ROUGH SEMI-UNIFORM SPACES AND ITS IMAGE PROXIMITIES

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Abstract. In this paper, we introduce the concept of rough semi-uniform spaces as a supercategory of rough pseudometric spaces and approximation spaces. A completion of approximation spaces has been constructed using rough semi-uniform spaces. Applications of rough semi-uniform spaces in the construction of proximities of digital images is also discussed.

1. **Introduction.** Various topological notions have been involved in the conceptual frameworks of data analysis theories and successfully applied in solving many problems and tasks in science and engineering, e.g., image processing, forgery detection, determining the age of fossils, pattern recognition, image classification, etc. Rough set theory is the quintessence of data processing methodology, which is infused with topological concepts (see [13]). Another great exemplar is given by near set theory, which has shifted from standard topologies towards uniformities. The near set theory was introduced by Peters and had applications in various fields [8, 9, 11]. Peters et al. used the concept of descriptive proximity to study digital images. In [10], Peters et al. used rough sets and nearness-like structures to describe and compare visual objects. Pessoa et al. [7] used the concept of rough sets theory for image classification. In this paper, we use semi-uniform spaces in the framework of rough sets to study nearness of digital images.

Zdzisław Pawlak used an equivalence relation on a non-empty set to introduce approximation spaces during the early 1980s for the classification of objects employing attributes of information systems [6]. The rough set theory is an extension of the set theory mainly used for the study of intelligent systems characterized by insufficient and incomplete information. There is a close similitude between rough set theory and general topology. Topology is a rich source of constructs that can be helpful to enrich the original modal of approximation spaces [14, 15]. The approximation operators (upper approximation and lower approximation) are topological operators (closure and interior, respectively). Conjoint study of topology and rough set theory has been widely applied to many real-world problems and in the development of the new mathematical structures [13, 14, 15].

Biswas firstly gave the idea of rough distance [14]. In [14, 15], we defined a proximity relation over an approximation space by using the concept of a rough metric

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and named it a rough proximity relation. In [15], we studied the compactification of rough proximity spaces. It is well known that every proximity δ is induced by a semi-uniformity, and among all semi-uniformities inducing proximity, there exists a unique smallest one, which is termed as proximally coarse semi-uniformity. If a proximity δ is induced by a uniformity, i.e., if δ is uniformizable, then the proximally coarse semi-uniformity inducing δ is a uniformity. Therefore, the study of proximities is equivalent to the study of proximally coarse semi-uniformities. Moreover, proximally coarse uniformities are totally bounded uniformities. Uniform spaces [5] are similar to metric spaces; however, the application area of uniform spaces is more substantial than that of metric spaces. Since every uniform space can be transformed to a topological space, therefore there exists a relation between uniform spaces and topological spaces. Consequently, it is vital to carry uniform spaces in the framework of rough sets. In [16], Vlach discussed a co-relation between approximation spaces and uniform spaces and showed that Pawlak's approximation spaces are uniform spaces whose uniform topologies coincide with partition topologies.

In practice, an information system describes a finite sample X of elements from the larger (may be infinite) universe U. The study of mathematical structures of rough sets in infinite universes of discourse has been done in [19]. In the field of image analysis, the classification of images of a complete universe becomes easier. Completion has always been a favorite extension problem of topologists. When we deal with the infinite universe of discourse, the study becomes easier if space is complete. For example, the completeness property plays a vital role in the Hahn-Banach theorem, Baire's category theorem, the uniform boundedness principle, and the open mapping theorem. If space is not complete, we can make it complete by the completion of the space. Many researchers gave the completion of quasi-uniform spaces (see [5]). Batíková is one of them, who constructed the completion of non-Hausdorff t-semi-uniform spaces [1]. Completion of quasi-uniform spaces has been studied by researchers, but the completion of semi-uniform spaces is still missing in the literature.

In this paper, we introduce the notion of rough semi-uniform spaces as a super-category of rough pseudometric spaces and construct its completion by using the concept of generalized Cauchy sequences [5]. The cardinal motive of this paper is the construction of a semi-uniform structure on an approximation space to study the completion of an approximation space. The category of rough semi-uniform spaces is shown to be a supercategory of the categories of metric spaces, ultrametric spaces and rough metric spaces with their respective mappings. Therefore, the study in this paper presents a unified study of all these spaces. Finally, the application of rough semi-uniform spaces in the study of proximities of digital images is also investigated. Image proximities are used in the classification of digital images that have applications in image analysis, pattern recognition, fluid dynamics and many more [2, 3, 4, 9, 17, 18].

2. **Preliminaries and basic results.** In this paper, we deal with the concept of rough sets defined by Yao [20], which are a generalized form of Pawlak's rough sets. Throughout this paper, let R be an arbitrary tolerance (reflexive and symmetric) relation in the approximation space (U, R), where U is a non-empty set, called the universe. For basic definitions of topological terms, we refer [5]. In this section, we collect some basic definitions of rough sets and uniform spaces.

2.1. **Rough set theory.** Let U be a non-empty set and R be a given binary relation. For $x \in U$ define $R(x) := \{y : xRy\}$, i.e., $R(x)$ consists of all elements of U which are R-related to the element 'x'. We may define two unary set theoretic operators $\overline{R}$ (Upper Approximation) and $\underline{R}$ (Lower Approximation) as follows:

$$\overline{R}(A) = \{x : R(x) \cap A \neq \emptyset\},$$
$$\underline{R}(A) = \{x : R(x) \subseteq A\}, \ A \subseteq U.$$

A set $A \subseteq U$ is said to be crisp if $\overline{R}(A) = \underline{R}(A)$, otherwise A is rough. For properties of upper and lower approximation operators and other basic results we refer [20].

Remark 2.1. Relation between $R(x)$ and $\overline{R}(\{x\})$: As defined above, $R(x) = \{y : xRy\}$ and $\overline{R}(\{x\}) = \{y : R(y) \cap \{x\} \neq \emptyset\}$. If R is reflexive, then $A \subseteq \overline{R}(A)$. So we have $\{x\} \subseteq \overline{R}(\{x\})$. Now, let $z \in \overline{R}(\{x\})$. Then $R(z) \cap \{x\} \neq \emptyset$ which yields that zRx. If R is symmetric also, then xRz which yields $\overline{R}(\{x\}) \subseteq R(x)$. Thus we have $R(x) = \overline{R}(\{x\})$, if R is symmetric and reflexive, *i.e.*, a tolerance relation. For convenience, we will use $\overline{R}(x)$ in place of $\overline{R}(\{x\})$. If R is a reflexive relation on U, then the family $\tau_R = \{X \subseteq U : \underline{R}(X) = X\}$ is a topology on U.

Throughout the paper, we restrict ourselves to the tolerance relation R in the approximation space (U, R).

2.2. **Topological spaces.**

Definition 2.1. [9]. Let U be a non-empty set and τ a subset of the power set of U. Then τ is said to be a topology on U if the following properties hold:

1. $\emptyset$ and U are in τ.
2. If $A_i \in \tau$, where $i \in \Lambda$, then $\bigcup_{i \in \Lambda} A_i \in \tau$.
3. If $A_1, A_2, \ldots, A_n \in \tau$, then $\bigcap_{j=1}^{n} A_j \in \tau$.

The pair (U, τ) is said to be a topological space.

Closure operation was proposed by Riesz as a primitive topological notion [14].

Definition 2.2. [14]. Let U be a non-empty set. An operator $cl : \mathcal{P}(U) \to \mathcal{P}(U)$ is called a Čech closure operator on U if it satisfies the following axioms, for all $A, B \subset U$:

1. $cl \ (\emptyset) = \emptyset$.
2. $A \subset cl(A)$.
3. $cl(A \cup B) = cl(A) \cup cl(B)$.

Further, cl is a Kuratowski closure operator on U if it satisfies, in addition, $cl(clA) = cl(A)$. Every Kuratowski closure operation on U induces a topology τ on U. Also, if cl_τ is the closure induced by τ, then $cl_\tau = cl$. Therefore, sometimes for convenience, we will use the term (U, cl) as a topological space in place of (U, τ).

2.3. **Proximity spaces.** In general way of representation, topology deals with structures "a point is near to a set", while proximity structures [5] are finer than topology and are based on the concept "one set is near to another set". The following definition axiomatize a proximity space.

Definition 2.3. [5]. Let U be a non-empty set, a binary relation δ on $\mathcal{P}(U)$ is called a basic proximity if the following axioms are satisfied for $A, B, C \subseteq U$:

1. $A\delta B \Rightarrow B\delta A$.
2. $(A \cup B)\delta C \Leftrightarrow A\delta C$ or $B\delta C$.
3. $A\delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.

4. $A \cap B \neq \emptyset \Rightarrow A\delta B$.

A binary relation δ is called an EF-proximity if it is a basic proximity and additionally satisfies the following axiom,

5. $A \not\delta B \Rightarrow \exists E \subset U$ such that $A \not\delta E$ and $B \not\delta E^c$,

where $A \not\delta B$ means that the set A is far (not near) from the set B, and A^c is the compliment of the set A. Further, δ is called separated if the following axiom holds:

6. $\{x\}\delta\{y\} \Rightarrow x = y$.

The pair (U, δ) is called a basic (EF, separated) proximity space.

If a basic proximity δ satisfies the following axiom, in addition:
$$A\delta B \text{ and } \{b\}\delta C, \text{ for each } b \in B \Rightarrow A\delta C;$$
then δ is called an LO-proximity and the pair (U, δ) is known as an LO-proximity space.

Let (U, δ) be a proximity space. For convenience we will write $a\delta A$ in place of $\{a\}\delta A$. Then we may define a closure operator cl_δ over U by:
$$cl_\delta(A) = \{x \in U : \{x\}\delta A\}.$$
Every proximity δ induces a unique topology τ_δ induced by the closure operator cl_δ. A topological space (U, τ) admits a proximity, if there exists a proximity δ on U such that $\tau = \tau_\delta$. Also, δ is called a compatible proximity on the topological space (U, τ).

2.4. Uniform spaces.

Definition 2.4. [5]. Let U be a non-empty set. Then a collection $\mathcal{U}$ of binary relations on U is called a semi-uniformity on U, if the following axioms hold for every $P, Q \subseteq U \times U$:

1. $\Delta \subseteq \bigcap \mathcal{U}$, where $\Delta = \{(x, x) : x \in U\}$.
2. $P, Q \in \mathcal{U} \Rightarrow P \cap Q \in \mathcal{U}$.
3. $P \subseteq Q$ such that $P \in \mathcal{U} \Rightarrow Q \in \mathcal{U}$.
4. If $P \in \mathcal{U}$, then $P^{-1} \in \mathcal{U}$.

A semi-uniform space is a struct $(U, \mathcal{U})$ such that U is a non-empty set and $\mathcal{U}$ is a semi-uniformity for U. The collection $\mathcal{U}$ of binary relations is called a uniformity if it is a semi-uniformity and additionally satisfies the following axiom:

5. For each $P \in \mathcal{U}$, there exists a $Q \in \mathcal{U}$ such that $Q \circ Q \subseteq P$, where $\circ$ stands for standard composition of relations.

Sometimes the axiom '5' is referred as "strong axiom". In our study, we will not consider the strong axiom.

Remark 2.2. There is a natural transformation from a uniform space to a proximity space, i.e., every uniform space gives rise to a proximity space.

3. **Rough semi-uniform spaces.** In this section, we define a rough semi-uniform space as a supercategory of approximation spaces, rough pseudometric spaces and pseudometric spaces. Some results on rough semi-uniform spaces are proved. Examples are constructed to well support the theory of rough semi-uniform spaces. Finally, we construct the completion of a rough semi-uniform space. The upper approximation of a set A is the set of elements of U which are near to the set A with respect to relation R. The closure of the set A has the similar definition. Therefore in the next definition, we give a connection between the upper approximation of a set and its closure in the framework of approximation spaces.

Definition 3.1. [14] Let (U, R) be an approximation space. A function $cl_R :$ $\mathcal{P}(U) \to \mathcal{P}(U)$ is said to be a Čech rough closure operator on U if it is a Čech closure operator and for $A \subseteq U$, $cl_R(A) \supseteq \overline{R}(A)$. The pair (U, cl_R) is said to be Čech rough closure space.

The Čech rough closure operator returns sets divorced from R. Approximation spaces give rise to uniform spaces in an easy and elegant way. So let us begin with defining the uniformity on an approximation space.

Definition 3.2. Let (U, R) be an approximation space, where R is a tolerance relation. Let $U^* := \{R(x) : x \in U\}$, $\mathcal{U}_R \subseteq \mathcal{P}(U^* \times U^*)$ and $\Delta_R := \{(R(x), R(x)) : x \in U\}$. Then The pair $(U, \mathcal{U}_R)$ is called a rough semi-uniform space if the pair $(U^*, \mathcal{U}_R)$ is a semi-uniform space, and $\mathcal{U}_R$ is called a rough semi-uniformity on U.

Let $(U, \mathcal{U}_R)$ be a rough semi-uniform space. Define $P[x] := \{y \in U : R(x)PR(y)\}$, for every $P \in \mathcal{U}_R$ and $x \in U$. Then $\mathcal{U}_R[x] = \{P[x] : P \in \mathcal{U}_R\}$ forms a neighbourhood system on the approximation space (U, R), i.e., $\mathcal{U}_R[x]$ is a local base at x in (U, cl_R), for each $x \in U$, where cl_R is a Čech rough closure operator on U defined by: $x \in cl_R(A)$ iff $P[x] \cap A \neq \emptyset$, for all $P \in \mathcal{U}_R$. The following statements can be easily verified:

• Let $(U, \mathcal{U}_R)$ be a rough semi-uniform space such that each $Q \in \mathcal{U}_R$ satisfies: $R(x)QR(y)$ iff $R(x) \cap R(y) \neq \emptyset$, $x, y \in U$. If $Q \subseteq \cap \mathcal{U}_R$, then cl_R (defined as above) becomes a Čech rough closure operator.

• Let $(U, \mathcal{U}_R)$ be a rough semi-uniform space such that $\cap \mathcal{U}_R \in \mathcal{U}_R$ and $\cap \mathcal{U}_R$ is an equivalence relation, i.e., $(U, \mathcal{U}_\mathcal{R})$ is a rough $\cap$-complete semi-uniform space. Then cl_R becomes a Kuratowski (topological) closure operator on U, i.e., $cl_R cl_R(A) = cl_R(A)$, $A \subseteq U$ (see [14]).

• The semi-uniform spaces can be embedded into rough semi-uniform spaces with their respective mappings (consider the relation R as the equality relation).

Let (U, cl) be a Čech closure space [14]. A set $X \subseteq U$ is said to be closed if $X = cl(X)$. A set $X \subseteq U$ is said to be open if $X = (cl(X^c))^c$. Further, $X \subseteq U$ is said to be dense in U if $cl(X) = U$. A set $X \subseteq U$ in a Čech closure space (U, cl) is dense iff every open set intersects X.

Now, we will show that the category of rough semi-uniform spaces is a super category of the category of rough pseudometric spaces. In the process, let us first define rough pseudometric spaces. Biswas defined a rough metric on Pawlak's approximation spaces [6] and discussed the properties of rough metric spaces (see [15]). Equivalently, a rough pseudometric on a Yao's approximation space can be defined as follows:

Definition 3.3. [14] Let U be a non-empty set and R be a tolerance relation defined on U. Then the function $d_R : U \times U \to [0, \infty)$ is called a rough pseudometric on U if the following conditions are true, for all $x, y, z \in U$:

1. $d_R(x, y) = 0$ if $R(x) = R(y)$.
2. $d_R(x, y) = d_R(y, x)$.
3. $d_R(x, y) + d_R(y, z) \geq d_R(x, z)$.

The pair (U, d_R) is called a rough pseudometric space.

If R is an equality relation on U, then d_R is a standard pseudometric space. Thus, the category of pseudometric spaces and metric spaces can be embedded into the category of rough pseudometric spaces, with their respective mappings.

Proposition 3.1. *Let (U, d_R) be a rough pseudometric space. Define $U_\varepsilon := \{(R(x), R(y)) : d_R(x, y) < \varepsilon\}$, for every real number $\varepsilon > 0$. Then U_ε is a filter base for a rough semi-uniformity $\mathcal{U}_{d_R}$ on U.*

Thus, the category of rough pseudometric spaces and hence the category of pseudometric spaces can be embedded into the category of rough semi-uniform spaces, with their respective mappings. We now present some examples of rough semi-uniform spaces.

Example 3.1. Let (U, R) be an approximation space. Define $\mathcal{U}_R := \{Q \subseteq U^* \times U^* : \Delta_R \subseteq Q\}$. Then $\mathcal{U}_R$ is a rough semi-uniformity on U. Further, $\mathcal{U}_R$ is the largest rough semi-uniformity on U and is called the discrete rough semi-uniformity on U. Moreover, $cl_R(A) = \overline{R}(A)$, $A \subseteq U$.

Remark 3.1. From Example 3.1, it is clear that the category of approximation spaces can be embedded into the category of rough semi-uniform spaces with their respective mappings.

Example 3.2. Let (U, R) be an approximation space and $\mathcal{U}_R := \{U^* \times U^*\}$. Then $\mathcal{U}_R$ is a rough semi-uniformity on U. Further, $\mathcal{U}_R$ is the smallest rough semi-uniformity on U and is called the indiscrete rough semi-uniformity on U.

Example 3.3. Let $U = \{x_1, x_2, \ldots, x_{20}\}$. Define a tolerance relation R on U such that $R(x_1) = \{x_1, x_2, x_3, x_6, x_{12}, x_{18}\}$; $R(x_2) = \{x_1, x_2, x_3, x_6, x_7, x_8, x_{18}, x_{19}\}$; $R(x_3) = \{x_1, x_2, x_3, x_8, x_9, x_{15}, x_{20}\}$; $R(x_4) = \{x_4, x_5, x_6, x_{10}\}$; $R(x_5) = \{x_4, x_5\}$; $R(x_6) = \{x_1, x_2, x_4, x_6, x_7, x_8, x_{12}, x_{18}, x_{20}\}$; $R(x_7) = \{x_2, x_6, x_7, x_{18}, x_{19}\}$; $R(x_8) = \{x_2, x_3, x_6, x_8, x_9, x_{10}, x_{15}\}$; $R(x_9) = \{x_3, x_8, x_9, x_{11}\}$; $R(x_{10}) = \{x_4, x_8, x_{10}, x_{12}, x_{13}, x_{14}\}$; $R(x_{11}) = \{x_9, x_{11}, x_{20}\}$; $R(x_{12}) = \{x_1, x_6, x_{10}, x_{12}, x_{13}, x_{14}\}$; $R(x_{13}) = \{x_{10}, x_{12}, x_{13}, x_{14}, x_{20}\}$; $R(x_{14}) = \{x_{10}, x_{12}, x_{13}, x_{14}, x_{20}\}$; $R(x_{15}) = \{x_3, x_8, x_{15}, x_{16}, \}$; $R(x_{16}) = \{x_{15}, x_{16}, x_{20}\}$; $R(x_{17}) = \{x_{17}, x_{18}, x_{19}\}$; $R(x_{18}) = \{x_1, x_2, x_6, x_7, x_{17}, x_{18}, x_{19}\}$; $R(x_{19}) = \{x_2, x_7, x_{17}, x_{18}, x_{19}, x_{20}\}$; $R(x_{20}) = \{x_3, x_6, x_{11}, x_{13}, x_{14}, x_{16}, x_{19}, x_{20}\}$.
Let $P \subseteq U^* \times U^*$ such that $R(x)PR(y)$ iff $R(x) \cap R(y) \neq \emptyset$. Define $\mathcal{U}_R := \{Q \subseteq U^* \times U^* : P \subseteq Q\}$, then $\mathcal{U}_R$ is a rough semi-uniformity on U.

In Example 3.3, we can easily verify that the closure operator cl_R generated by $\mathcal{U}_R$, is a Čech rough closure operator on U. Further, if $X = \{x_4, x_5\}$, then $cl_R(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{12}, x_{13}, x_{14}, x_{18}, x_{20}\}$. Therefore, $\overline{R}(cl_R(X)) = U$ and $\underline{R}(cl_R(X)) = \{x_4, x_5, x_6, x_{10}, x_{13}, x_{14}\}$. Thus $\overline{R}(cl_R(X)) \neq \underline{R}(cl_R(X))$. Hence, closure of a set need not to be crisp, in general. We can conclude that the Čech rough closure space on U generated by a rough semi-uniformity may consists of rough sets.

Example 3.4. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$. Define a tolerance relation R on U such that $R(x_1) = \{x_1, x_2, x_3\}$; $R(x_2) = \{x_1, x_2\}$; $R(x_3) = \{x_1, x_3\}$; $R(x_4) = \{x_4, x_5\}$; $R(x_5) = \{x_4, x_5\}$. Let $P \subseteq U^* \times U^*$ such that $P = \Delta_R \cup \{(R(x_1), R(x_2)), (R(x_2), R(x_1)), (R(x_3), R(x_4)), (R(x_4), R(x_3))\}$. Define $\mathcal{U}_R := \{Q \subseteq U^* \times U^* : P \subseteq Q\}$, then $\mathcal{U}_R$ is a rough semi-uniformity on U. Also, cl_R (the closure induced by $\mathcal{U}_R$) is a Kuratowski closure operator on U. Further, if $A = \{x_1\}$, then $\overline{R}(cl_R(A)) = \{x_1, x_2, x_3\} \neq \underline{R}(cl_R(A)) = \{x_1, x_2\}$. That is, the topology generated by $\mathcal{U}_R$ consists of rough sets also.

Keeping in mind the importance of complete (well-behaved) spaces (discussed in the Introduction section), we will construct the completion of a rough semi-uniform space. Since the category of approximation spaces, rough pseudometric

spaces, pseudometric spaces are embedded into the category of rough semi-uniform spaces, therefore, the theory developed in this section includes the study of all these aforementioned topological structures. In the process, we first define rough filters and rough Cauchy filters and use these concepts to construct the completion of a rough semi-uniform space.

Definition 3.4. A collection ζ_R of non-empty subsets of U is called a rough filter in U if it satisfies the following conditions, for all $A, B \subseteq U$:
1. $A \in \zeta_R$ and $\overline{R}(A) \subseteq \overline{R}(B) \Rightarrow B \in \zeta_R$.
2. $A, B \in \zeta_R \Rightarrow A \cap B \in \zeta_R$.

Definition 3.5. Let $(U, \mathcal{U}_R)$ be a rough semi-uniform space. A rough filter ζ_R on space U is called a rough Cauchy filter if for every $P \in \mathcal{U}_R$ there exists $F \in \zeta_R$ such that $F^* \times F^* \subseteq P$.

Definition 3.6. Let $(U, \mathcal{U}_R)$ be a rough semi-uniform space. Then $(U, \mathcal{U}_R)$ is said to be a complete rough semi-uniform space if every rough Cauchy filter converges.

Theorem 3.7. *Let $(U, \mathcal{U}_R)$ be an non-complete rough semi-uniform space. Let*
$$\hat{U} = U \cup \{\zeta_R : \zeta_R \text{ is a non-convergent rough Cauchy filter on } (U, \mathcal{U}_R)\}.$$
Let R' be a tolerance relation defined on $\hat{U}$ by:

$xR'y$ *iff* xRy *for all,* $x, y \in U$ *and* $\zeta_R R' \eta_R$ *iff* $\zeta_R = \eta_R$ *for all,* $\zeta_R, \eta_R \in \hat{U} - U$.

Let $\hat{\mathcal{U}}_{R'}$ be a filter on $\hat{U}^* \times \hat{U}^*$ having the base $\mathcal{B}$ consisting of symmetric sets V such that $V \cap (U^* \times U^*) \in \mathcal{U}_R$ and $V[\zeta_R] = \{\zeta_R\} \cup F$, where F is an open set from the filter ζ_R, for all $\zeta_R \in \hat{U} - U$. Then $\hat{\mathcal{U}}_{R'}$ is a complete rough semi-uniformity on $\hat{U}$.

Proof. Clearly $(\hat{U}, \hat{\mathcal{U}}_{R'})$ is a rough semi-uniformity and U is dense $\hat{U}$. We will prove that $(\hat{U}, \hat{\mathcal{U}}_{R'})$ is complete rough semi-uniform space. Let $\xi_{R'}$ be a rough Cauchy filter in $(\hat{U}, \hat{\mathcal{U}}_{R'})$. If $U \in \xi_{R'}$, then $\zeta_R = \{G \cap U : G \in \xi_{R'}\}$ converges to an element $x \in U$ and so $\xi_{R'}$ convergent rough Cauchy filter. Thus $\zeta_R \in \hat{U} - U$. Clearly $\xi_{R'}$ converges to ζ_R. If $U \notin \xi_{R'}$, then for every $V \in \mathcal{B}$, there exists $F_V \in \xi_{R'}$ such that $V[x] \in \xi_{R'}$, for each $x \in F_V$. Since $U \notin \xi_{R'}$, therefore F_V intersects $\hat{U} - U$. For every $\zeta_R \in F_V \cap (\hat{U} - U)$, we have $V[\zeta_R] = \{\zeta_R\} \cup G$, for some open set G and for every $V \in \mathcal{B}$. Hence $\xi_{R'}$ converges to ζ_R. So $(\hat{U}, \hat{\mathcal{U}}_{R'})$ is a complete rough semi-uniform space. $\square$

Remark 3.2. Let $(U, \mathcal{U}_R)$ be a rough semi-uniform space. Then $\hat{U}$, constructed in the above theorem, is called the completion of U. If $(U, \mathcal{U}_R)$ is a complete rough semi-uniform space, then $\hat{U}$ coincides with U.

4. **Applications of rough semi-uniform spaces.** The idea of perceptual nearness was introduced by Peters et al. when they used the concept of probe functions to classify digital images [11]. Later, probe functions were used to define descriptive nearness [2] between granules of a digital image. Probe functions work on feature vectors to perceptually differentiate digital objects. Descriptive nearness measures the perceptual nearness of objects, which are digital images that can be spatially far but may have similar feature vectors. Tolerance spaces and a perceptual approach in image analysis can be found in [8]. A set is a near perceptual set if and only if it is never empty, and it contains pairs of perceived objects that have descriptions that are within some tolerance of each other. From the beginning, the near set

approach to perception has had direct links to rough sets in its approach to the perception of objects and the classification of objects. This is evident in the early work on the nearness of objects and the extension of the approximation space model (see [2, 3, 4, 8, 9, 17, 18]). In [10], Peters et al. proved that two sets could be descriptively near if their upper approximations intersect. This motivated us to define Čech rough proximity spaces [14] for the purpose of studying image proximities. In this section, we gave a natural transformation of a rough semi-uniform space to a Čech rough proximity space. We used this transformation to give the perceptual applications of rough semi-uniform structures.

Example 4.2 gives the classification of digital images using a rough semi-uniformity. In Example 4.3, we use a rough semi-uniformity to distinguish the velocities of the fluid particles. First of all, let us define a Čech rough proximity space.

Definition 4.1. [14] Let (U, R) be an approximation space and δ_R be a binary relation defined on $\mathcal{P}(U)$ satisfying the following axioms, for $A, B, C \subseteq U$:

P1. $\overline{R}(A)\delta_R\overline{R}(B) \Rightarrow \overline{R}(B)\delta_R\overline{R}(A)$.
P2. $(\overline{R}(A) \cup \overline{R}(B))\delta_R\overline{R}(C) \Leftrightarrow \overline{R}(A)\delta_R\overline{R}(C)$ or $\overline{R}(B)\delta_R\overline{R}(C)$.
P3. $\overline{R}(A)\delta_R\overline{R}(B) \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.
P4. $\overline{R}(A) \cap \overline{R}(B) \neq \emptyset \Rightarrow A\delta_R B$.

The relation δ_R satisfying P1-P4 is called a Čech rough proximity on U and (U, δ_R) is called a Čech rough proximity space.

Proposition 4.1. (U, δ_R) be an Čech rough proximity space. For $A \subset U$, define $cl_{\delta_R} : \mathcal{P}(U) \to \mathcal{P}(U)$ as $cl_{\delta_R}(A) =\{x : R(x)\delta_R\overline{R}(A)\}$ $A \subseteq U$. Then cl_{δ_R} is a Čech closure operator on U. Clearly, $\overline{R}(A) \subset cl_{\delta_R}(A)$. If δ_R satisfies the following axiom, in addition,

P5. $\overline{R}(cl_{\delta_R}(A))\delta_R\overline{R}(cl_{\delta_R}(B)) \Rightarrow \overline{R}(A)\delta_R\overline{R}(B)$.

Then cl_{δ_R} is a Kuratowski closure operator on U. If the relation δ_R satisfies P1-P5 then it is called a rough proximity on U and (U, δ_R) is called a rough proximity space [15]. Therefore, $A\delta_R B$ means that "A is roughly near to B". Also $A \not\delta_R B$ means that A is not roughly near (or roughly far) to B.

There is a natural transformation of a rough semi-uniform space to a Čech rough proximity space which can be seen in the following theorem.

Theorem 4.2. Let (U, R) be an approximation space and let $\mathcal{U}_R$ be a rough semi-uniformity on U. Define relation δ_R on U by,

$$A\delta_R B \text{ iff } (A^* \times B^*) \cap P \neq \emptyset, \text{ for all } P \in \mathcal{U}_R,$$

where $A^* = \{R(x) : x \in A\}$; $B^* = \{R(y) : y \in B\}$. Then δ_R is a Čech rough proximity on U and the pair (U, δ_R) is a Čech rough proximity space.

Proof. Properties P1, P3 and P4 are obvious. For P2, if $(\overline{R}(A) \cup \overline{R}(B))\delta_R\overline{R}(C)$, then $(\overline{R}(A) \cup \overline{R}(B))^* \times (\overline{R}(C))^* \cap P \neq \emptyset$, $A, B, C \subset U$; for all $P \in \mathcal{U}_R$. This implies $((\overline{R}(A))^* \cup (\overline{R}(C))^*) \cap P \neq \emptyset$ or $((\overline{R}(B))^* \cup (\overline{R}(C))^*) \cap P \neq \emptyset$, for all $P \in \mathcal{U}_R$. Thus $\overline{R}(A)\delta_R\overline{R}(C)$ or $\overline{R}(B)\delta_R\overline{R}(C)$. Hence δ_R is a Čech rough proximity on U. $\square$

Example 4.1. Let (U, R) be an approximation space and d_R be a pseudo metric on (U, R). Define $\mathcal{U}_R = \{U_\varepsilon : \varepsilon \in \mathbb{R}^+\}$, where $U_\varepsilon = \{(R(x), R(y)) : d_R(x, y) < \varepsilon\}$. Then $\mathcal{U}_R$ is a rough uniformity. Further, let $\delta_{\mathcal{U}_R}$ be a relation defined as follows: $A\delta_{\mathcal{U}_R} B$ iff $(A^* \times B^*) \cap U_\varepsilon \neq \emptyset$, for all $U_\varepsilon \in \mathcal{U}_R$. Then $\delta_{\mathcal{U}_R}$ is a Čech rough proximity. Moreover, define a Čech rough proximity δ_{d_R} as: $A\delta_{d_R} B$ iff there exists $x \in A$ and

(A) Set of Pixels.   (B) Original Picture
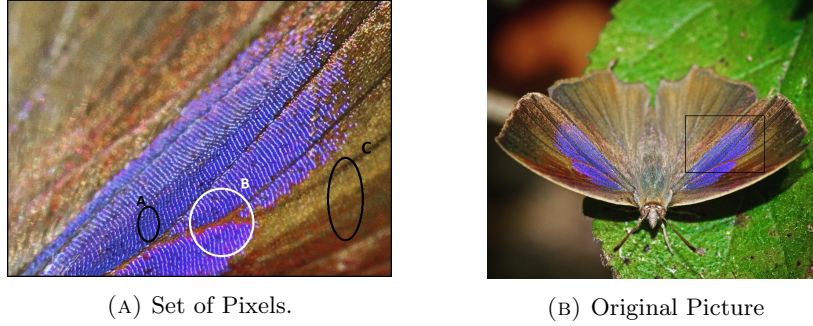
FIGURE 1. Digital Image of a Butterfly

$y \in B$ such that $d_R(x, y) < \varepsilon$, for all $\varepsilon > 0$. Then $\delta_{\mathcal{U}_R}$ and δ_{d_R} are compatible, i.e., $A\delta_{\mathcal{U}_R}B$ iff $A\delta_{d_R}B$.

Now, we present the illustration of rough semi-uniform spaces in finding the proximities of digital images. The illustration of image proximities in a butterfly and color thresholding in the context of nearness-like structures have been extensively used in [8, 9].

Example 4.2. Consider the image of a butterfly in Figure 1(B). An extracted part of it is shown in Figure 1(A), which we will consider as the universe U. We will consider the feature value - color for classification of images. Let U be the set of pixels in Figure 1(A). The color strength of each pixel p can be represented by the triplet $p := (p_r, p_g, p_b)$, where p_r, p_g, p_b represents the red, green and blue intensity values of the pixel p, respectively. Each intensity value is on a scale of 0 to 255. The co-ordinate of each pixel represents its RGB value. Define a map

$$d : U \times U \to \mathbb{R} \text{ as } : d(p, q) := \max\{\mid p_r - q_r \mid, \mid p_g - q_g \mid, \mid p_b - q_b \mid\}.$$

Define a tolerance relation R on U as:

$$p_1 R p_2 \text{ iff } d(p_1, p_2) \leq 5.$$

Thus the neighborhood $R(p)$ of a pixel p is set of all pixels which have perceptual distance less than or equal to 5. Define

$$U_\varepsilon := \{(R(x), R(y)) : d(x, y) < 5\}, \text{ for every real number } \varepsilon > 0.$$

Then U_ε is a filter base for a rough semi-uniformity $\mathcal{U}_R$ on U. The two subsets A and B of U are said to be perceptual near $(A\delta_{\mathcal{U}_R}B)$ if the following condition holds:

$$A\delta_{\mathcal{U}_R}B \text{ iff } (A^* \times B^*) \cap U_\varepsilon \neq \emptyset, \text{ for all } U_\varepsilon \in \mathcal{U}_R.$$

Here $\delta_{\mathcal{U}_R}$ is a Čech rough proximity on U.

Let A and B be two sets of pixels, as shown in Figure 1(A). The color strength of each pixel, say p, can be represented by the tuple value (p_r, p_g, p_b), where $p_r, p_g, p_b \in \{0, 1, 2, 3, \ldots, 255\}$. As we choose the RGB values as the feature values of elements (pixels), so the neighborhood of a given element (pixel) 'p' is $R(p) = \{x \in U : d(p, x) \leq 5\}$. The pixel p is perceptually similar to each element in $R(p)$ because the difference between the corresponding RGB values of p and any element in $R(p)$ is less than or equal to 5. In Figure 1(A), we may see that the sets A and B are perceptual near as there are some elements (pixels) which have close RGB values, that is their RGB values are within the difference of 5. Also, the sets A and C are far from each other as the RGB value of every element in C have a difference
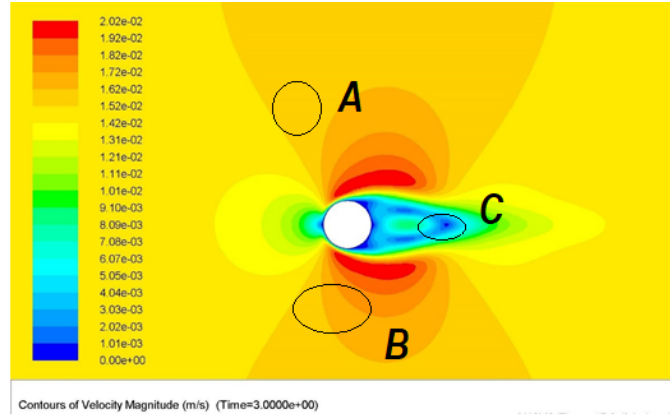
FIGURE 2. Velocity contours describes the velocity contours of fluid flow past a circular cylinder (Plotted in Ansys 15.0).

more than 5 from the RGB value of each element of A. In this way, we can classify digital images using different feature values. For example, by this method, we may re-design a given picture on lower resolution visual output devices. By using far (not near) relation $\delta_{\mathcal{U}_R}$, we can distinguish different kinds of objects in a picture on a digital platform.

In the next example, we discuss the application of a rough semi-uniformity to distinguish the velocity of a fluid at different time and space.

Example 4.3. Let us consider a two dimensional fluid flow past a cylinder. The contours in Figure 2 represents the velocity profile for the fluid flow with the variation of Reynolds number. Let U be the set of all pixels in Figure 2. The velocity at each co-ordinate is represented by a particular color and the color strength of each pixel p can be represented by the triplet $p := (p_r, p_g, p_b)$, where p_r, p_g, p_b represent the red, green and blue intensity values of the pixel p, respectively. Each intensity value is on a scale of 0 to 255. Define a map $f : U \times U \to \mathbb{R}$ as:

$$f(p, q) = \max \{| p_r - q_r |, | p_g - q_g |, | p_b - q_b |\}.$$

Using this function, define a tolerance relation R on U as:

$$p_1 R p_2 \text{ iff } f(p_1, p_2) \leq 7.$$

Thus the neighborhood $R(p)$ of a pixel p is set of all pixels which have perceptual distance less than or equal to value 7, i.e., all the elements which have almost similar velocities, belongs to $R(p)$. These neighborhoods are shown as contours (see Figure 2). Define a relation P on $\mathcal{P}(U)$ by:

$$R(x)PR(y) \text{ iff } R(x) \cap R(y) \neq \emptyset.$$

Define $\mathcal{U}_R = \{Q \subseteq U^* \times U^* : P \subseteq Q\}$. Then $\mathcal{U}_R$ is a rough semi-uniformity on U. We will say that the two subsets A and B of U are perceptually near $(A\delta_{\mathcal{U}_R}B)$ iff $(A^* \times B^*) \cap Q \neq \emptyset$, for all $Q \in \mathcal{U}_R$. Then $\delta_{\mathcal{U}_R}$ is a Čech rough proximity on U.

Let A and B be two sets of pixels as shown in Figure 2. The color strength of each pixel, say p, can be represented by the tuple value (p_r, p_g, p_b), where $p_r, p_g, p_b \in \{0, 1, 2, 3, \ldots, 255\}$. If we choose the RGB values as the feature values of elements (pixels), then the neighborhood of a given element (pixel) 'p' is $R(p) = \{x \in U : f(p, x) \leq 7\}$. The pixel p is very similar to each element in $R(p)$ because the

difference between the corresponding RGB values of 'p' and any element in $R(p)$ is less than or equal to 7. In Figure 2, the sets A and B are perceptually near. That is, the value of the velocity magnitudes of the fluid particles which passes through some part of the area of A and some part of the area of B are same and approx to $0.0162\ m/s$. Similarly, the sets B and C are perceptually far from each other as the RGB value of every element in C have a difference more than 7 from the RGB value of each element of B. That is, the fluid particles in sets B an C have velocity variations that cannot be neglected.

5. **Conclusion.** We have discussed here the concept of semi-rough uniformity over an approximation space, which provides us topology over the universe U, such that members of topology are not always crisp, in general. So it gives us a new way to study rough topology in the more generalized framework. Since the category of rough semi-uniform spaces is a generalization of uniform spaces; therefore the work done in this paper is a unified study of semi-uniform spaces, uniform spaces, approximation spaces, rough pseudometric spaces, and pseudometric spaces. Approximation spaces have a wider number of applications in various fields like image analysis, feature selection, artificial intelligence, neural networking, and many more. Therefore by completing an approximation space, we can simplify the study of all these fields of applications of approximation spaces. By using our method, we can get the completion of approximation spaces through a rough semi-uniformity on a non-empty universe of discourse. We have discussed the applications of the rough semi-uniform structures in the defining proximities of digital images, which have applications in various fields of image classification [2, 3, 4, 9, 17, 18].

Furthermore, completion is an essential requirement for the study of various topological problems, such as finding fixed points of a map. Therefore, completing a rough semi-uniform space opens the way of finding rough fixed points of a rough contraction mapping. This forms the content of our future research [12].

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