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GLOBAL BEHAVIOR OF THE SOLUTIONS TO NONLINEAR KLEIN-GORDON EQUATION WITH CRITICAL INITIAL ENERGY

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Abstract. Nonlinear Klein-Gordon equation with combined power type nonlinearity and critical initial energy is investigated. The qualitative properties of a new ordinary differential equation are studied and the concavity method of Levine is improved. Necessary and sufficient conditions for finite time blow up and global existence of the solutions are proved. New sufficient conditions on the initial data for finite time blow up, based on the necessary and sufficient ones, are obtained. The asymptotic behavior of the global solutions is also investigated.

1. **Introduction.** The aim of this paper is to study the global behavior of the solutions to the Cauchy problem for the nonlinear Klein-Gordon equation

$$\begin{aligned} &u_{tt} - \Delta u + u = f(u), && (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ &u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), && x \in \mathbb{R}^n, \\ &u_0(x) \in \mathrm{H}^1(\mathbb{R}^n),\ u_1(x) \in \mathrm{L}^2(\mathbb{R}^n) && \end{aligned} \tag{1}$$

with critical initial energy $E(0) = d$. The nonlinear term $f(u)$ has one of the following forms

$$\begin{aligned} f(u) &= \sum_{k=1}^{l} a_k |u|^{p_k - 1} u - \sum_{j=1}^{s} b_j |u|^{q_j - 1} u, \\ f(u) &= a_1 |u|^{p_1} + \sum_{k=2}^{l} a_k |u|^{p_k - 1} u - \sum_{j=1}^{s} b_j |u|^{q_j - 1} u, \end{aligned} \tag{2}$$

where the constants a_k, p_k $(k = 1, 2, \dots, l)$ and b_j, q_j $(j = 1, 2, \dots, s)$ fulfill the conditions

$$\begin{aligned} &a_1 > 0,\ a_k \geq 0,\ b_j \geq 0 \text{ for } k = 2, \dots, l,\ j = 1, \dots, s, \\ &1 < q_s < q_{s-1} < \cdots < q_1 < p_1 < p_2 < \cdots < p_{l-1} < p_l, \\ &p_l < \infty \text{ for } n = 1, 2; \quad p_l < \frac{n+2}{n-2} \text{ for } n \geq 3. \end{aligned} \tag{3}$$

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The combined power type nonlinearity (2) appears in numerous models of quantum mechanics, field theory, nonlinear optics and others. For example, the quadratic-cubic nonlinearity $f(u) = u^2 + u^3$ describes the dislocation of crystals, see [16], while the the cubic-quintic nonlinearity $f(u) = u^3 + u^5$ arises in particles physics, see e.g. [21, 14].

The global existence or finite time blow up of the solutions to (1) - (3) is fully investigated for nonpositive energy $E(0) \leq 0$ and for subcritical energy $0 < E(0) < d$ by means of the potential well method. Here d is the critical energy constant, defined in (8). Potential well method is suggested in [20] for the wave equation and further on is applied for wide class of nonlinear dispersive equations, e.g. for nonlinear Klein-Gordon equations see [1, 18, 19, 24, 30]. Within this method the sign of the Nehari functional $I(0)$, see (7), is crucial for the global behavior of the solutions to (1) - (3). More precisely, for $0 < E(0) < d$ the solutions blow up for a finite time if $I(0) < 0$ and they are globally defined if $I(0) \geq 0$.

The case of critical initial energy, i.e. $E(0) = d$, is treated in [5, 9, 26, 28] for the wave and damped wave equations in bounded domains and for the Klein-Gordon equation – in [8, 16, 19, 24]. In the above papers the global existence is proved under conditions $I(0) > 0$ or $\|u(t, \cdot)\|_{\mathrm{H}^1(\mathbb{R}^n)} = 0$ without any restrictions on the sign of the scalar product (u_0, u_1) of the initial data. In the same papers the finite time blow up is obtained when $I(0) < 0$ and $(u_0, u_1) \geq 0$. The case $E(0) = d$, $I(0) < 0$ and $(u_0, u_1) < 0$ is investigated only in [9, 16]. The asymptotic behavior of the global solutions to the wave equation in bounded domains is studied in [9], while for Klein-Gordon equation similar results are given in [16].

In the case of supercritical initial energy, i.e. $E(0) > d$, only partial results for global behavior of the solutions to (1) - (3) are reported in the literature. There are a few sufficient conditions on the initial data u_0 and u_1, which guarantee finite time blow up, see [3, 4, 8, 11, 12, 13, 19, 22, 23, 27, 29]. In these sufficient conditions the nonnegative sign of (u_0, u_1) is crucial.

In our previous paper [4] we prove a necessary and sufficient condition for finite time blow up of the solutions to (1) - (3) for arbitrary positive initial energy $E(0) > 0$. More precisely, if $u(t, x)$ is a solution to (1) - (3) defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \leq \infty$, then $u(t, x)$ blows up for a finite time $T_m < \infty$ if and only if there exists $b \in [0, T_m)$ such that

$$(u(b, \cdot), u_t(b, \cdot)) > 0 \text{ and } 0 < E(0) \leq \frac{p_1 - 1}{2(p_1 + 1)} \|u(b, \cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^n)}. \tag{4}$$

Let us emphasize once again that the sign condition $(u_0, u_1) \geq 0$ plays very important role in all known sufficient conditions for finite time blow up of the solutions to (1) - (3) with supercritical energy, as well as in the necessary and sufficient condition (4).

In the present paper we focus on the global behavior of the solutions to (1) - (3) with critical initial energy $E(0) = d$. We give new necessary and sufficient conditions for finite time blow up and global existence, which are based on the study of the qualitative properties to a new ordinary differential equation. This approach improves the concavity method of Levine. As a consequence of the necessary and sufficient conditions for finite time blow up, we get new, more general sufficient conditions on the initial data for finite time blow up. In the case $I(0) < 0$, new necessary conditions on the initial data for global existence are proved. The asymptotic behavior of the global solutions with $I(0) < 0$ is studied in a similar way as in [9], where the wave equation in bounded domains is considered.

The paper is organized in the following way. In Section 2 some preliminary results are given. Section 3 deals with the global behavior of the solutions to a new ordinary differential equation. The results are an improvement of the concavity method of Levine and allow us to formulate necessary and sufficient conditions for finite time blow up. The main results of the paper are formulated and proved in Section 4 and Section 5. In Section 4 the finite time blow up is treated, while Section 5 deals with the global existence of the solutions and their asymptotic behavior.

2. **Preliminary.** We will use the following short notations for the functions $u(t,x)$ and $v(t,x)$ depending on t and x

$$\|u\| = \|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^n)}, \quad \|u\|_1 = \|u(t,\cdot)\|_{\mathrm{H}^1(\mathbb{R}^n)},$$
$$(u,v) = (u(t,\cdot), v(t,\cdot)) = \int_{\mathbb{R}^n} u(t,x)v(t,x)\,dx.$$

We have the following local existence result to the Cauchy problem (1) - (3), see e.g. [2, 6, 7].

Theorem 2.1. *Problem (1) - (3) admits a unique local weak solution*

$$u(t,x) \in \mathrm{C}((0,T_m);\mathrm{H}^1(\mathbb{R}^n)) \cap \mathrm{C}^1((0,T_m);\mathrm{L}^2(\mathbb{R}^n)) \cap \mathrm{C}^2((0,T_m);\mathrm{H}^{-1}(\mathbb{R}^n))$$

in the maximal existence time interval $[0,T_m)$. *Moreover,*

(i)
$$\text{if} \qquad \limsup_{t\to T_m, t<T_m} \|u\|_1 < \infty, \qquad \text{then} \qquad T_m = \infty;$$

(ii) *for every* $t \in [0,T_m)$ *the solution* $u(t,x)$ *satisfies the conservation law*

$$E(0) = E(t), \tag{5}$$

where the energy functional $E(t)$ *is defined by*

$$E(t) := E(u(t,\cdot), u_t(t,\cdot)) = \frac{1}{2}\left(\|u_t\|^2 + \|u\|_1^2\right) - \int_{\mathbb{R}^n}\int_0^u f(y)\,dy\,dx. \tag{6}$$

Definition 2.2. The solution $u(t,x)$ to (1) - (3), defined in the maximal existence time interval $[0,T_m)$, $T_m \le \infty$, blows up at T_m if

$$\limsup_{t\to T_m, t<T_m} \|u\|_1 = \infty.$$

In order to prove a necessary and sufficient condition for finite time blow up of the solutions to (1) - (3), we use the following equivalence between the blow up of the H^1 and L^2 norms of $u(t,x)$.

Lemma 2.3. *Suppose* $u(t,x)$ *is the solution to (1) - (3) with* $p_l < (n+4)/n$ *in the maximal existence time interval* $[0,T_m)$, $0 < T_m \le \infty$. *Then the blow up of* H^1 *norm of* $u(t,x)$ *is equivalent to the blow up of the* L^2 *norm of* $u(t,x)$ *at* T_m, *i.e.*

$$\limsup_{t\to T_m, t<T_m} \|u\|_1 = \infty \qquad \text{if and only if} \qquad \limsup_{t\to T_m, t<T_m} \|u\| = \infty.$$

The proof of Lemma 2.3 is based on the Gagliardo - Nirenberg inequality. In one-dimensional case it is given in [4]. The multidimensional case is treated in a similar way and we omit the proof.

Let us recall some important functionals - the Nehari functional $I(u(t,\cdot))$ and the potential energy functional $J(u(t,\cdot))$, as well as the critical energy constant d. When u depends on x and t we use the short notations $I(u(t,\cdot)) = I(t)$ and $J(u(t,\cdot)) = J(t)$, i.e.

$$I(t) := I(u(t,\cdot)) = \|u\|_1^2 - \int_{\mathbb{R}^n} f(u)u\,dx, \tag{7}$$

$$J(t) := J(u(t,\cdot)) = \frac{1}{2}\|u\|_1^2 - \int_{\mathbb{R}^n}\int_0^u f(y)\,dy\,dx,$$

$$d = \inf_{u\in\mathcal{N}} J(u(t,\cdot)), \quad \mathcal{N} = \{u \in \mathrm{H}^1(\mathbb{R}^n) : \ \|u\|_1 \neq 0,\ I(u(t,\cdot)) = 0\}. \tag{8}$$

In the framework of the potential well method there are two important subsets of $\mathrm{H}^1(\mathbb{R}^n)$:

$$W = \left\{u \in \mathrm{H}^1(\mathbb{R}^n) : I(u) > 0\right\} \cup \{0\},\ V = \left\{u \in \mathrm{H}^1(\mathbb{R}^n) : I(u) < 0\right\}.$$

In the following theorem we formulate the sign preserving properties of $I(u)$, i.e. the invariance of V and W under the flow of (1) - (3) when $E(0) = d$.

Theorem 2.4. *Suppose $u(t,x)$ is the weak solution of (1) - (3) defined in the maximal existence time interval $[0,T_m)$ and $E(0) = d$.*

(i) *If $u_0 \in W$, then $u(t,x) \in W$ for every $t \in [0,T_m)$;*
(ii) *If $u_0 \in V$, then $u(t,x) \in V$ for every $t \in [0,T_m)$.*

Proof. (i) Suppose $u_0 \in W$ but the result in (i) fails. Then for some $t_0 \in (0,T_m)$ we have $u(t,x) \in W$ for $t \in [0,t_0)$ and $u(t_0,x) \in \partial W$, i.e. $I(t) > 0$ or $\|u\|_1 = 0$ for $t \in [0,t_0)$ but $I(t_0) = 0$ and $\|u(t_0,\cdot)\|_1 \neq 0$. Hence $u(t_0,\cdot) \in \mathcal{N}$ and from (5), (6), (8) it follows that $J(t_0) \geq d$ and the following inequalities hold

$$d = E(t_0) = \frac{1}{2}\|u_t(t_0,\cdot)\|^2 + J(t_0) \geq d.$$

Hence

$$\|u_t(t_0,\cdot)\| = 0,\ J(t_0) = d \text{ and } I(t_0) = 0. \tag{9}$$

If $\hat{u}(x)$ is a ground state solution of (1), then $\hat{u}(x)$ satisfies the equation

$$-\Delta\hat{u} + \hat{u} - f(\hat{u}) = 0 \text{ for } x \in \mathbb{R}^n.$$

Consequently, condition (9) means that the function $u(t_0,x)$ coincides with some ground state solution of (1). Without loss of generality we assume that $u(t_0,x) = \hat{u}(x)$. Since $\frac{\partial}{\partial t}\hat{u}(x) = \frac{\partial}{\partial t}u(t_0,x) = 0$, from the uniqueness of the weak solution to (1) - (3) we get $u(t,x) = u(t_0,x) = \hat{u}(x)$ for every $t \in [0,T_m)$, $x \in \mathbb{R}^n$. Hence from (9) it follows that $I(t) = I(t_0) = 0$ for every $t \in [0,T_m)$ and for $t = 0$ we get $I(0) = 0$, $\|u_0\|_1 = \|\hat{u}\|_1 \neq 0$, which contradicts the assumption $u_0 \in W$. Thus statement (i) in Theorem 2.4 is proved.

(ii) Suppose $u_0 \in V$. If $u(t_0,x) \notin V$ for some $t_0 \in (0,T_m)$, then either $u(t_0,x) \in W$ or $I(t_0) = 0$ and $\|u(t_0,\cdot)\|_1 \neq 0$. If $u(t_0,x) \in W$, then from (i) it follows that $u(t,x) \in W$ for every $t \in [0,T_m)$. When $t = 0$ we get $u_0(x) \in W$, which contradicts our assumption $u_0 \in V$. If $I(t_0) = 0$ and $\|u(t_0,\cdot)\|_1 \neq 0$, then $u(t_0,x)$ coincides with some ground state solution $\hat{u}(x)$ of (1). Since $\|u(t_0,\cdot)\|_1 \neq 0$ from the uniqueness result it follows that $u(t,x) = \hat{u}(x)$ for every $t \geq 0$. Hence $I(u(t,\cdot)) = I(\hat{u}) = 0$ for every $t \geq 0$, which contradicts our assumption $u_0 \in V$. Thus (ii) in Theorem 2.4 is proved. □

Remark 1. We rewrite the conservation law (5), (6) by means of (7) in the following way

$$E(0) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p_1+1}I(t) + \frac{p_1-1}{2(p_1+1)}\|u\|_1^2 + B(t), \tag{10}$$

where from (2) and (3)

$$\begin{aligned} B(t) = &\sum_{k=2}^{l} \frac{a_k(p_k-p_1)}{(p_k+1)(p_1+1)} \int_{\mathbb{R}^n} |u|^{p_k+1}\,dx \\ &+ \sum_{j=1}^{s} \frac{b_j(p_1-q_j)}{(q_j+1)(p_1+1)} \int_{\mathbb{R}^n} |u|^{q_j+1}\,dx \geq 0. \end{aligned} \tag{11}$$

Remark 2. If $E(0) = d$ then condition $I(0) < 0$ is equivalent to

$$\|u_0\|^2 > \frac{2(p_1+1)}{p_1-1}d - \frac{p_1+1}{p_1-1}\|u_1\|^2 - \|\nabla u_0\|^2 - \frac{2(p_1+1)}{p_1-1}B(0), \tag{12}$$

while condition $I(0) \geq 0$ is equivalent to

$$\|u_0\|^2 \leq \frac{2(p_1+1)}{p_1-1}d - \frac{p_1+1}{p_1-1}\|u_1\|^2 - \|\nabla u_0\|^2 - \frac{2(p_1+1)}{p_1-1}B(0).$$

For the proofs of our main results in Section 4 and Section 5 we need the following auxiliary statement.

Lemma 2.5. *Suppose $u(t,x)$ is the weak solution of (1) - (3) in the maximal existence time interval $[0, T_m)$, $0 < T_m \leq \infty$ and $E(0) = d$. If $I(0) < 0$, then*

$$I(t) < (p_1+1)\,(J(t) - d) \;\; \text{for } t \in [0, T_m). \tag{13}$$

The proof of Lemma 2.5 is identical with the proof of Lemma 2.3 in [25] and we omit it.

3. **Improved concavity method of Levine.** In the last decades the concavity method, introduced by Levine [15], is one of the powerful methods in the investigation of the finite time blow up of the solutions to nonlinear dispersive equations. The main idea of the concavity method is one to prove finite time blow up of the solutions to the ordinary differential inequality

$$\psi''(t)\psi(t) - \gamma\psi'^2(t) \geq 0,\;\; t \geq 0,\;\; \gamma > 1, \tag{14}$$

where $\psi(t)$ is a nonnegative, twice differentiable function for $t > 0$. When

$$\psi(0) > 0,\;\; \psi'(0) > 0 \tag{15}$$

then the solution $\psi(t)$ of (14) blows up for a finite time T^* and

$$T^* \leq \frac{\psi(0)}{(\gamma-1)\psi'(0)}.$$

In the applications to nonlinear dispersive equations usually $\psi(t)$ is some functional of the solution. For example, $\psi(t) = \int_{R^n} u^2(t,x)dx$ for Klein-Gordon equation. For fourth and sixth order double dispersive equations $\psi(t)$ is more complicated functional, including the L^2 norm of the solution and some additional terms.

Let us mention, that condition (15) is only sufficient one for finite time blow up of the solution $\psi(t)$ to (14). The question, which naturally arises, is whether a necessary and sufficient condition for blow up of $\psi(t)$ exists.

In order to give a satisfactory answer of this question, instead of inequality (14) we consider the following nonlinear ordinary differential equation

$$\psi''(t)\psi(t) - \gamma\psi'^2(t) = Q(t), \ t \in [0, T_m), \ 0 < T_m \le \infty, \ \gamma > 1, \tag{16}$$

$$Q(t) \in C([0, T_m)), \ Q(t) \ge 0, \ t \in [0, T_m). \tag{17}$$

Here the nonnegative, twice differentiable function $\psi(t)$ is defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \le \infty$. In the applications to nonlinear dispersive problems, equation (16) naturally appears instead of inequality (14). Since the nonnegative term $Q(t)$ can not be expressed by means of $\psi(t)$, this term has been neglected and (16) has been reduced to (14).

We recall the definition of blow up of a nonnegative function $\psi(t) \in C^2([0, T_m))$ at T_m.

Definition 3.1. The nonnegative function $\psi(t) \in C^2([0, T_m))$ blows up at T_m if

$$\limsup_{t \to T_m, t < T_m} \psi(t) = \infty. \tag{18}$$

Theorem 3.2. *Suppose $\psi(t) \in C^2([0, T_m))$ is a nonnegative solution to (16), (17) in the maximal existence time interval $[0, T_m)$, $0 < T_m \le \infty$. If $\psi(t)$ blows up at T_m, then $T_m < \infty$.*

Proof. **Step 1.** First we will show that

$$\text{there exists } \ b \in [0, T_m) \text{ such that } \psi'(b) > 0. \tag{19}$$

If not, then $\psi'(t) \le 0$ for every $t \in [0, T_m)$ and the estimate

$$0 \le \psi(t) \le \psi(0)$$

holds for every $t \in [0, T_m)$. Hence we get

$$\limsup_{t \to T_m, t < T_m} \psi(t) \le \psi(0),$$

which contradicts (18). Thus (19) holds.

Step 2. Now we will prove that $\psi(t) > 0$ for every $t \in [b, T_m)$. From (19) we have that $\psi(b) > 0$. Otherwise from (16) it follows that $\gamma\psi'^2(b) \le 0$, which contradicts (19). In order to prove that $\psi(t) > 0$ for every $t \in [b, T_m)$ we suppose by contradiction that there exists $t_0 \in (b, T_m)$ such that

$$\psi(t) > 0 \text{ for } t \in [b, t_0) \text{ and } \psi(t_0) = 0. \tag{20}$$

From (16), (17) and (20) we get

$$\psi''(t) = \left(\gamma\psi'^2(t) + Q(t)\right)\psi^{-1}(t) \ge 0 \text{ for } t \in [b, t_0),$$

i.e. $\psi(t)$ is a convex function for $t \in [b, t_0)$. Hence $\psi'(t) \ge \psi'(b) > 0$ for $t \in [b, t_0)$ and $\psi(t)$ is a strictly increasing function for $t \in [b, t_0)$. From the monotonicity of $\psi(t)$ we obtain the following impossible chain of inequalities $0 = \psi(t_0) > \psi(b) > 0$. Thus $\psi(t)$ is a positive function satisfying the estimate

$$\psi(t) \ge \psi(b) > 0 \text{ for every } t \in [b, T_m). \tag{21}$$

Additionally, from (16), (17) and (21) it follows that $\psi(t)$ is a convex function satisfying the inequality

$$\psi'(t) \ge \psi'(b) > 0 \text{ for } t \in [b, T_m). \tag{22}$$

Step 3. Let us prove that $T_m < \infty$. For this purpose we introduce the new function

$$z(t) = \psi^{1-\gamma}(t) \text{ for } t \in [b, T_m).$$

Straightforward computations give us

$$z'(t) = (1-\gamma)\psi^{-\gamma}(t)\psi'(t), \quad \frac{1}{1-\gamma}\psi^{1+\gamma}(t)z''(t) = \psi''(t)\psi(t) - \gamma\psi'^2(t) \geq 0 \tag{23}$$

and $z(t)$ satisfies the problem

$$\begin{aligned} &z''(t) = -(\gamma-1)Q(t)z^{\frac{\gamma+1}{\gamma-1}}(t) \text{ for } t \in [b, T_m), \\ &z(b) > 0, \ z'(b) < 0. \end{aligned} \tag{24}$$

Suppose that T_m is not finite, i.e. $T_m = \infty$. Then from (17), (24) it follows that

$$z''(t) \leq 0 \text{ for } t \geq b. \tag{25}$$

Integrating (25) twice from b to t we get

$$z(t) \leq z'(b)(t-b) + z(b).$$

Consequently, there exists a constant T^*,

$$b < T^* \leq b - \frac{z(b)}{z'(b)} = b + \frac{\psi(b)}{(\gamma-1)\psi'(b)} < \infty, \tag{26}$$

such that $z(T^*) = 0$, or equivalently $\psi(T^*) = \infty$, which contradicts our assumption that $T_m = \infty$. Theorem 3.2 is proved. □

The following necessary and sufficient condition for finite time blow up of the solution to the ordinary differential equation (16) is a key result in the investigation of the behavior of the solutions to nonlinear dispersive equations.

Theorem 3.3. *Suppose $\psi(t) \in C^2([0, T_m))$ is a nonnegative solution to (16) in the maximal existence time interval $[0, T_m)$, $0 < T_m \leq \infty$ and $Q(t) \in C([0, \infty))$, $Q(t) \geq 0$ for $t \geq 0$. Then $\psi(t)$ blows up at T_m if and only if (19) holds, i.e. there exists $b \in [0, T_m)$, such that $\psi'(b) > 0$. Moreover, the estimate*

$$T_m \leq b + \frac{\psi(b)}{(\gamma-1)\psi'(b)} < \infty \tag{27}$$

holds.

Proof. **(Necessity)** Suppose $\psi(t)$ blows up at T_m. Then (19) holds from *Step 1* in the proof of Theorem 3.2, while (27) is a consequence of the inequality (26) in *Step 3*.

(Sufficiency) Suppose (19) is satisfied. From *Step 2* and *Step 3* in the proof of Theorem 3.2 it follows that $T_m < \infty$.

If we assume by contradiction that $\psi(t)$ does not blow up at T_m, i.e. (18) fails, then

$$\limsup_{t \to T_m, t < T_m} \psi(t) < \infty. \tag{28}$$

From (22), (28) it follows that $\psi(t)$ is a strictly increasing and bounded function for $t \in [b, T_m)$ so that the limit of $\psi(t)$ for $t \to T_m$ exists and

$$\lim_{t \to T_m, t < T_m} \psi(t) = \psi_0 \geq 0, \ \psi_0 < \infty. \tag{29}$$

Integrating (24) from b to $t < T_m$ we get

$$z'(t) = z'(b) - (\gamma-1)\int_b^t Q(s)z^{\frac{\gamma+1}{\gamma-1}}(s)ds,$$

or equivalently, from (23)

$$\psi'(t)=\psi^{\gamma}(t)\left[\frac{\psi'(b)}{\psi^{\gamma}(b)}+(\gamma-1)^2\int_b^t Q(s)\psi^{-\gamma-1}(s)ds\right].$$

Thus from (21), (29) and the monotonicity of $\psi'(t)$ we have

$$\begin{aligned}\lim_{t\to T_m, t<T_m}\psi'(t)&=\psi_0^{\gamma}\left[\frac{\psi'(b)}{\psi^{\gamma}(b)}+(\gamma-1)^2\int_b^{T_m} Q(s)\psi^{-\gamma-1}(s)ds\right]\\&=\psi_1,\ 0<\psi_1<\infty.\end{aligned}$$

The initial value problem

$$\begin{aligned}&\varphi''(t)\varphi(t)-\gamma\varphi'^2(t)=Q(t) \text{ for } t\geq T_m,\\&\varphi(T_m)=\psi_0,\ \varphi'(T_m)=\psi_1\end{aligned}$$

has a classical solution $\varphi(t)\in C^2([T_m,T_m+\delta))$ for sufficiently small $\delta>0$. Hence the function

$$\tilde{\varphi}(t)=\begin{cases}\psi(t) & \text{for } t\in[0,T_m);\\ \varphi(t) & \text{for } t\in[T_m,T_m+\delta)\end{cases},$$

$\tilde{\varphi}(t)\in C^2([0,T_m+\delta))$, $\tilde{\varphi}(t)\geq 0$ for $t\in[0,T_m+\delta)$ is a classical, nonnegative solution of (16) in the interval $[0,T_m+\delta)$. This contradicts the choice of T_m. Hence $\psi(t)$ blows up at T_m and Theorem 3.3 is proved. □

4. **Finite time blow up of the solutions to nonlinear Klein-Gordon equation.** As a consequence of Theorem 3.2, Theorem 3.3 and Theorem 2.4 we have the following precise results for finite time blow up of the solutions to (1) - (3) in the critical case $E(0)=d$.

Theorem 4.1. *Suppose $u(t,x)$ is the weak solution of (1) – (3) with initial energy $E(0)=d$, defined in the maximal existence time interval $[0,T_m)$, $0<T_m\leq\infty$. If $I(0)<0$ and $p_l<(n+4)/n$ then $u(t,x)$ blows up at T_m if and only if*

$$\textit{there exists } \ b\in[0,T_m) \textit{ such that } (u(b,\cdot),u_t(b,\cdot))\geq 0. \tag{30}$$

Moreover, T_m is finite, i.e. $T_m<\infty$.

Proof. For the function $\psi(t)=\|u\|^2$, simple computations give us from (10) the identities

$$\begin{aligned}\psi'(t)&=2(u,u_t),\\ \psi''(t)&=2\|u_t\|^2-2I(t)\\&=(p_1+3)\|u_t\|^2-2(p_1+1)E(0)+(p_1-1)\|u\|_1^2+2(p_1+1)B(t).\end{aligned}\tag{31}$$

Hence $\psi(t)$ satisfies the following ordinary differential equation

$$\psi''(t)\psi(t)-\frac{p_1+3}{4}\psi'^2(t)=Q(t),\tag{32}$$

where

$$\begin{aligned}Q(t)=&(p_1+3)(\|u_t\|^2\|u\|^2-(u,u_t)^2)+2\left\{(p_1+1)(J(t)-d)-I(t)\right\}\|u\|^2\\&+2(p+1)B(t)\|u\|^2.\end{aligned}\tag{33}$$

From (11), (13) in Lemma 2.5 and the Cauchy-Schwartz inequality we have

$$Q(t)\geq 0 \text{ for } t\in[0,T_m).\tag{34}$$

Thus $\psi(t)$ is a solution to (16), (17) for $\gamma=(p_1+3)/4$ and $Q(t)\geq 0$ defined in (33).

(Necessity) Suppose $u(t,x)$ blows up at T_m. From Lemma 2.3 it follows that $\psi(t) = \|u\|^2$ blows up at T_m. Then from *Step 1* in the proof of Theorem 3.2 for $\psi(t) = \|u\|^2$, $\gamma = (p_1+3)/4$ and $Q(t) \geq 0$ defined in (33) there exists $b \in [0, T_m)$ such that $\psi'(b) = 2(u(b,\cdot), u_t(b,\cdot)) > 0$, i.e. (30) is satisfied.

(Sufficiency) Suppose (30) holds, but $u(t,x)$ does not blow up at T_m. From Theorem 2.1(i) it follows that $T_m = \infty$. From (31) and Theorem 2.4(ii) the function $\psi(t) = \|u\|^2$ is a strictly convex one, because $\psi''(t) = 2\|u_t\|^2 - 2I(t) > 0$. Thus (30) gives us the inequality

$$\psi'(t) > \psi'(b) \geq 0 \text{ for every } t \in (b, T_m). \tag{35}$$

From (35) there exists $b_1 \in (b, T_m)$, such that $\psi'(b_1) > 0$. According to Theorem 3.3 in the interval $[b_1, T_m)$ for $\psi(t) = \|u\|^2$, $\gamma = (p_1+3)/4$ and $Q(t) \geq 0$ defined in (33) it follows that $\|u\|^2$ blows up at T_m. Applying Theorem 3.2 we get $T_m < \infty$. Theorem 4.1 is proved. □

Remark 3. From the proof of Theorem 4.1 it is clear that the restriction $p_l < (n+4)/n$ for the nonlinear term (2), (3) is used only in the proof of the **(Necessity)** of Theorem 4.1. Let us note that the statement in the **(Sufficiency)** of Theorem 4.1 holds for every p_l satisfying (3), i.e. the assumption $p_l < (n+4)/n$ is superfluous.

Remark 4. Let us compare the condition (4) and the new one (30). The careful analysis of the necessary and sufficient conditions (30) in Theorem 4.1 and (4) shows that if (4) holds then (30) is also satisfied at the same time $t = b$. This conclusion follows from (12) in Remark 2 in case $t = b$. Conversely, if (30) holds at $t = b$ then necessarily (4) is satisfied at some time $b_1 \geq b$.

In the following theorem we give sufficient conditions for finite time blow up of the solutions to (1) - (3) in terms of the initial data u_0, u_1.

Theorem 4.2. *Suppose $u(t,x)$ is the weak solution of (1) - (3) with initial energy $E(0) = d$, defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \leq \infty$. Then the weak solution $u(t,x)$ blows up at T_m when one of the following conditions is fulfilled:*

(i) $(u_0, u_1) \geq 0$ *and* $I(0) < 0$;

(ii)

$$(u_0, u_1) < 0 \text{ and } \|u_0\|^2 \geq \frac{2(p_1+1)}{p_1-1} d - \frac{2}{\sqrt{p_1-1}} (u_0, u_1). \tag{36}$$

Moreover, T_m is finite, i.e. $T_m < \infty$.

Proof. (i) The proof of Theorem 4.2 (i) follows immediately from the sufficiency part of Theorem 4.1 and Remark 3 for $b = 0$ when $(u_0, u_1) > 0$. If $(u_0, u_1) = 0$ then from (31) and Theorem 2.4 we obtain that $\psi''(0) = 2\|u_1\|^2 - 2I(0) > 0$ and $\psi'(b) > \psi'(0) \geq 0$ for every $b > 0$. Since $\psi(t)$ satisfies (32) and (34) from Theorem 3.3 it follows that $\psi(t)$ blows up for a finite time, i.e. $u(t,x)$ blows up for a finite time.

(ii) Suppose (36) hold. Since

$$\begin{aligned}\|u_0\|^2 &\geq \frac{2(p_1+1)}{p_1-1} d - \frac{2}{\sqrt{p_1-1}} (u_0, u_1) \\ &> \frac{2(p_1+1)}{p_1-1} d - \frac{p_1+1}{p_1-1} \|u_1\|^2 - \|\nabla u_0\|^2 - \frac{2(p_1+1)}{p_1-1} B(0)\end{aligned}$$

from Remark 2 it follows that $I(0) < 0$.

In order to prove statement (ii) we suppose by contradiction that $u(t,x)$ does not blow up at T_m. Then from the local existence result, Theorem 2.1, it follows that $T_m = \infty$. Thus $u(t,x)$ is globally defined for every $t \geq 0$.

If (30) is satisfied, i.e. there exists $b \in [0, T_m)$ such that $(u(b,\cdot), u_t(b,\cdot)) \geq 0$, then from Theorem 4.1 $u(t,x)$ blows up at T_m, which contradicts our assumption. Hence $u(t,x)$ blows up at T_m and from Theorem 3.2 it follows that $T_m < \infty$. Thus statement (ii) is proved when (30) is fulfilled.

If (30) does not hold, then

$$\psi'(t) = 2(u, u_t) < 0 \text{ for every } t \geq 0. \tag{37}$$

From (31) the function $\psi(t) = \|u\|^2$ is a solution to the equation

$$\psi''(t) = \alpha\psi(t) - \beta + G(t) \text{ for } t \geq 0. \tag{38}$$

Here $\alpha = p_1 - 1 > 0$, $\beta = 2(p_1+1)E(0) > 0$ and

$$G(t) = (p_1+3)\|u_t\|^2 + (p_1-1)\|\nabla u\|^2 + 2(p_1+1)B(t) \geq 0,$$

because $B(t)$, given in (11), is a non negative function. Equation (38) has a unique classical solution

$$\begin{aligned} \psi(t) =& \frac{1}{2}\left(\psi(0) + \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha}\right) e^{\sqrt{\alpha}t} \\ &+ \frac{1}{2}\left(\psi(0) - \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha}\right) e^{-\sqrt{\alpha}t} \\ &+ \frac{\beta}{\alpha} + \frac{1}{\sqrt{\alpha}}\int_0^t G(s)\sinh(\sqrt{\alpha}(t-s))\,ds \end{aligned} \tag{39}$$

and

$$\begin{aligned} \psi'(t) =& \frac{\sqrt{\alpha}}{2}\left(\psi(0) + \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha}\right) e^{\sqrt{\alpha}t} \\ &- \frac{\sqrt{\alpha}}{2}\left(\psi(0) - \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha}\right) e^{-\sqrt{\alpha}t} \\ &+ \int_0^t G(s)\cosh(\sqrt{\alpha}(t-s))\,ds. \end{aligned} \tag{40}$$

From (39) and (40) we get

$$\psi(t) + \frac{1}{\sqrt{\alpha}}\psi'(t) - \frac{\beta}{\alpha} = \left(\psi(0) + \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha} + \frac{1}{\sqrt{\alpha}}\int_0^t G(s)e^{-\sqrt{\alpha}s}\,ds\right) e^{\sqrt{\alpha}t} \tag{41}$$

By means of (31) the function $h(t) = -(u, u_t)$ satisfies the equation

$$\begin{aligned} h'(t) + \varepsilon h(t) &= I(t) - \|u_t\|^2 - \varepsilon(u, u_t) \\ &= I(t) - \left(u_t + \frac{\varepsilon}{2}u, u_t + \frac{\varepsilon}{2}u\right) + \frac{\varepsilon^2}{4}\|u\|^2 =: g(t) \end{aligned}$$

Since

$$g(t) \leq \frac{\varepsilon^2}{4}\|u\|^2 \leq \frac{\varepsilon^2}{4}\|u_0\|^2,$$

we have the estimates

$$h(t) = h(0)e^{-\varepsilon t} + e^{-\varepsilon t}\int_0^t g(s)e^{\varepsilon s}\,ds \leq h(0)e^{-\varepsilon t} + \frac{\varepsilon^2}{4\varepsilon}\|u_0\|^2\left(1 - e^{-\varepsilon t}\right).$$

After the limit $t \to \infty$ in the above inequality, from (37), we get the inequalities

$$0 \le \limsup_{t\to\infty} h(t) \le \frac{\varepsilon}{4}\|u_0\|^2.$$

Thus we obtain

$$\lim_{t\to\infty}(u, u_t) = 0, \tag{42}$$

because ε is an arbitrary positive constant.

Since $\psi(t)$ is monotone decreasing and bounded from below with zero, after the limit $t \to \infty$ in (41) we get from (42) the identity

$$\lim_{t\to\infty}\psi(t) - \frac{\beta}{\alpha} = \lim_{t\to\infty}\left(\psi(0) + \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha} + \frac{1}{\sqrt{\alpha}}\int_0^t G(s)e^{-\sqrt{\alpha}s}\,ds\right)e^{\sqrt{\alpha}t} \tag{43}$$

Hence necessarily we have

$$\int_0^\infty G(s)e^{-\sqrt{\alpha}s}\,ds = -\sqrt{\alpha}\left(\psi(0) + \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha}\right)$$

and from L'Hospital's rule it follows that

$$\begin{aligned}\lim_{t\to\infty}\psi(t) - \frac{\beta}{\alpha} &= \lim_{t\to\infty}\left(\psi(0) + \frac{1}{\sqrt{\alpha}}\psi'(0) - \frac{\beta}{\alpha} + \frac{1}{\sqrt{\alpha}}\int_0^t G(s)e^{-\sqrt{\alpha}s}\,ds\right)e^{\sqrt{\alpha}t}\\ &= -\frac{1}{\alpha}\lim_{t\to\infty}G(t) \le 0,\end{aligned} \tag{44}$$

i.e.

$$\lim_{t\to\infty}\psi(t) \le \frac{\beta}{\alpha}.$$

Multiplying (38) with $\psi'(t)$ and integrating from 0 to t we obtain the identity

$$\psi'^2(t) = \alpha\left(\psi(t) - \frac{\beta}{\alpha}\right)^2 + 2\int_0^t G(s)\psi'(s)\,ds + K, \tag{45}$$

$$K = -\alpha\left(\psi(0) - \frac{\beta}{\alpha}\right)^2 + \psi'^2(0).$$

Since $\beta = 2(p_1+1)d$, $\alpha = p_1 - 1$, $\psi(0) = \|u_0\|^2$ and $\psi'(0) = 2(u_0, u_1)$, then the second inequality in (36) can be rewritten as

$$\psi(0) - \frac{\beta}{\alpha} \ge -\frac{1}{\sqrt{\alpha}}\psi'(0). \tag{46}$$

Thus from (37) it follows that $\psi'(0) < 0$ and (46) gives us

$$\psi(0) > \frac{\beta}{\alpha}. \tag{47}$$

Let us consider the case

$$\lim_{t\to\infty}\psi(t) < \frac{\beta}{\alpha}. \tag{48}$$

Since $\psi(t)$ is a strictly decreasing function for $t \in [0, \infty)$, from (47) and (48), there exists a point t_1, $t_1 \in (0, \infty)$, such that

$$\psi(t_1) = \frac{\beta}{\alpha}.$$

Then for $t = t_1$ in (45) we get

$$0 < \psi'^2(t_1) = 2\int_0^{t_1} G(s)\psi'(s)\,ds + K < K. \tag{49}$$

Now we consider the case

$$\lim_{t\to\infty}\psi(t)=\frac{\beta}{\alpha}.$$

After the limit $t\to\infty$ in (45) the equality (45) becomes

$$0=\lim_{t\to\infty}\psi'^2(t)=2\int_0^\infty G(s)\psi'(s)\,ds+K<K. \tag{50}$$

In both cases from (49) and (50) we have

$$K=-\alpha\left(\psi(0)-\frac{\beta}{\alpha}\right)^2+\psi'^2(0)>0.$$

The above inequality is satisfies if

$$\frac{\beta}{\alpha}+\frac{1}{\sqrt{\alpha}}\psi'(0)<\psi(0)<\frac{\beta}{\alpha}-\frac{1}{\sqrt{\alpha}}\psi'(0),$$

or equivalently

$$\frac{2(p_1+1)}{p_1-1}d+\frac{2}{\sqrt{p_1-1}}(u_0,u_1)<\|u_0\|^2<\frac{2(p_1+1)}{p_1-1}d-\frac{2}{\sqrt{p_1-1}}(u_0,u_1),$$

which contradicts condition (36). Thus $u(t,x)$ blows up at T_m and from Theorem 3.2 it follows that $T_m<\infty$. Theorem 4.2 is proved. □

Remark 5. The statement of Theorem 4.2(i) has been already proved in a different way for the nonlinear wave equation in a bounded domain, see e.g. [9, 26] and for nonlinear Klein-Gordon equation, see [16, 19]. In the present paper the proof of Theorem 4.2(i) is a consequence of Theorem 3.3.

Remark 6. Let the initial data satisfy conditions (36). Then from (12) it follows, that $I(0)<0$, i.e the assumption $I(0)<0$ is unnecessary in Theorem (4.2)(ii).

In the following corollary we reformulate the statements in Theorem 4.2. The requirement for the sign of the Nehari functional $I(0)$ is replaced by the assumptions on the initial data according to Remark 2.

Corollary 1. *Suppose $u(t,x)$ is the weak solution of (1) - (3) with initial energy $E(0)=d$, defined in the maximal existence time interval $[0,T_m)$, $0<T_m\le\infty$. Then the weak solution $u(t,x)$ blows up at T_m when the initial data satisfy one of the following conditions:*

(i)

$$(u_0,u_1)\ge 0 \text{ and } \|u_0\|^2>\frac{2(p_1+1)}{p_1-1}d-\frac{p_1+1}{p_1-1}\|u_1\|^2-\|\nabla u_0\|^2-\frac{2(p_1+1)}{p_1-1}B(0);$$

(ii)

$$(u_0,u_1)<0 \text{ and } \|u_0\|^2\ge\frac{2(p_1+1)}{p_1-1}d-\frac{2}{\sqrt{p_1-1}}(u_0,u_1).$$

Moreover, T_m is finite, i.e. $T_m<\infty$.

Below we compare the result in Theorem 4.2 (Corollary 1) with the result in [16] for the nonlinear term

$$f(u)=u^2+u^3. \tag{51}$$

Proposition 1. *Suppose $u(t,x)$ is the weak solution of (1) with initial energy $E(0)=d$, defined in the maximal existence time interval $[0,T_m)$, $0<T_m\le\infty$ and $f(u)=u^2+u^3$. Then $u(t,x)$ blows up at $T_m<\infty$ when one of the following conditions holds:*

(i) *(Theorem (4.2)(i),* [16, Theorem 1.3(3)]*)*

$$(u_0,u_1)\ge 0 \text{ and } \|u_0\|^2>6d-3\|u_1\|^2-\|\nabla u_0\|^2-\frac{1}{2}\int_{\mathbb{R}^n}u_0^4(x)\,dx;$$

(ii) *(Theorem (4.2)(ii))*

$$(u_0,u_1)<0 \text{ and } \|u_0\|^2\ge 6d-2(u_0,u_1);$$

(iii) *(*[16, Theorem 1.3(3)]*)*

$$(u_0,u_1)<0,$$

$$\frac{1}{2}\int_{\mathbb{R}^n}u_e^4(x)\,dx+\|\nabla u_e\|^2<3\|u_1\|^2+\|\nabla u_0\|^2+\frac{1}{2}\int_{\mathbb{R}^n}u_0^4(x)\,dx, \tag{52}$$

$$\begin{aligned}6d-3\|u_1\|^2-\|\nabla u_0\|^2-\frac{1}{2}\int_{\mathbb{R}^n}u_0^4(x)\,dx&<\|u_0\|^2\\ &\le 6d-\left(\frac{1}{2}\int_{\mathbb{R}^n}u_e^4(x)\,dx+\|\nabla u_e\|^2\right),\end{aligned} \tag{53}$$

where u_e satisfies conditions $I(u_e)=0$ and $J(u_e)=d$.

Proof. (i) and (ii) We apply Theorem 4.2 for $p_1=2$, $p_2=3$, $a_1=1$, $a_2=1$, $a_k=0$ for $k=3,\dots,l$, $b_j=0$ for $j=1,\dots,s$, $n\le 3$ and

$$B(t)=\frac{1}{12}\int_{\mathbb{R}^n}u^4(t,x)\,dx.$$

According to Theorem 4.2 and Corollary 1 the solution $u(t,x)$ of (1), (51) blows up for a finite time when the initial data satisfy one of the following conditions:

$$(u_0,u_1)\ge 0 \text{ and } \|u_0\|^2>6d-3\|u_1\|^2-\|\nabla u_0\|^2-\frac{1}{2}\int_{\mathbb{R}^n}u_0^4(x)\,dx;$$

$$(u_0,u_1)<0 \text{ and } \|u_0\|^2\ge 6d-2(u_0,u_1), \tag{54}$$

So the statements (i) and (ii) are proved. Note, that for $(u_0,u_1)\ge 0$ and $I(0)<0$ the result in Theorem 1.3(3) in [16] coincides with the statement in Proposition 1(i).

(iii) For $(u_0,u_1)<0$ Theorem 1.3(3) says that the solution blows up for finite time if

$$(u_0,u_1)<0,\ I(0)<0 \text{ and } \|u_0\|^2\le\|u_e\|^2, \tag{55}$$

where u_e satisfies conditions $I(u_e)=0$ and $J(u_e)=d$. Since the conservation law (6) gives us

$$d=\frac{1}{6}\|u_e\|_1^2+\frac{1}{12}\int_{\mathbb{R}^n}u_e^4(x)\,dx,$$

assumptions (55) are equivalent to

$$\begin{aligned}(u_0,u_1)<0,\ 6d-3\|u_1\|^2-\|\nabla u_0\|^2-\frac{1}{2}\int_{\mathbb{R}^n}u_0^4(x)\,dx&<\|u_0\|^2\\ &\le 6d-\left(\frac{1}{2}\int_{\mathbb{R}^n}u_e^4(x)\,dx+\|\nabla u_e\|^2\right).\end{aligned}$$

The above inequality holds only for

$$\frac{1}{2}\int_{\mathbb{R}^n}u_e^4(x)\,dx+\|\nabla u_e\|^2<3\|u_1\|^2+\|\nabla u_0\|^2+\frac{1}{2}\int_{\mathbb{R}^n}u_0^4(x)\,dx.$$

When the opposite inequality is satisfied, i.e.

$$6d - 3\|u_1\|^2 - \|\nabla u_0\|^2 - \frac{1}{2}\int_{\mathbb{R}^n} u_0^4(x)\,dx \geq 6d - \left(\frac{1}{2}\int_{\mathbb{R}^n} u_e^4(x)\,dx + \|\nabla u_e\|^2\right),$$

then the set of functions satisfying Theorem 1.3(3) in [16] is empty. In this case the finite time blow up of the solutions is possible only under conditions of Theorem (4.2)(ii), i.e. when (54) is satisfied.

However, if (52) holds, then the conditions for finite time of the solution to (1), (51) in Theorem (4.2)(ii) and Theorem 1.3(3) in [16] are completely different. Indeed from the inequality

$$6d - 2(u_0, u_1) > 6d - \left(\frac{1}{2}\int_{\mathbb{R}^n} u_e^4(x)\,dx + \|\nabla u_e\|^2\right),$$

it follows that the intervals for $\|u_0\|^2$ in assumptions (54) and (53) have no intersection points. Thus the result in Theorem (4.2)(ii) is a new one. □

5. Global existence and the asymptotic behavior of the solutions to the nonlinear Klein-Gordon equation.

5.1. Global existence of the solutions to the nonlinear Klein-Gordon equation.

Theorem 5.1. *Suppose $u(t,x)$ is the weak solution of (1) - (3) with initial energy $E(0) = d$, defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \leq \infty$, $p_l < (n+4)/n$ and $I(0) < 0$. Then $u(t,x)$ is globally defined for every $t \geq 0$, i.e. $T_m = \infty$, if and only if*

$$(u(t,\cdot), u_t(t,\cdot)) < 0 \text{ for every } t \geq 0. \tag{56}$$

Proof. **(Necessity)** Suppose $u(t,x)$ is defined for every $t \geq 0$, i.e. $T_m = \infty$. If (56) fails, then there exists $b \in [0,\infty)$ such that

$$(u(b,\cdot), u_t(b,\cdot)) \geq 0.$$

From Theorem 4.1 it follows that $u(t,x)$ blows up for finite time $T_m < \infty$, which contradicts our assumption $T_m = \infty$.

(Sufficiency) Suppose condition (56) is satisfied. Then $\|u(t,\cdot)\|$ is a strictly decreasing function and the following inequality holds

$$\|u(t,\cdot)\|^2 \leq \|u_0\|^2 \text{ for every } t \in [0, T_m).$$

From Lemma 2.3 it follows that

$$\|u(t,\cdot)\|_1^2 \leq K_0 < \infty \text{ for every } t \in [0, T_m)$$

and from the local existence result we get $T_m = \infty$. Thus $u(t,x)$ is defined for every $t \geq 0$ and Theorem 5.1 is proved. □

Remark 7. The growth condition $p_l < (n+4)/n$ for the nonlinear term (2), (3) is used only in the proof of the **(Sufficiency)** of Theorem 5.1. For the proof of the **(Necessity)** of Theorem 5.1, assumption $p_l < (n+2)/(n-2)$ in (3) is enough.

Let us formulate necessary conditions on the initial data for global existence of the solutions to (1) - (3).

Theorem 5.2. *Suppose $u(t,x)$ is the weak solution of (1) - (3) with initial energy $E(0) = d$, defined in the maximal existence time interval $[0, T_m)$, $0 < T_m \leq \infty$.*

(i) *If $I(0) \geq 0$ then $u(t,x)$ is globally defined for every $t \geq 0$, i.e. $T_m = \infty$;*

(ii) *If* $I(0) < 0$, *then a necessary condition for global existence of* $u(t,x)$ *for every* $t \geq 0$ *is*

$$(u_0, u_1) < 0 \text{ and } \|u_0\|^2 < \frac{2(p_1+1)}{p_1-1}d - \frac{2}{\sqrt{p_1-1}}(u_0, u_1). \tag{57}$$

Proof. The statement (i) in Theorem 5.2 has been already proved in [9, 16, 24, 27] when $u_0 \in W = \{u \in H^1(\mathbb{R}^n); I(0) > 0\} \cup \{0\}$. Since the condition $I(u_0) \geq 0$ is slightly more general then $u_0 \in W$, for completeness we give the proof.

(i) If $I(0) = 0$ and $\|u_0\|_1 \neq 0$ then $u_0 \in \mathcal{N}$ and $J(u_0) \geq d$. From (6) and (7) we get

$$d = E(0) = \frac{1}{2}\|u_1\|^2 + J(u_0) \geq d,$$

i.e. $\|u_1\| = 0$ and $J(u_0) = d$. Since the function $u_0(x)$ is a solution to (1) - (3), from the uniqueness result it follows that $u(t,x) = u_0(x)$ for every $t \geq 0$, i.e. $u(t,x)$ is a global solution.

If $I(0) > 0$ or $I(0) = 0$ but $\|u_0\|_1 = 0$, i.e. $u_0 = 0$ for a.e. $x \in \mathbb{R}^n$, then $u_0 \in W$ and from Theorem 2.4 $I(t) \in W$ for every $t \in [0, T_m)$. From the conservation law (5), see also (10), we have the estimate

$$\begin{aligned}\|u(t,\cdot)\|_1^2 &= \frac{2(p_1+1)}{(p_1-1)}d - \frac{(p_1+1)}{(p_1-1)}\|u_t(t,\cdot)\|^2 - \frac{2}{(p_1-1)}I(t) - \frac{2(p_1+1)}{(p_1-1)}B(t) \\ &\leq \frac{2(p_1+1)}{(p_1-1)}d < \infty.\end{aligned}$$

Thus from statement (i) in the local existence result Theorem 2.1 it follows that $T_m = \infty$.

(ii) Suppose that $u(t,x)$ is defined for every $t \geq 0$, i.e. $T_m = \infty$. If (57) fails, i.e. one of the following conditions is satisfied,

$$(u_0, u_1) \geq 0$$

or

$$(u_0, u_1) < 0 \text{ and } \|u_0\|^2 \geq \frac{2(p_1+1)}{p_1-1}d - \frac{2}{\sqrt{p_1-1}}(u_0, u_1),$$

then from Theorem 4.2 it follows that $u(t,x)$ blows up for finite time. Thus $T_m < \infty$, which contradicts our assumption $T_m = \infty$. Theorem 5.2 is proved. □

5.2. **Asymptotic behavior of the global solutions to the nonlinear Klein-Gordon equation.** Let us mention that the asymptotic behavior of the global solution to the nonlinear wave equation in bounded domains has been studied in [9] and to the Klein-Gordon equation with quadratic-cubic nonlinearity - in [16].

In the following theorem we study the asymptotic behavior for $t \to \infty$ of the global solutions to (1) - (3) with critical initial energy $E(0) = d$ and $I(0) < 0$, i.e. when condition (ii) of Theorem 5.2 is satisfied.

Theorem 5.3. *Suppose* $u(t,x)$ *is a weak solution of* (1) - (3) *with initial energy* $E(0) = d$, *defined for every* $t \in [0,\infty)$, $p_l < (n+4)/n$ *and* $I(0) < 0$. *Then there exist a sequence of time* $t_m \to \infty$, *functions* $\hat{u}(x) \in \mathrm{H}^1(\mathbb{R}^n)$ *and* $\hat{v}(x) \in \mathrm{L}^2(\mathbb{R}^n)$, *such that* $u(t_m, x) \rightharpoonup \hat{u}(x)$ *weakly in* $\mathrm{H}^1(\mathbb{R}^n)$ *and* $u_t(t_m, x) \rightharpoonup \hat{v}(x)$ *weakly in* $\mathrm{L}^2(\mathbb{R}^n)$ *when* $m \to \infty$. *Moreover,*

(i) $\|\hat{v}\| = 0$ *and* $\liminf_{t\to\infty} \|u_t(t,\cdot)\| = 0$;

(ii) $\lim_{t\to\infty} I(u(t,\cdot)) = 0$;

(iii) $\lim_{t\to\infty} t(u(t,\cdot), u_t(t,\cdot)) = 0.$

Proof. Since the solution $u(t,x)$ is defined for every t, it follows that

$$\|u(t,\cdot\|_1 \le C_0 \text{ for every } t \ge 0. \tag{58}$$

Indeed, if (58) fails, then from the local existence result, Theorem 2.1 (i), we get $\lim_{t\to\infty} \|u(t,\cdot)\|_1 = \infty$, which contradicts the results in Lemma 2.3 and Theorem 3.2. From the embedding of $\mathrm{H}^1(\mathbb{R}^n)$ into $\mathrm{L}^p(\mathbb{R}^n)$ for $p > 2$ we obtain

$$\left| \int_{\mathbb{R}^n} \int_0^u f(y)\,dy\,dx \right| \le C_1 \text{ for } t \ge 0 \tag{59}$$

and from the conservation law (6)

$$\|u_t(t,\cdot)\| \le C_2 \text{ for } t \ge 0, \tag{60}$$

where the constants C_1 and C_2 depend on the parameters of the nonlinearity $f(u)$ as well as on the initial data. As a consequence of (58), (59) and (60), there exist a sequence $t_m \to \infty$ and functions $\hat{u}(x) \in \mathrm{H}^1(\mathbb{R}^n)$ and $\hat{v}(x) \in \mathrm{L}^2(\mathbb{R}^n)$, such that for every $\Phi \in \mathrm{H}^1(\mathbb{R}^n)$ and every $\Theta \in \mathrm{L}^2(\mathbb{R}^n)$ the following equations are true

$$\lim_{t_m\to\infty} \int_{\mathbb{R}^n} \int_0^{u(t_m,\cdot)} f(y)\,dy\ \Phi(x)\,dx = \int_{\mathbb{R}^n} \int_0^{\hat{u}(x)} f(y)\,dy\ \Phi(x)\,dx,$$

$$(u(t_m,\cdot), \Phi) \to (\hat{u}, \Phi),\ (u_t(t_m,\cdot), \Theta) \to (\hat{v}, \Theta) \text{ when } t_m \to \infty, \tag{61}$$

i.e. $u(t,x) \rightharpoonup \hat{u}(x)$ and $u_t(t,x) \rightharpoonup \hat{v}(x)$ for $t \to \infty$.

(i) We will prove that $\|\hat{v}\| = 0$. Since $I(t) < 0$ for every $t \ge 0$ (see Theorem 2.4), integrating (31) for $s > t \ge 0$ we obtain

$$0 < \int_t^s \left(\|u_t(\tau,\cdot)\|^2 - I(\tau)\right)\,d\tau = (u(s,\cdot), u_t(s,\cdot)) - (u(t,\cdot), u_t(t,\cdot)) \tag{62}$$

$$\begin{aligned} 0 < &\int_t^{t+1} \|u_t(\tau,\cdot)\|^2\,d\tau < \int_t^{t+1} \left(\|u_t(\tau,\cdot)\|^2 - I(\tau)\right)\,d\tau \\ =&(u(t+1,\cdot), u_t(t+1,\cdot)) - (u(t,\cdot), u_t(t,\cdot)). \end{aligned} \tag{63}$$

After the change of the variable $\tau = \lambda + t$ (63) becomes

$$0 < \int_0^1 \|u_t(\lambda + t,\cdot)\|^2\,d\lambda < (u(t+1,\cdot), u_t(t+1,\cdot)) - (u(t,\cdot), u_t(t,\cdot)). \tag{64}$$

Thus for every $s_m \to \infty$ from (42) and (64) it follows that

$$\lim_{m\to\infty} \int_0^1 \|u_t(s_m + \lambda,\cdot)\|^2\,d\lambda = 0.$$

As a consequence of Fatou's lemma we get

$$\liminf_{m\to\infty} \|u_t(s_m + \lambda,\cdot)\|^2 = 0 \text{ for a.e. } \lambda \in [0,1]. \tag{65}$$

By means of the weak convergence of $u_t(t,x)$ to $\hat{v}(x)$ in $\mathrm{L}^2(\mathbb{R}^n)$, i.e. (61), and the lower semicontinuity of the norm of $\mathrm{L}^2(\mathbb{R}^n)$, we get for some $\lambda_0 \in [0,1]$, $s_m = t_m - \lambda_0$ and (65) the final inequality

$$0 \le \|\hat{v}\|^2 \le \liminf_{m\to\infty} \|u_t(t_m,\cdot)\|^2 = 0,$$

i.e.

$$\|\hat{v}\|^2 = 0.$$

Thus we proved that every weak limit of $u_t(t_m,x)$ for $m\to\infty$ is zero and

$$\liminf_{t\to\infty}\|u_t(t,\cdot)\|=0.$$

(ii) From (44) and (38) it follows that

$$\begin{aligned}\lim_{t\to\infty}\psi(t)-&\frac{\beta}{\alpha}+\frac{1}{\alpha}\lim_{t\to\infty}G(t)=0,\\ \lim_{t\to\infty}\psi''(t)=&\alpha\left(\lim_{t\to\infty}\psi(t)+\frac{1}{\alpha}\lim_{t\to\infty}G(t)-\frac{\beta}{\alpha}\right)=0.\end{aligned}$$

Thus (31) and (i) in Theorem 5.3 give us

$$0\ge\limsup_{t\to\infty}I(t)\ge\liminf_{t\to\infty}I(t)=\liminf_{t\to\infty}\|u_t(t,\cdot)\|^2-\frac{1}{2}\lim_{t\to\infty}\psi''(t)=0.$$

Hence $\lim_{t\to\infty}I(t)=0$ and (ii) is proved.

(iii) From (42) and (62) after the limit $s\to\infty$ we obtain

$$\int_t^\infty\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)\,d\tau=-(u(t,\cdot),u_t(t,\cdot)) \tag{66}$$

and integrating (66) from 0 to ∞ it follows that

$$\begin{aligned}\int_0^\infty\int_t^\infty\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau dt=&-\int_0^\infty(u(t,\cdot),u_t(t,\cdot))dt\\ =&\frac{1}{2}\left(\|u_0\|^2-\lim_{t\to\infty}\|u(t,\cdot)\|^2\right).\end{aligned} \tag{67}$$

Applying Fubini's theorem, (67) becomes

$$\begin{aligned}\frac{1}{2}\left(\|u_0\|^2-\lim_{t\to\infty}\|u(t,\cdot)\|^2\right)=&\int_0^\infty\int_t^\infty\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau dt\\ =&\int_0^\infty\tau\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau.\end{aligned} \tag{68}$$

After the integration of (66) from 0 to s we have

$$\int_0^s\int_t^\infty\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau dt=\frac{1}{2}\left(\|u_0\|^2-\|u(s,\cdot)\|^2\right)$$

and Fubini's theorem implies

$$\begin{aligned}&\frac{1}{2}\left(\|u_0\|^2-\|u(s,\cdot)\|^2\right)=\\ &\int_0^s\int_0^\tau\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)dtd\tau+\int_s^\infty\int_0^s\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)dtd\tau\\ &=\int_0^s\tau\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau+s\int_s^\infty\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau.\end{aligned} \tag{69}$$

After the limit $s\to\infty$ in (69), from (66) and (68) it follows that

$$0=\lim_{s\to\infty}s\int_s^\infty\left(\|u_t(\tau,\cdot)\|^2-I(\tau)\right)d\tau=-\lim_{s\to\infty}s(u(s,\cdot),u_t(s,\cdot),$$

hence (iii) holds. Theorem 5.3 is proved. □

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