ELECTRONIC RESEARCH ARCHIVE
Volume **28**, Number **2**, June **2020**

doi:10.3934/era.2020034
pp. **651–669**

KIRCHHOFF-TYPE DIFFERENTIAL INCLUSION PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN AND STRONG DAMPING

Mingqi Xiang

College of Science, Civil Aviation University of China
Tianjin 300300, China

Binlin Zhang*

College of Mathematics and System Science
Shandong University of Science and Technology
Qingdao 266590, China

Die Hu

College of Science, Civil Aviation University of China
Tianjin 300300, China

Abstract. The aim of this paper is to investigate the existence of weak solutions for a Kirchhoff-type differential inclusion wave problem involving a discontinuous set-valued term, the fractional p-Laplacian and linear strong damping term. The existence of weak solutions is obtained by using a regularization method combined with the Galerkin method.

1. **Introduction.** In this paper, we consider the following initial boundary value problem

$$\begin{cases} u_{tt} + M([u]_{s,p}^p)(-\Delta)_p^s u + (-\Delta)^\alpha u_t + \mu(x,t) = f, \quad (x,t) \in Q_T, \\ \mu \in \Phi\big(x,t,u_t\big), \quad \text{a.e. } (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $s, \alpha \in (0,1)$, $1 < p < N/s$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < T < \infty$ is a given constant and $Q_T = \Omega \times (0,T)$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and Φ is a discontinuous and nonlinear set valued mapping by filling in jumps of a function $a(x,t,s) : Q_T \times \mathbb{R} \to \mathbb{R}$, $[u]_{s,p}$ is the Gagliardo seminorm defined by

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x,t) - u(y,t)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}.$$

2010 *Mathematics Subject Classification.* 35L86, 35L70, 35L20.
Key words and phrases. Fractional p-Laplacian, Kirchhoff-type problem, differential inclusion, Galerkin approximation.
* Corresponding author: Binlin Zhang.

Here $(-\Delta)^s_p$ is the fractional p-Laplace operator which, up to a normalization constant, is defined as

$$(-\Delta)^s_p\varphi(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dy, \quad x\in\mathbb{R}^N,$$

along functions $\varphi\in C_0^\infty(\mathbb{R}^N)$. Henceforward $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x\in\mathbb{R}^N$ and radius $\varepsilon>0$. In particular, if $p=2$ the fractional p-Laplacian $(-\Delta)^s_p$ reduces to the fractional Laplacian $(-\Delta)^s$.

Furthermore, we assume a and Φ satisfy the following assumptions.

(H1) For each $\xi\in\mathbb{R}$, a is a continuous function with respect to $(x,t)\in Q_T$ and for each $(x,t)\in Q_T$, $a\in L^\infty_{\rm loc}(\mathbb{R})$. Moreover, there exist positive constants a_0,a_1,a_2 such that

$$a_0|\xi|^q-a_1\le a(x,t,\xi)\xi,\quad |a(x,t,\xi)|\le a_2(|\xi|^{q-1}+1),\quad \text{for each } (x,t,\xi)\in Q_T\times\mathbb{R},$$

where $q\in(1,p_s^*)$ and $p_s^*=Np/(N-sp)$.

(H2) The set valued function $\Phi: Q_T\times\mathbb{R}\to 2^{\mathbb{R}}$ is obtained by filling in jumps of the function a in (H1) by means of the functions $\underline{a}_\varepsilon,\overline{a}_\varepsilon,\underline{a},\overline{a}:\mathbb{R}\to\mathbb{R}$ as follows

$$\underline{a}_\varepsilon(x,t,\xi)=\inf_{|\eta-\xi|\le\varepsilon}a(x,t,\eta),\quad \overline{a}_\varepsilon(x,t,\xi)=\sup_{|\eta-\xi|\le\varepsilon}a(x,t,\eta),\quad \underline{a}(x,t,\xi)=\lim_{\varepsilon\to0^+}\underline{a}_\varepsilon(x,t,\xi),$$
$$\overline{a}(x,t,\xi)=\lim_{\varepsilon\to0^+}\overline{a}_\varepsilon(x,t,\xi),\quad \Phi(x,t,\xi)=[\underline{a}(x,t,\xi),\overline{a}(x,t,\xi)].$$

(H3) $u_0\in W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$, $u_1\in L^2(\Omega)$, $f\in L^{q'}(Q_T)$, where $q'=\dfrac{q}{q-1}$.

Throughout the paper, without explicit mention, we assume that $M:[0,\infty)\to\mathbb{R}^+$ is continuous and verifies

(M_1) *there exists* $m_0>0$ *such that* $M(\tau)\ge m_0$ *for all* $\tau\ge0$.

A typical example of M is given by $M(t)=m_0+m_1t$ for $t\ge0$, where $m_0>0, m_1\ge0$. When M is of this type, problem (1) is said to be degenerate if $a=0$, while it is called non–degenerate if $a>0$. As for some recent existence results on Kirchhoff-type problems, we refer the interested readers to [5, 11]. Recently, the fractional Kirchhoff problems have received more and more attention. Some new existence results for fractional Kirchhoff problems were given, for example, in [18, 19, 20, 24, 25, 26, 27, 34].

In recent years, fractional Laplacian operator and related equations have an increasingly wide utilization in many important fields, as explained by Caffarelli in [4], Laskin in [15] and Vázquez in [36]. In [8], a stationary Kirchhoff variational equation was first proposed by Fiscella and Valdinoci as a model to study the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. Indeed, the stationary problem of (1) is a fractional version of a model, the so-called stationary Kirchhoff equation, which was introduced by Kirchhoff in [12] as a model to study elastic string vibrations. The literature on elliptic type problems involving fractional Laplacian and its variant is rich and very vast, see for example [8, 25, 26, 27, 35] and the references cited there.

Recently, the fractional hyperbolic problems with continuous nonlinearities have been studied by many researchers. For example, Pan, Pucci and Zhang [29] studied the initial-boundary value problem of degenerate Kirchhoff-type

$$u_{tt}+[u]_s^{2(\theta-1)}(-\Delta)^s u=|u|^{p-1}u \;\text{ in } \Omega\times\mathbb{R}^+, \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domian with Lipshcitz boundary, $\theta \in [1, 2_s^*/2)$, $p \in (2\theta - 1, 2_s^* - 1)$ and $[u]_s$ is the Gagliardo seminorm of u defined by

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x,t) - u(y,t)|^2}{|x - y|^{N+2s}} dxdy \right)^{1/2}.$$

Under some appropriate conditions, the authors obtained the global existence, vacuum isolating and blow up of solutions for (2) by using the Galerkin method combined with the potential wells theory. Moreover, the authors also investigated the global existence of solutions under the critical initial conditions. Furthermore, Pan *et al.* in [28] considered the following degenerate Kirchhoff equation with nonlinear damping term

$$u_{tt} + [u]_s^{2(\theta-1)}(-\Delta)^s u + |u_t|^{a-2}u_t + u = |u|^{b-2}u \ \text{ in } \Omega \times \mathbb{R}^+, \tag{3}$$

$2 < a < 2\theta < b < 2_s^*$. Under some natural assumptions, the authors obtained the global existence, vacuum isolating, asymptotic behavior and blow up of solutions for (3) by combining the Galerkin method with potential wells theory. In [20], Lin *et al.* studied the initial-boundary value problem of Kirchhoff wave equation

$$u_{tt} + [u]_s^{2(\theta-1)}(-\Delta)^s u = f(u) \ \text{ in } \Omega \times \mathbb{R}^+.$$

The authors established some sufficient conditions on initial data such that the solutions blow up in finite time for arbitrary positive initial energy by using an modified concavity method. Moreover, when $f(u) = |u|^{p-2}u$, the authors obtained the upper and lower bounds for blow up time. Concerning the related diffusion problems, for instance, we refer to [30, 31, 33, 39] for more results and methods.

It is worth mentioning that problem (1) can be regard as a fractional version of the initial-boundary value problem of the following equation

$$\begin{cases} u_{tt} - M(\|\nabla u\|^p_{L^p(\mathbb{R}^N)})\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t + \pi = f, & (x,t) \in Q_T, \\ \pi \in \Phi\big(x, t, u_t\big), \quad \text{a.e. } (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{4}$$

In [32], Park and Kim studied the existence of solutions for problem (4) without the Kirchhoff function M. The research on differential inclusions is an interesting topic in recent years. These problems arise mainly from physics and optimization, especially continuum mechanics, where non-monotone, multi-valued constitutive laws lead to a class of differential inclusions (variational inequalities). For a brief account of works on such variational inequalities we refer to [21] for the details. For the analysis of nonlinear second order or fourth order or six order hyperbolic partial differential equations with damping, we refer to the seminal work of Lions and Strauss [22], see also [9, 16, 17, 40, 41] for more recent results. In recent years, partial differential equations of hyperbolic type with variable exponent growth conditions were studied by Antontsev [1], see also Autuori and Pucci [2, 3] for Kirchhoff systems with $p(x)$-growth.

However, to the best of our knowledge, there are no papers that deal with the global existence and blow-up results for problems like (1). Inspired by above papers, we study the existence of global solutions that vanish at infinity or solutions that blow up in finite time for problem (1) involving the fractional p-Laplacian and discontinuous nonlinearity. Since our problem is nonlocal and the diffusion coefficient $M([u]_{s,p}^p)$ is a function, our discussion is more elaborate than the papers in the literature.

The rest of this paper is organized as follows. In section 2, we will give some necessary definitions and properties of fractional Sobolev spaces. In section 3, we obtain the existence of weak solutions for problem (1.1) by Galerkin's approximation method. In section 4, we give an example to illustrate our result.

2. Preliminaries.

To study the existence of solutions for equation (1), let us first recall some results related to the fractional Sobolev space $W_0^{s,p}(\Omega)$. For convenience, we shortly denote by $\|\cdot\|_2$ the norm of $L^2(\Omega)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 < p < \infty$ and set

$$W_0^{s,p}(\Omega) = \{u \in L^p(\Omega) : [u]_{s,p} < \infty,\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

where the Gagliardo seminorm $[u]_{s,p}$ is defined as

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p}.$$

Equipped with the norm

$$\|u\| = [u]_{s,p},$$

$W_0^{s,p}(\Omega)$ is a uniformly convex Banach space, and hence reflexive. The fractional critical exponent is defined by

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N; \\ \infty & \text{if } sp \geq N. \end{cases}$$

Moreover, the fractional Sobolev embedding states that $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous if $sp < N$ and $1 \leq q < p_s^*$. For more detailed account on the properties of $W_0^{s,p}(\Omega)$, we refer to [7].

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We define

$$W(Q_T) = \left\{ u \in L^q(Q_T) : u \in L^p(0, T; W_0^{s,p}(\Omega)) \right\},$$

with the norm

$$\|u\|_{W(Q_T)} = \|u\|_{L^q(Q_T)} + \left(\int_0^T \iint_{\mathbb{R}^{2N}} \frac{|u(x,t) - u(y,t)|^p}{|x - y|^{N+ps}} dxdydt \right)^{1/p}.$$

Remark 1. Following the standard proof of Sobolev spaces, we can prove that $W(Q_T)$ is a reflexive Banach space if $1 < p, q < \infty$.

In the following, we give a useful result which will be used to get the existence of solutions for problem (1).

Proposition 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $\{\omega_i\}_{i=1}^{\infty}$ be an orthogonal basis in $L^2(\Omega)$. Then for any $\varepsilon > 0$, there exists a constant $N_\varepsilon > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq \left(\sum_{i=1}^{N_\varepsilon} \left(\int_\Omega u\omega_i dx \right)^2 \right)^{\frac{1}{2}} + \varepsilon \|u\|_{W_0^{s,p}(\Omega)}$$

for all $u \in W_0^{s,p}(\Omega)$ where $2N/(N + 2s) \leq p < N/s$.

Proof. Following the idea of [13], we first show that for any $\delta > 0$ and $\varepsilon > 0$ there exists positive integer number $N_{\varepsilon,\delta}$ such that

$$\|u\|_2 \le (1+\delta)\left(\sum_{i=1}^{N_{\varepsilon,\delta}} \int_\Omega u(x)\omega_i(x)dx\right)^{1/2} + \varepsilon[u]_{s,p}. \tag{5}$$

Arguing by contradiction, we assume that there exists an $\epsilon_0 > 0$ and a sequence $\{u_n\}_{n=1}^\infty \subset W_0^{s,p}(\Omega)$ such that

$$\|u_n\|_2 > (1+\delta)\left(\sum_{i=1}^{n}\left(\int_\Omega u_n(x)\omega_i(x)dx\right)^2\right)^{1/2} + \varepsilon_0[u_n]_{s,p},$$

for any n. Let $v_n = u_n/\|u_n\|_2$. Then

$$1 = \|v_n\|_2 > (1+\delta)\left(\sum_{i=1}^{n}\left(\int_\Omega v_n(x)\omega_i(x)dx\right)^2\right)^{1/2} + \varepsilon_0[v_n]_{s,p}, \tag{6}$$

which means that

$$[v_n]_{s,p} \le \frac{1}{\varepsilon_0} \quad \text{for any } n = 1, 2, \cdots .$$

Since $2N/(N+2s) \le p < N/s$ and the embedding of $W_0^{s,p}(\Omega)$ into $L^2(\Omega)$ is compact, there exists a subsequence $\{v_{n_k}\}_{k=1}^\infty$ such that $\{v_{n_k}\}_{k=1}^\infty$ strongly converges to some function v in $L^2(\Omega)$. Hence $\|v\|_2 = \|v_n\|_2 = 1$. Since $\{\omega_i\}_{i=1}^\infty$ is an orthogonal basis of $L^2(\Omega)$, we also have

$$P_{n_k}v_{n_k} := \sum_{i=1}^{n_k}\left(\int_\Omega v_{n_k}(x)\omega_i(x)dx\right)\omega_i$$

converges strongly to v. Indeed, we have

$$\|v - P_{n_k}v_{n_k}\|_2 = \|P_{n_k}(v - v_{n_k}) + v - P_{n_k}v\|_2 \le \|v - v_{n_k}\|_2 + \|v - P_{n_k}v\|_2 \to 0$$

as $k \to \infty$. Using (6) and letting $k \to \infty$, we deduce $1 \ge 1 + \delta$, which is absurd. Thus, (5) holds true.

By (5), we obtain

$$\begin{aligned}
\|u\|_2 &\le \left(\sum_{i=1}^{N_{\varepsilon,\delta}} \int_\Omega u(x)\omega_i(x)dx\right)^{1/2} + \delta\left(\sum_{i=1}^{N_{\varepsilon,\delta}} \int_\Omega u(x)\omega_i(x)dx\right)^{1/2} + \varepsilon[u]_{s,p} \\
&\le \left(\sum_{i=1}^{N_{\varepsilon,\delta}} \int_\Omega u(x)\omega_i(x)dx\right)^{1/2} + \delta\|u\|_2 + \varepsilon[u]_{s,p} \\
&\le \left(\sum_{i=1}^{N_{\varepsilon,\delta}} \int_\Omega u(x)\omega_i(x)dx\right)^{1/2} + (C\delta + \varepsilon)[u]_{s,p},
\end{aligned}$$

where $C > 0$ is the embedding constant of $W_0^{s,p}(\Omega)$ into $L^2(\Omega)$. Therefore, the proof is complete. □

3. **Existence of weak solutions.** In this section, we prove the main result of this paper.

Definition 3.1. A pair of functions $u, \mu : Q_T \to \mathbb{R}$ is called a weak solution of (1.1), if

$$\begin{cases} u \in W(Q_T) \bigcap L^\infty(0,T;W_0^{s,p}(\Omega)) \bigcap C(0,T;W_0^{\alpha,2}(\Omega)), \\ u_t \in L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;W_0^{\alpha,2}(\Omega)) \bigcap L^q(Q_T), \\ f \in L^{q'}(Q_T), \quad \mu \in \Phi(x,t,u_t) \text{ a.e. } (x,t) \in Q_T, \end{cases}$$

and

$$\int_\Omega u_t(x,T)\varphi(x,T)dx - \int_{Q_T} u_t\varphi_t dxdt + \int_0^T M([u]_{s,p}^p)\langle u,\varphi\rangle_{s,p}dt$$
$$+ \int_0^T \langle u_t,\varphi\rangle_{\alpha,2}dt + \int_{Q_T}\mu\varphi dxdt = \int_{Q_T} f\varphi dxdt + \int_\Omega u_1\varphi(x,0)dx$$

hold for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$. Here $\langle u,\varphi\rangle_{s,p}$ and $\langle u_t,\varphi\rangle_{\alpha,2}$ are defined as

$$\langle u,\varphi\rangle_{s,p} := \iint_{\mathbb{R}^{2N}} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{N+ps}}(\varphi(x,t)-\varphi(y,t))dxdy$$

and

$$\langle u_t,\varphi\rangle_{\alpha,2} = \iint_{\mathbb{R}^{2N}} \frac{(u_t(x,t)-u_t(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2\alpha}}dxdy.$$

We need a regularization of a defined by

$$a_n(x,t,\xi) = n\int_{-\infty}^{\infty} a(x,t,\xi-\tau)\rho(n\tau)d\tau,$$

where $\rho \in C_0^\infty(-1,1)$, $\rho \geq 0$ and $\int_{-1}^1 \rho(\tau)d\tau = 1$.

Lemma 3.2. *The function a_n is continuous and satisfies the following inequalities*

$$a_n(x,t,\xi)\xi \geq \frac{a_0}{2^q}|\xi|^q - C_0$$

and

$$|a_n(x,t,\xi)| \leq 2^q a_2(|\xi|^{q-1}+1),$$

for each $(x,t,\xi) \in Q_T \times \mathbb{R}$, where $C_0 = \frac{a_0}{2} + a_1 + a_2 + a_2(\frac{2a_2}{a_0})^{q-1}$.

Proof. Since for each $(x,t) \in Q_T$, $a \in L^\infty_{\rm loc}(\mathbb{R})$, it's easy to show that $a_n \in C(Q_T \times \mathbb{R})$ for each $n \in \mathbb{N}$. From assumption (H2) and Young's inequality, for each $(x,t,s) \in Q_T \times \mathbb{R}$, we have

$$\begin{aligned} a_n(x,t,\xi)\xi &= n\int_{-\infty}^{\infty} a(x,t,\xi-\tau)\xi\rho(n\tau)d\tau \\ &= \int_{-1}^1 a\left(x,t,\xi-\frac{\tau}{n}\right)\xi\rho(\tau)d\tau \\ &\geq a_0\int_{-1}^1 \left|\xi-\frac{\tau}{n}\right|^q \rho(\tau)d\tau - a_2\int_{-1}^1 \left|\xi-\frac{\tau}{n}\right|^{q-1}\rho(\tau)d\tau - a_1 - a_2 \end{aligned}$$

$$\geq \frac{a_0}{2}\int_{-1}^{1}\left|\xi-\frac{\tau}{n}\right|^q\rho(\tau)d\tau - a_2\left(\frac{2a_2}{a_0}\right)^{q-1} - a_1 - a_2$$
$$= \frac{a_0}{2}\left|\xi-\frac{\tau}{n}\right|^q - a_2\left(\frac{2a_2}{a_0}\right)^{q-1} - a_1 - a_2$$

where $\tau_0 \in [-1,1]$. From the inequality

$$\left|\xi\right|^q \leq 2^{q-1}\left(\left|\xi-\frac{\tau}{n}\right|^q + \left|\frac{\tau_0}{n}\right|^q\right),$$

we obtain

$$a_n(x,t,\xi)\xi \geq \frac{a_0}{2^q}|\xi|^q - C_0, \tag{7}$$

where $C_0 = \frac{a_0}{2} + a_1 + a_2 + a_2(\frac{2a_2}{a_0})^{q-1}$. Similarly, by (H2), we get

$$|a_n(x,t,\xi)| \leq a_2\int_{-1}^{1}\left|\xi-\frac{\tau}{n}\right|^{q-1}\rho(\tau)d\tau + a_2 \leq 2^q a_2\big(|\xi|^{q-1}+1\big). \tag{8}$$

Thus, the proof is finished. □

We choose a sequence $\{\omega_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\Omega)$ such that $C_0^{\infty}(\Omega) \subset \overline{\bigcup_{n=1}^{\infty} V_n}^{C^1(\bar{\Omega})}$ and $\{\omega_j\}_{j=1}^{\infty}$ is a complete orthonormal basis in $L^2(\Omega)$, where $V_n = \operatorname{span}\{\omega_1,\omega_2,\ldots,\omega_n\}$, see [10, 24].

Since $C_0^{\infty}(\Omega) \subset \overline{\bigcup_{n=1}^{\infty} V_n}^{C^1(\bar{\Omega})}$, we have the following lemma.

Lemma 3.3. *For the function $u_0 \in W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$, there exists a sequence $\{\psi_n\}$ with $\psi_n \in V_n$ such that $\psi_n \to u_0$ in $W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$ as $n\to\infty$.*

Proof. For $u_0 \in W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$, there exists a sequence $\{v_n\}$ in $C_0^{\infty}(\Omega)$ such that $v_n \to u_0$ in $W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$. Since $\{v_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega) \subset \overline{\bigcup_{m=1}^{\infty} V_m}^{C^1(\overline{\Omega})}$, we can find a sequence $\{v_n^k\} \subset \bigcup_{m=1}^{\infty} V_m$ such that for each $n\in\mathbb{N}$, there holds $v_n^k \to u_n$ in $C^1(\overline{\Omega})$ as $k\to\infty$. For $\frac{1}{2^n}$, there exists $k_n \geq 1$ such that $\|v_n^{k_n} - u_n\|_{C^1(\overline{\Omega})} \leq \frac{1}{2^n}$. Thus

$$\|v_n^{k_n} - u_0\|_{W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)} \leq C\|v_n^{k_n} - v_n\|_{C^1(\overline{\Omega})} + \|v_n - u_0\|_{W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)}.$$

That is $v_n^{k_n} \to u_0$ in $W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$ as $n\to\infty$. Denote $u_n = v_n^{k_n}$. Since $u_n \in \bigcup_{m=1}^{\infty} V_m$, there exists V_{m_n}such that $u_n \in V_{m_n}$, without lost of generality, we assume that $V_{m_1} \subset V_{m_2}$ as $m_1 \leq m_2$. We suppose that $m_1 > 1$ and define ψ_n as follows: $\psi_n(x) = 0$, $n = 1,\cdots,m_1-1$; $\psi_n = u_1, n = m_1,\cdots,m_2-1$; $\psi_n = u_2, n = m_2,\cdots,m_3-1;\cdots$, then we obtain the sequence $\{\psi_n\}$ and $\psi_n \to u_0$ in $W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$ as $n\to\infty$. □

The existence of weak solutions for problem (1.1) is proved by Galerkin's approximation method. We shall find the sequence of approximate solutions with the form

$$u_n(x,t) = \sum_{j=1}^{n}(\eta_n(t))_j\omega_j(x).$$

The unknown functions $(\eta_n(t))_j$ are determined by an ordinary differential system as follows:

$$\begin{cases}\eta''(t) + P_n(t, \eta(t), \eta'(t)) = F_n(t),\\ \eta(0) = U_{0n},\ \eta'(0) = U_{1n},\end{cases} \tag{9}$$

where $(U_{0n})_i = \int_\Omega \psi_n \omega_i dx$, $(U_{1n})_i = \int_\Omega \phi_n \omega_i dx$, $(F_n)_i = \int_\Omega f_n \omega_i dx$, $\psi_n \in V_n, \phi_n \in V_n, f_n \in C_0^\infty(Q_T)$, and $\psi_n \to u_0$ strongly in $W^{s,p}(\Omega) \bigcap W_0^{\alpha,2}(\Omega)$, $\phi_n \to u_1$ strongly in $L^2(\Omega)$, $f_n \to f$ strongly in $L^{q'(x,t)}(Q_T)$. Here the vector-valued function $P_n(t, \mu, \nu) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as:

$$\begin{aligned}&\big(P_n(t,\mu,\nu)\big)_i\\ &= M\Big([\sum_{j=1}^n \mu_j\omega_j]_{s,p}^p\Big)\langle\sum_{j=1}^n \mu_j\omega_j, \omega_i\rangle_{s,p} + \langle\sum_{j=1}^n \nu_j\omega_j, \omega_i\rangle_{\alpha,2} + \int_\Omega a_n\big(x, t, \sum_{j=1}^n \nu_j\omega_j\big)\omega_i dx,\\ &i = 1, \cdots, n,\end{aligned}$$

where $\mu = (\mu_1, \cdots, \mu_n)$ and $\nu = (\nu_1, \cdots, \nu_n)$.

Let $\eta'(t) = X(t), Y(t) = \big(\eta(t), X(t)\big)$ and $H_n(t, Y) = \big(X, F_n - P_n(t, \eta, X)\big)$. Then the problem (9) is transformed into the following problem

$$\begin{cases}Y'(t) = H_n(t, Y(t)),\\ Y(0) = (U_{0n}, U_{1n}).\end{cases} \tag{10}$$

The inequality (7) implies

$$\begin{aligned}P_n(t,\eta,X)X =& P_n(t,\eta,\eta')\eta'\\ =& M([u_n]_{s,p}^p)\langle u_n, \frac{\partial u_n}{\partial t}\rangle_{s,p} + \langle\frac{\partial u_n}{\partial t}, \frac{\partial u_n}{\partial t}\rangle_{\alpha,2} + \int_\Omega a_n\big(x, t, \frac{\partial u_n}{\partial t}\big)\frac{\partial u_n}{\partial t} dx\\ \geq& \frac{1}{p}\frac{d}{dt}[u_n]_{s,p}^p + [\frac{\partial u_n}{\partial t}]_{\alpha,2}^2 + \frac{a_0}{2^q}\int_\Omega |\frac{\partial u_n}{\partial t}|^q dx - C_0.\end{aligned}$$

From (10) and Young's inequality, we obtain

$$\begin{aligned}&Y'Y + \frac{1}{p}\frac{d}{dt}\widetilde{M}([u_n]_{s,p}^p) + [\frac{\partial u_n}{\partial t}]_{\alpha,2}^2 + \frac{a_0}{2^q}\int_\Omega |\frac{\partial u_n}{\partial t}|^q dx\\ \leq& |X||\eta(t)| + |F_n(t)||X| + C_0\\ \leq& \frac{1}{2}|X|^2 + \frac{1}{2}|\eta(t)|^2 + \frac{1}{2}|X|^2 + \frac{1}{2}|F_n(t)|^2 + C_0\\ \leq& |Y|^2 + \frac{1}{2}|F_n(t)|^2 + C_0,\end{aligned}$$

where $\widetilde{M}([u_n]_{s,p}^p) = \int_0^{[u_n]_{s,p}^p} M(\tau)d\tau$. Denote $E_n(t) = \frac{1}{2}|Y|^2 + \frac{1}{p}\widetilde{M}([u_n]_{s,p}^p)$. Then,

$$E_n'(t) \leq 2E_n(t) + \frac{1}{2}|F_n(t)|^2 + C_0.$$

This together with Gronwall's inequality yields that

$$\begin{aligned}E_n(t) \leq& E_n(0)e^{2t} + e^{2t}\int_0^t |F_n(t)|^2 e^{-2\tau}d\tau + \frac{C_0}{2}\left(e^{2t} - 1\right)\\ \leq& E_n(0)e^{2T} + e^{2T}\int_0^T |F_n(t)|^2 dt + \frac{C_0}{2}\left(e^{2T} - 1\right)\\ :=& C_n(T), \quad \text{for each } t \in [0, T].\end{aligned}$$

Thus, $|Y(t) - Y(0)| \le 2\sqrt{2C_n(T)}$. Denote

$$L_n = \max_{(t,Y)\in[0,T]\times B\left(Y(0),2\sqrt{2C_n(T)}\right)} |H_n(t,Y)|,\ \tau_n = \min\left\{T, \frac{2\sqrt{2C_n(T)}}{L_n}\right\},$$

where $B\big(Y(0), 2\sqrt{2C_n(T)}\big)$ is the ball of radius $2\sqrt{2C_n(T)}$ with center at the point $Y(0)$ in $\mathbb{R}^{2n}$. From the definition of $H(t,Y)$, $H(t,Y)$ is continuous with respect to (t,Y). By Peano's Theorem, we know that (10) admits a C^1 solution on $[0,\tau_n]$, that is, (9) has a C^2 solution on $[0,\tau_n]$ denoted by $\eta_n^1(t)$. Let $\eta(\tau_n), \dfrac{\partial\eta(\tau_n)}{\partial t}$ be the initial value of problem (9), then we can repeat the above process and get a C^2 solution $\eta_n^2(t)$ on $[\tau_n, 2\tau_n]$. Without lost of generality, we assume that $T = [\frac{T}{\tau_n}]\tau_n + (\frac{T}{\tau_n})\tau_n, 0 < (\frac{T}{\tau_n}) < 1$, where $[\frac{T}{\tau_n}]$ is the integer part of $\frac{T}{\tau_n}$, $(\frac{T}{\tau_n})$ is the decimal part of $\frac{T}{\tau_n}$. We can divide $[0,T]$ into $[(i-1)\tau_n, i\tau_n], i = 1, ..., L$ and $[L\tau_n, T]$ where $L = [\frac{T}{\tau_n}]$, then there exist C^2 solution $\eta_n^i(t)$ in $[(i-1)\tau_n, i\tau_n], i = 1, ..., L$ and $\eta_n^{L+1}(t)$ in $[L\tau_n, T]$. Therefore, we get a solution $\eta_n(t) \in C^2([0,T])$ defined by

$$\eta_n(t) = \begin{cases} \eta_n^1(t), & \text{if } t \in [0,\tau_n], \\ \eta_n^2(t), & \text{if } t \in (\tau_n, 2\tau_n], \\ \dots \\ \eta_n^L(t), & \text{if } t \in ((L-1)\tau_n, L\tau_n], \\ \eta_n^{L+1}(t), & \text{if } t \in (L\tau_n, T]. \end{cases}$$

Lemma 3.4. *(A priori estimate) There exists $C(T) > 0$ independent of n such that the following estimates*

$$\int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2 dx + [u_n(x,t)]_{s,p}^p + [u_n(x,t)]_{\alpha,2}^2 \le C(T), \quad \forall t \in [0,T],$$
$$\int_{Q_T} \Big|\frac{\partial u_n}{\partial t}\Big|^q dxdt + \int_0^T [u_n]_{s,p}^p dt + \int_0^T [\frac{\partial u_n}{\partial t}]_{\alpha,2}^2 dt \le C(T),$$

hold.

Proof. By (9), for each $1 \le i \le n$, we have

$$\int_\Omega \frac{\partial^2 u_n}{\partial t^2}\omega_i dx + M([u_n]_{s,p}^p)\langle u_n, \omega_i\rangle_{s,p} + \langle \frac{\partial u_n}{\partial t}, \omega_i\rangle_{\alpha,2} + \int_\Omega a_n(x,t,\frac{\partial u_n}{\partial t})\omega_i dx = \int_\Omega f_n\omega_i dx. \tag{11}$$

Multiplying (11) by $\frac{d}{dt}(\eta_n(t))_i$, then summing up i from 1 to n, we obtain

$$\int_\Omega \frac{\partial^2 u_n}{\partial t^2}\frac{\partial u_n}{\partial t} dx + M([u_n]_{s,p}^p)\langle u_n, \frac{\partial u_n}{\partial t}\rangle_{s,p} + \langle \frac{\partial u_n}{\partial t}, \frac{\partial u_n}{\partial t}\rangle_{\alpha,2} + \int_\Omega a_n(x,t,\frac{\partial u_n}{\partial t})\frac{\partial u_n}{\partial t} dx = \int_\Omega f_n \frac{\partial u_n}{\partial t} dx.$$

The inequality (7), (11) and Young's inequality imply

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2 dx + \frac{1}{p}\frac{d}{dt}\widetilde{M}([u_n]_{s,p}^p) + [\frac{\partial u_n}{\partial t}]_{\alpha,2}^2 + \frac{a_0}{2^{q+1}}\int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^q dx \le \left(\frac{2^{q+1}}{a_0}\right)^{\frac{1}{q-1}} \int_\Omega |f_n(x,t)|^{q'} dx + C_0|\Omega|. \tag{12}$$

Further,

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega}|\frac{\partial u_n(x,t)}{\partial t}|^2dx+\frac{1}{p}\widetilde{M}([u_n]_{s,p}^p)\right)$$
$$\leq\left(\frac{2^{q+1}}{a_0}\right)^{\frac{1}{q-1}}\int_{\Omega}|f_n(x,t)|^{q'}dx+C_0|\Omega|.$$

Thus,

$$\frac{1}{2}\int_{\Omega}|\frac{\partial u_n(x,t)}{\partial t}|^2dx+\frac{1}{p}\widetilde{M}([u_n]_{s,p}^p)$$
$$\leq\left(\frac{1}{2}\int_{\Omega}|\frac{\partial u_n(x,0)}{\partial t}|^2dx+\frac{1}{p}[u_n(x,0)]_{s,p}^p\right)\left(\frac{2^{q+1}}{a_0}\right)^{\frac{1}{q-1}}\int_0^t\int_{\Omega}|f_n(x,t)|^qdxdt$$
$$+C_0|\Omega|T. \tag{13}$$

Since $u_n(x,0)=\psi_n\to u_0$ strongly in $W^{s,p}(\Omega)\bigcap W_0^{\alpha,2}(\Omega)$, $\frac{\partial u_n(x,0)}{\partial t}=\phi_n\to u_1$ strongly in $L^2(\Omega)$ and $f_n\to f$ strongly in $L^{q'}(Q_T)$, we deduce from (13) that

$$\frac{1}{2}\int_{\Omega}|\frac{\partial u_n(x,t)}{\partial t}|^2dx+\frac{1}{p}\widetilde{M}([u_n]_{s,p}^p)\leq C(T),$$

which together with assumption (M_1) yields that

$$\frac{1}{2}\int_{\Omega}|\frac{\partial u_n(x,t)}{\partial t}|^2dx+[u_n]_{s,p}^p\leq C(T)\quad\text{for all } t\in[0,T].$$

Moreover, integrating (12) with respect to t over $(0,T)$, we have

$$\int_{Q_T}|\frac{\partial u_n}{\partial t}|^qdxdt+\int_0^T[\frac{\partial u_n}{\partial t}]_{\alpha,2}^2dt\leq C(T).$$

Furthermore, for each $t\in[0,T]$, by Hölder's inequality, we get

$$[u_n(x,t)]_{\alpha,2}^2=\iint_{\mathbb{R}^{2N}}\frac{\left(\int_0^t(\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial u_n(y,\tau)}{\partial\tau})d\tau+(\frac{\partial u_n(x,0)}{\partial\tau}-\frac{\partial u_n(y,0)}{\partial\tau})\right)^2}{|x-y|^{N+2\alpha}}dxdy$$
$$\leq 2T\int_0^T[\frac{\partial u_n}{\partial t}]_{\alpha,2}^2dt+2[u_n(x,0)]_{\alpha,2}^2.$$

Thus, we obtain that $[u_n(x,t)]_{\alpha,2}^2\leq C(T)$ for each $t\in[0,T]$. □

By Lemma 3.4, we have

Lemma 3.5. *The estimate*

$$\|u_n\|_{W(Q_T)}+\|a_n(x,t,\frac{\partial u_n}{\partial t})\|_{L^{q'}(Q_T)}\leq C(T),$$

holds uniformly with respect to n.

Proof. By (8), we have

$$\|a_n(x,t,\frac{\partial u_n}{\partial t})\|_{L^{q'(x,t)}(Q_T)}\leq C(T).$$

From the fractional Sobolev inequality, there holds

$$\|u_n\|_{L^q(\Omega)}\leq C[u_n]_{s,p}^p\leq C(T).$$

Furthermore, $\int_{Q_T}|u_n|^qdxdt\leq C(T)$. □

Theorem 3.6. *Under the conditions* (H1)–(H3)*, problem (1.1) has a weak solution.*

Proof. By Lemma 3.4 and Lemma 3.5, there exist a subsequence of $\{u_n\}_{n=1}^{\infty}$ still denoted by $\{u_n\}_{n=1}^{\infty}$ and $u, \mu : Q_T \to \mathbb{R}$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly} * \text{in } L^{\infty}(0,T;W_0^{s,p}(\Omega)) \bigcap L^{\infty}(0,T;W_0^{\alpha,2}(\Omega)),\\ u_n \rightharpoonup u & \text{weakly in } W(Q_T),\\ \dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly} * \text{in } L^{\infty}(0,T;L^2(\Omega)),\\ \dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly in } L^{q}(Q_T),\\ \dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly in } L^2(0,T;W_0^{\alpha,2}(\Omega)),\\ a_n(x,t,\dfrac{\partial u_n}{\partial t}) \rightharpoonup \pi & \text{weakly in } L^{q'(x,t)}(Q_T). \end{cases}$$

First, we prove that there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$ (still denoted by $\{u_n\}_{n=1}^{\infty}$) such that $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ strongly in $L^2(Q_T)$ and $u_n \to u$ strongly in $L^q(Q_T)$.

Since $(\eta_n'(t))_j = \int_\Omega \frac{\partial u_n}{\partial t}\omega_j dx$, by Lemma 3.4, $(\eta_n'(t))_j$ is uniformly bounded on $[0,T]$. For all $0 \le t_1 < t_2 \le T$, integrating (11) with respect to t from t_1 to t_2, we have

$$\int_\Omega \frac{\partial u_n(x,t_1)}{\partial t}\omega_j dx - \int_\Omega \frac{\partial u_n(x,t_2)}{\partial t}\omega_j dx + \int_{t_1}^{t_2}\left(\langle u_n,\omega_j\rangle_{s,p} + \langle \frac{\partial u_n(x,t_1)}{\partial t},\omega_j\rangle_{\alpha,2}\right)dt$$
$$+\int_{t_1}^{t_2}\int_\Omega a_n(x,t,\frac{\partial u_n}{\partial t})\omega_j dxdt = \int_{t_1}^{t_2}\int_\Omega f_n\omega_j dxdt.$$

Hölder's inequality, Lemmas 3.4–3.5 yield

$$\begin{aligned} &|(\eta_n'(t_1))_j - (\eta_n'(t_2))_j|\\ \le&\Big(\int_{t_1}^{t_2}[u_n]_{s,p}^{p-1}[\omega_j]_{s,p}dt + \int_{t_1}^{t_2}[u_n]_{\alpha,2}[\omega_j]_{\alpha,2}dt\\ &+\|a_n(x,t,\frac{\partial u_n}{\partial t})\|_{L^{q'}(Q_T)}\|\omega_j\|_{L^q(Q_{t_1}^{t_2})} + \|f_n\|_{L^{q'}(Q_T)}\|\omega_j\|_{L^q(Q_{t_1}^{t_2})}\Big)\\ \le&C(T)\left(\left(\int_{t_1}^{t_2}[\omega_j]_{s,p}^p dt\right)^{1/p} + \left(\int_{t_1}^{t_2}[\omega_j]_{\alpha,2}^p dt\right)^{1/2} + \|\omega_j\|_{L^{q(x,t)}(Q_{t_1}^{t_2})}\right)\\ \le&C(j,T)\max\left\{|t_1-t_2|^{\frac{1}{p}}, |t_1-t_2|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{q}}\right\}, \end{aligned}$$

where $Q_{t_1}^{t_2} = \Omega \times (t_1,t_2)$. Thus, the sequence $\{(\eta_n'(t))_j\}_{n=1}^{\infty}$ is uniformly bounded and equi-continuous for fixed j. By Ascoli-Arzela Theorem, for $j=1$, there exists a subsequence of $\{n\}$ denoted by $\{n_{1,k}\}$ such that $\{(\eta_{n_{1,k}}'(t))_1\}$ converges uniformly on $[0,T]$ to some continuous function $\zeta_1(t)$; for $j=2$, there exists a subsequence of $\{n_{1,k}\}$ denoted by $\{n_{2,k}\}$ such that $\{(\eta_{n_{2,k}}'(t))_2\}$ converges uniformly on $[0,T]$ to $\zeta_2(t)$;generally, for j, there exists a subsequence of $\{n_{j-1,k}\}$ denoted by $\{n_{j,k}\}$ such that $\{(\eta_{n_{j,k}}'(t))_j\}$ converges uniformly on $[0,T]$ to $\zeta_j(t)$;.... The diagonal procedure imply that there exists a sequence of $\{(\eta_{n_{k,k}}'(t))_j\}_{k=1}^{\infty}$ still denoted by $\{(\eta_n'(t))_j\}_{n=1}^{\infty}$ such that $\{(\eta_n'(t))_j\}$ converges uniformly on $[0,T]$ to $\zeta_j(t)$ for each $j = 1,2,\cdots$.

For $r \le n$ with $r \in \mathbb{N}$, by Lemma 3.4, we have

$$\sum_{j=1}^{r}|(\eta_n'(t))_j|^2 \le \int_\Omega |\frac{\partial u_n}{\partial t}|^2 dx \le C(T), \text{ for each } t \in [0,T].$$

Letting $n \to \infty$, we get

$$\sum_{j=1}^{r} |\zeta_j(t)|^2 \le C(T), \text{ for each } t \in [0,T].$$

Letting $r \to \infty$, we obtain

$$\sum_{j=1}^{\infty} |\zeta_j(t)|^2 \le C(T), \text{ for each } t \in [0,T].$$

Denote $\overline{u}(x,t) = \sum_{j=1}^{\infty} \zeta_j(t)\omega_j(x)$, then $\sup_{0\le t\le T} \|\overline{u}(x,t)\|_{L^2(\Omega)} \le C(T)$ and for each $j \in \mathbb{N}$, there holds

$$\lim_{n\to\infty} \int_\Omega \frac{\partial u_n}{\partial t}\omega_j dx = \int_\Omega \overline{u}\omega_j dx. \tag{14}$$

uniformly on $[0,T]$. For each $\varepsilon_1 > 0$ and $\phi \in L^2(\Omega)$, by the completeness of $\{\omega_j\}$, there exists a $m_0 > 0$ such that $\|\phi - \sum_{i=1}^{m_0}(\int_\Omega \phi\omega_i dx)\omega_i\|_{L^2(\Omega)} \le \varepsilon_1$. Thus,

$$\begin{aligned}
\left|\int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}\right)\phi dx\right| &\le \left\|\frac{\partial u_n}{\partial t} - \bar{u}\right\|_{L^2(\Omega)} \left\|\phi - \sum_{i=1}^{m_0}\left(\int_\Omega \phi\omega_i dx\right)\omega_i\right\|_{L^2(\Omega)} \\
&\quad + \left|\int_\Omega \left(\frac{\partial u}{\partial t} - \bar{u}\right)\sum_{i=1}^{m_0}\left(\int_\Omega \phi\omega_i dx\right)\omega_i dx\right| \\
&\le C(T)\varepsilon_1 + \left|\int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}\right)\sum_{i=1}^{m_0}\left(\int_\Omega \phi\omega_i dx\right)\omega_i dx\right|.
\end{aligned} \tag{15}$$

For the $\varepsilon_1 > 0$, by (14), there exists a $n_{\varepsilon_1} > 0$ such that

$$\left|\int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}\right)\omega_i dx\right| \le \frac{\varepsilon_1}{m_0}, \quad \text{for } n > n_{\varepsilon_1} \text{ and } i = 1,\dots,m_0.$$

From (15) and Hölder's inequality, we have

$$\begin{aligned}
\left|\int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}\right)\phi dx\right| &\le \left\|\frac{\partial u_n}{\partial t} - \bar{u}\right\|_{L^2(\Omega)} \left\|\phi - \sum_{i=1}^{m_0}\left(\int_\Omega \phi\omega_i dx\right)\omega_i\right\|_{L^2(\Omega)} \\
&\quad + \left|\int_\Omega \left(\frac{\partial u}{\partial t} - \bar{u}\right)\sum_{i=1}^{m_0}\left(\int_\Omega \phi\omega_i dx\right)\omega_i dx\right| \\
&\le C(T)\varepsilon_1 + \sum_{i=1}^{m_0}\left|\int_\Omega \phi\omega_i dx\right|\left|\int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}\right)\omega_i dx\right| \\
&\le \left(C(T) + \|\phi\|_{L^2(\Omega)}\right)\varepsilon_1, \quad \text{for } n > n_{\varepsilon_1}.
\end{aligned} \tag{16}$$

It follows from (16) that

$$\frac{\partial u_n}{\partial t} \rightharpoonup \overline{u} \text{ weakly in } L^2(\Omega). \tag{17}$$

uniformly on $[0,T]$ as $n \to \infty$. For each $\varphi \in C_0^\infty(Q_T)$, by Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_{Q_T} \left(\frac{\partial u_n}{\partial t} - \overline{u}\right)\varphi dxdt = 0.$$

By integration by parts, we get

$$\int_{Q_T} \frac{\partial u_n}{\partial t}\varphi dxdt = -\int_{Q_T} u_n \frac{\partial \varphi}{\partial t} dxdt.$$

Letting $n \to \infty$, we have

$$\int_{Q_T} \overline{u}\varphi dxdt = -\int_{Q_T} u\frac{\partial \varphi}{\partial t} dxdt, \quad \text{for } \varphi \in C_0^\infty(Q_T).$$

Thus, we obtain that $\overline{u} = \dfrac{\partial u}{\partial t}$. Moreover, for each $j \in \mathbb{N}$, Lemma 3.3 and Lebesgue's dominated convergence theorem yield

$$\lim_{n\to\infty} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right) \omega_j dx \right)^2 dt = 0.$$

Thus, for $\varepsilon > 0$, by Proposition 1, there exists a positive number N_ε independent of n such that

$$\begin{aligned} &\left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \\ \le & 2\sum_{i=1}^{N_\varepsilon} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right) \omega_i dx \right)^2 dt + 2\varepsilon^2 \int_0^T \left[\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right]^2_{\alpha,2} dt. \end{aligned}$$

From a discussion similar to that of (17), there is a $\widetilde{n}(\varepsilon) > 0$ such that

$$\left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \le C\varepsilon^2, \quad \text{for } n > \widetilde{n}(\varepsilon).$$

Thus, $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ strongly in $L^2(Q_T)$. Further, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ a.e. on Q_T.

As $u_n \in L^\infty(0,T; W_0^{\alpha,2}(\Omega))$ and $\frac{\partial u_n}{\partial t} \in L^2(Q_T)$, by the Lions-Aubin Lemma (see Lions [21]), there exists a subsequence of $\{u_n\}$ still labelled by $\{u_n\}$ such that $u_n \to u$ strongly in $L^2(Q_T)$ and a.e. on Q_T. Since $1 \le q < p_s^* := Np/(N-sp)$, by Lemma 3.4 we have

$$\|u_n\|_{L^{p_s^*}(\Omega)} \le C\|u\|_{W^{s,p}(\Omega)} \le C(T).$$

Furthermore, $\int_0^T \int_\Omega |u_n|^{p_s^*} dxdt \le C(T)$. For each measurable subset $U \subset Q_T$ with $|U| \le 1$, Hölder's inequality yields

$$\begin{aligned} \int_U |u_n|^q dxdt \le & 2\||u_n|\|_{L^{\frac{p_s^*}{q}}(Q_T)} \|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(U)} \\ \le & C(T)\|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(U)} \\ \le & C(T)|U|^{\frac{p_s^*-q}{p_s^*}}. \end{aligned}$$

Thus, the sequence $\{|u_n|^q\}_{n=1}^\infty$ is equi-integrable on $L^1(Q_T)$. Vitali Theorem implies that $\lim_{n\to\infty} \int_{Q_T} |u_n - u|^q dxdt = 0$, that is to say, $u_n \to u$ strongly in $L^q(Q_T)$.

For each $u \in L^p(0,T; W_0^{s,p}(\Omega))$, we define a linear functional $\mathcal{L} : L^p(0,T; W_0^{s,p}(\Omega)) \to \mathbb{R}$ as:

$$\langle \mathcal{L}(u), v\rangle = \int_0^T M([u]^p_{s,p}) \langle u, v\rangle^p_{s,p} dt$$

for all $v \in L^p(0,T;W_0^{s,p}(\Omega))$. By Hölder's inequality, we have

$$|\langle \mathcal{L}(u), v\rangle| \le C \left(\int_0^T [u]_{s,p}^p dt\right)^{(p-1)/p} \left(\int_0^T [v]_{s,p}^p dt\right)^{1/p}.$$

This means that $\mathcal{L}(u)$ is a bounded linear functional on $L^p(0,T;W_0^{s,p}(\Omega))$. Thus,

$$\|\mathcal{L}(u_n)\| \le C,$$

where $C>0$ independent of n. Further, up to a subsequence we assume that there exists $\chi \in (L^p(0,T;W_0^{s,p}(\Omega)))^*$ such that

$$\mathcal{L}(u_n) \rightharpoonup \chi \text{ weakly} * \text{ in } (L^p(0,T;W_0^{s,p}(\Omega)))^*.$$

Here $(L^p(0,T;W_0^{s,p}(\Omega)))^*$ denotes the dual space of $L^p(0,T;W_0^{s,p}(\Omega))$. Then,

$$\lim_{n\to\infty} \langle \mathcal{L}(u_n), v\rangle = \langle \chi, v\rangle$$

for all $v \in L^p(0,T;W_0^{s,p}(\Omega))$. From (11), for $\varphi \in C^1(0,T,V_k)$ $(k \le n)$, we have

$$\begin{aligned}&\int_\Omega \frac{\partial u_n(x,\tau)}{\partial t}\varphi(x,\tau)dx - \int_\Omega \frac{\partial u_n(x,0)}{\partial t}\varphi(x,0)dx - \int_{Q_0^\tau} \frac{\partial u_n}{\partial t}\frac{\partial \varphi}{\partial t}dxdt\\ &+ \int_0^\tau \left(M([u_n]_{s,p}^p)\langle u_n,\varphi\rangle_{s,p} + \langle \frac{\partial u_n}{\partial t},\varphi\rangle_{\alpha,2}\right)dt + \int_{Q_0^\tau} a_n(x,t,\frac{\partial u_n}{\partial t})\varphi dxdt\\ &= \int_{Q_0^\tau} f_n\varphi dxdt, \end{aligned} \tag{18}$$

where $0<\tau\le T$. Letting $n\to\infty$ in (18), we obtain

$$\begin{aligned}&\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi(x,\tau)dx - \int_\Omega u_1\varphi(x,0)dx - \int_{Q_0^\tau}\frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}dxdt + \langle \chi,\varphi\rangle + \int_0^\tau \langle \frac{\partial u}{\partial t},\varphi\rangle_{\alpha,2}dt\\ &+ \int_{Q_0^\tau} \mu(x,t)\varphi dxdt = \int_{Q_0^\tau} f\varphi dxdt,\end{aligned} \tag{19}$$

where $\varphi \in C^1(0,T;V_k)$ $(k\in\mathbb{N})$. Since $C_0^\infty(\Omega) \subset \overline{\bigcup_{n=1}^\infty V_n}^{C^1(\overline{\Omega})}$, for each $\varphi \in C_0^\infty(\Omega)$, there exists a sequence $\{\varphi_{n_k}\}_{k=1}^\infty$ with $\varphi_{n_k}\in V_{n_k}$ such that $\varphi_{n_k}\to\varphi$ in $C^1(\overline{\Omega})$. Taking φ_{n_k} in (19) and letting $k\to\infty$, we get

$$\begin{aligned}&\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi dx - \int_\Omega u_1\varphi dx - \int_{Q_0^\tau}\frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}dxdt + \langle \chi,\varphi\rangle + \int_0^\tau \langle \frac{\partial u}{\partial t},\varphi\rangle_{\alpha,2}dt\\ &= \int_{Q_0^\tau} f\varphi dxdt, \quad \text{for } \varphi\in C_0^\infty(\Omega).\end{aligned} \tag{20}$$

Letting $\tau\to 0$, then we have

$$\lim_{\tau\to 0}\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi dx = \int_\Omega u_1\varphi dx, \quad \text{for } \varphi \in C_0^\infty(\Omega).$$

Similarly, for $t_0\in[0,T]$, there holds

$$\lim_{\tau\to t_0}\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi dx = \int_\Omega \frac{\partial u(x,t_0)}{\partial t}\varphi dx, \quad \text{for } \varphi\in C_0^\infty(\Omega).$$

Furthermore, we obtain that $\frac{\partial u(x,0)}{\partial t} = u_1(x)$ for $x\in\Omega$.

Since $u\in L^\infty(0,T;W_0^{\alpha,2}(\Omega))$ and $\frac{\partial u}{\partial t}\in L^2(0,T;W_0^{\alpha,2}(\Omega))$, we can assume that $u\in C(0,T;W_0^{\alpha,2}(\Omega))$ (see Lions [21]). By Lemma 3.4 and the embedding $W_0^{\alpha,2}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, we deduce that $\int_\Omega u_n^2(x,T)dx \le C(T)$. Thus, there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and a function $\widehat{u}$ such that $u_n(x,T) \rightharpoonup \widehat{u}$ weakly in $L^2(\Omega)$. For each $\varphi \in C_0^\infty(\Omega)$ and $\eta \in C^1([0,T])$, we have

$$\int_{Q_T} \frac{\partial u_n}{\partial t}\varphi\eta dxdt = \int_\Omega u_n(x,T)\varphi\eta(T) - u_n(x,0)\varphi\eta(0)dx - \int_{Q_T} u_n\varphi\eta'(t)dxdt.$$

Letting $n \to \infty$, we get

$$\int_{Q_T} \frac{\partial u}{\partial t}\varphi\eta dxdt = \int_\Omega \widehat{u}\varphi\eta(T) - u_0\varphi\eta(0)dx - \int_{Q_T} u\varphi\eta'(t)dxdt.$$

Integration by parts yields

$$\int_\Omega (u(x,T) - \widehat{u})\varphi\eta(T)dx = \int_\Omega (u(x,0) - u_0)\varphi\eta(0)dx.$$

Choosing $\eta(T) = 1, \eta(0) = 0$ or $\eta(T) = 0, \eta(0) = 1$, we obtain that $\widehat{u} = u(x,T)$ and $u(x,0) = u_0(x)$ for $x \in \Omega$. Similarly, we can prove that $u_n(x,T) \rightharpoonup u(x,T)$ weakly in $W_0^{\alpha,2}(\Omega)$ and

$$[u(x,T)]^2_{\alpha,2} \le \liminf_{n\to\infty}[u_n(x,T)]^2_{\alpha,2}. \tag{21}$$

Further, by the compactness of embedding $W_0^{\alpha,2}(\Omega)$ to $L^2(\Omega)$, we get $u_n(x,T) \to u(x,T)$ strongly in $L^2(\Omega)$. Taking $\varphi = u_k$ in (19), then letting $k \to \infty$, we obtain

$$\begin{aligned}\int_\Omega \frac{\partial u(x,T)}{\partial t}u(x,T) - u_1u_0dx - \int_{Q_T} |\frac{\partial u}{\partial t}|^2dxdt + \langle\chi, u\rangle + \int_0^T \langle\frac{\partial u}{\partial t}, u\rangle_{\alpha,2}dt \\ + \int_{Q_T} \mu udxdt = \int_{Q_T} fudxdt.\end{aligned} \tag{22}$$

Finally, we prove that $\pi \in \Phi(x,t,\frac{\partial u}{\partial t})$ a.e. on Q_T and $u_n \to u$ strongly in $L^p(0,T;W_0^{s,p}(\Omega))$. Since $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ a.e. on Q_T, for each $\delta > 0$, by Lusin's theorem and Egoroff's theorem, we can choose a subset $E_\delta \subset Q_T$ such that meas$(E_\delta) < \delta$, $\frac{\partial u}{\partial t} \in L^\infty(Q_T \setminus E_\delta)$ and $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ uniformly on $Q_T \setminus E_\delta$. Then, for each $0 < \varepsilon < 1$, there exists a $K > 0$ such that

$$\left|\frac{\partial u_n(x,t)}{\partial t} - \frac{\partial u(x,t)}{\partial t}\right| < \frac{\varepsilon}{2}, \quad \text{for all } n > K \text{ and } (x,t) \in Q_T \setminus E_\delta.$$

If $|\frac{\partial u_n(x,t)}{\partial t} - \tau| < \frac{1}{n}$, then we have $|\frac{\partial u(x,t)}{\partial t} - \tau| < \varepsilon$, for all $n > \max\{K, \frac{2}{\varepsilon}\}$ and $(x,t) \in Q_T \setminus E_\delta$. From the definition of a_n, there holds

$$\begin{aligned}a_n\Big(x,t,\frac{\partial u_n(x,t)}{\partial t}\Big) &= n\int_{-\infty}^{\infty} a\big(x,t,\frac{\partial u_n(x,t)}{\partial t} - \tau\big)\rho(n\tau)d\tau \\ &= n\int_{-\infty}^{\infty} a(x,t,\tau)\rho\big(n(\frac{\partial u_n(x,t)}{\partial t} - \tau)\big)d\tau.\end{aligned}$$

Thus, $|\frac{\partial u_n(x,t)}{\partial t} - \tau| < \frac{1}{n}$, for each $(x,t) \in Q_T \setminus E_\delta$. Furthermore,

$$\underline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t}) \le a_n(x,t,\frac{\partial u_n}{\partial t}) \le \overline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t}),$$

for all $n > \max\{K, \frac{2}{\varepsilon}\}$ and $(x,t) \in Q_T \setminus E_\delta$. Let $\varphi \in L^\infty(Q_T)$ with $\varphi \ge 0$. Then,

$$\int_{Q_T\setminus E_\delta} \underline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t})\varphi dxdt \le \int_{Q_T\setminus E_\delta} a_n(x,t,\frac{\partial u_n}{\partial t})\varphi dxdt \le \int_{Q_T\setminus E_\delta} \overline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t})\varphi dxdt.$$

Letting $n \to \infty$ and using the weak convergence of $a_n(x,t,\frac{\partial u_n}{\partial t})$, we get

$$\int_{Q_T\setminus E_\delta} \underline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t})\varphi dxdt \le \int_{Q_T\setminus E_\delta} \mu\varphi dxdt \le \int_{Q_T\setminus E_\delta} \overline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t})\varphi dxdt.$$

By (H1), the definition of $\underline{a}_\varepsilon$ and $\overline{a}_\varepsilon$, we have

$$|\underline{a}_\varepsilon(x,t,\xi)| \le a_2\big(|\xi|^{q-1}+1\big)$$

and

$$|\overline{a}_\varepsilon(x,t,\xi)| \le a_2\big(|\xi|^{q-1}+1\big),$$

for each $x,t,\xi \in Q_T\times\mathbb{R}$. Thus, $\underline{a}_\varepsilon(x,t,\frac{\partial u_n}{\partial t})$ and $\overline{a}_\varepsilon(x,t,\frac{\partial u_n}{\partial t})$ are bounded on $Q_T\setminus E_\delta$. By $\underline{a} = \lim\limits_{\varepsilon\to 0}\underline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t})$, $\overline{a} = \lim\limits_{\varepsilon\to 0}\overline{a}_\varepsilon(x,t,\frac{\partial u}{\partial t})$ and Lebesgue's dominated convergence theorem, we obtain

$$\int_{Q_T\setminus E_\delta} \underline{a}(x,t,\frac{\partial u}{\partial t})\varphi dxdt \le \int_{Q_T\setminus E_\delta} \mu(x,t)\varphi dxdt \le \int_{Q_T\setminus E_\delta} \overline{a}(x,t,\frac{\partial u}{\partial t})\varphi dxdt. \quad (23)$$

It's easy to check that $\underline{a}$ and $\overline{a}$ still satisfy the same conditions imposed on a in (H2). Thus, $\underline{a}(x,t,\frac{\partial u}{\partial t})$ and $\underline{a}(x,t,\frac{\partial u}{\partial t})$ belong to the space $L^{q'}(Q_T)$. Letting $\delta\to 0$ in (23), we have

$$\int_{Q_T} \underline{a}(x,t,\frac{\partial u}{\partial t})\varphi dxdt \le \int_{Q_T} \mu(x,t)\varphi dxdt \le \int_{Q_T} \overline{a}(x,t,\frac{\partial u}{\partial t})\varphi dxdt. \quad (24)$$

Without loss of generality, we assume that $\big|\{(x,t)\in Q_T : \pi(x,t) < \underline{a}(x,t,\frac{\partial u}{\partial t})\}\big| > 0$. Taking

$$\varphi = \begin{cases} 1, & \mu(x,t) < \underline{a}(x,t,\frac{\partial u}{\partial t}),\\ 0, & \mu(x,t) \ge \underline{a}(x,t,\frac{\partial u}{\partial t}),\end{cases}$$

in the first inequality of (24), we have

$$0 < \int_{Q_T} (\underline{a}(x,t,\frac{\partial u}{\partial t}) - \mu(x,t))^+ dxdt \le 0,$$

where $(\underline{a}(x,t,\frac{\partial u}{\partial t}) - \mu(x,t))^+ = \max\{\underline{a}(x,t,\frac{\partial u}{\partial t}) - \mu(x,t), 0\}$. Thus, from the contradiction above, $\underline{a}(x,t,\frac{\partial u}{\partial t}) \le \mu(x,t)$ a.e. on Q_T. Similarly, we can deduce that $\mu(x,t) \le \overline{a}(x,t,\frac{\partial u}{\partial t})$ a.e. on Q_T.

Multiplying (11) by $(\eta_n(t))_j$ and summing up j from 1 to n, and integrating with respect to t from 0 to T, we have

$$\int_0^T\int_\Omega \frac{\partial^2 u_n}{\partial t^2} u_n dxdt + \int_0^T M([u_n]_{s,p}^p)[u_n]_{s,p}^p dt + \int_0^T \langle \frac{\partial u_n}{\partial t}, u_n\rangle_{\alpha,2} dt$$
$$+ \int_0^T\int_\Omega a_n\big(x,t,\frac{\partial u_n}{\partial t}\big)u_n dxdt = \int_0^T\int_\Omega f_n(x,t)u_n dxdt.$$

Thus,

$$0 \le \int_0^T M([u_n]_{s,p}^p)\left(\langle u_n, u_n-u\rangle_{s,p} - \langle u, u_n-u\rangle_{s,p}\right) dt$$
$$= \int_0^T\int_\Omega (f_n u_n - a_n(x,t,\frac{\partial u_n}{\partial t})u_n) dxdt - \int_0^T \langle \frac{\partial u_n}{\partial t}, u_n\rangle_{\alpha,2} dt - \int_\Omega \frac{\partial u_n(x.T)}{\partial t} u_n(x,T) dx$$

$$+\int_\Omega \frac{\partial u_n(x,0)}{\partial t} u_n(x,0)dx + \int_0^T \int_\Omega |\frac{\partial u_n}{\partial t}|^2 dxdt$$
$$-\int_0^T M([u_n]_{s,p}^p)\langle u_n, u\rangle_{s,p} dt + \int_0^T M([u_n]_{s,p}^p)\langle u, u_n - u\rangle_{s,p} dt.$$

By (21), (22), the weak convergence of $\frac{\partial u_n(x,T)}{\partial t} \rightharpoonup \frac{\partial u(x,T)}{\partial t}$ in $L^2(\Omega)$ and the strong convergence of $u_n(x,T) \to u(x,T)$ in $L^2(\Omega)$, we get

$$\limsup_{n\to\infty} \int_0^T M([u_n]_{s,p}^p)\left(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\right) dt$$
$$\le \int_0^T \int_\Omega (fu - \mu u) dxdt - \frac{1}{2}[u(x,T)]_{\alpha,2}^2 + \frac{1}{2}[u(x,0)]_{\alpha,2}^2 - \int_\Omega \frac{\partial u(x,T)}{\partial t} u(x,T)dx$$
$$+ \int_\Omega u_1 u_0 dx + \int_0^T \int_\Omega |\frac{\partial u}{\partial t}|^2 dxdt - \langle \chi, u\rangle = 0.$$

This together with (M_1) implies that

$$\lim_{n\to\infty} \int_0^T \left(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\right) dt = 0.$$

Similar to the discussion as in [37, 26], we get $u_n \to u$ strongly in $L^p(0,T; W_0^{s,p}(\Omega))$ and $\langle \chi, u\rangle = M([u]_{s,p}^p)[u]_{s,p}^p$. It follows from (19) that the theorem is proved. □

Remark 2. Obviously, if the function $a : Q_T \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $(x,t,s) \in Q_T \times \mathbb{R}$, then $\mu(x,t) = a(x,t,\frac{\partial u}{\partial t})$.

4. An example.

We consider the following problem

$$\begin{cases} u_{tt} + (m_0 + m_1[u]_{s,p}^p)(-\Delta)_p^s u + (-\Delta)^\alpha u_t + \mu = f, & (x,t) \in Q_T, \\ \mu \in \Phi\big(x,t,u_t\big), \quad \text{a.e. } (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{25}$$

where $m_0, m_1 > 0$ and the set valued function Φ is obtained by filling in jumps of the function a defined by

$$a(x,t,\xi) = \begin{cases} |\xi|^{q-2}\xi + \xi \ln \xi, & |\xi| \le 1, \\ |\xi|^{q-2}\xi + e^{-|\xi|}\xi, & |\xi| > 1, \end{cases}$$

After a simple calculation, we have

$$\underline{a}(x,t,\xi) = \begin{cases} |\xi|^{q-2}\xi + \xi \ln \xi, & |\xi| < 1, \\ 1, & \xi = 1, \\ -1 - e^{-1}, & \xi = -1, \\ |\xi|^{q-2}\xi + e^{-|\xi|}\xi, & |\xi| > 1, \end{cases}$$

and

$$\overline{a}(x,t,\xi) = \begin{cases} |\xi|^{q-2}\xi + \xi \ln \xi, & |\xi| < 1, \\ 1 + e^{-1}, & \xi = 1, \\ -1, & \xi = -1, \\ |\xi|^{q-2}\xi + e^{-|\xi|}\xi, & |\xi| > 1. \end{cases}$$

Then $\Phi(x,t,u_t) = [\underline{a}(x,t,u_t), \overline{a}(x,t,u_t)]$ a.e. on Q_T. Suppose that (H1) and (H4) are satisfied and let q be defined as in (H3). Then by Theorem 3.6, there exists a weak solution for problem (25).

Acknowledgments. The authors would like to express their gratitude to the referees for valuable comments and suggestions. M. Xiang was supported by the Natural Science Foundation of China (No. 11601515) and the Tianjin Youth Talent Special Support Program. B. Zhang was supported by the National Natural Science Foundation of China (No. 11871199), the Heilongjiang Province Postdoctoral Startup Foundation (LBH-Q18109), and the Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

REFERENCES

[1] S. Antontsev, Wave equation with $p(x,t)$-Laplacian and damping term: Existence and blow-up, *Differ. Equations Appl.*, **3** (2011), 503–525.
[2] F. V. Atkinson and F. A. Peletier, Elliptic equations with nearly critical growth, *J. Differential Equations*, **70** (1987), 349–365.
[3] G. Autuori, P. Pucci and M. C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, *Arch. Rational Mech. Anal.*, **196** (2010), 489–516.
[4] L. Caffarelli, Non-local diffusions, drifts and games, *Nonlinear Partial Differential Equations, Abel Symposia, Springer, Heidelberg*, **7** (2012), 37–52.
[5] S. T. Chen, B. L. Zhang and X. H. Tang, Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity, *Adv. Nonlinear Anal.*, **9** (2020), 148–167.
[6] J. Clements, Existence theorems for a quasilinear evolution equation, *SIAM J. Appl. Math.*, **26** (1974), 745–752.
[7] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.
[8] A. Fiscella and E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170.
[9] A. Friedman and J. Nečas, Systems of nonlinear wave equations with nonlinear viscosity, *Pac. J. Math.*, **135** (1988), 29–55.
[10] Y. Q. Fu and N. Pan, Existence of solutions for nonlinear parabolic problems with $p(x)$-growth, *J. Math. Anal. Appl.*, **362** (2010), 313–326.
[11] C. Ji, F. Fang and B. L. Zhang, A multiplicity result for asymptotically linear Kirchhoff equations, *Adv. Nonlinear Anal.*, **8** (2019), 267–277.
[12] G. Kirchhoff, *Vorlesungen über Mathematische Physik*, Mechanik, Teubner, Leipzig, 1883.
[13] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasi-Linear Equations of Parabolic Type*, Translations of Mathematical Monographs, 23. American Mathematical Society, Providence, R.I., 1968.
[14] R. Landes, On the existence of weak solutions for quasilinear parabolic initial boundary value problem, *Proc. Roy. Soc. Edinburgh Sect. A*, **89** (1981), 217–237.
[15] N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A*, **268** (2000), 298–305.
[16] W. Lian and R. Z. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632.
[17] W. Lian, R. Z. Xu, V. Rădulescu, Y. B. Yang and N. Zhao, Global well-posedness for a class of fourth order nonlinear strongly damped wave equations, *Adv. Calc. Var.*, (2019).
[18] S. H. Liang, D. D. Repovš and B. L. Zhang, Fractional magnetic Schrödinger-Kirchhoff problems with convolution and critical nonlinearities, *Math. Method. Appl. Sci.*, **43** (2020), 2473–2490.
[19] S. H. Liang, D. Repovš and B. L. Zhang, On the fractional Schrödinger-Kirchhoff equations with electromagnetic fields and critical nonlinearity, *Comput. Math. Appl.*, **75** (2018), 1778–1794.
[20] Q. Lin, X. T. Tian, R. Z. Xu and M. N. Zhang, Blow up and blow up time for degenerate Kirchhoff-type wave problems involving the fractional Laplacian with arbitrary positive initial energy, *Discrete Contin. Dyn. Syst. Ser. S*, (2019).

[21] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[22] J.-L. Lions and W. A. Strauss, Some non-linear evolution equations, *Bull. Soc. Math. Fr.*, **93** (1965), 43–96.

[23] T. F. Ma and J. A. Soriano, On weak solutions for an evolution equation with exponent nonlinearities, *Nonlinear Anal.*, **37** (1977), 1029–1038.

[24] M. Q. Xiang, V. D. Rădulescu and B. L. Zhang, Nonlocal Kirchhoff diffusion problems: Local existence and blow-up of solutions, *Nonlinearity*, **31** (2018), 3228–3250.

[25] G. Molica Bisci, V. D. Radulescu and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, 162. Cambridge University Press, Cambridge, 2016.

[26] M. Q. Xiang, V. D. Rădulescu and B. L. Zhang, A critical fractional Choquard-Kirchhoff problem with magnetic field, *Comm. Contem. Math.*, **21** (2019), 1850004, 36 pp.

[27] M. Q. Xiang, V. Rădulescu and B. L. Zhang, Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity, *Cal. Var. Partial Differential Equations*, **58** (2019), 27 pp.

[28] N. Pan, P. Pucci, R. Z. Xu and B. L. Zhang, Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms, *J. Evolution Equations*, **19** (2019), 615–643.

[29] N. Pan, P. Pucci and B. L. Zhang, Degenerate Kirchhoff-type hyperbolic problems involving the fractional Laplacian, *J. Evolution Equations*, **18** (2018), 385–409.

[30] N. Pan, B. L. Zhang and J. Cao, Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian, *Nonlinear Anal. Real World Appl.*, **37** (2017), 56–70.

[31] N. S. Papageorgiou, V. Rădulescu and D. Repovš, Relaxation methods for optimal control problems, *Bull. Math. Sci.*, (2020).

[32] J. Y. Park, H. M. Kim and S. H. Park, On weak solutions for hyperbolic differential inclusion with discontinuous nonlinearities, *Nonlinear Anal.*, **55** (2003), 103–113.

[33] P. Pucci, M. Q. Xiang and B. L. Zhang, A diffusion problem of Kirchhoff type involving the nonlocal fractional p-Laplacian, *Discrete Contin. Dyn. Syst.*, **37** (2017), 4035–4051.

[34] P. Pucci, M. Q. Xiang and B. L. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.*, **5** (2016), 27–55.

[35] G. Singh, Nonlocal perturbations of the fractional Choquard equation, *Adv. Nonlinear Anal.*, **8** (2019), 694–706.

[36] J. L. Vázquez, Nonlinear diffusion with fractional Laplacian operators, *Nonlinear Partial Differential Equations, Abel Symp., Springer, Heidelberg*, **7** (2012), 271–298.

[37] J. Clments, Existence theorems for a quasilinear evolution equation, *SIAM J. Appl. Math.*, **26** (1974), 745–752.

[38] M. Q. Xiang, B. L. Zhang and D. Yang, Multiplicity results for variable-order fractional Laplacian equations with variable growth, *Nonlinear Anal.*, **178** (2019), 190–204.

[39] R. Z. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763.

[40] R. Z. Xu, X. C. Wang, Y. B. Yang and S. H. Chen, Global solutions and finite time blow-up for fourth order nonlinear damped wave equation, *J. Math. Phys.*, **59** (2018), 061503, 27 pp.

[41] R. Z. Xu, M. Y. Zhang, S. H. Chen, Y. B. Yang and J. H. Shen, The initial-boundary value problems for a class of six order nonlinear wave equation, *Discrete Contin. Dyn. Syst.*, **37** (2017), 5631–5649.

Received December 2019; revised March 2020.

E-mail address: xiangmingqi_hit@163.com
E-mail address: zhangbinlin2012@163.com
E-mail address: 970640972@qq.com