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TWO CONGRUENCES CONCERNING APÉRY NUMBERS CONJECTURED BY Z.-W. SUN

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Abstract. Let n be a nonnegative integer. The n-th Apéry number is defined by

$$A_n := \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2.$$

Z.-W. Sun investigated the congruence properties of Apéry numbers and posed some conjectures. For example, Sun conjectured that for any prime $p \geq 7$

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6}$$

and for any prime $p \geq 5$

$$\sum_{k=0}^{p-1} (2k+1)^3 A_k \equiv p^3 + 4p^4 H_{p-1} + \frac{6}{5}p^8 B_{p-5} \pmod{p^9},$$

where $H_n = \sum_{k=1}^{n} 1/k$ denotes the n-th harmonic number and $B_0, B_1, \ldots$ are the well-known Bernoulli numbers. In this paper we shall confirm these two conjectures.

1. Introduction.

The well-known Apéry numbers given by

$$A_n := \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, \ldots\}),$$

were first introduced by Apéry to prove the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (see [2, 12]). It is known that the Apéry numbers have close connections to modular forms (cf. [11]). Recall that the Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau},$$

where $\tau \in \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Beukers [3] conjectured that for any prime $p > 3$

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

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where $a(n)$ $(n = 1, 2, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n=1}^{\infty}a(n)q^n.$$

This conjecture was later confirmed by Ahlgren and Ono [1].

In 2012, Z.-W. Sun [16] introduced the Apéry polynomials

$$A_n(x) = \sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2 x^k \quad (n \in \mathbb{N})$$

and deduced various congruences involving sums of such polynomials. (Clearly, $A_n(1) = A_n$.) For example, for any odd prime p and integer x, he obtained that

$$\sum_{k=0}^{p-1}(2k+1)A_k(x) \equiv p\left(\frac{x}{p}\right) \pmod{p^2}, \tag{1}$$

where $(-)$ denotes the Legendre symbol. For $x = 1$ and any prime $p \geq 5$, Sun established the following generalization of (1):

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}, \tag{2}$$

where $B_0, B_1, \ldots$ are the well-known Bernoulli numbers defined as follows:

$$B_0 = 0,\ \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \quad (n = 2, 3, \ldots).$$

In 1850 Kummer (cf. [9]) proved that for any odd prime p and even number b with $b \not\equiv 0 \pmod{p-1}$

$$\frac{B_{k(p-1)+b}}{k(p-1)+b} \equiv \frac{B_b}{b} \pmod{p} \quad \text{for } k \in \mathbb{N}. \tag{3}$$

For $m \in \mathbb{Z}^+ = \{1, 2, \ldots\}$ the n-th harmonic numbers of order m are defined by

$$H_n^{(m)} := \sum_{k=1}^{n}\frac{1}{k^m} \quad (n = 1, 2, \ldots)$$

and $H_0^{(m)} := 0$. For the sake of convenience we use H_n to denote $H_n^{(1)}$. From [5] we know that $H_{p-1} \equiv -p^2 B_{p-3}/3 \pmod{p^3}$ for any prime $p \geq 5$. Thus (2) has the following equivalent form

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^5}. \tag{4}$$

Via some numerical computation, Sun [16, Conjecture 4.2] conjectured that (4) also holds modulo p^6 provided that $p \geq 7$. This is our first theorem.

Theorem 1.1. *For any prime $p \geq 7$ we have*

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6}. \tag{5}$$

Motivated by Sun's work, Guo and Zeng [6, 7] studied some congruences for sums involving Apéry polynomials. Particularly, they obtained

$$\sum_{k=0}^{n-1}(2k+1)^3A_k \equiv 0 \pmod{n^3} \tag{6}$$

and

$$\sum_{k=0}^{p-1}(2k+1)^3A_k \equiv p^3 \pmod{2p^6}, \tag{7}$$

where $p \geq 5$ is a prime. Strengthening (7), Sun [14, Conjecture A65] proposed the following challenging conjecture.

Conjecture 1. *For any prime $p \geq 5$ we have*

$$\sum_{k=0}^{p-1}(2k+1)^3A_k \equiv p^3 + 4p^4H_{p-1} + \frac{6}{5}p^8B_{p-5} \pmod{p^9}.$$

This is our second theorem.

Theorem 1.2. *Conjecture 1 is true.*

Proofs of Theorems 1.1 and 1.2 will be given in Sections 2 and 3, respectively.

2. Proof of Theorem 1.1.

The proofs in this paper strongly depend on the congruence properties of harmonic numbers and the Bernoulli numbers. For their properties, the reader is referred to [9, 13, 15, 17]. Below we first list some congruences involving harmonic numbers and the Bernoulli numbers which will be used later.

Lemma 2.1. [4, Remark 3.2] *For any prime $p \geq 5$ we have*

$$2H_{p-1} + pH_{p-1}^{(2)} \equiv \frac{2}{5}p^4B_{p-5} \pmod{p^5}.$$

From [13, Theorems 5.1 and 5.2], we have the following congruences.

Lemma 2.2. *For any prime $p \geq 7$ we have*

$$H_{(p-1)/2} \equiv -2q_p(2) \pmod{p},$$

$$H_{p-1}^{(2)} \equiv \left(\frac{4}{3}B_{p-3} - \frac{1}{2}B_{2p-4}\right)p + \left(\frac{4}{9}B_{p-3} - \frac{1}{4}B_{2p-4}\right)p^2 \pmod{p^3},$$

$$H_{(p-1)/2}^{(2)} \equiv \left(\frac{14}{3}B_{p-3} - \frac{7}{4}B_{2p-4}\right)p + \left(\frac{14}{9}B_{p-3} - \frac{7}{8}B_{2p-4}\right)p^2 \pmod{p^3},$$

$$H_{p-1}^{(3)} \equiv -\frac{6}{5}p^2B_{p-5} \pmod{p^3}, \quad H_{(p-1)/2}^{(3)} \equiv 6\left(\frac{2B_{p-3}}{p-3} - \frac{B_{2p-4}}{2p-4}\right) \pmod{p^2},$$

$$H_{p-1}^{(4)} \equiv \frac{4}{5}pB_{p-5} \pmod{p^2}, \quad H_{(p-1)/2}^{(4)} \equiv 0 \pmod{p}, \quad H_{p-1}^{(5)} \equiv 0 \pmod{p^2},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Remark 1. By Kummer's congruence (3), we know $B_{2p-4} \equiv 4B_{p-3}/3 \pmod{p}$. Then the congruences of $H_{p-1}^{(2)}$ and $H_{(p-1)/2}^{(2)}$ can be reduced to

$$H_{p-1}^{(2)} \equiv \left(\frac{4}{3}B_{p-3} - \frac{1}{2}B_{2p-4}\right)p + \frac{1}{9}p^2B_{p-3} \pmod{p^3}$$

and

$$H_{(p-1)/2}^{(2)} \equiv \left(\frac{14}{3}B_{p-3} - \frac{7}{4}B_{2p-4}\right)p + \frac{7}{18}p^2 B_{p-3} \pmod{p^3}$$

respectively. By Lemma 2.1, we immediately obtain that $H_{p-1} \equiv -pH_{p-1}^{(2)}/2 \pmod{p^4}$. Thus

$$H_{p-1} \equiv \left(\frac{1}{4}B_{2p-4} - \frac{2}{3}B_{p-3}\right)p^2 - \frac{1}{18}p^3 B_{p-3} \pmod{p^4}. \tag{8}$$

Recall that the Bernoulli polynomials $B_n(x)$ are defined as

$$B_n(x) := \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n \in \mathbb{N}). \tag{9}$$

Clearly, $B_n = B_n(0)$. Also, we have

$$\sum_{k=1}^{n-1} k^{m-1} = \frac{B_m(n) - B_m}{m} \tag{10}$$

for any positive integer n and m.

Let $d > 0$ and $\mathbf{s} := (s_1, \ldots, s_d) \in (\mathbb{Z}\backslash\{0\})^d$. The alternating multiple harmonic sum [18] is defined as follows

$$H(\mathbf{s}; n) := \sum_{1 \le k_1 < k_2 < \cdots k_d \le n} \prod_{i=1}^{d} \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}.$$

Clearly, $H_n^{(m)} = H(m; n)$.

Let A, B, D, E, F be defined as in [18, Section 6], i.e.,

$$A := \sum_{k=2}^{p-3} B_k B_{p-3-k}, \quad B := \sum_{k=2}^{p-3} 2^k B_k B_{p-3-k}, \quad D := \sum_{k=2}^{p-3} \frac{B_k B_{p-3-k}}{k},$$

$$E := \sum_{k=2}^{p-3} \frac{2^k B_k B_{p-3-k}}{k}, \quad F := \sum_{k=2}^{p-3} \frac{2^{p-3-k} B_k B_{p-3-k}}{k}.$$

Lemma 2.3. *For any prime $p \ge 7$ we have*

$$D - 4F \equiv 2B - 2A - q_p(2)B_{p-3} \pmod{p}.$$

Proof. In [18, Section 6], Tauraso and Zhao proved that

$$H(1, -3; p-1) \equiv B - A \equiv 2E - 2D + 2q_p(2)B_{p-3} \pmod{p}$$

and

$$\frac{5}{2}D - 2E - 2F - \frac{3}{2}q_p(2)B_{p-3} \equiv 0 \pmod{p}.$$

Combining the above two congruences we immediately obtain the desired result. □

Lemma 2.4. *Let $p \ge 7$ be a prime. Then we have*

$$H(3, 1; (p-1)/2) \equiv H_{(p-1)/2}^{(3)} H_{(p-1)/2} - 4B + 4A \pmod{p}. \tag{11}$$

Proof. By Lemma 2.2, it is easy to check that

$$\begin{aligned} H_{(p-1)/2}^{(3)} H_{(p-1)/2} &= H(1, 3; (p-1)/2) + H(3, 1; (p-1)/2) + H_{(p-1)/2}^{(4)} \\ &\equiv H(1, 3; (p-1)/2) + H(3, 1; (p-1)/2) \pmod{p}. \end{aligned} \tag{12}$$

Thus it suffices to evaluate $H(1,3;(p-1)/2)$ modulo p. By Fermat's little theorem, (9) and (10) we arrive at

$$\begin{aligned}
&H(1,3;(p-1)/2) = \sum_{1\le j<k\le (p-1)/2} \frac{1}{jk^3} \\
&\equiv \sum_{1\le j<k\le (p-1)/2} \frac{j^{p-2}}{k^3} = \sum_{1\le k\le (p-1)/2} \frac{B_{p-1}(k)-B_{p-1}}{k^3(p-1)} \\
&= \sum_{1\le k\le (p-1)/2} \frac{\sum_{i=1}^{p-1}\binom{p-1}{i}k^{i-3}B_{p-1-i}}{p-1} = \sum_{i=1}^{p-1}\frac{\binom{p-1}{i}B_{p-1-i}}{p-1}\sum_{k=1}^{(p-1)/2}k^{i-3} \\
&\equiv \frac{\binom{p-1}{2}B_{p-3}}{p-1}H_{(p-1)/2} + \sum_{i=4}^{p-1}\frac{\binom{p-1}{i}B_{p-1-i}}{p-1}\cdot\frac{B_{i-2}\left(\frac12\right)-B_{i-2}}{i-2} \pmod p,
\end{aligned}$$

where the last step follows from the fact $B_n=0$ for any odd $n\ge 3$. By [9, p. 248] we know that $B_n(1/2)=(2^{1-n}-1)B_n$. Thus

$$\begin{aligned}
H(1,3;(p-1)/2) &\equiv -B_{p-3}H_{(p-1)/2} - \sum_{i=4}^{p-1}\frac{(2^{3-i}-2)B_{p-1-i}B_{i-2}}{i-2} \\
&= -B_{p-3}H_{(p-1)/2} - \sum_{i=2}^{p-3}\frac{(2^{1-i}-2)B_{p-3-i}B_i}{i} \\
&\equiv -B_{p-3}H_{(p-1)/2} - 8F + 2D \pmod p.
\end{aligned}$$

With help of Lemmas 2.2 and 2.3, we have

$$H(1,3;(p-1)/2) \equiv 4B-4A \pmod p.$$

Combining this with (12), we have completed the proof of Lemma 2.4. □

Lemma 2.5. *Let $p\ge 7$ be a prime. Then we have*

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k} \equiv& \frac{3}{2p^2}H_{p-1} + \frac12 H_{(p-1)/2}^{(3)} + \frac12 H_{p-1}^{(2)}H_{(p-1)/2} - pH_{(p-1)/2}^{(3)}H_{(p-1)/2} \\
&+ 4p(B-A) \pmod{p^2}.
\end{aligned}$$

Proof. By [10, Eq. (3.13)] we know that for any odd prime p

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k} \equiv \frac{3}{p^2}H_{p-1} \pmod{p^2}. \tag{13}$$

On the other hand,

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k} = \sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k} + \sum_{k=1}^{(p-1)/2}\frac{H_{p-k}^{(2)}}{p-k}.$$

For $k=1,2,\ldots,(p-1)/2$ we have

$$H_{p-k}^{(2)} = \sum_{j=k}^{p-1}\frac{1}{(p-j)^2} \equiv \sum_{j=k}^{p-1}\left(\frac{1}{j^2}+\frac{2p}{j^2}\right) \equiv H_{p-1}^{(2)} - H_{k-1}^{(2)} - 2pH_{k-1}^{(3)} \pmod{p^2}$$

by Lemma 2.2. Thus

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}\equiv 2\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}-H_{(p-1)/2}^{(3)}+pH(2,2;(p-1)/2)$$
$$-H_{p-1}^{(2)}H_{(p-1)/2}+2pH(3,1;(p-1)/2) \pmod{p^2}.$$

In view of Lemma 2.2, we have

$$H(2,2;(p-1)/2)=\frac{\left(H_{(p-1)/2}^{(2)}\right)^2}{2}-\frac{H_{(p-1)/2}^{(4)}}{2}\equiv 0 \pmod{p}.$$

This together with Lemma 2.4 proves Lemma 2.5. □

Lemma 2.6. [16, Lemma 2.1] *Let $k\in\mathbb{N}$. Then for $n\in\mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2=\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2.$$

Proof of Theorem 1.1. By Lemma 2.6 it is routine to check that

$$\begin{aligned}\sum_{m=0}^{p-1}(2m+1)A_m&=\sum_{m=0}^{p-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\sum_{m=0}^{p-1}(2m+1)\binom{m+k}{2k}^2\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{(p-k)^2}{2k+1}\binom{p+k}{2k}^2\\&=p^2\sum_{k=0}^{p-1}\frac{1}{2k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2.\end{aligned}$$

Note that

$$\binom{p-1}{k}^2\binom{p+k}{k}^2=\prod_{j=1}^{k}\left(1-\frac{p^2}{j^2}\right)^2\equiv\prod_{j=1}^{k}\left(1-\frac{2p^2}{j^2}+\frac{p^4}{j^4}\right)$$
$$\equiv 1-2p^2H_k^{(2)}+p^4H_k^{(4)}+4p^4H(2,2;k) \pmod{p^5}.$$

Since $H_{(p-1)/2}^{(4)}\equiv 0 \pmod{p}$ and $H(2,2;(p-1)/2)\equiv 0 \pmod{p}$, we have

$$\sum_{m=0}^{p-1}(2m+1)A_m\equiv p^2\Sigma_1-2p^4\Sigma_2 \pmod{p^6}, \tag{14}$$

where

$$\Sigma_1:=\sum_{k=0}^{p-1}\frac{1}{2k+1}\quad\text{and}\quad\Sigma_2:=\sum_{k=0}^{p-1}\frac{H_k^{(2)}}{2k+1}.$$

We first consider Σ_1 modulo p^4. Clearly,

$$\begin{aligned}\sum_{k=(p+1)/2}^{p-1}\frac{1}{2k+1}=&\sum_{k=0}^{(p-3)/2}\frac{1}{2(p-1-k)+1}\\ \equiv&-8p^3\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^4}-2p\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^2}\\ &-4p^2\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^3}-\sum_{k=0}^{(p-3)/2}\frac{1}{2k+1}\pmod{p^4}.\end{aligned}$$

For $r\in\{2,3,4\}$,

$$\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^r}=H_{p-1}^{(r)}-\frac{1}{2^r}H_{(p-1)/2}^{(r)}.$$

By the above and Lemma 2.2,

$$\begin{aligned}\Sigma_1=&\frac{1}{p}+\sum_{k=0}^{(p-3)/2}\frac{1}{2k+1}+\sum_{k=(p+1)/2}^{p-1}\frac{1}{2k+1}\\ \equiv&\frac{1}{p}-2p\left(H_{p-1}^{(2)}-\frac{1}{4}H_{(p-1)/2}^{(2)}\right)+\frac{1}{2}p^2H_{(p-1)/2}^{(3)}\pmod{p^4}.\end{aligned}\tag{15}$$

Now we turn to Σ_2 modulo p^2. By Lemma 2.2,

$$\begin{aligned}\sum_{k=(p+1)/2}^{p-1}\frac{H_k^{(2)}}{2k+1}=&\sum_{k=0}^{(p-3)/2}\frac{H_{p-1-k}^{(2)}}{2(p-1-k)+1}\\ \equiv&\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}+2p\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{(2k+1)^2}\\ &+\frac{1}{2}H_{p-1}^{(2)}H_{(p-1)/2}+2p\sum_{k=0}^{(p-3)/2}\frac{H_k^{(3)}}{2k+1}\pmod{p^2}.\end{aligned}$$

Thus

$$\Sigma_2\equiv\frac{H_{(p-1)/2}^{(2)}}{p}+2\sigma_1+\frac{1}{2}H_{p-1}^{(2)}H_{(p-1)/2}+2p\sigma_2\pmod{p^2},$$

where

$$\sigma_1:=\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}+p\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{(2k+1)^2}$$

and

$$\sigma_2:=\sum_{k=0}^{(p-3)/2}\frac{H_k^{(3)}}{2k+1}.$$

It is easy to see that

$$\begin{aligned}\sigma_1 \equiv & -\sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{p-1-2k} = -\sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}^{(2)}}{2k} \\ \equiv & -\frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(2)} + \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{j=0}^{k-1} \left(\frac{4}{(2j+1)^2} + \frac{8p}{(2j+1)^3} \right) \\ = & \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left(4H_{2k}^{(2)} - H_k^{(2)} + 8pH_{2k}^{(3)} - pH_k^{(3)} \right) \\ & - \frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(2)} \pmod{p^2}.\end{aligned}$$

Also,

$$\begin{aligned}\sigma_2 \equiv & -\sum_{k=0}^{(p-3)/2} \frac{H_k^{(3)}}{p-1-2k} = -\sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}^{(3)}}{2k} \\ \equiv & -\frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(3)} + \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{j=0}^{k-1} \frac{-8}{(2j+1)^3} \\ = & -\frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(3)} - 4 \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left(H_{2k}^{(3)} - \frac{1}{8} H_k^{(3)} \right) \pmod{p}.\end{aligned}$$

Combining the above we deduce that

$$\begin{aligned}\Sigma_2 \equiv & \frac{H_{(p-1)/2}^{(2)}}{p} - H_{(p-1)/2} H_{(p-1)/2}^{(2)} + 4 \sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{k} - \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \\ & + \frac{1}{2} H_{p-1}^{(2)} H_{(p-1)/2} - pH_{(p-1)/2} H_{(p-1)/2}^{(3)} \pmod{p^2}.\end{aligned} \tag{16}$$

Note that

$$\sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{k} = \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} + H(2,-1;p-1) + \frac{1}{4} H_{(p-1)/2}^{(3)} - H_{p-1}^{(3)}. \tag{17}$$

By [18, Proposition 7.3] we know that

$$H(2,-1;p-1) \equiv -\frac{3}{2} X - \frac{7}{6} pq_p(2) B_{p-3} + p(B-A) \pmod{p^2}, \tag{18}$$

where $X := B_{p-3}/(p-3) - B_{2p-4}/(4p-8)$. Now combining (16)–(18), Lemmas 2.2 and 2.5 we obtain that

$$\Sigma_2 \equiv \frac{H_{(p-1)/2}^{(2)}}{p} + \frac{21 H_{p-1}}{2p^2} \pmod{p^2}. \tag{19}$$

Substituting (15) and (19) into (14) and using (13) and Lemma 2.2 we have

$$\begin{aligned}\sum_{m=0}^{p-1} (2m+1) A_m \equiv & p - 2p^3 H_{p-1}^{(2)} - \frac{3}{2} p^3 H_{(p-1)/2}^{(2)} + \frac{1}{2} p^4 H_{(p-1)/2}^{(3)} - 21 p^2 H_{p-1} \\ \equiv & p - \frac{7}{2} p^2 H_{p-1} \pmod{p^6}.\end{aligned}$$

The proof of Theorem 1.1 is complete now. □

3. **Proof of Theorem 1.2.** In order to show Theorem 1.2, we need the following results.

Lemma 3.1. *Let $p \geq 7$ be a prime. Then we have*

$$\sum_{k=1}^{p-1} \frac{H(2,2;k)}{k} \equiv -\frac{1}{2} B_{p-5} \pmod{p}.$$

Proof. Clearly,

$$\sum_{k=1}^{p-1} \frac{H(2,2;k)}{k} = H(2,2,1;p-1) + H(2,3;p-1).$$

By [19, Theorems 3.1 and 3.5] we have

$$H(2,3;p-1) \equiv -2B_{p-5} \pmod{p} \quad \text{and} \quad H(2,2,1;p-1) \equiv \frac{3}{2} B_{p-5} \pmod{p}.$$

Combining the above we obtain the desired result. □

Lemma 3.2. *For any prime $p \geq 7$ we have*

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \equiv \frac{3H_{p-1}}{p^2} - \frac{1}{2} p^2 B_{p-5} \pmod{p^3}. \tag{20}$$

Proof. Note that

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} = H(2,1;p-1) + H_{p-1}^{(3)}.$$

By [8, Lemma 3] we know that

$$H(1,2;p-1) \equiv -\frac{3H_{p-1}}{p^2} + \frac{1}{2} p^2 B_{p-5} \pmod{p^3}$$

for any prime $p > 3$. Therefore, by (8) and Lemma 2.2 we have

$$\begin{aligned} H(2,1;p-1) =& H_{p-1} H_{p-1}^{(2)} - H(1,2;p-1) - H_{p-1}^{(3)} \\ \equiv& \frac{3H_{p-1}}{p^2} - \frac{1}{2} p^2 B_{p-5} - H_{p-1}^{(3)} \pmod{p^3}. \end{aligned}$$

Then (20) follows at once. □

Lemma 3.3. *For any prime $p \geq 7$ we have*

$$\sum_{k=0}^{p-1} H(2,2;k) \equiv -\frac{p}{2} H_{p-1}^{(4)} - \frac{3H_{p-1}}{p^2} + H_{p-1}^{(3)} + \frac{1}{2} p^2 B_{p-5} \pmod{p^3}, \tag{21}$$

$$\sum_{k=0}^{p-1} (H(2,4;k) + H(4,2;k)) \equiv 3B_{p-5} \pmod{p}, \tag{22}$$

$$\sum_{k=0}^{p-1} H(2,2,2;k) \equiv -\frac{3}{2} B_{p-5} \pmod{p}. \tag{23}$$

Proof. By Lemma 2.2 we arrive at

$$\begin{aligned}\sum_{k=0}^{p-1}H(2,2;k)=&\sum_{k=1}^{p-1}\sum_{1\le i<j\le k}\frac{1}{i^2j^2}=\sum_{1\le i<j\le p-1}\frac{p-j}{i^2j^2}\\ =&\frac{p}{2}\left(\left(H_{p-1}^{(2)}\right)^2-H_{p-1}^{(4)}\right)-\sum_{1\le i<j\le p-1}\frac{1}{i^2j}\\ \equiv&-\frac{p}{2}H_{p-1}^{(4)}+\sum_{k=1}^{p-1}\frac{H_k}{k^2}\pmod{p^3}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k}{k^2}=&\sum_{k=1}^{p-1}\frac{1}{k^2}\sum_{j=1}^{k}\frac{1}{j}=\sum_{j=1}^{p-1}\frac{1}{j}\sum_{k=j}^{p-1}\frac{1}{k^2}\\ \equiv&-\sum_{j=1}^{p-1}\frac{H_j^{(2)}}{j}+H_{p-1}^{(3)}\pmod{p^3}.\end{aligned}$$

From the above and Lemma 3.2, we obtain (21).

Now we turn to prove (22). It is easy to see that

$$\begin{aligned}\sum_{k=0}^{p-1}(H(2,4;k)+H(4,2;k))=&\sum_{1\le i<j\le p-1}\frac{p-j}{i^2j^4}+\sum_{1\le i<j\le p-1}\frac{p-j}{i^4j^2}\\ \equiv&-H(2,3;p-1)-H(4,1;p-1)\pmod{p}.\end{aligned}$$

By [18, Theorem 3.1] we have $H(2,3;p-1)\equiv-2B_{p-5}\pmod{p}$ and $H(4,1;p-1)\equiv-B_{p-5}\pmod{p}$ for $p\ge7$. Then (22) follows at once.

Finally, we consider (23). Clearly,

$$\sum_{k=0}^{p-1}H(2,2,2;k)=\sum_{1\le i_1<i_2<i_3\le p-1}\frac{p-i_3}{i_1^2i_2^2i_3^2}\equiv-H(2,2,1;p-1)\pmod{p}.$$

By [19, Theorem 3.5], we have

$$H(2,2,1;p-1)\equiv\frac{3}{2}B_{p-5}\pmod{p}.$$

The proof of Lemma 3.3 is now complete. □

Lemma 3.4. *Let $k\in\mathbb{N}$. Then for $n\in\mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1}(2m+1)^3\binom{m+k}{2k}^2=\frac{(n-k)^2(2n^2-k-1)}{k+1}\binom{n+k}{2k}^2.$$

Proof. It can be verified directly by induction on n. □

Proof of Theorem 1.2. The case $p=5$ can be verified directly. Below we assume that $p\ge7$. By Lemma 3.4 we have

$$\begin{aligned}\sum_{m=0}^{p-1}(2m+1)^3A_m &= \sum_{m=0}^{p-1}(2m+1)^3\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2\\ &=\sum_{k=0}^{p-1}\binom{2k}{k}^2\sum_{m=0}^{p-1}(2m+1)^3\binom{m+k}{2k}^2\\ &=\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{(p-k)^2(2p^2-k-1)}{k+1}\binom{p+k}{2k}^2\\ &=p^2\sum_{k=0}^{p-1}\frac{2p^2-k-1}{k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2.\end{aligned}$$

Noting that

$$\begin{aligned}\binom{p-1}{k}^2\binom{p+k}{k}^2 &= \prod_{j=1}^{k}\left(1-\frac{p^2}{j^2}\right)^2 \equiv \prod_{j=1}^{k}\left(1-\frac{2p^2}{j^2}+\frac{p^4}{j^4}\right)\\ \equiv&1-2p^2H_k^{(2)}+p^4H_k^{(4)}+4p^4H(2,2;k)-2p^6\left(H(2,4;k)+H(4,2;k)\right)\\ &-8p^6H(2,2,2;k) \pmod{p^7},\end{aligned}$$

we arrive at

$$\begin{aligned}&\sum_{m=0}^{p-1}(2m+1)^3A_m\\ \equiv&\ 2p^4\sum_{k=1}^{p-1}\frac{1}{k+1}\Big(1-2p^2H_k^{(2)}+p^4H_k^{(4)}+4p^4H(2,2;k)\Big)\\ &-p^2\sum_{k=0}^{p-1}\Big(1-2p^2H_k^{(2)}+p^4H_k^{(4)}+4p^4H(2,2;k)\\ &-2p^6\left(H(2,4;k)+H(4,2;k)\right)-8p^6H(2,2,2;k)\Big) \pmod{p^9}.\end{aligned}$$

It is clear that

$$\sum_{k=0}^{p-1}\frac{1}{k+1}=H_{p-1}+\frac{1}{p}.$$

With the help of Lemma 3.2 we obtain that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{H_k^{(2)}}{k+1}=&\sum_{k=1}^{p}\frac{H_{k-1}^{(2)}}{k}=\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}-H_{p-1}^{(3)}+\frac{H_{p-1}^{(2)}}{p}\\ \equiv&\frac{3}{p^2}H_{p-1}+\frac{H_{p-1}^{(2)}}{p}-H_{p-1}^{(3)}-\frac{1}{2}p^2B_{p-5} \pmod{p^3}.\end{aligned}$$

Clearly,

$$\sum_{k=0}^{p-1}\frac{H_k^{(4)}}{k+1}=H(4,1;p-1)+\frac{H_{p-1}^{(4)}}{p}\equiv -B_{p-5}+\frac{H_{p-1}^{(4)}}{p} \pmod{p}.$$

Furthermore,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{H(2,2;k)}{k+1}=&\frac{1}{2}\sum_{k=1}^{p}\frac{1}{k}\left(\left(H_{k-1}^{(2)}\right)^2-H_{k-1}^{(4)}\right)\\=&\frac{1}{2}\sum_{k=1}^{p}\frac{1}{k}\left(\left(H_{k}^{(2)}\right)^2-H_{k}^{(4)}-\frac{2H_k^{(2)}}{k^2}+\frac{2}{k^4}\right)\\=&\sum_{k=1}^{p-1}\frac{H(2,2;k)}{k}-H(2,3;p-1)+\frac{1}{2p}\left(\left(H_{p-1}^{(2)}\right)^2-H_{p-1}^{(4)}\right).\end{aligned}$$

Then by Lemmas 2.2 and 3.1 we arrive at

$$\sum_{k=0}^{p-1}\frac{H(2,2;k)}{k+1}\equiv\frac{3}{2}B_{p-5}-\frac{1}{2p}H_{p-1}^{(4)}\pmod p.$$

For $r=2,4$ we have

$$\sum_{k=0}^{p-1}H_k^{(r)}=\sum_{k=1}^{p-1}\sum_{l=1}^{k}\frac{1}{l^r}=\sum_{l=1}^{p-1}\frac{p-l}{l^r}=pH_{p-1}^{(r)}-H_{p-1}^{(r-1)}.$$

Combining the above and in view of Lemma 3.3 we obtain

$$\begin{aligned}&\sum_{m=0}^{p-1}(2m+1)^3A_m\\\equiv&2p^4H_{p-1}+2p^3-12p^4H_{p-1}+4p^6H_{p-1}^{(3)}-4p^5H_{p-1}^{(2)}+2p^8B_{p-5}-2p^8B_{p-5}\\&+2p^7H_{p-1}^{(4)}+12p^8B_{p-5}-4p^7H_{p-1}^{(4)}-p^3+2p^5H_{p-1}^{(2)}-2p^4H_{p-1}-p^7H_{p-1}^{(4)}\\&+p^6H_{p-1}^{(3)}+2p^7H_{p-1}^{(4)}+12p^4H_{p-1}-2p^8B_{p-5}-4p^6H_{p-1}^{(3)}-6p^8B_{p-5}\\=&p^3-2p^4H_{p-1}-2p^5H_{p-1}^{(2)}+p^6H_{p-1}^{(3)}-p^7H_{p-1}^{(4)}+4p^8B_{p-5}\pmod{p^9}.\end{aligned}$$

Then Theorem 1.2 follows from Lemmas 2.1 and 2.2. □

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