}x_1\mathrm{d}x_2\\&\le\int_{\mathbb{R}_0^2}\Phi\left[\frac{C}{\lambda n}\Big(|h'(x_1)|+\varphi^2(x_1)|h''(x_1)|\Big)\right]\mathrm{d}\boldsymbol{x}\\&\le\frac{1}{2}\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}|h'(x_1)|\right)\mathrm{d}\boldsymbol{x}+\frac{1}{2}\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\varphi^2(x_1)|h''(x_1)|\right)\mathrm{d}\boldsymbol{x}\\&\le\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}|h(x_1)|\right)\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\varphi^2(x_1)|h''(x_1)|\right)\mathrm{d}\boldsymbol{x}.\end{aligned}$$

When denoting $\varphi_{12}(\boldsymbol{x})=\varphi_{21}(\boldsymbol{x})\triangleq\sqrt{x_1x_2}$, $D_{12}^2=\frac{\partial^2}{\partial x_1\partial x_2}$, and $D_{21}^2=\frac{\partial^2}{\partial x_2\partial x_1}$, we can write

$$\begin{aligned}|\varphi^2(u)h''(u)|&=\bigg|u(1+u)\bigg[D_1^2f+\frac{x_2}{1+x_1}D_{12}^2f+\frac{x_2}{1+x_1}D_{21}^2f\\&\quad+\frac{x_2^2}{(1+x_1)^2}D_{22}^2f\bigg]\left(u,(1+u)\frac{x_2}{1+x_1}\right)\bigg|\\&=\bigg|\bigg(\frac{1+x_1}{1+x_1+x_2}\varphi_1^2D_1^2f+\varphi_{12}^2D_{12}^2f+\varphi_{21}^2D_{21}^2f\\&\quad+\frac{u}{1+u}\frac{x_2}{1+x_1+x_2}\varphi_2^2D_2^2f\bigg)\left(u,(1+u)\frac{x_2}{1+x_1}\right)\bigg|.\end{aligned}$$

By virtue of the facts that $\varphi_{12}(\boldsymbol{x})$ is not bigger than $\varphi_1(\boldsymbol{x})$ or $\varphi_2(\boldsymbol{x})$ and that

$$\big|D_{12}^2f(\boldsymbol{x})\big|\le\sup\big\{\big|D_1^2f(\boldsymbol{x})\big|,\big|D_2^2f(\boldsymbol{x})\big|\big\}$$

in [7, Lemma 2.1], we obtain

$$\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}(\varphi^2(x_1)|h''(x_1)|)\right)\mathrm{d}\boldsymbol{x}\le\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\sum_{i=1}^2\big|\varphi_i^2D_i^2f\big|\right)\mathrm{d}\boldsymbol{x}$$

and

$$\int_{\mathbb{R}_0^2}\Phi\left(\frac{1}{\lambda}|J_2|\right)\mathrm{d}\boldsymbol{x}\le\frac{1}{2}\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}|f(x_1,x_2)|\right)\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\sum_{i=1}^2\big|\varphi_i^2D_i^2f\big|\right)\mathrm{d}\boldsymbol{x},$$

where we used the convexity of $\Phi(t)$. Combining these two inequalities with (3) and (6) and paying attention to computation of norm and the inequality (1) yield

$$\|V_{n,2}(f)-f\|_\Phi\le\|J_1\|_\Phi+\|J_2\|_\Phi\le\frac{C}{n}\left(\|f\|_\Phi+\sum_{i=1}^2\|\varphi_i^2D_i^2f\|_\Phi\right).$$

The proof of Lemma 2.4 is complete. □

3. **Equivalent theorems.** There exist the following equivalent theorems between the modulus of smoothness and the K-functional.

Theorem 3.1. *Let $f\in L_\Phi^*(\mathbb{R}_0^m)$ and $r\in\mathbb{N}$. Then there exist some constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_\Phi}{C}\le K_{r,\varphi}(f,t^r)_\Phi\le C\omega_{r,\varphi}(f,t)_\Phi,\quad 0<t\le t_0.\tag{8}$$

Proof. We shall reduce the proof to the one dimension. Some ideas come from [6]. For $\boldsymbol{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}_0^m$, we write $\boldsymbol{x}^* = (x_2, \dots, x_m)$ and $\mathbb{R}_0^{m*} = \{\boldsymbol{x}^* : \boldsymbol{x} = (x_1, \boldsymbol{x}^*) \in \mathbb{R}_0^m\}$. Let $x_1 = (1 + |\boldsymbol{x}^*|)z$ for $0 \le z < \infty$ and $F(z) = F(z, \boldsymbol{x}^*) = f\big((1 + |\boldsymbol{x}^*|)z, \boldsymbol{x}^*\big)$. Then

$$\varphi_1(\boldsymbol{x}) = (1 + |\boldsymbol{x}^*|)\varphi(z), \quad D_1^r f(\boldsymbol{x}) = \frac{F^{(r)}(z)}{(1 + |\boldsymbol{x}^*|)^r}, \quad \Delta^r_{h\varphi_1(\boldsymbol{x})\boldsymbol{e}_1} f(\boldsymbol{x}) = \Delta^r_{h\varphi(z)} F(z).$$

It was shown in [25] that

$$\int_0^\infty \Phi\left(\frac{|\Delta^r_{h\varphi(z)} F(z)|}{\lambda}\right) \mathrm{d}z \le \begin{cases} \displaystyle\int_0^\infty \Phi\left(\frac{C|F(z)|}{\lambda}\right) \mathrm{d}z, & F \in L_\Phi^*[0, \infty); \\ \displaystyle\int_0^\infty \Phi\left(\frac{C|h^r \varphi^r(z) F^{(r)}(z)|}{\lambda}\right) \mathrm{d}z, & F \in W_\varphi^{r,\Phi}[0, \infty). \end{cases}$$

Consequently, it follows that, for $f \in L_\Phi^*(\mathbb{R}_0^m)$,

$$\begin{aligned} \|\Delta^r_{h\varphi_1\boldsymbol{e}_1} f\|_{(\Phi)} &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} \mathrm{d}\boldsymbol{x}^* \int_0^\infty \Phi\left(\frac{1}{\lambda}\big|\Delta^r_{h\varphi_1(\boldsymbol{x})\boldsymbol{e}_1} f(\boldsymbol{x})\big|\right) \mathrm{d}x_1 \le 1\right\} \\ &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} (1 + |\boldsymbol{x}^*|) \int_0^\infty \Phi\left(\frac{1}{\lambda}\big|\Delta^r_{h\varphi(z)} F(z)\big|\right) \mathrm{d}z \mathrm{d}\boldsymbol{x}^* \le 1\right\} \\ &\le \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} (1 + |\boldsymbol{x}^*|) \int_0^\infty \Phi\left(\frac{C}{\lambda}|F(z)|\right) \mathrm{d}z \mathrm{d}\boldsymbol{x}^* \le 1\right\} \\ &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} \int_0^\infty \Phi\left(\frac{C}{\lambda}|F(z)|\right) \mathrm{d}x_1 \mathrm{d}\boldsymbol{x}^* \le 1\right\} \\ &= C\|f\|_{(\Phi)}. \end{aligned}$$

For $f \in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)$, we arrive at

$$\begin{aligned} \|\Delta^r_{h\varphi_1\boldsymbol{e}_1} f\|_{(\Phi)} &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} \mathrm{d}\boldsymbol{x}^* \int_0^\infty \Phi\left(\frac{1}{\lambda}\big|\Delta^r_{h\varphi_1(\boldsymbol{x})\boldsymbol{e}_1} f(\boldsymbol{x})\big|\right) \mathrm{d}x_1 \le 1\right\} \\ &\le \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} (1 + |\boldsymbol{x}^*|) \int_0^\infty \Phi\left(\frac{C}{\lambda} h^r |\varphi^r(z) F^{(r)}(z)|\right) \mathrm{d}z \mathrm{d}\boldsymbol{x}^* \le 1\right\} \\ &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}} \int_0^\infty \Phi\left(\frac{C}{\lambda} h^r |(\varphi_1^r D_1^r) f(x_1, \boldsymbol{x}^*)|\right) \mathrm{d}x_1 \mathrm{d}\boldsymbol{x}^* \le 1\right\} \\ &= Ch^r \|\varphi_1^r D_1^r(f)\|_{(\Phi)}. \end{aligned}$$

Similarly, for $i = 2, 3, \dots, m$, we have

$$\|\Delta^r_{h\varphi_i\boldsymbol{e}_i} f\|_{(\Phi)} \le C \begin{cases} \|f\|_{(\Phi)}, & f \in L_\Phi^*(\mathbb{R}_0^m); \\ h^r \|\varphi_i^r D_i^r(f)\|_{(\Phi)}, & f \in W_\varphi^{r,\Phi}(\mathbb{R}_0^m). \end{cases}$$

Adding these inequalities and applying (1) yields the lower bound in (8).

To estimate the upper bound in (8), we shall again reduce it to the one-dimensional case. First we note that, for fixed $\boldsymbol{x}^*$ and $t > 0$, there exists a function $G_t \in W_\varphi^{r,\Phi}(\mathbb{R}_0^1)$ such that

$$\int_0^\infty \Phi\left(\frac{1}{\lambda}|F(z) - G_t(z)|\right) \mathrm{d}z \le \frac{C}{t} \int_0^t \int_0^\infty \Phi\left(\frac{C}{\lambda}\big|\Delta^r_{\tau\varphi(z)} F(z)\big|\right) \mathrm{d}z \mathrm{d}\tau$$

and

$$\int_0^\infty \Phi\left(\frac{1}{\lambda}\left|t^r\varphi^r(z)G_t^{(r)}(z)\right|\right)\mathrm{d}z \le \frac{C}{t}\int_0^t\int_0^\infty \Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau\varphi(z)}F(z)\right|\right)\mathrm{d}z\mathrm{d}\tau,$$

where G_t depends on F continuously. See [25]. Accordingly, for $F(z) = F(z, \boldsymbol{x}^*)$, we have $G_t(z) = G_t(z, \boldsymbol{x}^*)$ as well. Let

$$g_t(\boldsymbol{x}) = G_t\left(\frac{x_1}{1+|\boldsymbol{x}^*|}, \boldsymbol{x}^*\right), \quad \boldsymbol{x} \in \mathbb{R}_0^m.$$

Then $g_t \in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)$ and

$$\begin{aligned}
\|f-g_t\|_{(\Phi)} &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \Phi\left(\frac{1}{\lambda}|F(z)-G_t(z)|\right)\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&\le \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\frac{C}{t}\int_0^t\int_0^\infty \Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau\varphi(z)}F(z)\right|\right)\mathrm{d}z\mathrm{d}\tau\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \frac{C}{t}\int_0^t \Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau\varphi(z)}F(z)\right|\right)\mathrm{d}\tau\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty C\Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau_1\varphi(z)}F(z)\right|\right)\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&\le \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^m}\Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau_1\varphi_1(\boldsymbol{x})\boldsymbol{e}_1}f(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x} \le 1\right\}\\
&= C\|\Delta^r_{\tau_1\varphi_1(\boldsymbol{x})\boldsymbol{e}_1}f\|_{(\Phi)}, \quad 0 \le \tau_1 \le t
\end{aligned}$$

and

$$\begin{aligned}
t^r\|\varphi_1^r D_1^r g_t\|_{(\Phi)} &= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \Phi\left(\frac{t^r}{\lambda}\left|\varphi^r(z)G_t^{(r)}(z)\right|\right)\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&\le \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\frac{C}{t}\int_0^t\int_0^\infty \Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau\varphi(z)}F(z)\right|\right)\mathrm{d}z\mathrm{d}\tau\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \frac{C}{t}\int_0^t \Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau\varphi(z)}F(z)\right|\right)\mathrm{d}\tau\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&\le \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau_1\varphi(z)}F(z)\right|\right)\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\right\}\\
&= \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{R}_0^m}\Phi\left(\frac{C}{\lambda}\left|\Delta^r_{\tau_1\varphi_1(\boldsymbol{x})\boldsymbol{e}_1}f(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x} \le 1\right\}\\
&= C\|\Delta^r_{\tau_1\varphi_1(\boldsymbol{x})\boldsymbol{e}_1}f\|_{(\Phi)}, \quad 0 \le \tau_1 \le t.
\end{aligned}$$

Similarly, we can prove that, for each i and $t>0$, there are functions $g_t \in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)$ and $\tau_i \in [0,t]$ such that

$$\|f-g_t\|_{(\Phi)} \le C\|\Delta^r_{\tau_i\varphi_i(\boldsymbol{x})\boldsymbol{e}_i}f\|_{(\Phi)} \quad \text{and} \quad t^r\|\varphi_i^r D_i^r g_t\|_{(\Phi)} \le C\|\Delta^r_{\tau_i\varphi_i(\boldsymbol{x})\boldsymbol{e}_i}f\|_{(\Phi)}.$$

By the double inequality (1), we obtain

$$\|f-g_t\|_\Phi \le C\|\Delta^r_{\tau_i\varphi_i(\boldsymbol{x})\boldsymbol{e}_i}f\|_\Phi \quad \text{and} \quad t^r\|\varphi_i^r D_i^r g_t\|_\Phi \le C\|\Delta^r_{\tau_i\varphi_i(\boldsymbol{x})\boldsymbol{e}_i}f\|_\Phi. \tag{9}$$

Adding these inequalities results in the upper bound in (8). □

Remark 3.1. For $m=1$, Theorem 3.1 coincides with corresponding 1-dimensional one in [25].

Theorem 3.2. *Let $f \in L^*_\Phi[0,\infty)$ and $r \in \mathbb{N}$. Then there exist some constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_\Phi}{C} \le \bar{K}_{r,\varphi}(f,t^r)_\Phi \le C\omega_{r,\varphi}(f,t)_\Phi, \quad 0<t\le t_0. \tag{10}$$

Proof. Since $K_{r,\varphi}(f,t^r)_\Phi \le \overline{K}_{r,\varphi}(f,t^r)_\Phi$, we only need to prove the upper estimate. By those inequalities in (9), we only need to prove

$$t^{2r}\|D_i^r g_t\|_{(\Phi)} \le C\|\Delta^r_{\tau_i'\varphi_i(\boldsymbol{x})\boldsymbol{e}_i} f\|_{(\Phi)}$$

for each i.

In [23], it was obtained that

$$\int_0^\infty \Phi\bigg(\frac{1}{\lambda}\big|t^{2r}G_t^{(r)}(z)\big|\bigg)\mathrm{d}z \le \frac{C}{t}\int_0^t\int_0^\infty \Phi\bigg(\frac{C}{\lambda}\big|\Delta^r_{\tau\varphi(z)}F(z)\big|\bigg)\mathrm{d}z\mathrm{d}\tau.$$

For $F(z)=F(z,\boldsymbol{x}^*)$, we have $G_t(z)=G_t(z,\boldsymbol{x}^*)$ as well. Let

$$g_t(\boldsymbol{x}) = G_t\bigg(\frac{x_1}{1+|\boldsymbol{x}^*|},\boldsymbol{x}^*\bigg), \quad \boldsymbol{x}\in\mathbb{R}_0^m.$$

Then $g_t \in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)$ and

$$\begin{aligned} t^{2r}\|D_1^r g_t\|_{(\Phi)} &= \inf_{\lambda>0}\bigg\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \Phi\bigg(\frac{1}{\lambda}\big|t^{2r}G_t^{(r)}(z)\big|\bigg)\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\bigg\} \\ &\le \inf_{\lambda>0}\bigg\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\frac{C}{t}\int_0^t\int_0^\infty \Phi\bigg(\frac{C}{\lambda}\big|\Delta^r_{\tau\varphi(z)}F(z)\big|\bigg)\mathrm{d}z\mathrm{d}\tau\mathrm{d}\boldsymbol{x}^* \le 1\bigg\} \\ &\le \inf_{\lambda>0}\bigg\{\lambda : \int_{\mathbb{R}_0^{m*}}(1+|\boldsymbol{x}^*|)\int_0^\infty \Phi\bigg(\frac{C}{\lambda}\big|\Delta^r_{\tau_1'\varphi(z)}F(z)\big|\bigg)\mathrm{d}z\mathrm{d}\boldsymbol{x}^* \le 1\bigg\} \\ &= C\|\Delta^r_{\tau_1'\varphi_1(\boldsymbol{x})\boldsymbol{e}_1}f\|_{(\Phi)}, \quad 0\le\tau_1'\le t. \end{aligned}$$

Similarly, we can prove that, for each i and $t>0$, there are functions $g_t \in W_\varphi^{r,\Phi}(\mathbb{R}_0^m)$ and $\tau_i' \in [0,t]$ such that

$$t^{2r}\|D_i^r g_t\|_{(\Phi)} \le C\|\Delta^r_{\tau_i'\varphi_i(\boldsymbol{x})\boldsymbol{e}_i} f\|_{(\Phi)}.$$

Using (1), it follows that

$$t^{2r}\|D_i^r g_t\|_\Phi \le C\|\Delta^r_{\tau_i'\varphi_i(\boldsymbol{x})\boldsymbol{e}_i} f\|_\Phi.$$

Combining this inequality with (9) and adding these inequalities lead to (10). The proof of Theorem 3.2 is complete. □

Remark 3.2. For $m=1$, Theorem 3.2 coincides with corresponding 1-dimensional one in [23].

4. **A direct theorem.**

We now in a position to state and prove the direct theorem.

Theorem 4.1 (Direct theorem)**.** *Let $f \in L^*_\Phi(\mathbb{R}_0^m)$, $n>m$, and $\Psi\in\Delta_2$. Then*

$$\|V_{n,m}(f)-f\|_\Phi \le C\bigg[\omega_{2,\varphi}\bigg(f,\frac{1}{n^{1/2}}\bigg)_\Phi + \frac{\|f\|_\Phi}{n}\bigg]. \tag{11}$$

Proof. Our proof is based on induction on the dimension m and on a decomposition for Baskakov–Kantorovich's operator. For $m=1$, the inequality (11) can be rewritten as

$$\|V_{n,1}(f)-f\|_\Phi \le C\omega_{2,\varphi}\left(f,\frac{1}{n^{1/2}}\right)_\Phi \le C\left[\omega_{2,\varphi}\left(f,\frac{1}{n^{1/2}}\right)_\Phi + \frac{\|f\|_\Phi}{n}\right]$$

which has been proved in [21].

For $m\ge 2$, the proof of Theorem 4.1 follows from combining Lemmas 2.1 and 2.4 with the estimates

$$\big\|V_{n,m}(f)-f\big\|_\Phi \le C\begin{cases}\|f\|_\Phi, & f\in L_\Phi^*(\mathbb{R}_0^m);\\ \dfrac{1}{n}\left(\displaystyle\sum_{i=1}^m \|\varphi_i^2D_i^2f\|_\Phi+\|f\|_\Phi\right), & f\in W_\varphi^{2,\Phi}(\mathbb{R}_0^m).\end{cases} \tag{12}$$

The first estimate in (12) can be derived from Lemma 2.1.

By (2), the second estimate in (12) is valid for $m=2$. If the second estimate in (12) is valid for $m=r\ge 2$, that is,

$$\|V_{n,r}(f)-f\|_\Phi \le \frac{C}{n}\left(\sum_{i=1}^r \|\varphi_i^2D_i^2f\|_\Phi+\|f\|_\Phi\right), \tag{13}$$

then we have to further verify its validity for $m=r+1$. We claim that the decomposition formula

$$\begin{aligned}V_{n,r+1}(f,\boldsymbol{x}) &= \sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\sum_{\boldsymbol{k}^*=0}^\infty p_{n+k_1,\boldsymbol{k}^*}\left(\frac{\boldsymbol{x}^*}{1+x_1}\right)n^r\\ &\quad\times\int_{\boldsymbol{k}^*/n}^{(\boldsymbol{k}^*+1)/n} f(u_1,\boldsymbol{u}^*)\mathrm{d}\boldsymbol{u}^*\mathrm{d}u_1\\ &= \sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\sum_{\boldsymbol{k}^*=0}^\infty p_{n+k_1,\boldsymbol{k}^*}\left(\frac{\boldsymbol{x}^*}{1+x_1}\right)n^r\\ &\quad\times\int_{\boldsymbol{k}^*/[n(1+u_1)]}^{(\boldsymbol{k}^*+1)/[n(1+u_1)]} f\big(u_1,(1+u_1)\boldsymbol{u}^*\big)(1+u_1)\mathrm{d}\boldsymbol{u}^*\mathrm{d}u_1\end{aligned} \tag{14}$$

is valid, where $\boldsymbol{x}^*=(x_2,x_3,\dots,x_{r+1})$, $x=(x_1,\boldsymbol{x}^*)\in\mathbb{R}_0^{r+1}$, $\boldsymbol{k}^*=(k_2,k_3,\dots,k_{r+1})$, $\boldsymbol{k}=(k_1,\boldsymbol{k}^*)\in\mathbb{N}_0^{r+1}$, and $\sum_{\boldsymbol{k}^*=0}^\infty=\sum_{k_2=0}^\infty\sum_{k_3=0}^\infty\cdots\sum_{k_{r+1}=0}^\infty$. This formula can be directly checked up and will take an important role in the following proof. Let

$$g_{u_1}(t)\triangleq f\big(u_1,(1+u_1)t\big),\quad 0\le t<\infty$$

and

$$\boldsymbol{z}=(z_1,z_2,\dots,z_r)=\left(\frac{x_2}{1+x_1},\frac{x_3}{1+x_1},\cdots,\frac{x_{r+1}}{1+x_1}\right)=\frac{\boldsymbol{x}^*}{1+x_1}.$$

From the formula (14), it follows that

$$\begin{aligned}V_{n,r+1}(f,\boldsymbol{x})-f(\boldsymbol{x}) &= \sum_{k_1=0}^\infty p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\big[V_{n+k_1,r}\big(g_{u_1}(\cdot),z\big)-g_{u_1}(z)\big]\mathrm{d}u_1\\ &\quad+\big[V_{n,1}\big(h(\cdot),x_1\big)-h(x_1)\big]\triangleq T_1+T_2,\end{aligned} \tag{15}$$

where

$$h(u_1)\triangleq h(u_1,\boldsymbol{x})\triangleq f\left(u_1,(1+u_1)\frac{\boldsymbol{x}^*}{1+x_1}\right),\quad 0\le u_1<\infty.$$

By the inequality

$$\begin{aligned}
&\int_{\mathbb{R}_0^r} \Phi\left(\frac{1}{\lambda}|V_{n,r}(f,\boldsymbol{x})-f(\boldsymbol{x})|\right)\mathrm{d}\boldsymbol{x}\\
&\le \int_{\mathbb{R}_0^r} \Phi\left(\frac{C}{\lambda n}\sum_{i=1}^{r}|(\varphi_i^2 D_i^2 f)(\boldsymbol{x})|\right)\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}_0^r} \Phi\left(\frac{C}{\lambda n}|f(\boldsymbol{x})|\right)\mathrm{d}\boldsymbol{x},
\end{aligned}$$

which can be obtained from (13), and Jensen's inequality, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{1}{\lambda}|T_1|\right)\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}_0^{r+1}} \Phi\Bigg(\frac{1}{\lambda}\Bigg|\sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n\\
&\quad\times \int_{k_1/n}^{(k_1+1)/n}\big[V_{n+k_1,r}\big(g_{u_1}(\cdot),z\big)-g_{u_1}(z)\big]\mathrm{d}u_1\Bigg|\Bigg)\mathrm{d}\boldsymbol{x}\\
&\le \int_{\mathbb{R}_0^{r+1}}\sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n\int_{k_1/n}^{(k_1+1)/n}\Phi\left(\frac{1}{\lambda}\big|V_{n+k_1,r}\big(g_{u_1}(\cdot),z\big)-g_{u_1}(z)\big|\right)\mathrm{d}u_1\mathrm{d}\boldsymbol{x}\\
&= \sum_{k_1=0}^{\infty}\int_0^{\infty} p_{n,k_1}(x_1)(1+x_1)\mathrm{d}x_1 n\int_{k_1/n}^{(k_1+1)/n}\mathrm{d}u_1\\
&\quad\times\int_{\mathbb{R}_0^r}\Phi\left(\frac{1}{\lambda}\big|V_{n+k_1,r}\big(g_{u_1}(\cdot),z\big)-g_{u_1}(z)\big|\right)\mathrm{d}\boldsymbol{z}\\
&\le \sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\Bigg[\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{\lambda(n+k_1)}\sum_{i=1}^{r}|(\varphi_i^2D_i^2 g_{u_1})(\boldsymbol{z})|\right)\mathrm{d}\boldsymbol{z}\\
&\quad+\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{\lambda(n+k_1)}|g_{u_1}(\boldsymbol{z})|\right)\mathrm{d}\boldsymbol{z}\Bigg]\mathrm{d}u_1.
\end{aligned}$$

On the other hand, by definition, we can deduce

$$\begin{aligned}
\varphi_i^2(\boldsymbol{x})D_i^2 g_{u_1}(\boldsymbol{x}) &= x_i(1+|\boldsymbol{x}|)(1+u_1)D_{i+1}^2 f\big(u_1,(1+u_1)\boldsymbol{x}\big)\\
&= (\varphi_{i+1}^2 D_{i+1}^2 f)\big(u_1,(1+u_1)\boldsymbol{x}\big).
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\int_{\mathbb{R}_0^{r+1}}\Phi\left(\frac{1}{\lambda}|T_1|\right)\mathrm{d}\boldsymbol{x} &\le \sum_{k_1=0}^{\infty}\frac{n(n+k_1-1)}{(n-1)(n-2)}\int_{k_1/n}^{(k_1+1)/n}\Bigg[\int_{\mathbb{R}_0^r}\Phi\Bigg(\frac{C}{\lambda(n+k_1)}\\
&\quad\times\sum_{i=1}^{r+1}|(\varphi_i^2D_i^2 f)\big(u_1,(1+u_1)\boldsymbol{z}\big)|\Bigg)\mathrm{d}\boldsymbol{z}\\
&\quad+\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{\lambda(n+k_1)}|f\big(u_1,(1+u_1)\boldsymbol{z}\big)|\right)\mathrm{d}\boldsymbol{z}\Bigg]\mathrm{d}u_1\\
&\le \sum_{k_1=0}^{\infty}\int_{k_1/n}^{(k_1+1)/n}\frac{1}{1+u_1}\Bigg[\int_{\mathbb{R}_0^r}\Phi\Bigg(\frac{Cn(n+k_1-1)}{(n+k_1)(n-1)(n-2)\lambda}\\
&\quad\times\sum_{i=1}^{r+1}|(\varphi_i^2D_i^2 f)\big(u_1,(1+u_1)\boldsymbol{z}\big)|\Bigg)\mathrm{d}\big((1+u_1)\boldsymbol{z}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}_0^r}\Phi\bigg(\frac{Cn(n+k_1-1)}{(n+k_1)(n-1)(n-2)\lambda}\big|f\big(u_1,(1+u_1)\boldsymbol{z}\big)\big|\bigg)\mathrm{d}\big((1+u_1)\boldsymbol{z}\big)\bigg]\mathrm{d}u_1\\
&\le\sum_{k_1=0}^{\infty}\int_{k_1/n}^{(k_1+1)/n}\bigg[\int_{\mathbb{R}_0^r}\Phi\bigg(\frac{C}{n\lambda}\sum_{i=1}^{r+1}\big|(\varphi_i^2D_i^2f)\big(u_1,(1+u_1)\boldsymbol{z}\big)\big|\bigg)\mathrm{d}\big((1+u_1)\boldsymbol{z}\big)\\
&\qquad+\int_{\mathbb{R}_0^r}\Phi\bigg(\frac{C}{n\lambda}\big|f\big(u_1,(1+u_1)\boldsymbol{z}\big)\big|\bigg)\mathrm{d}\big((1+u_1)\boldsymbol{z}\big)\bigg]\mathrm{d}u_1 \qquad (16)\\
&\le\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}\sum_{i=1}^{r+1}|(\varphi_i^2D_i^2f)(\boldsymbol{u})|\bigg)\mathrm{d}\boldsymbol{u}+\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}|f(\boldsymbol{u})|\bigg)\mathrm{d}\boldsymbol{u}.
\end{aligned}
$$

By Lemma 2.2, the inequality (7), and the convexity of $\Phi(t)$, we acquire

$$
\begin{aligned}
\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{1}{\lambda}|T_2|\bigg)\mathrm{d}\boldsymbol{x}&=\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{1}{\lambda}\big|V_{n,1}\big(h(\cdot),x_1\big)-h(x_1)\big|\bigg)\mathrm{d}\boldsymbol{x}\\
&\le\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}\big(|h'(x_1)|+\varphi^2(x_1)|h''(x_1)|\big)\bigg)\mathrm{d}\boldsymbol{x}\\
&\le\frac12\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}|h'(x_1)|\bigg)\mathrm{d}\boldsymbol{x}+\frac12\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}\varphi^2(x_1)|h''(x_1)|\bigg)\mathrm{d}\boldsymbol{x}\\
&\le\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}|h(x_1)|\bigg)\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}\varphi^2(x_1)|h''(x_1)|\bigg)\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Denoting $\varphi_{ij}(\boldsymbol{x})=\sqrt{x_ix_j}$ for $1\le i<j<r+1$ and $D_{ij}^2=\frac{\partial^2}{\partial x_i\partial x_j}$, we have

$$
\begin{aligned}
\varphi^2(u)h''(u)&=u(1+u)\bigg[D_1^2f+\sum_{i=2}^{r+1}\frac{x_i}{1+x_1}D_{1i}^2f+\sum_{i=2}^{r+1}\frac{x_i}{1+x_1}D_{i1}^2f\\
&\qquad+\sum_{i=2}^{r+1}\sum_{j=2}^{r+1}\frac{x_ix_j}{(1+x_1)^2}D_{ij}^2f\bigg]\bigg(u,(1+u)\frac{\boldsymbol{x}^*}{1+x_1}\bigg)\\
&=\bigg(\frac{1+x_1}{1+|\boldsymbol{x}|}\varphi_1^2D_1^2f+\sum_{i=2}^{r+1}\varphi_{1i}^2D_{1i}^2f+\sum_{i=2}^{r+1}\varphi_{i1}^2D_{i1}^2f+\sum_{i=2}^{r+1}\frac{u}{1+u}\frac{x_i}{1+|\boldsymbol{x}|}\varphi_i^2D_i^2f\\
&\qquad+\sum_{i,j=2,i\ne j}^{r+1}\frac{u}{1+u}\varphi_{ij}^2D_{ij}^2f\bigg)\bigg(u,(1+u)\frac{\boldsymbol{x}^*}{1+x_1}\bigg).
\end{aligned}
$$

Recalling that $\varphi_{ij}(\boldsymbol{x})$ is not bigger than $\varphi_i(\boldsymbol{x})$ or $\varphi_j(\boldsymbol{x})$ and the fact

$$
|D_{ij}^2f(\boldsymbol{x})|\le\sup_{1\le i\le r+1}|D_i^2f(\boldsymbol{x})|
$$

in [7, Lemma 2.1], we obtain

$$
\begin{aligned}
\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{1}{\lambda}|T_2|\bigg)\mathrm{d}\boldsymbol{x}&\le\frac12\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}|f(\boldsymbol{x})|\bigg)\mathrm{d}\boldsymbol{x}\\
&\qquad+\int_{\mathbb{R}_0^{r+1}}\Phi\bigg(\frac{C}{n\lambda}\sum_{i=1}^{r+1}|(\varphi_i^2D_i^2f)(\boldsymbol{x})|\bigg)\mathrm{d}\boldsymbol{x}. \qquad (17)
\end{aligned}
$$

Combining (15), (16), and (17) and paying attention to computation of norm and the inequality (1), we obtain the second estimate of (12) for any $m\ge 2$.

For $g \in W_{\varphi}^{2,\Phi}(\mathbb{R}_0^m)$, combining (12) with Lemma 2.1 and Theorem 3.1 gives

$$\begin{aligned}
\|V_{n,m}(f)-f\|_\Phi &\le \|V_{n,m}(f)-V_{n,m}(g)\|_\Phi + \|V_{n,m}(g)-g\|_\Phi + \|f-g\|_\Phi \\
&\le C\|f-g\|_\Phi + \frac{C}{n}\left(\|g\|_\Phi + \sum_{i=1}^m \|\varphi_i^2 D_i^2 g\|_\Phi\right) \\
&\le C\left(\|f-g\|_\Phi + \frac{1}{n}\sum_{i=1}^m \|\varphi_i^2 D_i^2 g\|_\Phi\right) + \frac{C}{n}\|f\|_\Phi \\
&\le C\left[\omega_{2,\varphi}\left(f, \frac{1}{n^{1/2}}\right)_\Phi + \frac{1}{n}\|f\|_\Phi\right].
\end{aligned}$$

The proof of Theorem 4.1 is complete. □

5. **Conclusions.** In this paper, using K-functional and a decomposition technique and considering some properties of multivariate Baskakov–Kantorovich operators in the form of Lemmas 2.1 to 2.4, we presented two equivalent theorems, Theorems 3.1 and 3.2, between the K-functional and modulus of smoothness, and obtained a direct theorem, Theorem 4.1, in the Orlicz spaces $L_\Phi^*(\mathbb{R}_0^m)$.

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REFERENCES

[1] A. M. Acu and V. Gupta, Direct results for certain summation-integral type Baskakov–Szász operators, *Results Math.*, **72** (2017), 1161–1180.

[2] P. N. Agrawal, V. Gupta and A. Sathish Kumar, Generalized Baskakov–Durrmeyer type operators, *Rend. Circ. Mat. Palermo (2)*, **63** (2014), 193–209.

[3] V. A. Baskakov, An instance of a sequence of linear positive operators in the space of continuous functions, *Dokl. Akad. Nauk SSSR (N.S.)*, **113** (1957), 249–251. (Russian)

[4] F. L. Cao and Y. F. An, L^p approximation by multivariate Baskakov–Durrmeyer operator, *J. Inequal. Appl.*, **2011** (2011), Art. ID 158219, 7 pp.

[5] F. L. Cao and C. M. Ding, L^p approximation by multivariate Baskakov–Kantorovich operators, *J. Math. Anal. Appl.*, **348** (2008), 856–861.

[6] F. L. Cao, C. M. Ding and Z. B. Xu, On multivariate Baskakov operators, *J. Math. Anal. Appl.*, **307** (2005), 274–291.

[7] W. Chen and Z. Ditzian, Mixed and directional derivatives, *Proc. Amer. Math. Soc.*, **108** (1990), 177–185.

[8] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer Series in Computational Mathematics, 9. Springer-Verlag, New York, 1987.

[9] A. Erençin, Durrmeyer type modification of generalized Baskakov operators, *Appl. Math. Comput.*, **218** (2011), 4384–4390.

[10] I. Gadjev, Approximation of functions by Baskakov–Kantorovich operator, *Results Math.*, **70** (2016), 385–400.

[11] I. Gadjev, Strong converse result for Baskakov operator, *Serdica Math. J.*, **40** (2014), 273–318.

[12] R. B. Gandhi, M. Deepmala and V. N. Mishra, Local and global results for modified Szász-Mirakjan operators, *Math. Methods Appl. Sci.*, **40** (2017), 2491–2504.

[13] M. Goyal and P. N. Agrawal, Bézier variant of the generalized Baskakov Kantorovich operators, *Boll. Unione Mat. Ital.*, **8** (2016), 229–238.

[14] S. S. Guo, Q. L. Qi and G. F. Liu, The central approximation theorems for Baskakov–Bézier operators, *J. Approx. Theory*, **147** (2007), 112–124.

[15] V. Gupta and A. M. Acu, On Baskakov–Szász–Mirakyan-type operators preserving exponential type functions, *Positivity*, **22** (2018), 919–929.

[16] F. Gürbüz, Generalized weighted Morrey estimates for Marcinkiewicz integrals with rough kernel associated with Schrödinger operator and their commutators, *Chin. Ann. Math. Ser. B*, **41** (2020), 77–98.

[17] F. Gürbüz, Local campanato estimates for multilinear commutator operators with rough kernel on generalized local morrey spaces, *J. Coupled Syst. Multiscale Dyn.*, **6** (2018), 71–79.

[18] F. Gürbüz, Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces, *Canad. Math. Bull.*, **60** (2017), 131–145.

[19] F. Gürbüz, Sublinear operators with rough kernel generated by Calderón-Zygmund operators and their commutators on generalized Morrey spaces, *Math. Notes*, **101** (2017), 429–442.

[20] F. Gürbüz, Weighted Morrey and weighted fractional Sobolev-Morrey spaces estimates for a large class of pseudo-differential operators with smooth symbols, *J. Pseudo-Differ. Oper. Appl.*, **7** (2016), 595–607.

[21] L.-X. Han, B.-N. Guo and F. Qi, Equivalent theorem of approximation by linear combination of weighted Baskakov–Kantorovich operators in Orlicz spaces, *J. Inequal Appl.*, **2019** (2019), 18 pp.

[22] L.-X. Han and F. Qi, On approximation by linear combinations of modified summation operators of integral type in Orlicz spaces, *Mathematics*, **7** (2019), Art. 6, 10 pp.

[23] L.-X. Han and G. Wu, Approximation by modified summation integral type operators in Orlicz spaces, *Math. Appl. (Wuhan)*, **30** (2017), 613–622.

[24] L.-X. Han and G. Wu, Strong converse inequality of weighted simultaneous approximation for Gamma operators in Orlicz spaces $L^*_\Phi(0,\infty)$, *Appl. Math. J. Chinese Univ. Ser. A*, **31** (2016), 366–378. (Chinese)

[25] L.-X. Han, G. Wu and G. Liu, The equivalence of the smooth modulus with weights and a K-functional in Orlicz spaces and it's application, *Acta Math. Sci. Ser. A*, **34** (2014), 95–108. (Chinese)

[26] A. S. Kumar and T. Acar, Approximation by generalized Baskakov–Durrmeyer–Stancu type operators, *Rend. Circ. Mat. Palermo (2)*, **65** (2016), 411–424.

[27] V. N. Mishra, H. H. Khan, K. Khatri and L. N. Mishra, Hypergeometric representation for Baskakov–Durrmeyer–Stancu type operators, *Bull. Math. Anal. Appl.*, **5** (2013), 18–26.

[28] V. N. Mishra, K. Khatri and L. N. Mishra, On simultaneous approximation for Baskakov–Durrmeyer–Stancu type operators, *J. Ultra Scientist Phys. Sci. A*, **24** (2012), 567–577.

[29] V. N. Mishra, K. Khatri and L. N. Mishra, Some approximation properties of q-Baskakov–Beta–Stancu type operators, *Journal of Calculus of Variations*, **2013** (2013), Art. ID 814824, 8 pp.

[30] V. N. Mishra, K. Khatri, L. N. Mishra and Deepmala, Inverse result in simultaneous approximation by Baskakov–Durrmeyer–Stancu operators, *J. Inequal. Appl.*, **2013** (2013), 11 pp.

[31] M. M. Rao and Z. D. Ren, *Theory of Orlicz Space*, Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.

[32] V. Totik, An interpolation theorem and its applications to positive operators, *Pacific J. Math.*, **111** (1984), 447–481.

[33] V. Totik, Approximation by Bernstein polynomials, *Amer. J. Math.*, **116** (1994), 995–1018.

[34] D. K. Verma, V. Gupta and P. N. Agrawal, Some approximation properties of Baskakov–Durrmeyer–Stancu operators, *Appl. Math. Comput.*, **218** (2012), 6549–6556.

[35] C. G. Zhang and Z. H. Zhu, Preservation properties of the Baskakov–Kantorovich operators, *Comput. Math. Appl.*, **57** (2009), 1450–1455.

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