REVISITED BILINEAR SCHRÖDINGER ESTIMATES WITH APPLICATIONS TO GENERALIZED BOUSSINESQ EQUATIONS

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Abstract. In this paper, our goal is to improve the local well-posedness theory for certain generalized Boussinesq equations by revisiting bilinear estimates related to the Schrödinger equation. Moreover, we propose a novel, automated procedure to handle the summation argument for these bounds.

1. **Introduction.** The focus of this article is to develop a local well-posedness[1] (LWP) theory for the Cauchy problem given by

$$\begin{cases} u_{tt} - \Delta u + \Delta^2 u \pm \Delta(u^2) = 0, \qquad u = u(x,t) : \mathbb{R}^n \times I \to \mathbb{R}, \\[2mm] u(x,0) \,=\, u_0(x), \qquad u_t(x,0) \,=\, u_1(x), \end{cases} \tag{1}$$

where $0 \in I \subseteq \mathbb{R}$ is an open interval and $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$. The differential equation above belongs to a family of equations called *generalized Boussinesq equations*, with the $1+1$-dimensional version being known as the *"good" Boussinesq equation*.

In fact, the $1+1$-dimensional Cauchy problem is the best understood so far, with Kishimoto [6] showing that it is LWP for $s \geq -1/2$ and ill-posed (IP) for $s < -1/2$. This result capped a sustained drive for this problem with contributors like Bona-Sachs [1], Linares [8], Fang-Grillakis [3], Farah [5], and Kishimoto-Tsugawa [7]. Thus, our interest here is in investigating the high-dimensional (i.e., $n \geq 2$) case of 1, for which, to our knowledge, the only available results are due to Farah [4] and Okamoto [9].

The former states that 1 is LWP for $u_0 \in H^s(\mathbb{R}^n)$, $u_1 = \Delta\phi$ with $\phi \in H^s(\mathbb{R}^n)$, and

$$s \geq \max\left\{ 0, \frac{n-4}{2} \right\}.$$

We make the remark that the index $(n-4)/2$ appears naturally in connection to our problem since, by ignoring the lower order term Δu, the equation is scale-invariant under the transformation

$$u \mapsto u_\lambda(x,t) = \lambda^{-2} u(\lambda^{-1}x, \lambda^{-2}t)$$

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[1]Here, well-posedness is meant in the Hadamard sense: existence, uniqueness, and continuity of the data-to-solution map in appropriate topologies.

and one has

$$\|u_\lambda(0)\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{\frac{n-4}{2}-s}\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}.$$

For the second result, Okamoto proved that 1 is IP for $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ when $s < -1/2$, in the sense that norm inflation occurs and, as a consequence, the associated flow map is discontinuous everywhere. Hence, based on this picture, one is naturally led to study what happens in the regime when

$$-\frac{1}{2} \le s < \max\left\{0, \frac{n-4}{2}\right\}.$$

In particular, is it the case that 1 is LWP for $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ with $s < 0$ when $n \ge 2$? Our main result provides a partial positive answer to this question.

Theorem 1.1. *If $n = 2$ or $n = 3$, then 1 is LWP for $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ with $-1/4 < s < 0$.*

The argument for this theorem is inspired by an approach due to Kishimoto-Tsugawa [7] (see also [6] and [9]), in which the first step consists in reformulating 1 as the Cauchy problem for a nonlinear Schrödinger equation with initial data in $H^s(\mathbb{R}^n)$. This is followed by setting up a contraction scheme for the integral version of this new Cauchy problem, where we use Bourgain functional spaces and corresponding linear and bilinear estimates.

The structure of the paper is as follows. In the next section, we start by introducing the notation and terminology used throughout the article and by performing the reformulation step. Also there, we detail the contraction scheme and reduce it to the proof of a family of bilinear estimates related to the Schrödinger equation. In section 3, we revisit work by Colliander-Delort-Kenig-Staffilani [2] and Tao [11] for this type of bounds, provide a unitary framework to tackle them, and derive results in previously unknown scenarios. In the final section, we discuss an innovative, automated method, based on a Python code, to deal with the summation component of the proof for the bilinear estimates, which might also be of independent interest.

2. **Preliminaries.**

2.1. **Notational conventions and terminology.** First, we agree to write $A \lesssim B$ in a certain setting when $A \le CB$ and $C > 0$ is a constant depending only upon fixed parameters which may change from one setting to another. Moreover, we write $A \sim B$ to denote that both $A \lesssim B$ and $B \lesssim A$ are valid. Next, we recall the notations $\langle a \rangle = (1 + |a|^2)^{1/2}$ (for any $a \in \mathbb{R}^n$),

$$\widehat{z}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} z(x)\,dx \quad \text{and} \quad \widehat{w}(\xi, \tau) = \int_{\mathbb{R}^n \times \mathbb{R}} e^{-i(x\xi + t\tau)} w(x, t)\,dx\,dt,$$

the last two representing the Fourier transform of $z = z(x)$ and the spacetime Fourier transform of $w = w(x, t)$, respectively. Finally, we let $\varphi = \varphi(t)$ denote the classical, smooth cutoff function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi \equiv 1$ on $[-1, 1]$ and $\mathrm{supp}(\varphi) \subseteq [-2, 2]$.

Following this, we define the Sobolev and Bourgain norms[2]

$$\|z\|_{H^s(\mathbb{R}^n)} := \|\langle\xi\rangle^s \widehat{z}(\xi)\|_{L^2_\xi(\mathbb{R}^n)}, \tag{2}$$

$$\|w\|_{X^{s,\theta}(\mathbb{R}^n\times\mathbb{R})} := \|\langle\xi\rangle^s \langle\tau-|\xi|^2\rangle^\theta \widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}(\mathbb{R}^n\times\mathbb{R})}, \tag{3}$$

for arbitrary $s, \theta \in \mathbb{R}$. For $T > 0$, we will also use the truncated norm

$$\|z\|_{X^{s,\theta}_T} := \inf_{w=z \text{ on } [0,T]} \|w\|_{X^{s,\theta}}.$$

Working directly with these norms, one can easily prove the classical bound

$$\|w\|_{L^\infty_t H^s_x} \lesssim \|w\|_{X^{s,\theta}} \tag{4}$$

and the inclusion $X^{s,\theta} \subset C(\mathbb{R}, H^s)$, both for all $s \in \mathbb{R}$ and $\theta > 1/2$.

2.2. **Reformulation step.** As mentioned in the introduction, we start the argument for Theorem 1.1 by rewriting 1 in the form of a Cauchy problem for a Schrödinger equation. For this purpose, we define as in [7]

$$v := u - i(1-\Delta)^{-1}u_t \quad \text{and} \quad v_0 := u_0 - i(1-\Delta)^{-1}u_1.$$

Straightforward calculations reveal that

$$\begin{cases} iv_t - \Delta v = H(v,\overline{v}) := \frac{\overline{v}-v}{2} \pm \omega(D)\left(\frac{v+\overline{v}}{2}\right)^2, & v = v(x,t): \mathbb{R}^n \times I \to \mathbb{C}, \\ v(x,0) = v_0(x), \end{cases} \tag{5}$$

where $\omega = \omega(D)$ is the spatial multiplier operator with symbol

$$\omega(\xi) = \frac{|\xi|^2}{1+|\xi|^2}.$$

Moreover, for an arbitrary $T > 0$, the map $(u, u_0, u_1) \mapsto (v, v_0)$ from

$$U := (C([0,T], H^s) \cap C^1([0,T], H^{s-2})) \times H^s \times H^{s-2}$$

to

$$V := C([0,T]; H^s) \times H^s$$

is Lipschitz continuous. Conversely, if v and v_0 satisfy 5, then, by letting

$$u = \frac{v+\overline{v}}{2}, \quad u_0 = \frac{v_0 + \overline{v_0}}{2}, \quad \text{and} \quad u_1 = (1-\Delta)\left(\frac{\overline{v_0}-v_0}{2i}\right),$$

it is easy to check that that u, u_0, and u_1 are all real-valued and they satisfy 1. Furthermore, noticing that

$$-2iu_t = (1-\Delta)(v-\overline{v}),$$

one deduces that the map $(v, v_0) \in V \mapsto (u, u_0, u_1) \in U$ is also Lipschitz continuous. Thus, LWP in $H^s \times H^{s-2}$ for 1 is equivalent to LWP in H^s for 5.

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[2]From here on out, for a functional space Y, we write either $Y = Y(\mathbb{R}^n)$ or $Y = Y(\mathbb{R}^n \times \mathbb{R})$ as the majority of such norms refers to these two particular situations.

2.3. **Setting up the contraction argument and reducing it to the proof of bilinear Schrödinger estimates.** In proving that 5 is LWP for $v_0 \in H^s$, we adopt the standard procedure and, using Duhamel's formula, write its integral version

$$v(t) = S(t)v_0 - i \int_0^t S(t-t')H(v(t'), \overline{v}(t')) \, dt', \qquad (6)$$

for which we set up a contraction argument using suitable $X^{s,\theta}$ spaces. Above, $S(t) = e^{-it\Delta}$ is the propagator for the linear Schrödinger equation $iw_t - \Delta w = 0$, i.e.,

$$w(t) = S(t)w(0), \qquad (\forall) \, t \in \mathbb{R}.$$

Remark 1. By comparison, Farah [4] writes the main equation as

$$u_{tt} + \Delta^2 u = \Delta(u \mp u^2)$$

and, using the Fourier transform and Duhamel's formula, derives

$$u(t) = \frac{S(t) + S(-t)}{2} \, u(0) + \frac{S(t) - S(-t)}{-2i\Delta} \, u_t(0)$$
$$+ \int_0^t \frac{S(t-t') - S(-t+t')}{2i} \left(-u(t') \pm u^2(t') \right) dt'.$$

Following this, he proves LWP for 1 by running a contraction argument for this integral formulation in functional spaces related to Strichartz-type estimates for the Schrödinger group $(S(t))_{t \in \mathbb{R}}$.

The next statement is our LWP result for 5, which, as we argued, implies Theorem 1.1.

Theorem 2.1. *For $n = 2$ or $n = 3$, if $\theta > 1/2$, $(\theta - 1)/2 < s < 0$, and $r \geq 1$, then, for any $\|v_0\|_{H^s} \leq r$, there exist $T \sim r^{-4/(2s-n+4)}$ and $v \in X_T^{s,\theta} \cap C([0,T], H^s)$ solving the integral equation 6 on $[0,T]$ with the data-to-solution map*

$$v_0 \in \{z; \, \|z\|_{H^s} \leq r\} \mapsto v \in C([0,T], H^s) \cap X_T^{s,\theta}$$

being Lipschitz continuous. Moreover, this solution is unique in the class of $X_T^{s,\theta} \cap C([0,T], H^s)$ solutions for 6.

As is always the case with this type of results, they are the joint outcome of a set of estimates which are used in the context of a contraction scheme. For the above theorem, these bounds are

$$\|z_\lambda\|_{H^s} \lesssim \lambda^{\frac{n}{2}-s-2}\|z\|_{H^s}, \qquad (7)$$

$$\|w\|_{X^{s,\theta-1}} + \|\overline{w}\|_{X^{s,\theta-1}} \lesssim \|w\|_{X^{s,\theta}}, \qquad (8)$$

$$\left\| \varphi(t) \left(S(t)z - i \int_0^t S(t-t')F(\cdot, t') \, dt' \right) \right\|_{X^{s,\theta}} \lesssim \|z\|_{H^s} + \|F\|_{X^{s,\theta-1}}, \qquad (9)$$

and

$$\|\omega_\lambda(D)(\overline{u}\,\overline{v})\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}}, \qquad (10)$$

$$\|\omega_\lambda(D)(u\,v)\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}}, \qquad (11)$$

$$\|\omega_\lambda(D)(\overline{u}\,v)\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}}, \qquad (12)$$

where $\lambda \geq 1$ is an arbitrary scaling parameter, $z_\lambda = z_\lambda(x) = \lambda^{-2}z(\lambda^{-1}x)$, and the multiplier operator $\omega_\lambda = \omega_\lambda(D)$ has the symbol $\omega_\lambda(\xi) = \omega(\lambda\xi)$. With the exception of the bilinear estimates, the other ones are by now somewhat classical

with 7 and 8 being directly argued from 2 and 3, while 9 appeared in a more general setting in Tao's monograph [12] (Proposition 2.12). Furthermore, the way in which we combine 7-12 to derive Theorem 2.1 mirrors closely the path followed by Kishimoto-Tsugawa in [7] to prove their respective results. This is why we provide here only an outline of the argument for Theorem 2.1 and refer the interested reader to [7] for more details.

Sketch of proof for Theorem 2.1. By letting $\lambda \geq 1$ denote an arbitrary scaling parameter and taking

$$v_\lambda(x,t) = \lambda^{-2} v(\lambda^{-1}x, \lambda^{-2}t) \quad \text{and} \quad v_{0\lambda}(x) = \lambda^{-2} v_0(\lambda^{-1}x),$$

it follows that

$$v_\lambda(t) = S(t)v_{0\lambda} - i \int_0^t S(t-t') H_\lambda(v_\lambda(t'), \overline{v_\lambda}(t')) \, dt', \tag{13}$$

where

$$H_\lambda(w, \overline{w}) := \lambda^{-2} \frac{\overline{w} - w}{2} \pm \omega_\lambda(D) \left(\frac{w + \overline{w}}{2} \right)^2.$$

It is clear that v solves 6 on the interval $[0, T]$ if and only if v_λ solves 13 on $[0, \lambda^2 T]$. The goal is to show that 13 admits a unique local solution on the time interval $[0, 1]$ if λ is chosen sufficiently large.

For this reason, one works with the following modified version of 13,

$$v_\lambda(t) = \varphi(t) S(t) v_{0\lambda} - i\varphi(t) \int_0^t S(t-t') H_\lambda(v_\lambda(t'), \overline{v_\lambda}(t')) \, dt', \tag{14}$$

and proves that it has a unique global-in-time solution. If we denote the right-hand side of this integral equation, with $v_{0\lambda}$ fixed, by $I_\lambda = I_\lambda(v_\lambda)$, then an application of 8-12 yields

$$\|I_\lambda(v_\lambda)\|_{X^{s,\theta}}$$
$$\lesssim \|v_{0\lambda}\|_{H^s} + \|H_\lambda(v_\lambda, \overline{v_\lambda})\|_{X^{s,\theta-1}}$$
$$\lesssim \|v_{0\lambda}\|_{H^s} + \lambda^{-2} \left(\|v_\lambda\|_{X^{s,\theta-1}} + \|\overline{v_\lambda}\|_{X^{s,\theta-1}} \right) + \left\| \omega_\lambda(D) \left(v_\lambda + \overline{v_\lambda} \right)^2 \right\|_{X^{s,\theta-1}}$$
$$\lesssim \|v_{0\lambda}\|_{H^s} + \lambda^{-2} \|v_\lambda\|_{X^{s,\theta}} + \|v_\lambda\|_{X^{s,\theta}}^2.$$

Similarly, one obtains

$$\|I_\lambda(v_\lambda) - I_\lambda(w_\lambda)\|_{X^{s,\theta}} \lesssim \left(\lambda^{-2} + \|v_\lambda\|_{X^{s,\theta}} + \|w_\lambda\|_{X^{s,\theta}} \right) \|v_\lambda - w_\lambda\|_{X^{s,\theta}}.$$

Based on these two estimates, we argue that for $R \sim \|v_{0\lambda}\|_{H^s}$ the mapping

$$I_\lambda : \{ \|w\|_{X^{s,\theta}} \leq R \} \to \{ \|w\|_{X^{s,\theta}} \leq R \}$$

is a contraction if we can choose λ large enough and, at the same time, have[3] $\|v_{0\lambda}\|_{H^s} \lesssim 1$. This is feasible by taking $\lambda \sim r^{2/(2s-n+4)}$ and using 7. Moreover, with this choice, we also obtain that the time of existence for solutions to 6 satisfies $T \sim \lambda^{-2} \sim r^{-4/(2s-n+4)}$.

The uniqueness claim follows by comparable arguments (also relying on 4), for which we point to the proof of Proposition 4.1 in [7].   □

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[3]It is precisely the role of the scaling procedure to make the size of $\|v_{0\lambda}\|_{H^s}$ small enough to be amenable for the contraction argument.

3. **Bilinear estimates.** In this section, we focus our attention on proving 10-12 and, for this purpose, we first revisit related results obtained by Colliander-Delort-Kenig-Staffilani [2] (see also earlier work addressing similar issues by Staffilani [10]) and Tao [11]. The former paper provided a sharp geometric analysis for bilinear bounds of the type

$$\|\overline{u}\,\overline{v}\|_{X^{\sigma,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{15}$$

$$\|u\,v\|_{X^{\sigma,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{16}$$

$$\|\overline{u}\,v\|_{X^{\sigma,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{17}$$

on $\mathbb{R}^{2+1}$ and then used them in the context of LWP for Schrödinger equations with quadratic nonlinearities. The article by Tao took up the more general issue of multilinear estimates for arbitrary $X^{s,\theta}$ spaces and developed an abstract framework for proving them, which is now referred to in the literature as the $[k;Z]$-*multiplier norm method*. As an application of this method, the same paper established the bilinear estimate

$$\|\overline{u}\,v\|_{X^{s,-1/2+\epsilon}} \lesssim \|u\|_{X^{s,1/2-\epsilon}} \|v\|_{X^{s,1/2-\epsilon}} \tag{18}$$

on $\mathbb{R}^{n+1}$ with $1 \le n \le 3$, $\epsilon > 0$, and $\epsilon \lesssim s + 1/4 \le 1/4$, and made the claim that similar arguments lead to

$$\|\overline{u}\,\overline{v}\|_{X^{s,-1/2+\epsilon}} \lesssim \|u\|_{X^{s,1/2-\epsilon}} \|v\|_{X^{s,1/2-\epsilon}}, \tag{19}$$

$$\|uv\|_{X^{s,-1/2+\epsilon}} \lesssim \|u\|_{X^{s,1/2-\epsilon}} \|v\|_{X^{s,1/2-\epsilon}}, \tag{20}$$

on $\mathbb{R}^{n+1}$ when either $n = 2$ and $s + 3/4 \gtrsim \epsilon$ or $n = 3$ and $s + 1/2 \gtrsim \epsilon$.

In line with our main goal, we investigate the validity of 10-12 on $\mathbb{R}^{n+1}$ with $n = 2$ or 3 for pairs of indices (s, θ) satisfying $s < 0$ and $\theta > 1/2$. Using the trivial observation

$$\left| \widehat{\omega_\lambda(D)w}(\tau,\xi) \right| = \frac{\lambda^2 |\xi|^2}{1 + \lambda^2 |\xi|^2} \, |\widehat{w}(\tau,\xi)| \le |\widehat{w}(\tau,\xi)|,$$

which yields

$$\|\omega_\lambda(D)w\|_{X^{\tilde{s},\tilde{\theta}}} \le \|w\|_{X^{\tilde{s},\tilde{\theta}}}$$

for an arbitrary pair $(\tilde{s}, \tilde{\theta})$, it follows that it is enough to look at

$$\|\overline{u}\,\overline{v}\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{21}$$

$$\|u\,v\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{22}$$

$$\|\overline{u}\,v\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{23}$$

under the same conditions for n, s and θ.

Even though one can argue that whatever is needed for proving Theorem 1.1 in terms of bilinear estimates is already covered by 15-17 and 18-20, we choose to provide a stand-alone proof of 21-23 for a number of reasons. One is that we have a unitary argument for both $n = 2$ and $n = 3$. Another is that we are able to prove 15-16 for indices σ, s, and θ not covered in [2]. Finally, our proof suggests that, in principle, the pairs of indices (s, θ) for which 10-12 hold true coincide with the ones available for the validity of 21-23. Thus, it is very likely that the functional spaces on which we run the contraction argument need to be modified in order for the Sobolev regularity in Theorem 1.1 to be lowered.

In arguing for 21-23, we rely on Tao's methodology, which is directly specialized to our setting. We denote

$$\Gamma_3(\mathbb{R}^n \times \mathbb{R}) = \{((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3)) \in (\mathbb{R}^n \times \mathbb{R})^3; (\xi_1,\tau_1)+(\xi_2,\tau_2)+(\xi_3,\tau_3) = 0\}$$

and define

$$\int_{\Gamma_3(\mathbb{R}^n \times \mathbb{R})} f := \int_{(\mathbb{R}^n \times \mathbb{R})^2} f((\xi_1, \tau_1), (\xi_2, \tau_2), (-\xi_1 - \xi_2, -\tau_1 - \tau_2)) \, d\xi_1 d\tau_1 d\xi_2 d\tau_2.$$

Any function $m : \Gamma_3(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{C}$ is called a $[3; \mathbb{R}^n \times \mathbb{R}]$-multiplier and we let $\|m\|_{[3;\mathbb{R}^n \times \mathbb{R}]}$ denote the best constant for which

$$\left| \int_{\Gamma_3(\mathbb{R}^n \times \mathbb{R})} m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3)) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) \right|$$

$$\leq \|m\|_{[3;\mathbb{R}^n \times \mathbb{R}]} \|f_1\|_{L^2(\mathbb{R}^n \times \mathbb{R})} \|f_2\|_{L^2(\mathbb{R}^n \times \mathbb{R})} \|f_3\|_{L^2(\mathbb{R}^n \times \mathbb{R})}$$

is valid for all test functions $(f_i)_{1 \leq i \leq 3}$ on $\mathbb{R}^n \times \mathbb{R}$.

If we take for example 21, then, by applying duality and Plancherel's theorem, we can rewrite it equivalently as

$$\left| \int_{\Gamma_3(\mathbb{R}^n \times \mathbb{R})} \widehat{\overline{u}}(\xi_1, \tau_1) \widehat{\overline{v}}(\xi_2, \tau_2) \widehat{\overline{w}}(\xi_3, \tau_3) \right|$$

$$\sim \left| \int_{\mathbb{R}^n \times \mathbb{R}} \overline{u}(x, t) \overline{v}(x, t) \overline{w}(x, t) \, dx dt \right|$$

$$\lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}} \|w\|_{X^{-s,1-\theta}}$$

$$= \|\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^\theta \widehat{u}(\xi, \tau)\|_{L^2_{\xi,\tau}} \|\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^\theta \widehat{v}(\xi, \tau)\|_{L^2_{\xi,\tau}}$$

$$\cdot \|\langle \xi \rangle^{-s} \langle \tau - |\xi|^2 \rangle^{1-\theta} \widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}},$$

which can be easily turned into

$$\left| \int_{\Gamma_3(\mathbb{R}^n \times \mathbb{R})} \frac{\langle \xi_3 \rangle^s \langle \tau_3 + |\xi_3|^2 \rangle^{\theta-1}}{\langle \xi_1 \rangle^s \langle \tau_1 + |\xi_1|^2 \rangle^\theta \langle \xi_2 \rangle^s \langle \tau_2 + |\xi_2|^2 \rangle^\theta} f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_3(\xi_3, \tau_3) \right|$$

$$\lesssim \|f_1\|_{L^2(\mathbb{R}^n \times \mathbb{R})} \|f_2\|_{L^2(\mathbb{R}^n \times \mathbb{R})} \|f_3\|_{L^2(\mathbb{R}^n \times \mathbb{R})}.$$

Thus, according to the above definitions, proving 21 is identical to showing that

$$\left\| \frac{\langle \xi_3 \rangle^s \langle \tau_3 + |\xi_3|^2 \rangle^{\theta-1}}{\langle \xi_1 \rangle^s \langle \tau_1 + |\xi_1|^2 \rangle^\theta \langle \xi_2 \rangle^s \langle \tau_2 + |\xi_2|^2 \rangle^\theta} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim 1 \tag{24}$$

holds true, with similar multiplier-norm estimates being available for both 22 and 23. In fact, these new bounds can be stated generically in the form

$$\left\| \frac{\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \langle \xi_3 \rangle^s}{\langle \tau_1 - h_1(\xi_1) \rangle^\theta \langle \tau_2 - h_2(\xi_2) \rangle^\theta \langle \tau_3 - h_3(\xi_3) \rangle^{1-\theta}} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim 1, \tag{25}$$

where $h_i(\xi) = \pm |\xi|^2$ for all $1 \leq i \leq 3$.

At this point, Tao introduces the notation

$$\lambda_i = \tau_i - h_i(\xi_i), \qquad 1 \leq i \leq 3,$$

and defines the *resonance function* $h : \Gamma_3(\mathbb{R}^n) \to \mathbb{R}$ by

$$h(\xi_1, \xi_2, \xi_3) := h_1(\xi_1) + h_2(\xi_2) + h_3(\xi_3). \tag{26}$$

It is easy to see that on the support of the multiplier in 25 we have

$$\xi_1 + \xi_2 + \xi_3 = 0 \qquad \text{and} \qquad \lambda_1 + \lambda_2 + \lambda_3 + h(\xi_1, \xi_2, \xi_3) = 0. \tag{27}$$

Next, it is argued that one can reduce the proof of 25 to the case when

$$\min\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \gtrsim 1 \qquad \text{and} \qquad \max\{|\xi_1|, |\xi_2|, |\xi_3|\} \gtrsim 1.$$

Following this, a dyadic decomposition for $(\xi_i)_{1 \leq i \leq 3}$, $(\lambda_i)_{1 \leq i \leq 3}$, and h is performed and one infers

(LHS) of 25

$$\lesssim \left\| \sum_{\max N_i \gtrsim 1} \sum_H \sum_{\min L_i \gtrsim 1} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}} X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]}$$

where

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3} = X_{N_1,N_2,N_3;H;L_1,L_2,L_3}((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3))$$
$$:= \chi_{|h(\xi_1,\xi_2,\xi_3)| \sim H} \prod_{1 \leq i \leq 3} \left(\chi_{|\xi_i| \sim N_i} \chi_{|\lambda_i| \sim L_i} \right) \tag{28}$$

and $(N_i)_{1 \leq i \leq 3}$, $(L_i)_{1 \leq i \leq 3}$, and $H \in 2^\mathbb{Z}$. If we let $N_{max} \geq N_{med} \geq N_{min}$ denote the values of N_1, N_2, and N_3 in decreasing order, with a similar notation for the values of L_1, L_2, and L_3, then, based on 27, we deduce that

$$N_{max} \sim N_{med} \qquad \text{and} \qquad L_{max} \sim \max\{H, L_{med}\} \tag{29}$$

need to be valid in order for $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ not to vanish.

Using also the relative orthogonality of the dyadic decomposition, Tao is able to derive initially that

(LHS) of 25

$$\lesssim \sup_{N \gtrsim 1} \left\| \sum_{N_{max} \sim N_{med} \sim N} \sum_H \sum_{L_{max} \sim \max\{H,L_{med}\}} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}} \right.$$
$$\left. \cdot X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]}$$

where the summation in the inner and the outer sums is in fact performed over all L_i's and N_i's, respectively, obeying the restriction listed under the sums[4]. Jointly with the triangle inequality, this implies that, for some $N \gtrsim 1$, at least one of the estimates

(LHS) of 25 $\lesssim \displaystyle\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{min} \gtrsim 1} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}}$
$$\cdot \left\| X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]}$$

and

(LHS) of 25 $\lesssim \displaystyle\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \ll L_{max}} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}}$
$$\cdot \left\| X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]}$$

holds true. In this way, 25 would follow if one shows that

$$\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{min} \gtrsim 1} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}}$$
$$\cdot \left\| X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim 1 \tag{30}$$

---

[4]Similar summation conventions are used throughout this section. See also Section 2 in [11].

and

$$\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \ll L_{max}} \frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}}$$
$$\cdot \|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim 1, \tag{31}$$

for all values of $N \gtrsim 1$. Tao calls the setting of the first bound (i.e., $H \sim L_{max}$) the *low modulation* case and the one for the second bound (i.e., $L_{max} \sim L_{med} \gg H$) the *high modulation* case.

The first part of the argument for proving 30 and 31 consists in estimating the two multiplier norms and this has been achieved by Tao in a sharp manner. Given 26, 28, and the existing symmetries, the analysis is reduced to two scenarios. The so-called $(+ + +)$ case happens when $h_1(\xi) = h_2(\xi) = h_3(\xi) = |\xi|^2$ and, hence,

$$H \sim |h| = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 \sim N_{max}^2. \tag{32}$$

The other instance, named the $(+ + -)$ case, takes place when $h_1(\xi) = h_2(\xi) = -h_3(\xi) = |\xi|^2$ and, due to 27, one has

$$H \sim |h| = \left| |\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 \right| = 2|\xi_1 \cdot \xi_2| \lesssim N_1 N_2. \tag{33}$$

The following are the combined outcomes of Propositions 11.1 and 11.2 in [11] when $n \geq 2$.

Lemma 3.1. *Let $n \geq 2$ and take N_1, N_2, N_3, L_1, L_2, L_3, and H to be positive numbers satisfying 29.*

- $(+ + +)$ ***case:*** *both 32 and*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} N_{min}^{\frac{n-1}{2}} \min\{N_{max} N_{min}, L_{med}\}^{\frac{1}{2}} \quad (34)$$

  are valid.
- $(+ + -)$ ***case:*** *33 holds true and*
  1. *if $N_1 \sim N_2 \gg N_3$, the multiplier norm vanishes unless $H \sim N_1^2$ and, in this situation,*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} N_{min}^{\frac{n-1}{2}} \min\{N_{max} N_{min}, L_{med}\}^{\frac{1}{2}} \quad (35)$$

     is valid;
  2. *if $N_1 \sim N_3 \gg N_2$ and $H \sim L_2 \gg L_1, L_3, N_2^2$, then*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} N_{min}^{\frac{n-1}{2}} \min\left\{H, \frac{H}{N_{min}^2} L_{med}\right\}^{\frac{1}{2}} \quad (36)$$

     *is valid. The same estimate holds true if the roles of indices 1 and 2 are reversed. This is also called the **coherence subcase**;*
  3. *in all other instances not covered above and for $\epsilon > 0$,*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} N_{min}^{\frac{n-1}{2}}$$
$$\cdot \min\{H, L_{med}\}^{\frac{1}{2}} \min\left\{1, \frac{H}{N_{min}^2}\right\}^{\frac{1}{2} - \epsilon} \quad (37)$$

     is valid, with the implicit constant depending on ϵ. If $n = 2$, ϵ can be removed.

The second part of the proof for 30 and 31 consists in using the multiplier norm bounds from the previous lemma and performing the two summations. This is where we start, in earnest, our own argument. The following definition describes the indices s and θ relevant to our analysis.

Definition 3.2. We say that the triplet (n, s, θ) is admissible if either

$$n = 2, \qquad \frac{1}{2} < \theta \neq \frac{3}{4}, \qquad \max\left\{\theta - \frac{5}{4}, 2\theta - 2\right\} \leq s < 0, \qquad (38)$$

or

$$n = 2, \qquad \theta = \frac{3}{4}, \qquad -\frac{1}{2} < s < 0, \qquad (39)$$

or

$$n = 3, \qquad \theta > \frac{1}{2}, \qquad 2\theta - \frac{3}{2} \leq s < 0. \qquad (40)$$

Remark 2. It is easy to verify that if (n, s, θ) is admissible then

$$s \geq 2\theta + \frac{n-6}{2} > \frac{n-4}{2}. \qquad (41)$$

Moreover, if

$$n = 2 \text{ or } n = 3, \qquad \theta > 1/2, \qquad \frac{\theta - 1}{2} < s < 0, \qquad (42)$$

then a direct argument shows that (n, s, θ) is admissible.

Proposition 1. *The bilinear estimate 21 is valid if (n, s, θ) is admissible.*

Proof. As argued before, the bound to be proven is equivalent to 24 which, by using the compatible transformation $(\tau_1, \tau_2, \tau_3) \mapsto (-\tau_1, -\tau_2, -\tau_3)$, becomes

$$\left\| \frac{\langle \xi_3 \rangle^s \langle \tau_3 - |\xi_3|^2 \rangle^{\theta-1}}{\langle \xi_1 \rangle^s \langle \tau_1 - |\xi_1|^2 \rangle^\theta \langle \xi_2 \rangle^s \langle \tau_2 - |\xi_2|^2 \rangle^\theta} \right\|_{[3, \mathbb{R}^n \times \mathbb{R}]} \lesssim 1.$$

We are in the $(+++)$ case and we would be done if we show that 30 and 31 hold true in this setting. According to 32, we can assume $H \sim N_{max}^2 \sim N^2$ and, since $s < 0$ and $\theta > 1/2$, we deduce

$$\frac{\langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \langle N_3 \rangle^s}{L_1^\theta L_2^\theta L_3^{1-\theta}} \lesssim \frac{N^{-2s} \langle N_{min} \rangle^s}{L_{min}^\theta L_{med}^\theta L_{max}^{1-\theta}}. \qquad (43)$$

We treat first 30, for which one has $L_{max} \sim H \sim N^2$. If we take advantage jointly of 34, 43, and $\theta > 1/2$, then we can estimate the left-hand side of 30 by

(LHS) of 30

$$\lesssim N^{-2s+2\theta-\frac{5}{2}} \sum_{N_{min} \lesssim N} \sum_{1 \lesssim L_{min} \leq L_{med} \lesssim N^2} \left(\langle N_{min} \rangle^s N_{min}^{\frac{n-1}{2}} \right.$$
$$\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{-\theta} \min\{N N_{min}, L_{med}\}^{\frac{1}{2}} \right)$$

$$\lesssim N^{-2s+2\theta-2} \sum_{N_{min}\lesssim N^{-1}} \sum_{1\lesssim L_{med}\lesssim N^2} N_{min}^{\frac{n}{2}} L_{med}^{-\theta}$$

$$+ N^{-2s+2\theta-\frac{5}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{1\lesssim L_{med}\lesssim NN_{min}} \langle N_{min}\rangle^s N_{min}^{\frac{n-1}{2}} L_{med}^{\frac{1}{2}-\theta}$$

$$+ N^{-2s+2\theta-2} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{NN_{min}\lesssim L_{med}\lesssim N^2} \langle N_{min}\rangle^s N_{min}^{\frac{n}{2}} L_{med}^{-\theta}$$

$$\lesssim N^{-2s+2\theta-2-\frac{n}{2}} + N^{-2s+2\theta-\frac{5}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \langle N_{min}\rangle^s N_{min}^{\frac{n-1}{2}} \left(1 + (NN_{min})^{\frac{1}{2}-\theta}\right)$$

$$\sim N^{-2s+2\theta-\frac{5}{2}} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+\frac{n-1}{2}}\right).$$

A simple analysis based on how $s+(n-1)/2$ compares to 0 yields that

$$N^{-2s+2\theta-\frac{5}{2}} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+\frac{n-1}{2}}\right) \lesssim 1$$

if and only if (n,s,θ) is admissible.

Next, we address 31, for which we work with $L_{max}\sim L_{med} \gg H \sim N^2$. This implies

$$NN_{min} \lesssim N^2 \ll L_{med}, \tag{44}$$

which leads to

$$\min\{NN_{min}, L_{med}\} \sim NN_{min}. \tag{45}$$

Together with 34,43, and $\theta > 1/2$, this fact allows us to infer

(LHS) of 31

$$\lesssim N^{-2s} \sum_{N_{min}\lesssim N} \sum_{\substack{1\lesssim L_{min}\leq L_{med}\sim L_{max}\\ N^2\ll L_{max}}} \langle N_{min}\rangle^s N_{min}^{\frac{n}{2}} L_{min}^{\frac{1}{2}-\theta} L_{max}^{-1}$$

$$\lesssim N^{-2s-2} \sum_{N_{min}\lesssim N} \langle N_{min}\rangle^s N_{min}^{\frac{n}{2}} \sim N^{-2s-2} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+\frac{n}{2}}\right).$$

Using now 41, we deduce

$$N^{-2s-2} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+\frac{n}{2}}\right) \sim N^{-s+\frac{n-4}{2}} \lesssim 1$$

and the argument is concluded.   □

Proposition 2. *The bilinear estimate 22 is valid if (n,s,θ) is admissible.*

Proof. Following the blueprint of deriving 24, we argue first that 22 is equivalent to

$$\left\| \frac{\langle\xi_3\rangle^s \langle\tau_3 + |\xi_3|^2\rangle^{\theta-1}}{\langle\xi_1\rangle^s \langle\tau_1 - |\xi_1|^2\rangle^\theta \langle\xi_2\rangle^s \langle\tau_2 - |\xi_2|^2\rangle^\theta} \right\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \lesssim 1.$$

Thus, we need to prove that both 30 and 31 hold true in the $(+ + -)$ case. We know that we can rely on 33 and, for each of the bounds, we have to go through all the three subcases covered in Lemma 3.1.

We start with the analysis for 30 and consider first the instance when $N_1 \sim N_2 \gg N_3$, which also forces $H \sim N_1^2$. Then, based on 35, we see that we can estimate the left-hand side of 30 in identical fashion to the way we estimated it in the previous proposition. Hence, we obtain

(LHS) of 30

$$
\lesssim N^{-2s+2\theta-\frac{5}{2}} \sum_{N_{min} \lesssim N} \sum_{1 \lesssim L_{min} \leq L_{med} \lesssim N^2} \left(\langle N_{min} \rangle^s N_{min}^{\frac{n-1}{2}} \right.
$$
$$
\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{-\theta} \min\{NN_{min}, L_{med}\}^{\frac{1}{2}} \right)
$$
$$
\lesssim N^{-2s+2\theta-\frac{5}{2}} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{s+\frac{n-1}{2}} \right)
$$

and, consequently, 30 is valid in this instance if (n, s, θ) is admissible.

If we are in the second scenario of Lemma 3.1, by the symmetry of 30 in the indices 1 and 2, it is enough to work under the assumption that $N_1 \sim N_3 \gg N_2$ and $H \sim L_2 \gg L_1, L_3, N_2^2$. Using 36, $1/2 < \theta < 1$, and 41, we infer

(LHS) of 30

$$
\lesssim N^{-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{min} \leq L_{med} \leq L_{max} \\ N_{min}^2 \ll L_{max} \lesssim NN_{min}}} \left(\langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} \right.
$$
$$
\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{\theta-1} L_{max}^{\frac{1}{2}-\theta} \min\left\{ 1, \frac{L_{med}}{N_{min}^2} \right\}^{\frac{1}{2}} \right)
$$
$$
\lesssim N^{-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim 1} \sum_{1 \lesssim L_{med} \leq L_{max} \lesssim NN_{min}} N_{min}^{\frac{n-1}{2}} L_{med}^{\theta-1} L_{max}^{\frac{1}{2}-\theta}
$$
$$
+ N^{-\frac{1}{2}} \sum_{1 \lesssim N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{med} \lesssim N_{min}^2 \\ N_{min}^2 \ll L_{max} \lesssim NN_{min}}} N_{min}^{-s+\frac{n-3}{2}} L_{med}^{\theta-\frac{1}{2}} L_{max}^{\frac{1}{2}-\theta}
$$
$$
+ N^{-\frac{1}{2}} \sum_{1 \lesssim N_{min} \lesssim N} \sum_{\substack{N_{min}^2 \lesssim L_{med} \leq L_{max} \\ N_{min}^2 \ll L_{max} \lesssim NN_{min}}} N_{min}^{-s+\frac{n-1}{2}} L_{med}^{\theta-1} L_{max}^{\frac{1}{2}-\theta}
$$
$$
\lesssim N^{-\frac{1}{2}} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+\frac{n-3}{2}} \right) \lesssim N^{-\frac{1}{2}} + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+\frac{n-4}{2}} \sim 1,
$$

which proves 30 in this scenario.

To finish the argument for 30, we need to consider the third subcase of the $(+ + -)$ case in Lemma 3.1, which, reduced by symmetry, comes down to either $N_1 \sim N_2 \sim N_3$ or $N_1 \sim N_3 \gtrsim N_2$. For each of them, since $H \sim L_{max}$, we have

$$
\min\{H, L_{med}\} \sim L_{med} \qquad \text{and} \qquad \min\left\{1, \frac{H}{N_{min}^2}\right\} \sim \min\left\{1, \frac{L_{max}}{N_{min}^2}\right\}. \qquad (46)
$$

Moreover, since $\theta > 1/2$, it follows that

$$
\frac{1}{L_1^\theta L_2^\theta L_3^{1-\theta}} \leq \frac{1}{L_{min}^\theta L_{med}^\theta L_{max}^{1-\theta}}. \qquad (47)
$$

Therefore, when $N_1 \sim N_2 \sim N_3$, these two facts together with 37 and $\theta > 1/2$ allow us to deduce that

$$(\text{LHS}) \text{ of } 30 \lesssim N^{-s+\frac{n-4}{2}+2\epsilon} \sum_{1 \lesssim L_{min} \le L_{med} \le L_{max} \lesssim N^2} L_{min}^{\frac{1}{2}-\theta} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-\frac{1}{2}-\epsilon}$$

$$\lesssim N^{-s+\frac{n-4}{2}+2\epsilon} \sum_{1 \lesssim L_{max} \lesssim N^2} L_{max}^{\theta-\frac{1}{2}-\epsilon}.$$

By choosing $0 < \epsilon < \theta - 1/2$, we argue based on 41 that

$$N^{-s+\frac{n-4}{2}+2\epsilon} \sum_{1 \lesssim L_{max} \lesssim N^2} L_{max}^{\theta-\frac{1}{2}-\epsilon} \sim N^{-s+2\theta+\frac{n-6}{2}} \lesssim 1,$$

which yields the desired result.

On the other hand, if we have $N_1 \sim N_3 \gtrsim N_2$, then, on the basis of 46, 47, 37, $1/2 < \theta < 1$, and with the same choice for ϵ, we obtain

$(\text{LHS}) \text{ of } 30$

$$\lesssim N^{-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{min} \le L_{med} \le L_{max} \\ L_{max} \lesssim N N_{min}}} \left(\langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} \right.$$

$$\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-1} \min\left\{1, \frac{L_{max}}{N_{min}^2}\right\}^{\frac{1}{2}-\epsilon} \right)$$

$$\lesssim N^{-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim N} \sum_{\langle N_{min} \rangle^2 \lesssim L_{max} \lesssim N N_{min}} \langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} L_{max}^{\theta-1}$$

$$+ N^{-\frac{1}{2}} \sum_{1 \lesssim N_{min} \lesssim N} \sum_{1 \lesssim L_{max} \lesssim N_{min}^2} N_{min}^{-s+\frac{n-3}{2}+2\epsilon} L_{max}^{\theta-\frac{1}{2}-\epsilon}$$

$$\lesssim N^{-\frac{1}{2}} \left(\sum_{N^{-1} \lesssim N_{min} \lesssim N} \langle N_{min} \rangle^{-s+2\theta-2} N_{min}^{\frac{n-1}{2}} + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+2\theta+\frac{n-5}{2}} \right)$$

$$\sim N^{-\frac{1}{2}} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+2\theta+\frac{n-5}{2}} \right).$$

It can be checked easily that if (n, s, θ) is admissible, then $-s + 2\theta + (n-5)/2 \ne 0$. Thus, we derive

$$N^{-\frac{1}{2}} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+2\theta+\frac{n-5}{2}} \right) \lesssim N^{-\frac{1}{2}} + N^{-s+2\theta+\frac{n-6}{2}} \lesssim 1, \qquad (48)$$

where the last bound follows according to 41. This finishes the proof of 30.

Next, we address 31, for which the scenario $N_1 \sim N_2 \gg N_3$ and $H \sim N_1^2$ implies 44 and, hence, 45. Then, we can estimate the left-hand side of 31 in exactly the

same way as we estimated it in the previous proposition. Thus, we infer

$$\text{(LHS) of } 31 \lesssim N^{-2s} \sum_{N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{min} \leq L_{med} \sim L_{max} \\ N^2 \ll L_{max}}} \langle N_{min} \rangle^s N_{min}^{\frac{n}{2}} L_{min}^{\frac{1}{2} - \theta} L_{max}^{-1}$$

$$\lesssim N^{-2s-2} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{s + \frac{n}{2}} \right) \sim N^{-s + \frac{n-4}{2}} \lesssim 1.$$

The second subcase of the $(++-)$ case in Lemma 3.1 does not apply here because $H \ll L_{max}$. The last one can be reduced by symmetry to the instances when either $N_1 \sim N_2 \sim N_3$ or $N_1 \sim N_3 \gtrsim N_2$. For each of them, we have

$$\min\{H, L_{med}\} \sim H, \tag{49}$$

while for the former we can also rely on

$$\min \left\{ 1, \frac{H}{N_{min}^2} \right\} \sim \frac{H}{N_{min}^2}, \tag{50}$$

due to 33. Thus, when $N_1 \sim N_2 \sim N_3$, we argue based on 37, applicable to $0 < \epsilon < \theta - 1/2$, and 41 that

(LHS) of 31

$$\lesssim N^{-s + \frac{n-4}{2} + 2\epsilon} \sum_{1 \lesssim L_{min} \leq L_{med} \sim L_{max}} \sum_{H \lesssim \min\{L_{max}, N^2\}} L_{min}^{\frac{1}{2} - \theta} L_{max}^{-1} H^{1-\epsilon}$$

$$\lesssim N^{-s + \frac{n-4}{2} + 2\epsilon} \sum_{H \lesssim N^2} \sum_{\langle H \rangle \lesssim L_{max}} L_{max}^{-1} H^{1-\epsilon}$$

$$\lesssim N^{-s + \frac{n-4}{2} + 2\epsilon} \sum_{H \lesssim N^2} \langle H \rangle^{-1} H^{1-\epsilon} \sim N^{-s + \frac{n-4}{2} + 2\epsilon} \lesssim N^{-s + 2\theta + \frac{n-6}{2}} \lesssim 1.$$

For the case when $N_1 \sim N_3 \gtrsim N_2$, we use again 37 with $0 < \epsilon < \theta - 1/2$ and 48 to deduce

(LHS) of 31

$$\lesssim N^{-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{1 \lesssim L_{min} \leq L_{med} \sim L_{max}} \sum_{H \lesssim \min\{L_{max}, NN_{min}\}} \left(\langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} L_{min}^{\frac{1}{2} - \theta} \right.$$

$$\left. \cdot L_{max}^{-1} H^{\frac{1}{2}} \min \left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2} - \epsilon} \right)$$

$$\lesssim N^{-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{H \lesssim NN_{min}} \sum_{\langle H \rangle \lesssim L_{max}} \langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} L_{max}^{-1} H^{\frac{1}{2}} \min \left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2} - \epsilon}$$

$$\lesssim N^{-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{H \lesssim NN_{min}} \langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} \langle H \rangle^{-1} H^{\frac{1}{2}} \min \left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2} - \epsilon}$$

$$\lesssim N^{-\frac{1}{2}} \sum_{N_{min}\lesssim N^{-1}} N_{min}^{\frac{n-1}{2}} \left(\sum_{H\lesssim N_{min}^2} \frac{H^{1-\epsilon}}{N_{min}^{1-2\epsilon}} + \sum_{N_{min}^2\lesssim H\lesssim NN_{min}} H^{\frac{1}{2}} \right)$$

$$+ N^{-\frac{1}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim 1} N_{min}^{\frac{n-1}{2}} \left(\sum_{H\lesssim N_{min}^2} \frac{H^{1-\epsilon}}{N_{min}^{1-2\epsilon}} + \sum_{N_{min}^2\lesssim H\lesssim 1} H^{\frac{1}{2}} \right.$$

$$\left. + \sum_{1\lesssim H\lesssim NN_{min}} H^{-\frac{1}{2}} \right)$$

$$+ N^{-\frac{1}{2}} \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{-s+\frac{n-1}{2}} \left(\sum_{H\lesssim 1} \frac{H^{1-\epsilon}}{N_{min}^{1-2\epsilon}} + \sum_{1\lesssim H\lesssim N_{min}^2} \frac{H^{-\epsilon}}{N_{min}^{1-2\epsilon}} \right.$$

$$\left. + \sum_{N_{min}^2\lesssim H\lesssim NN_{min}} H^{-\frac{1}{2}} \right)$$

$$\lesssim \sum_{N_{min}\lesssim N^{-1}} N_{min}^{\frac{n}{2}} + N^{-\frac{1}{2}} \left(\sum_{N^{-1}\lesssim N_{min}\lesssim 1} N_{min}^{\frac{n-1}{2}} + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{-s+\frac{n-3}{2}+2\epsilon} \right)$$

$$\lesssim N^{-\frac{1}{2}} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{-s+2\theta+\frac{n-5}{2}} \right) \lesssim 1.$$

This finishes the proof of this proposition. $\qquad\square$

Remark 3. Following up on our rationale to argue for 21-23, by comparison to what is proved in [2] for 15-16, one can see that Propositions 1 and 2 cover the previously unknown case for which

$$\frac{1}{2} < \theta \neq \frac{3}{4} \quad \text{and} \quad \sigma = s = \max\left\{\theta - \frac{5}{4}, 2\theta - 2\right\} < 0.$$

Proposition 3. *The bilinear estimate 23 is valid if (n, s, θ) satisfy 42.*

Proof. As in the case of the previous two results, one recognizes first that the above claim is equivalent to the multiplier norm bound

$$\left\| \frac{\langle\xi_3\rangle^s \langle\tau_3 + |\xi_3|^2\rangle^{\theta-1}}{\langle\xi_1\rangle^s \langle\tau_1 + |\xi_1|^2\rangle^\theta \langle\xi_2\rangle^s \langle\tau_2 - |\xi_2|^2\rangle^\theta} \right\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \lesssim 1.$$

By using the compatible transformation $(\tau_1, \tau_2, \tau_3) \mapsto (-\tau_1, -\tau_2, -\tau_3)$ and relabeling the indices according to $(1, 2, 3) \mapsto (1, 3, 2)$, this can be rewritten as

$$\left\| \frac{\langle\xi_3\rangle^{-s} \langle\tau_3 + |\xi_3|^2\rangle^{-\theta}}{\langle\xi_1\rangle^s \langle\tau_1 - |\xi_1|^2\rangle^\theta \langle\xi_2\rangle^{-s} \langle\tau_2 - |\xi_2|^2\rangle^{1-\theta}} \right\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \lesssim 1. \tag{51}$$

As in the derivation of 30 and 31, the previous estimate would follow if we show that

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_1,L_2,L_3\gtrsim 1} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^s\langle N_3\rangle^{-s}}{L_1^\theta L_2^{1-\theta} L_3^\theta}$$

$$\cdot \|X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3}\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \lesssim 1 \tag{52}$$

and

$$\sum_{N_{max}\sim N_{med}\sim N} \sum_{L_{max}\sim L_{med}} \sum_{H\ll L_{max}} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^s\langle N_3\rangle^{-s}}{L_1^\theta L_2^{1-\theta} L_3^\theta}$$
$$\cdot \|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \lesssim 1 \tag{53}$$

hold true for any $N \gtrsim 1$.

From 51, we see that we operate in the $(++-)$ case and, as such, we can rely on 33 and we perform an analysis based on the subcases described in Lemma 3.1. Furthermore, due to 42 and Remark 2, we can also take advantage of 41.

For the low modulation estimate 52, if we are in the $N_1 \sim N_2 \gg N_3$ scenario, we also have that $H \sim L_{max} \sim N_1^2$. Thus, based on 35, $1/2 < \theta < 1$, $s < 0$, and 41, we infer

(LHS) of 52

$$\lesssim N^{2\theta-\frac{5}{2}} \sum_{N_{min}\lesssim N} \sum_{1\lesssim L_{min}\leq L_{med}\lesssim N^2} \left(\langle N_{min}\rangle^{-s} N_{min}^{\frac{n-1}{2}}\right.$$
$$\left.\cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{-\theta} \min\{NN_{min},L_{med}\}^{\frac{1}{2}}\right)$$
$$\lesssim N^{2\theta-2} \sum_{N_{min}\lesssim N^{-1}} \sum_{1\lesssim L_{med}\lesssim N^2} N_{min}^{\frac{n}{2}} L_{med}^{-\theta}$$
$$+ N^{2\theta-\frac{5}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{1\lesssim L_{med}\lesssim NN_{min}} \langle N_{min}\rangle^{-s} N_{min}^{\frac{n-1}{2}} L_{med}^{\frac{1}{2}-\theta}$$
$$+ N^{2\theta-2} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{NN_{min}\lesssim L_{med}\lesssim N^2} \langle N_{min}\rangle^{-s} N_{min}^{\frac{n}{2}} L_{med}^{-\theta}$$
$$\lesssim N^{2\theta-\frac{n+4}{2}} + N^{2\theta-\frac{5}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \langle N_{min}\rangle^{-s} N_{min}^{\frac{n-1}{2}} \left(1 + (NN_{min})^{\frac{1}{2}-\theta}\right)$$
$$\sim N^{2\theta-\frac{n+4}{2}} + N^{-s+2\theta+\frac{n-6}{2}} \lesssim 1.$$

Next, if $N_1 \sim N_3 \gg N_2$ and $H \sim L_2 \gg L_1, L_3, N_2^2$, then, using 36 and $\theta > 1/2$, we derive that

(LHS) of 52

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{\substack{N^{-1}\lesssim N_{min}\lesssim N}} \sum_{\substack{1\lesssim L_{min}\leq L_{med}\leq L_{max} \\ N_{min}^2\ll L_{max}\lesssim NN_{min}}} \left(\langle N_{min}\rangle^s N_{min}^{\frac{n-1}{2}}\right.$$
$$\left.\cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{-\theta} L_{max}^{\theta-\frac{1}{2}} \min\left\{1, \frac{L_{med}}{N_{min}^2}\right\}^{\frac{1}{2}}\right)$$
$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim 1} \sum_{1\lesssim L_{med}\leq L_{max}\lesssim NN_{min}} N_{min}^{\frac{n-1}{2}} L_{med}^{-\theta} L_{max}^{\theta-\frac{1}{2}}$$
$$+ N^{-2s-\frac{1}{2}} \sum_{1\lesssim N_{min}\lesssim N} \sum_{\substack{1\lesssim L_{med}\lesssim N_{min}^2 \\ N_{min}^2\ll L_{max}\lesssim NN_{min}}} N_{min}^{s+\frac{n-3}{2}} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-\frac{1}{2}}$$

$$+ N^{-2s-\frac{1}{2}} \sum_{1 \lesssim N_{min} \lesssim N} \sum_{\substack{N_{min}^2 \lesssim L_{med} \leq L_{max} \\ N_{min}^2 \ll L_{max} \lesssim N N_{min}}} N_{min}^{s+\frac{n-1}{2}} L_{med}^{-\theta} L_{max}^{\theta-\frac{1}{2}}$$

$$\lesssim N^{-2s+\theta-1} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{s+\theta+\frac{n-4}{2}} \right).$$

When $n = 2$, we argue that $s < 0$ and $\theta < 1$ imply

$$s + \theta + \frac{n-4}{2} < 0$$

and, thus, 52 is valid if $s \geq (\theta - 1)/2$. When $n = 3$ and (n, s, θ) is admissible, it is easy to check that $s + \theta + (n-4)/2$ can be either negative, positive, or equal to zero. If it is negative, then, as above, $s \geq (\theta - 1)/2$ is a sufficient condition for 52 to hold true. If it is positive, then we deduce with the help of 41 that

$$\text{(LHS) of } 52 \lesssim N^{-2s+\theta-1} + N^{-s+2\theta+\frac{n-6}{2}} \lesssim N^{-2s+\theta-1} + 1$$

and, yet again, 52 is valid if $s \geq (\theta - 1)/2$. If $s + \theta + (n-4)/2 = 0$, then we infer that

$$\text{(LHS) of } 52 \lesssim N^{-2s+\theta-1} \ln N$$

and we need to impose the stricter condition $s > (\theta - 1)/2$ for 52 to hold true.

Given that, unlike 30, 52 is not symmetric in the indices 1 and 2, we also need to consider the scenario when $N_2 \sim N_3 \gg N_1$ and $H \sim L_1 \gg L_2, L_3, N_1^2$. In this situation, an application of 36 yields

(LHS) of 52

$$\lesssim N^{-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{min} \leq L_{med} \leq L_{max} \\ N_{min}^2 \ll L_{max} \lesssim N N_{min}}} \left(\langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} \right.$$

$$\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{\theta-1} L_{max}^{\frac{1}{2}-\theta} \min \left\{ 1, \frac{L_{med}}{N_{min}^2} \right\}^{\frac{1}{2}} \right),$$

which is identical with the estimate satisfied by the left-hand side of 30 for the subcase when $N_1 \sim N_3 \gg N_2$ and $H \sim L_2 \gg L_1, L_3, N_2^2$. Hence,

$$\text{(LHS) of } 52 \lesssim N^{-\frac{1}{2}} + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+\frac{n-4}{2}} \sim 1.$$

In order to conclude the proof of 52, we need to investigate the third subcase, which can be reduced to $N_1 \sim N_2 \sim N_3$, $N_2 \sim N_3 \gtrsim N_1$, and $N_1 \sim N_3 \gtrsim N_2$, without making extra assumptions. As in the previous proposition, in addition to $L_{max} \sim H \lesssim N_1 N_2$, we can rely on 46 and, since $\theta > 1/2$, on

$$\frac{1}{L_1^\theta L_2^{1-\theta} L_3^\theta} \leq \frac{1}{L_{min}^\theta L_{med}^\theta L_{max}^{1-\theta}} \tag{54}$$

for either of these scenarios.

If $N_1 \sim N_2 \sim N_3$, then 37 implies

$$\text{(LHS) of } 52 \lesssim N^{-s+\frac{n-4}{2}+2\epsilon} \sum_{1 \lesssim L_{min} \leq L_{med} \leq L_{max} \lesssim N^2} L_{min}^{\frac{1}{2}-\theta} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-\frac{1}{2}-\epsilon},$$

which coincides with the initial bound satisfied by the left-hand side of 30 in the same situation. Thus, with the appropriate choice for ϵ (i.e., $0 < \epsilon < \theta - 1/2$), we obtain

$$\text{(LHS) of } 52 \lesssim N^{-s+2\theta+\frac{n-6}{2}} \lesssim 1.$$

When $N_2 \sim N_3 \gtrsim N_1$, we use 46, 54, and 37 to derive that

$$\text{(LHS) of } 52 \lesssim N^{-\frac{1}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{\substack{1\lesssim L_{min}\leq L_{med}\leq L_{max} \\ L_{max}\lesssim NN_{min}}} \left(\langle N_{min}\rangle^{-s} N_{min}^{\frac{n-1}{2}} \right.$$
$$\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-1} \min\left\{1, \frac{L_{max}}{N_{min}^2}\right\}^{\frac{1}{2}-\epsilon} \right).$$

This estimate is identical to the one satisfied by the left-hand side of 30 when $N_1 \sim N_3 \gtrsim N_2$ and, thus, 52 holds true if (n, s, θ) is admissible.

If $N_1 \sim N_3 \gtrsim N_2$, then we can apply 46, 54, 37, and $1/2 < \theta < 1$, and take $0 < \epsilon < \theta - 1/2$ to argue that

(LHS) of 52

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{\substack{1\lesssim L_{min}\leq L_{med}\leq L_{max} \\ L_{max}\lesssim NN_{min}}} \left(\langle N_{min}\rangle^{s} N_{min}^{\frac{n-1}{2}} \right.$$
$$\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-1} \min\left\{1, \frac{L_{max}}{N_{min}^2}\right\}^{\frac{1}{2}-\epsilon} \right)$$

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim N} \sum_{\langle N_{min}\rangle^2\lesssim L_{max}\lesssim NN_{min}} \langle N_{min}\rangle^{s} N_{min}^{\frac{n-1}{2}} L_{max}^{\theta-1}$$
$$+ N^{-2s-\frac{1}{2}} \sum_{1\lesssim N_{min}\lesssim N} \sum_{1\lesssim L_{max}\lesssim N_{min}^2} N_{min}^{s+\frac{n-3}{2}+2\epsilon} L_{max}^{\theta-\frac{1}{2}-\epsilon}$$

$$\lesssim N^{-2s-\frac{1}{2}} \left(\sum_{N^{-1}\lesssim N_{min}\lesssim N} \langle N_{min}\rangle^{s+2\theta-2} N_{min}^{\frac{n-1}{2}} + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+2\theta+\frac{n-5}{2}} \right)$$

$$\sim N^{-2s-\frac{1}{2}} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+2\theta+\frac{n-5}{2}} \right).$$

It is easy to verify that, when (n, s, θ) is admissible, $s + 2\theta + (n-5)/2$ can be either positive, negative, or equal to zero. As such

$$N^{-2s-\frac{1}{2}} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+2\theta+\frac{n-5}{2}} \right) \sim N^{-s+2\theta+\frac{n-6}{2}}, N^{-2s-\frac{1}{2}}, \text{ or } N^{-2s-\frac{1}{2}} \ln N,$$

respectively. Due to 41, we see that 52 would be valid in this case if we ask for $s > -1/4$, which is a weaker condition than $s > (\theta - 1)/2$ imposed before. With this, the argument for 52 is finished.

Next, we turn to the proof of 53, which is quite similar to the one for 31. If $N_1 \sim N_2 \gg N_3$ and, hence, $H \sim N_1^2$, then we can rely on 45. Jointly with 54, 35,

$\theta > 1/2$, and 41, it yields

$$\text{(LHS) of } 53 \lesssim \sum_{N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{min} \leq L_{med} \sim L_{max} \\ N^2 \ll L_{max}}} \langle N_{min} \rangle^{-s} N_{min}^{\frac{n}{2}} L_{min}^{\frac{1}{2}-\theta} L_{max}^{-1}$$

$$\lesssim N^{-2} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{-s+\frac{n}{2}} \right) \sim N^{-s+\frac{n-4}{2}} \lesssim 1.$$

We have no coherence case to explore since $H \ll L_{max}$. Thus, all we are left to analyze is the stand-alone scenarios $N_1 \sim N_2 \sim N_3$, $N_2 \sim N_3 \gtrsim N_1$, and $N_1 \sim N_3 \gtrsim N_2$. First, we note that we can use 49 in all three of these cases. When $N_1 \sim N_2 \sim N_3$, 50 is also available. If we bring 54 and 37 into the mix, then we deduce

(LHS) of 53

$$\lesssim N^{-s+\frac{n-4}{2}+2\epsilon} \sum_{1 \lesssim L_{min} \leq L_{med} \sim L_{max}} \sum_{H \lesssim \min\{L_{max}, N^2\}} L_{min}^{\frac{1}{2}-\theta} L_{max}^{-1} H^{1-\epsilon},$$

which coincides with the estimate satisfied by the left-hand side of 31 in the same situation. Accordingly, by choosing $0 < \epsilon < \theta - 1/2$ and applying 41, we infer that 53 holds true in this instance.

If $N_2 \sim N_3 \gtrsim N_1$, then, with the help of 54, 37, and 49, we obtain

(LHS) of 53

$$\lesssim N^{-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{1 \lesssim L_{min} \leq L_{med} \sim L_{max}} \sum_{H \lesssim \min\{L_{max}, NN_{min}\}} \left(\langle N_{min} \rangle^{-s} N_{min}^{\frac{n-1}{2}} L_{min}^{\frac{1}{2}-\theta} \right.$$

$$\left. \cdot L_{max}^{-1} H^{\frac{1}{2}} \min\left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2}-\epsilon} \right).$$

This is identical to the bound satisfied by the left-hand side of 31 when $N_1 \sim N_3 \gtrsim N_2$ and, hence, 53 is seen to be valid by taking ϵ as above and relying on 48.

When $N_1 \sim N_3 \gtrsim N_2$, a very similar argument leads to

(LHS) of 53

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{1 \lesssim L_{min} \leq L_{med} \sim L_{max}} \sum_{H \lesssim \min\{L_{max}, NN_{min}\}} \left(\langle N_{min} \rangle^{s} N_{min}^{\frac{n-1}{2}} L_{min}^{\frac{1}{2}-\theta} \right.$$

$$\left. \cdot L_{max}^{-1} H^{\frac{1}{2}} \min\left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2}-\epsilon} \right)$$

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{H \lesssim NN_{min}} \sum_{\langle H \rangle \lesssim L_{max}} \langle N_{min} \rangle^{s} N_{min}^{\frac{n-1}{2}} L_{max}^{-1} H^{\frac{1}{2}} \min\left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2}-\epsilon}$$

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N_{min} \lesssim N} \sum_{H \lesssim NN_{min}} \langle N_{min} \rangle^{s} N_{min}^{\frac{n-1}{2}} \langle H \rangle^{-1} H^{\frac{1}{2}} \min\left\{ 1, \frac{H}{N_{min}^2} \right\}^{\frac{1}{2}-\epsilon}$$

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N_{min} \lesssim N^{-1}} N_{min}^{\frac{n-1}{2}} \left(\sum_{H \lesssim N_{min}^2} \frac{H^{1-\epsilon}}{N_{min}^{1-2\epsilon}} + \sum_{N_{min}^2 \lesssim H \lesssim NN_{min}} H^{\frac{1}{2}} \right)$$

$$+ N^{-2s-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim 1} N_{min}^{\frac{n-1}{2}} \left(\sum_{H \lesssim N_{min}^2} \frac{H^{1-\epsilon}}{N_{min}^{1-2\epsilon}} + \sum_{N_{min}^2 \lesssim H \lesssim 1} H^{\frac{1}{2}} \right.$$

$$\left. + \sum_{1 \lesssim H \lesssim N N_{min}} H^{-\frac{1}{2}} \right)$$

$$+ N^{-2s-\frac{1}{2}} \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{s+\frac{n-1}{2}} \left(\sum_{H \lesssim 1} \frac{H^{1-\epsilon}}{N_{min}^{1-2\epsilon}} + \sum_{1 \lesssim H \lesssim N_{min}^2} \frac{H^{-\epsilon}}{N_{min}^{1-2\epsilon}} \right.$$

$$\left. + \sum_{N_{min}^2 \lesssim H \lesssim N N_{min}} H^{-\frac{1}{2}} \right)$$

$$\lesssim N^{-2s} \sum_{N_{min} \lesssim N^{-1}} N_{min}^{\frac{n}{2}} + N^{-2s-\frac{1}{2}} \left(\sum_{N^{-1} \lesssim N_{min} \lesssim 1} N_{min}^{\frac{n-1}{2}} + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{s+\frac{n-3}{2}+2\epsilon} \right)$$

$$\lesssim N^{-2s-\frac{1}{2}} \left(1 + \sum_{1 \lesssim N_{min} \lesssim N} N_{min}^{s+2\theta+\frac{n-5}{2}} \right),$$

which coincides with the estimate derived for 52 in the same scenario. It follows that 53 holds true if we impose $s > -1/4$. This concludes the proof of 53 and of the entire proposition. $\square$

For the purpose of obtaining LWP results using the framework in our paper, we notice that both 21 and 22 require $s > -3/4$ and $s > -1/2$ when $n = 2$ and $n = 3$, respectively. On the other hand, 23 asks for $s > -1/4$ when either $n = 2$ or $n = 3$. Hence, a natural question is whether the actual bilinear estimates needed for the fixed point argument (i.e., 10-12) would be valid for lower values of s than the ones above. We next address comments made earlier that, in our judgement, this is not the case. We take a look at 12 with $\lambda = 1$ chosen for convenience, which, arguing as in the derivation of 51, is equivalent to

$$\left\| \frac{|\xi_2|^2 \langle \xi_3 \rangle^{-s} \langle \tau_3 + |\xi_3|^2 \rangle^{-\theta}}{\langle \xi_1 \rangle^s \langle \tau_1 - |\xi_1|^2 \rangle^{\theta} \langle \xi_2 \rangle^{2-s} \langle \tau_2 - |\xi_2|^2 \rangle^{1-\theta}} \right\|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim 1.$$

The corresponding low modulation estimate is given by

$$\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_1,L_2,L_3 \gtrsim 1} \frac{\langle N_1 \rangle^{-s} N_2^2 \langle N_2 \rangle^{s-2} \langle N_3 \rangle^{-s}}{L_1^{\theta} L_2^{1-\theta} L_3^{\theta}} \tag{55}$$

$$\cdot \| X_{N_1,N_2,N_3;L_{max};L_1,L_2,L_3} \|_{[3,\mathbb{R}^n \times \mathbb{R}]} \lesssim 1$$

and we consider the coherence scenario where, in addition to 33, one has $N_1 \sim N_3 \gg N_2$ and $H \sim L_2 \gg L_1, L_3, N_2^2$. By applying 36 and $\theta > 1/2$, we derive that

(LHS) of 55

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N^{-1} \lesssim N_{min} \lesssim N} \sum_{\substack{1 \lesssim L_{min} \leq L_{med} \leq L_{max} \\ N_{min}^2 \ll L_{max} \lesssim N N_{min}}} \left(\langle N_{min} \rangle^{s-2} N_{min}^{\frac{n+3}{2}} \right.$$

$$\left. \cdot L_{min}^{\frac{1}{2}-\theta} L_{med}^{-\theta} L_{max}^{\theta-\frac{1}{2}} \min \left\{ 1, \frac{L_{med}}{N_{min}^2} \right\}^{\frac{1}{2}} \right)$$

$$\lesssim N^{-2s-\frac{1}{2}} \sum_{N^{-1}\lesssim N_{min}\lesssim 1} \sum_{1\lesssim L_{med}\leq L_{max}\lesssim NN_{min}} N_{min}^{\frac{n+3}{2}} L_{med}^{-\theta} L_{max}^{\theta-\frac{1}{2}}$$

$$+ N^{-2s-\frac{1}{2}} \sum_{1\lesssim N_{min}\lesssim N} \sum_{\substack{1\lesssim L_{med}\lesssim N_{min}^2 \\ N_{min}^2 \ll L_{max}\lesssim NN_{min}}} N_{min}^{s+\frac{n-3}{2}} L_{med}^{\frac{1}{2}-\theta} L_{max}^{\theta-\frac{1}{2}}$$

$$+ N^{-2s-\frac{1}{2}} \sum_{1\lesssim N_{min}\lesssim N} \sum_{\substack{N_{min}^2 \lesssim L_{med}\leq L_{max} \\ N_{min}^2 \ll L_{max}\lesssim NN_{min}}} N_{min}^{s+\frac{n-1}{2}} L_{med}^{-\theta} L_{max}^{\theta-\frac{1}{2}}$$

$$\lesssim N^{-2s+\theta-1} \left(1 + \sum_{1\lesssim N_{min}\lesssim N} N_{min}^{s+\theta+\frac{n-4}{2}}\right),$$

which coincides with the bound obtained in the same setting in the previous proposition. As argued there, one would still need to impose $s > (\theta - 1)/2$ (and, thus, $s > -1/4$) for 55 to hold true.

4. **Alternative method for the summation argument.** In this section, we propose an alternative way to perform the summation component for the proofs of 30 and 31 (as well as for the ones of 52 and 53). It is based on a Python code which streamlines the summation process and, in our opinion, has the potential to be readily adaptable to other similar problems.

In order to explain the idea behind this method, let us discuss first some elementary examples. As in the previous section, we adopt the convention that all variables involved in summations assume only dyadic values. Clearly, for B fixed, one has

$$\sum_{A\lesssim B} A \sim B.$$

However, when slightly more involved conditional inequalities are introduced in the summation, e.g.,

$$\sum_A \sum_{B\lesssim\min\{1,A^{-2}\}} AB,$$

the situation is less straightforward. In fact, for the above sum, one needs to split it into two pieces corresponding to the two possible values of the minimum. As such, it follows that

$$\sum_A \sum_{B\lesssim\min\{1,A^{-2}\}} AB = \sum_{A\leq 1} \left(\sum_{B\lesssim 1} B\right) A + \sum_{A>1} \left(\sum_{B\lesssim A^{-2}} B\right) A$$

$$\sim \sum_{A\leq 1} A + \sum_{A>1} A^{-1} \sim 1.$$

What we want to stress here is that in order to perform the summation in B, we had to split the values of A into two complementary sets.

When dealing with a summation like the one in 31, which is performed over seven variables (i.e., $(N_i)_{1\leq i\leq 3}$, $(L_i)_{1\leq i\leq 3}$, and H), with each one being involved in at least one conditional inequality, the process is obviously much more complex. This is why a computer-assisted analysis makes sense in this type of situation. The way in which we conduct the analysis is as follows:

1. write the full summation as an iterated summation over each present variable;

2. allow first for the variables to vary independently;

3. let the computer perform the summation;

4. in case the summation yields an infinite result, use one or more conditional inequalities to impose restrictions on the ranges of the variables and repeat the previous step.

To illustrate the efficacy of this procedure, we take as a case study the low modulation scenario for 21 with $(n, s, \theta) = (2, -1/2, 5/8)$. Hence, the variables involved in 30 satisfy the conditional inequalities

$$N_{max} \sim N_{med} \gtrsim 1 \gtrsim N_{min}, \tag{56}$$

$$N_{max} \sim N, \tag{57}$$

$$L_{max} \gg L_{med} \geq L_{min} \gtrsim 1, \tag{58}$$

$$H \sim N_{max}^2 \sim L_{max}, \tag{59}$$

while, according to 34,

$$\left\| X_{N_1,N_2,N_3;H;L_1,L_2,L_3} \right\|_{[3,\mathbb{R}^2 \times \mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} N_{min}^{\frac{1}{2}} \min\{N_{max} N_{min}, L_{med}\}^{\frac{1}{2}}.$$

To be able to work with a summand which is as explicit as possible, we make two assumptions. First, we let

$$\min\{N_{max} N_{min}, L_{med}\} = L_{med}. \tag{60}$$

Secondly, by taking into account 43, we specialize to the more challenging case when $N_{min} = N_3$ and $L_{max} = L_3$. Thus, the summand has the formula

$$S = \langle N_{min} \rangle^{-\frac{1}{2}} N_{min}^{\frac{1}{2}} \langle N_{med} \rangle^{\frac{1}{2}} \langle N_{max} \rangle^{\frac{1}{2}} N_{max}^{-\frac{1}{2}} L_{min}^{-\frac{1}{8}} L_{med}^{-\frac{1}{8}} L_{max}^{-\frac{3}{8}}.$$

This is the moment when we initiate the procedure described above, for which the first iteration trivially yields that

$$\sum_{N_{max}=0}^{\infty} \sum_{N_{med}=0}^{\infty} \sum_{N_{min}=0}^{\infty} \sum_{L_{max}=0}^{\infty} \sum_{L_{med}=0}^{\infty} \sum_{L_{min}=0}^{\infty} \sum_{H=0}^{\infty} S = \infty.$$

Next, we implement 56 and 58 jointly with $H \sim L_{max}$ to infer that

$$S \sim N_{min}^{\frac{1}{2}} N_{max}^{\frac{1}{2}} L_{min}^{-\frac{1}{8}} L_{med}^{-\frac{1}{8}} L_{max}^{-\frac{3}{8}}$$

and write the summation as

$$\sum_{N_{max}=2}^{\infty} \sum_{N_{med}=\frac{N_{max}}{2}}^{N_{max}} \sum_{N_{min}=0}^{1} \sum_{L_{max}=8}^{\infty} \sum_{L_{med}=1}^{\frac{L_{max}}{8}} \sum_{L_{min}=1}^{L_{med}} \sum_{H=\frac{L_{max}}{2}}^{2L_{max}} S.$$

However, another iteration of the third step in the procedure still produces an infinite sum. Following this, we use 59 and 60 to argue that $N_{max} N_{min}$ is a better upper bound for L_{med} than $L_{max}/8$. Since $L_{med} \geq 1$, this change also brings about N_{max}^{-1} and $N_{max} N_{min}$ as new, improved lower bounds for N_{min} and L_{max}. Consequently, the summation takes the form

$$\sum_{N_{max}=2}^{\infty} \sum_{N_{med}=\frac{N_{max}}{2}}^{N_{max}} \sum_{N_{min}=N_{max}^{-1}}^{1} \sum_{L_{max}=N_{max} N_{min}}^{\infty} \sum_{L_{med}=1}^{N_{max} N_{min}} \sum_{L_{min}=1}^{L_{med}} \sum_{H=\frac{L_{max}}{2}}^{2L_{max}} S.$$

Unfortunately, by running again the computation step, we obtain infinity for an answer. Finally, if we rely on the unused part of 59 (i.e., $L_{max} \sim N_{max}^2$), we can

modify, with better lower and upper bounds, the sums with respect to L_{max} and H. Hence, we are dealing with

$$\sum_{N_{max}=2}^{\infty} \sum_{N_{med}=\frac{N_{max}}{2}}^{N_{max}} \sum_{N_{\min}=N_{max}^{-1}}^{1} \sum_{L_{\max}=\frac{N_{max}^2}{2}}^{2N_{max}^2} \sum_{L_{med}=1}^{N_{max}N_{min}} \sum_{L_{min}=1}^{L_{med}} \sum_{H=\frac{N_{max}^2}{4}}^{4N_{\max}^2} S$$

and another iteration of the third step in our procedure yields a result which is both finite and comparable to 1. It is worth noticing that we did not make use of 57 in the process.

As final comments, let us say that our code is easily adapted to cover the summation arguments for the other types of bilinear estimates proved by Tao in [11] (e.g., bounds related to the KdV and wave equations). Moreover, we see no reason not to believe that it can accommodate even general multilinear estimates involving dyadic decompositions.

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