

ASYMPTOTIC BEHAVIOUR OF A NEURAL FIELD LATTICE MODEL WITH DELAYS

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ABSTRACT. The asymptotic behaviour of an autonomous neural field lattice system with delays is investigated. It is based on the Amari model, but with the Heaviside function in the interaction term replaced by a sigmoidal function. First, the lattice system is reformulated as an infinite dimensional ordinary delay differential equation on weighted sequence state space ℓ_p^2 under some appropriate assumptions. Then the global existence and uniqueness of its solution and its formulation as a semi-dynamical system on a suitable function space are established. Finally, the asymptotic behaviour of solution of the system is investigated, in particular, the existence of a global attractor is obtained.

1. Introduction. Neural field models are often represented as evolution equations generated as continuum limits of computational models of neural fields theory. They are tissue level models that describe the spatio-temporal evolution of coarse grained variables such as synaptic or firing rate activity in populations of neurons. See Coombes et al. [4] and the literature therein. A particularly influential model is that proposed by S. Amari in [1] (see also Chapter 3 of Coombes et al. [4] by Amari):

$$\partial_t u(t, x) = -u(t, x) + \int_{\Omega} K(x - y) H(u(t, y) - \theta) dy, \quad x \in \Omega \subset \mathbb{R},$$

where $\theta > 0$ is a given threshold and $H : \mathbb{R} \rightarrow \mathbb{R}$ is the Heaviside function.

The continuum neural models may lose their validity in capturing detailed dynamics at discrete sites when the discrete structures of neural systems become dominant. Lattice models, e.g., [2, 6, 9, 10], can be used to describe dynamics at each

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site of the neural field. Han & Kloeden [7] introduced and investigated the following lattice version of the Amari model:

$$\frac{d}{dt}u_i(t) = f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j}H(u_j(t) - \theta) + g_i(t), \quad i \in \mathbb{Z}^d.$$

Delays are often included in neural field models to account for the transmission time of signals between neurons. In addition, to facilitate the analysis, the Heaviside function can be replaced by a simplifying sigmoidal function such as

$$\sigma_\varepsilon(x) = \frac{1}{1 + e^{-x/\varepsilon}}, \quad x \in \mathbb{R}, \quad 0 < \varepsilon < 1.$$

In this paper we consider the autonomous neural field lattice system with delays

$$\frac{d}{dt}u_i(t) = f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j}\sigma_\varepsilon(u_j(t - \tau_j) - \theta) + g_i, \quad i \in \mathbb{Z}^d. \tag{1}$$

Throughout this paper we assume that the delays $\tau_j > 0$ are uniformly bounded, i.e., satisfy

Assumption 1. *There exists a constant $h \in (0, \infty)$ that $0 \leq \tau_i \leq h$ for all $i \in \mathbb{Z}^d$.*

and that the interconnection matrix $(k_{i,j})_{i,j \in \mathbb{Z}^d}$ satisfies

Assumption 2. *$k_{i,j} \geq 0$ for all $i, j \in \mathbb{Z}^d$ and there exists a constant $\kappa > 0$ such that $\sum_{j \in \mathbb{Z}^d} k_{i,j} \leq \kappa$ for each $i \in \mathbb{Z}^d$.*

The main goal of this paper is to investigate asymptotic behaviour of solutions to the neural lattice system with delays (1), in particular, the attractor for the semidynamical system generated by its solutions. The initial conditions for such delay systems have the form

$$u_i(s) = \psi_i(s), \quad \forall s \in [-h, 0], \quad i \in \mathbb{Z}^d, \tag{2}$$

for appropriate functions ψ_i .

2. Preliminaries. We follow Han & Kloeden [7] and consider a weighted space of bi-infinite real valued sequences with vectorial indices $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$.

In particular, given a positive sequence of weights $(\rho_i)_{i \in \mathbb{Z}^d}$, we consider the separable Hilbert space

$$\ell_\rho^2 := \left\{ \mathbf{u} = (u_i)_{i \in \mathbb{Z}^d} : \sum_{i \in \mathbb{Z}^d} \rho_i u_i^2 < \infty \right\}$$

with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i \in \mathbb{Z}^d} \rho_i u_i v_i \text{ for } \mathbf{u} = (u_i)_{i \in \mathbb{Z}^d}, \mathbf{v} = (v_i)_{i \in \mathbb{Z}^d} \in \ell_\rho^2$$

and norm

$$\|\mathbf{u}\|_\rho := \sqrt{\sum_{i \in \mathbb{Z}^d} \rho_i u_i^2}.$$

We assume that the ρ_i satisfy the following assumption.

Assumption 3. *$\rho_i > 0$ for all $i \in \mathbb{Z}^d$ and $\rho_\Sigma := \sum_{i \in \mathbb{Z}^d} \rho_i < \infty$.*

The appropriate function space for the solutions of the lattice system with delays (1) is the Banach space $\mathcal{C}([-h, 0], \ell_\rho^2)$ of continuous functions by $\mathbf{v} : [-h, 0] \rightarrow \ell_\rho^2$ with the norm

$$\|\mathbf{v}\|_{\mathcal{C}([-h, 0], \ell_\rho^2)} = \max_{s \in [-h, 0]} \|\mathbf{v}(s)\|_\rho.$$

For a solution $\mathbf{u}(t) = (u_i(t))_{i \in \mathbb{Z}^d} \in \ell_\rho^2$ of (1) we denote by \mathbf{u}_t the segment of the solution in $\mathcal{C}([-h, 0], \ell_\rho^2)$ defined by $\mathbf{u}_t(s) = \mathbf{u}(t + s)$ for each $s \in [-h, 0]$. The corresponding initial condition (2) must then satisfy $(\psi_i(\cdot))_{i \in \mathbb{Z}^d} \in \mathcal{C}([-h, 0], \ell_\rho^2)$.

2.1. The reaction term. For any $\mathbf{u} = (u_i)_{i \in \mathbb{Z}^d} \in \ell_\rho^2$, we define the operator f by

$$f(\mathbf{u}) := (f_i(u_i))_{i \in \mathbb{Z}^d}.$$

To ensure that the $f(\mathbf{u})$ take values in ℓ_ρ^2 for every $\mathbf{u} \in \ell_\rho^2$ and has necessary dissipative properties, we make the following standing assumptions on the f_i throughout the rest of the paper.

Assumption 4. *The functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ are continuously differential with weighted equi-locally bounded derivatives, i.e., there exists a non-decreasing function $L(\cdot) \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$\sup_{i \in \mathbb{Z}^d} \max_{s \in [-r, r]} |f'_i(s)| \leq L(\rho_i r), \quad \forall r \in \mathbb{R}^+, i \in \mathbb{Z}^d;$$

Assumption 5. $f_i(0) = 0$ for all $i \in \mathbb{Z}^d$;

Assumption 6. *There exist constants $\alpha > 0$ and β_i with $\beta = (\beta_i)_{i \in \mathbb{Z}^d} \in \ell_\rho^2$ such that*

$$s f_i(s) \leq -\alpha |s|^2 + \beta_i^2, \quad \forall s \in \mathbb{R}, \quad \forall i \in \mathbb{Z}^d.$$

It was shown in [7] that Assumption 4 implies that f_i is locally Lipschitz with

$$|f_i(x) - f_i(y)| \leq L(\rho_i(|x| + |y|)) |x - y|, \quad \forall i \in \mathbb{Z}^d, x, y \in \mathbb{R}.$$

Since

$$\rho_i |u_i| \leq \sqrt{\rho_\Sigma} \left(\sum_{i \in \mathbb{Z}^d} \rho_i u_i^2 \right)^{1/2} = \sqrt{\rho_\Sigma} \|\mathbf{u}\|_\rho,$$

it follows

$$|f_i(u_i) - f_i(v_i)| \leq L(\rho_i(|u_i| + |v_i|)) \cdot |u_i - v_i| \leq L(\sqrt{\rho_\Sigma}(\|\mathbf{u}\|_\rho + \|\mathbf{v}\|_\rho)) \cdot |u_i - v_i|$$

for every $\mathbf{u} = (u_i)_{i \in \mathbb{Z}^d}$ and $\mathbf{v} = (v_i)_{i \in \mathbb{Z}^d}$. The following lemma from [7] states the Lipschitz and dissipative properties of the operator f .

Lemma 2.1. *Assume that Assumptions 4-6 hold. Then $f : \ell_\rho^2 \rightarrow \ell_\rho^2$ is locally Lipschitz and satisfies the dissipativity condition*

$$\langle f(\mathbf{u}), \mathbf{u} \rangle \leq -\alpha \|\mathbf{u}\|_\rho^2 + \|\beta\|_\rho^2.$$

2.2. The interaction term. For any $\mathbf{v} \in \mathcal{C}([-h, 0], \ell_\rho^2)$ we define the operator K_τ by $K_\tau(\mathbf{v}) = (K_{\tau, i}(\mathbf{v}))_{i \in \mathbb{Z}^d}$ by

$$K_{\tau, i}(\mathbf{v}) = \sum_{j \in \mathbb{Z}^d} k_{i, j} \sigma_\varepsilon(v_j(-\tau_j) - \theta), \quad \forall i \in \mathbb{Z}^d.$$

Lemma 2.2. *The operator K_τ maps $\mathcal{C}([-h, 0], \ell_\rho^2)$ to ℓ_ρ^2 .*

Proof. The function σ_ε takes values in the unit interval $[0,1]$, so

$$|K_{\tau,i}(\mathbf{v})| \leq \sum_{j \in \mathbb{Z}^d} k_{i,j} \leq \kappa, \quad \forall i \in \mathbb{Z}^d, \mathbf{v} \in \mathcal{C}([-h, 0], \ell_\rho^2).$$

Then

$$\|K_\tau(\mathbf{v})\|_\rho^2 = \sum_{i \in \mathbb{Z}^d} \rho_i |K_{\tau,i}(\mathbf{v})|^2 \leq \kappa^2 \rho_\Sigma < \infty.$$

□

Remark 1. The function σ_ε is differentiable with a uniformly bounded derivative

$$\frac{d}{dx} \sigma_\varepsilon(x) \leq \frac{1}{\varepsilon} \quad \text{for all } x \in \mathbb{R}.$$

Hence it is globally Lipschitz with the Lipschitz constant $L_\sigma = \frac{1}{\varepsilon}$.

2.3. The forcing term. Finally, we suppose that the constant forcing term $\mathbf{g} := (g_i)_{i \in \mathbb{Z}^d}$ satisfies the following assumption.

Assumption 7. $\mathbf{g} \in \ell_\rho^2$.

3. Existence and uniqueness of solutions. The lattice differential equation (1) can be rewritten as an infinitely dimensional ordinary differential equation on ℓ_ρ^2 ,

$$\frac{d}{dt} \mathbf{u}(t) = G_\tau(t, \mathbf{u}_t) := f(\mathbf{u}) + K_\tau(\mathbf{u}_t) + \mathbf{g}, \tag{3}$$

where $G_\tau(t, \mathbf{u}_t) := (G_{\tau,i}(t, \mathbf{u}_t))_{i \in \mathbb{Z}^d}$.

In this section we study the existence and uniqueness of solutions of the differential equation (3). To this end, we will need the following auxiliary Lemma 3.1.

Let $\mathbb{Z}_N^d := \{\mathbf{i} = (i_1, i_2, \dots, i_d) \in \mathbb{Z}^d : |i_1|, |i_2|, \dots, |i_d| \leq N\}$ and define

$$K_{\tau,i}^N(\mathbf{v}) := \sum_{j \in \mathbb{Z}_N^d} k_{i,j} \sigma_\varepsilon(v_j(-\tau_j) - \theta), \quad \mathbf{i} \in \mathbb{Z}^d.$$

Lemma 3.1. *The mapping $\mathbf{v} \mapsto K_{\tau,i}^N(\mathbf{v})$ is continuous from $\mathcal{C}([-h, 0], \ell_\rho^2)$ to \mathbb{R} for every $\mathbf{i} \in \mathbb{Z}^d$.*

Proof. Let $\mathbf{v}^n \rightarrow \mathbf{v}^0$ in $\mathcal{C}([-h, 0], \ell_\rho^2)$. Since Assumption 1: $0 \leq \tau_j \leq h$ for each $j \in \mathbb{Z}^d$, we see that $(v_j^n(-\tau_j))_{j \in \mathbb{Z}^d} \rightarrow (v_j^0(-\tau_j))_{j \in \mathbb{Z}^d}$ in ℓ_ρ^2 . Thus for every $\varepsilon > 0$ there exist an $M(\varepsilon) > 0$ such that

$$\sum_{j \in \mathbb{Z}^d} \rho_j |v_j^n(-\tau_j) - v_j^0(-\tau_j)|^2 < \varepsilon^2, \quad \forall n \geq M(\varepsilon).$$

Considering only the $j \in \mathbb{Z}_N^d$ appearing in the sum defining $K_{\tau,i}^N$, we obtain

$$|v_j^n(-\tau_j) - v_j^0(-\tau_j)| < \varepsilon / \sqrt{\rho_N}, \quad \forall n \geq M(\varepsilon), j \in \mathbb{Z}_N^d,$$

where $\rho_N := \min_{j \in \mathbb{Z}_N^d} \rho_j$.

The mapping $x \mapsto \sigma_\varepsilon(x - \theta)$ is continuous for all $x \in \mathbb{R}$. Since there are a finite number of terms in the sum in the definition of $K_{\tau,i}^N$, it follows from the elementary inequality

$$|(a_1 + b_1) - (a_2 + b_2)| \leq |a_1 - a_2| + |b_1 - b_2|, \quad a_1, a_2, b_1, b_2 \in \mathbb{R}^1$$

that the mapping $\mathbf{v} \rightarrow K_{\tau,i}^N(\mathbf{v})$ is continuous. □

3.1. Existence of solutions.

Theorem 3.2. *Suppose that Assumptions 1–7 hold. Then for each $r > 0$ there exists $a(r) > 0$ such that for every $\psi \in \mathcal{C}([-h, 0], \ell_\rho^2)$ satisfying $\|\psi\|_{\mathcal{C}([-h, 0], \ell_\rho^2)} \leq r$, the lattice delay equation (3) has at least one solution defined on $[0, a(r)]$. Moreover, the solution $u(\cdot) \in \mathcal{C}^1([0, a], \ell_\rho^2)$.*

Proof. Step 1. First, we claim that $G_\tau(t, \mathbf{u}_t)$ is well defined and bounded.

It is easy to see that $G_\tau(t, \mathbf{u}_t)$ is well defined since $f(\mathbf{u})$, $K_\tau(\mathbf{u}_t)$ and \mathbf{g} are all well defined. As for the boundedness, we denote that

$$|G_{\tau,i}(\mathbf{u}_t)| \leq |f_i(u_i(t))| + |K_{\tau,i}(\mathbf{u}_t)| + |g_i|. \tag{4}$$

Since f_i is locally Lipschitz and satisfies $f_i(0) = 0$ by Assumption 4-5, we see that

$$|f_i(u_i(t))| \leq L(\rho_i |u_i(t)|) \cdot |u_i(t)| \leq L(\sqrt{\rho_\Sigma} \|\mathbf{u}(t)\|_\rho) \cdot |u_i(t)|.$$

Then we obtain

$$\left(\sum_{i \in \mathbb{Z}^d} \rho_i |f_i(u_i(t))|^2\right)^{\frac{1}{2}} \leq L(\sqrt{\rho_\Sigma} \|\mathbf{u}(t)\|_\rho) \|\mathbf{u}(t)\|_\rho. \tag{5}$$

For the second term with delay, we have $|K_{\tau,i}(\mathbf{u}_t)| \leq \kappa$ by Assumption 2, which gives

$$\left(\sum_{i \in \mathbb{Z}^d} \rho_i |K_{\tau,i}(\mathbf{u}_t)|^2\right)^{\frac{1}{2}} \leq \sqrt{\rho_\Sigma} \kappa, \tag{6}$$

where we have used Assumption 3.

Finally, for the last term \mathbf{g} , Assumption 7 gives

$$\|\mathbf{g}\|_\rho < \infty. \tag{7}$$

Using (5), (6) and (7) in (4) we conclude that G_τ is well defined and bounded.

Step 2. Next, we claim that the maps $G_{\tau,i} : \mathcal{C}([-h, 0], \ell_\rho^2) \rightarrow \mathbb{R}$ are continuous for all $i \in \mathbb{Z}^d$.

We consider $\{\mathbf{u}_t^n\}_{n \in \mathbb{N}} \subset \mathcal{C}([-h, 0], \ell_\rho^2)$ and $\mathbf{u}_t^0 \in \mathcal{C}([-h, 0], \ell_\rho^2)$ such that $\mathbf{u}_t^n \rightarrow \mathbf{u}_t^0$ in $\mathcal{C}([-h, 0], \ell_\rho^2)$. Then

$$|G_{\tau,i}(\mathbf{u}_t^n) - G_{\tau,i}(\mathbf{u}_t^0)| \leq |f_i(u_i^n(t)) - f_i(u_i^0(t))| + |K_{\tau,i}(\mathbf{u}_t^n) - K_{\tau,i}(\mathbf{u}_t^0)|. \tag{8}$$

By the local Lipschitz continuity of f_i ,

$$|f_i(u_i^n(t)) - f_i(u_i^0(t))| \leq L(\sqrt{\rho_\Sigma} (\|\mathbf{u}_t^n(0)\|_\rho + \|\mathbf{u}_t^0(0)\|_\rho)) \cdot |u_i^n(t) - u_i^0(t)|, \tag{9}$$

which shows that this term converges to zero.

Next for the second term on the right-hand side

$$\begin{aligned} |K_{\tau,i}(\mathbf{u}_t^n) - K_{\tau,i}(\mathbf{u}_t^0)| &= \left| \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j^n(t - \tau_j) - \theta) - \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j^0(t - \tau_j) - \theta) \right| \\ &= \sum_{j \in \mathbb{Z}^d} k_{i,j} |\sigma_\varepsilon(u_j^n(t - \tau_j) - \theta) - \sigma_\varepsilon(u_j^0(t - \tau_j) - \theta)| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{j \in \mathbb{Z}_N^d} k_{i,j} |\sigma_\varepsilon(u_j^n(t - \tau_j) - \theta) - \sigma_\varepsilon(u_j^0(t - \tau_j) - \theta)| \\ &\quad + \sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}_N^d} k_{i,j} |\sigma_\varepsilon(u_j^n(t - \tau_j) - \theta) - \sigma_\varepsilon(u_j^0(t - \tau_j) - \theta)| \tag{10} \\ &\leq L_\sigma \sum_{j \in \mathbb{Z}_N^d} k_{i,j} |u_j^n(t - \tau_j) - u_j^0(t - \tau_j)| + 2 \sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}_N^d} k_{i,j}. \end{aligned}$$

On one hand, since $\sum_{j \in \mathbb{Z}^d} k_{i,j} \leq \kappa$, for every $\varepsilon > 0$, there exists a $N_1(\varepsilon)$ such that $\sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}_N^d} k_{i,j} \leq 4\varepsilon$ when $N \geq N_1(\varepsilon)$: we assume that the N in (10) is such an N . On the other hand, $\sum_{j \in \mathbb{Z}_N^d} k_{i,j} |u_j^n(t - \tau_j) - u_j^0(t - \tau_j)| < \varepsilon$ when $n > M(\varepsilon)$ by Lemma 3.1 for all N . Thus $K_{\tau,i}$ is continuous.

Using (9) and (10) in (8), we complete the proof of the claim.

Having the two steps above, by Theorem 10 in Caraballo *et al.* [3], for each $r > 0$ there exists $a(r) > 0$ such that if $\boldsymbol{\psi} \in \mathcal{C}([-h, 0], \ell_\rho^2)$ and $\|\boldsymbol{\psi}\|_{\mathcal{C}([-h, 0], \ell_\rho^2)} \leq r$, then the problem (3) has at least one solution defined on $[0, a(r)]$.

Step 3. Finally, we claim the following inequality holds:

$$\sum_{|i| \geq K} \rho_i |G_{\tau,i}(\mathbf{u}_t)|^2 \leq C(\sqrt{\rho_\Sigma} \|\mathbf{u}(t)\|_\rho) \left(\max_{s \in [-h, 0]} \sum_{|i| \geq K} \rho_i u_i^2(t + s) + b_K \right),$$

where $b_K \rightarrow 0^+$ as $K \rightarrow \infty$, and $C(\cdot) > 0$ is a continuous non-decreasing function.

The proof is as follows.

$$\begin{aligned} &\sum_{|i| \geq K} \rho_i |G_{\tau,i}(\mathbf{u}_t)|^2 \\ &\leq 3 \sum_{|i| \geq K} \rho_i |f_i(u_i(t))|^2 + 3 \sum_{|i| \geq K} \rho_i \left| \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j(t - \tau_j) - \theta) \right|^2 + 3 \sum_{|i| \geq K} \rho_i |g_i|^2 \\ &\leq 3 \sum_{|i| \geq K} \rho_i L^2(\rho_i |u_i(t)|) |u_i(t)|^2 + 3 \sum_{|i| \geq K} \rho_i \cdot \kappa^2 + 3 \sum_{|i| \geq K} \rho_i |g_i|^2 \\ &\leq 3L^2(\sqrt{\rho_\Sigma} \|\mathbf{u}(t)\|_\rho) \sum_{|i| \geq K} \rho_i |u_i(t)|^2 + 3 \sum_{|i| \geq K} \rho_i \cdot \kappa^2 + 3 \sum_{|i| \geq K} \rho_i |g_i|^2 \\ &\leq C(\sqrt{\rho_\Sigma} \|\mathbf{u}(t)\|_\rho) \left(\max_{s \in [-h, 0]} \sum_{|i| \geq K} \rho_i u_i^2(t + s) + b_K \right). \end{aligned}$$

By Corollary 13 in [3], we also conclude that the solution $\mathbf{u}(\cdot) \in \mathcal{C}^1([0, a], \ell_\rho^2)$. \square

3.2. A prior estimate of solutions. Here we will establish some estimates of the solutions, which imply that the solutions are bounded uniformly with respect to bounded sets of initial conditions and all positive values of time.

Proposition 1. *Suppose that Assumptions 1-7 hold. Then every solution $\mathbf{u}(\cdot)$ of (3) with $\mathbf{u}_0 = \boldsymbol{\psi} \in \mathcal{C}([-h, 0], \ell_\rho^2)$ verifies*

$$\|\mathbf{u}_t\|_{\mathcal{C}([-h, 0], \ell_\rho^2)}^2 \leq R_1 e^{-\alpha t} \|\boldsymbol{\psi}\|_{\mathcal{C}([-h, 0], \ell_\rho^2)}^2 + R_2, \tag{11}$$

where $R_j > 0$, $j = 1, 2$, are constants depending on the parameters of the problem.

Proof. We multiply the i th component of (1) by $\rho_i u_i(t)$ and sum over i to obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_\rho^2 = \sum_{i \in \mathbb{Z}^d} \rho_i u_i f_i(u_i) + \sum_{i \in \mathbb{Z}^d} \left(\rho_i u_i \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j(t - \tau_j) - \theta) \right) + \sum_{i \in \mathbb{Z}^d} \rho_i u_i g_i. \quad (12)$$

By Assumption 6 and $\rho_i > 0$ we have

$$\rho_i f_i(u_i(t)) u_i(t) \leq -\alpha \rho_i u_i^2(t) + \rho_i \beta_i^2,$$

so

$$\sum_{i \in \mathbb{Z}^d} \rho_i f_i(u_i(t)) u_i(t) \leq -\alpha \|\mathbf{u}(t)\|_\rho^2 + \|\boldsymbol{\beta}\|_\rho^2.$$

Since function σ_ε takes values in the unit interval, using Young’s inequality we obtain

$$\begin{aligned} \left| \rho_i u_i \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j(t - \tau_j) - \theta) \right| &\leq \left| \rho_i u_i \sum_{j \in \mathbb{Z}^d} k_{i,j} \right| \\ &\leq \frac{\alpha}{4} \rho_i u_i^2 + \frac{1}{\alpha} \rho_i \left(\sum_{j \in \mathbb{Z}^d} k_{i,j} \right)^2 \\ &\leq \frac{\alpha}{4} \rho_i u_i^2 + \frac{1}{\alpha} \rho_i \kappa^2, \end{aligned}$$

so

$$\sum_{i \in \mathbb{Z}^d} \left| \rho_i u_i \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j(t - \tau_j) - \theta) \right| \leq \frac{\alpha}{4} \|\mathbf{u}(t)\|_\rho^2 + \frac{1}{\alpha} \rho_\Sigma \kappa^2.$$

The last term on the right hand of (12) satisfies

$$\begin{aligned} \sum_{i \in \mathbb{Z}^d} \rho_i g_i u_i(t) &\leq \frac{\alpha}{4} \sum_{i \in \mathbb{Z}^d} \rho_i u_i^2(t) + \frac{1}{\alpha} \sum_{i \in \mathbb{Z}^d} \rho_i g_i^2 \\ &\leq \frac{\alpha}{4} \|\mathbf{u}(t)\|_\rho^2 + \frac{1}{\alpha} \|\mathbf{g}\|_\rho^2. \end{aligned}$$

In summary, collecting the inequalities above, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_\rho^2 \leq -\frac{1}{2} \alpha \|\mathbf{u}(t)\|_\rho^2 + \|\boldsymbol{\beta}\|_\rho^2 + \frac{1}{\alpha} (\rho_\Sigma \kappa^2 + \|\mathbf{g}\|_\rho^2).$$

Integrating both sides of this differential inequality yields

$$\|\mathbf{u}(t)\|_\rho^2 \leq \|\mathbf{u}(0)\|_\rho^2 e^{-\alpha t} + \frac{2}{\alpha} \left(\|\boldsymbol{\beta}\|_\rho^2 + \frac{1}{\alpha} (\rho_\Sigma \kappa^2 + \|\mathbf{g}\|_\rho^2) \right) (1 - e^{-\alpha t}). \quad (13)$$

Let $\theta \in [-h, 0]$. Replacing t by $t + \theta$ in (13) and using

$$\|\mathbf{u}(t + \theta)\|_\rho = \|\boldsymbol{\psi}(t + \theta)\|_\rho \leq \|\boldsymbol{\psi}\|_{C([-h, 0], \ell_\rho^2)}, \quad t + \theta < 0,$$

we obtain

$$\|\mathbf{u}(t + \theta)\|_\rho^2 \leq \|\boldsymbol{\psi}\|_\rho^2 e^{-\alpha(t+\theta)} + \frac{2}{\alpha} \left(\|\boldsymbol{\beta}\|_\rho^2 + \frac{1}{\alpha} (\rho_\Sigma \kappa^2 + \|\mathbf{g}\|_\rho^2) \right) (1 - e^{-\alpha(t+\theta)}).$$

Finally, using that $\theta \in [-h, 0]$ and neglecting the negative terms yields

$$\begin{aligned} \|\mathbf{u}_t\|_{C([-h, 0], \ell_\rho^2)}^2 &\leq e^{\alpha h} e^{-\alpha t} \|\boldsymbol{\psi}\|_{C([-h, 0], \ell_\rho^2)}^2 + \frac{2}{\alpha} \left(\|\boldsymbol{\beta}\|_\rho^2 + \frac{1}{\alpha} (\rho_\Sigma \kappa^2 + \|\mathbf{g}\|_\rho^2) \right) \\ &= R_1 e^{-\alpha t} \|\boldsymbol{\psi}\|_{C([-h, 0], \ell_\rho^2)}^2 + R_2, \end{aligned} \quad (14)$$

where

$$R_1 := e^{\alpha h}, \quad R_2 := \frac{2}{\alpha} \left(\|\boldsymbol{\beta}\|_\rho^2 + \frac{1}{\alpha} (\rho_\Sigma \kappa^2 + \|\mathbf{g}\|_\rho^2) \right).$$

□

3.3. Uniqueness of solutions. Having the existence of the solution of problem (3), moreover, we now establish the uniqueness of the solution with the additional assumption that

Assumption 8. *There exists a constant $\tilde{\kappa} > 0$ such $\sum_{j \in \mathbb{Z}^d} \frac{k_{i,j}^2}{\rho_j} \leq \tilde{\kappa}$ for each $i \in \mathbb{Z}^d$.*

Lemma 3.3. *Suppose that Assumptions 1–8 hold. Then the solution \mathbf{u} of problem (3) is unique.*

Proof. Assumption 8 implies that the operator $K_\tau : C([-h, 0], \ell_\rho^2) \rightarrow \ell_\rho^2$ is Lipschitz. In fact,

$$\begin{aligned} & \sum_{i \in \mathbb{Z}^d} \rho_i \left| K_{\tau,i}(\mathbf{u}_t) - K_{\tau,i}(\mathbf{v}_t) \right|^2 \\ & \leq \sum_{i \in \mathbb{Z}^d} \rho_i \left| \sum_{j \in \mathbb{Z}^d} k_{i,j} \left(\sigma_\varepsilon(u_j(t - \tau_j) - \theta) - \sigma_\varepsilon(v_j(t - \tau_j) - \theta) \right) \right|^2 \\ & \leq \sum_{i \in \mathbb{Z}^d} \rho_i \left(\sum_{j \in \mathbb{Z}^d} k_{i,j} \left| \sigma_\varepsilon(u_j(t - \tau_j) - \theta) - \sigma_\varepsilon(v_j(t - \tau_j) - \theta) \right| \right)^2 \\ & \leq \sum_{i \in \mathbb{Z}^d} \rho_i \left(L_\sigma \sum_{j \in \mathbb{Z}^d} k_{i,j} |u_j(t - \tau_j) - v_j(t - \tau_j)| \right)^2 \\ & \leq \sum_{i \in \mathbb{Z}^d} \rho_i L_\sigma^2 \left(\sum_{j \in \mathbb{Z}^d} \frac{k_{i,j}}{\sqrt{\rho_j}} \sqrt{\rho_j} |u_j(t - \tau_j) - v_j(t - \tau_j)| \right)^2 \\ & \leq \rho_\Sigma \tilde{\kappa} L_\sigma^2 \|\mathbf{u}_t - \mathbf{v}_t\|_{C([-h,0], \ell_\rho^2)}^2. \end{aligned}$$

Hence, suppose that we have two different solutions \mathbf{u}, \mathbf{v} of problem (3) with the same initial condition $\mathbf{u}(s) = \mathbf{v}(s) = \boldsymbol{\psi}(s), \forall s \in [-h, 0]$.

Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$, we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|_\rho^2 \\ & \leq L(\sqrt{\rho_\Sigma}(\|\mathbf{u}\|_\rho + \|\mathbf{v}\|_\rho)) \|\mathbf{w}\|_\rho^2 + \|\mathbf{w}\|_\rho \sqrt{C_1 \rho_\Sigma \tilde{\kappa} L_\sigma} \|\mathbf{w}_t\|_{C([-h,0], \ell_\rho^2)} \\ & \leq L(2\sqrt{\rho_\Sigma}(R_1 \|\boldsymbol{\psi}\|_{C([-h,0], \ell_\rho^2)}^2 + R_2)) \|\mathbf{w}\|_\rho^2 + \|\mathbf{w}\|_\rho \sqrt{C_1 \rho_\Sigma \tilde{\kappa} L_\sigma} \|\mathbf{w}_t\|_{C([-h,0], \ell_\rho^2)} \\ & \leq C \|\mathbf{w}_t\|_{C([-h,0], \ell_\rho^2)}^2, \end{aligned}$$

where $C = \max \left\{ L(2\sqrt{\rho_\Sigma}(R_1 \|\boldsymbol{\psi}\|_{C([-h,0], \ell_\rho^2)}^2 + R_2)), \sqrt{C_1 \rho_\Sigma \tilde{\kappa} L_\sigma} \right\}$.

Integrating from 0 to t then gives

$$\|\mathbf{w}(t)\|_\rho^2 \leq 2C \int_0^t \|\mathbf{w}_\tau\|_{C([-h,0], \ell_\rho^2)}^2 d\tau + \|\mathbf{w}(0)\|_\rho^2.$$

Let $\theta \in [-h, 0]$. Replacing t by $t + \theta$ in the inequality above and using $\|\mathbf{w}(t + \theta)\|_\rho = 0$ when $t + \theta < 0$. We obtain

$$\|\mathbf{w}(t + \theta)\|_\rho^2 \leq 2C \int_0^{t+\theta} \|\mathbf{w}_\tau\|_{C([-h,0], \ell_\rho^2)}^2 d\tau + \|\mathbf{w}(0)\|_\rho^2.$$

Then take the supremum on θ ,

$$\|\mathbf{w}_t\|_{C([-h,0], \ell_\rho^2)}^2 \leq 2C \int_0^t \|\mathbf{w}_\tau\|_{C([-h,0], \ell_\rho^2)}^2 d\tau + \|\mathbf{w}(0)\|_\rho^2.$$

By Gronwall’s inequality, we have

$$\|\mathbf{w}_t\|_{\mathcal{C}([-h,0],\ell_\rho^2)}^2 \leq 2Cte^{2Ct}\|\mathbf{w}(0)\|_\rho^2 + \|\mathbf{w}(0)\|_\rho^2. \tag{15}$$

Since $\mathbf{w}(0) = 0$, we obtain that $\mathbf{w} \equiv 0$. □

The proof of the next corollary follows easily using (15).

Corollary 1. *The map $(t, \psi) \mapsto \mathbf{u}_t$ is continuous.*

Proposition 1 implies that every local solution of (1) can be extended globally, which, with the uniqueness of the solution, will allow us to define a semigroup in terms of the solution mapping and to conclude that it has a bounded absorbing set.

4. Asymptotic behaviour. When Assumptions 1–8 hold, Theorem 3.2 and Lemma 3.3 ensure the local existence and uniqueness of solutions of the delayed lattice system (3), while Proposition 1 shows that the solutions are, in fact, globally defined.

We can thus define a semigroup of operators $\mathcal{S} : \mathbb{R}^+ \times \mathcal{C}([-h, 0], \ell_\rho^2) \rightarrow \mathcal{C}([-h, 0], \ell_\rho^2)$ by

$$\mathcal{S}(t, \psi) = \mathbf{u}_t,$$

where \mathbf{u}_t is the unique solution to (3) with $\mathbf{u}_0 = \psi$. The semigroup map \mathcal{S} is continuous in its variables by Corollary 1.

It also follows from inequality (11) that the semigroup has a bounded absorbing set.

Corollary 2. *The bounded set defined by*

$$B_0 := \left\{ \psi \in \mathcal{C}([-h, 0], \ell_\rho^2) : \|\psi\|_{\mathcal{C}([-h,0],\ell_\rho^2)} \leq R_0 \right\},$$

with $R_0 := \sqrt{1 + R_2}$, is absorbing for the semigroup \mathcal{S} .

Our aim is to study the asymptotic behaviour of solution of problem (1). In particular, we will show the existence of a global attractor. For this we will apply the following well-known results about the existence of global attractors, see [8] and [5].

Theorem 4.1. *Let $x \rightarrow \mathcal{S}(t, x)$ be continuous for any $t \geq 0$. Assume that \mathcal{S} is asymptotically compact and possesses a bounded absorbing set B_0 . Then there exists a global compact attractor \mathcal{A} , which is the minimal closed set attracting any bounded set. If, moreover, the space X is connected and the map $t \rightarrow \mathcal{S}(t, x)$ is continuous for any $x \in X$, then the set \mathcal{A} is connected.*

4.1. Tail estimate. To show the asymptotic compactness of the semigroup, we need to estimate the tails of solutions of (3), i.e., their higher dimensional components, see [2].

Lemma 4.2. *Suppose that Assumptions 1–8 hold and let B be a bounded set of $\mathcal{C}([-h, 0], \ell_\rho^2)$. Then, for any $\varepsilon > 0$ there exist $T(\varepsilon, B)$ and $M(\varepsilon, B)$ such that*

$$\max_{s \in [-h, 0]} \sum_{|i| > 2M(\varepsilon, B)} \rho_i |u_i(t + s)|^2 < \varepsilon, \quad t \geq T,$$

for any initial condition $\psi \in B$ and the corresponding solution $\mathbf{u}(\cdot)$ of (3) with $\mathbf{u}_0 = \psi$.

Proof. Define a smooth function ξ satisfying

$$\xi(s) \begin{cases} = 0, & 0 \leq s \leq 1, \\ \in [0, 1], & 1 \leq s \leq 2, \\ = 1, & s \geq 2. \end{cases}$$

Let M be a fixed (and large) integer to be specified later, and set

$$v_i(t) = \xi_M(|i|)u_i(t) \quad \text{with} \quad \xi_M(|i|) = \xi\left(\frac{|i|}{M}\right), \quad i \in \mathbb{Z}^d,$$

where $|\cdot|$ denotes the Euclidean norm. We multiply the i th component of (1) by $\rho_i v_i$, then summing over $i \in \mathbb{Z}^d$, and since $\mathbf{u}(\cdot) \in \mathcal{C}^1([0, \infty], \ell_\rho^2)$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) |u_i(t)|^2 &= \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i(t) \frac{du_i(t)}{dt} \\ &= \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i(t) f_i(u_i(t)) + \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i(t) g_i \\ &\quad + \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i(t) \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j(t - \tau_j) - \theta). \end{aligned} \tag{16}$$

First, by Assumption 6,

$$\rho_i \xi_M(|i|) u_i(t) f_i(u_i) \leq -\alpha \rho_i \xi_M(|i|) u_i^2(t) + \rho_i \xi_M(|i|) \beta_i^2. \tag{17}$$

Then, since function σ_ε takes values in the unit interval, using Young's inequality,

$$\begin{aligned} \left| \rho_i \xi_M(|i|) u_i(t) \sum_{j \in \mathbb{Z}^d} k_{i,j} \sigma_\varepsilon(u_j(t - \tau_j) - \theta) \right| &\leq \left| \rho_i \xi_M(|i|) u_i(t) \kappa \right| \\ &\leq \frac{\alpha}{4} \rho_i \xi_M(|i|) u_i^2(t) + \frac{\kappa^2}{\alpha} \rho_i \xi_M(|i|). \end{aligned} \tag{18}$$

And using Young's inequality again,

$$\begin{aligned} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) g_i u_i(t) &= \sum_{|i| \geq M} \rho_i \xi_M(|i|) g_i u_i(t) \\ &\leq \frac{\alpha}{4} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t) + \frac{1}{\alpha} \sum_{|i| \geq M} \rho_i g_i^2. \end{aligned} \tag{19}$$

Inserting the estimations (17), (18) and (19) into (16), then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t) &\leq -\frac{1}{2} \alpha \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t) + \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) \beta_i^2 \\ &\quad + \frac{\kappa^2}{\alpha} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) + \frac{1}{\alpha} \sum_{|i| \geq M} \rho_i g_i^2. \end{aligned} \tag{20}$$

We now estimate each term on the right hand side of the above inequality. Note that

$$\sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) \beta_i^2 = \sum_{|i| \geq M} \rho_i \xi_M(|i|) \beta_i^2 \leq \sum_{|i| \geq M} \rho_i \beta_i^2.$$

Since $\beta = (\beta_i)_{i \in \mathbb{Z}^d} \in \ell_\rho^2$, then for every $\varepsilon > 0$ there exists $I_1(\varepsilon) > 0$ such that

$$\sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) \beta_i^2 \leq \sum_{|i| \geq M} \rho_i \beta_i^2 < \frac{1}{6} \varepsilon \quad \text{when} \quad M \geq I_1(\varepsilon). \tag{21}$$

Similarly, since $\rho_\Sigma = \sum_{i \in \mathbb{Z}^d} \rho_i < \infty$, then for every $\varepsilon > 0$, there exists $I_2(\varepsilon) > 0$ such that

$$\sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) = \sum_{|i| \geq M} \rho_i \xi_M(|i|) \leq \sum_{|i| \geq M} \rho_i < \frac{\alpha}{6\kappa^2} \varepsilon \text{ when } M \geq I_2(\varepsilon). \tag{22}$$

In addition, since $\mathbf{g} = (g_i)_{i \in \mathbb{Z}^d} \in \ell_\rho^2$ by Assumption 7, for every $\varepsilon > 0$ there exists $I_3(\varepsilon) > 0$ such that

$$\sum_{|i| \geq M} \rho_i g_i^2 \leq \frac{\alpha}{6} \varepsilon, \quad \forall t \in \mathbb{R}, \text{ when } M \geq I_3(\varepsilon). \tag{23}$$

Finally, for any $\varepsilon > 0$, choosing $I(\varepsilon) := \max\{I_1(\varepsilon), I_2(\varepsilon), I_3(\varepsilon)\}$, inserting the estimations (21), (22) and (23) into (20) results in

$$\frac{d}{dt} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t) \leq -\alpha \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t) + \varepsilon, \quad \forall M \geq I(\varepsilon).$$

It follows immediately from Gronwall’s lemma that

$$\sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t) \leq e^{-\alpha t} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(0) + \frac{\varepsilon}{\alpha}.$$

In a similar way as in Proposition 1 we have

$$\max_{s \in [-h, 0]} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) u_i^2(t + s) \leq e^{\alpha h} e^{-\alpha t} \max_{s \in [-h, 0]} \sum_{i \in \mathbb{Z}^d} \rho_i \xi_M(|i|) \psi_i^2(s) + \frac{\varepsilon}{\alpha}.$$

Thus, there exist $T(\varepsilon, B)$ and $M(\varepsilon, B)$ such that

$$\max_{s \in [-h, 0]} \sum_{|i| > 2M} \rho_i |u_i(t + s)|^2 \leq \varepsilon \text{ if } t \geq T.$$

□

4.2. Existence of the global attractor. In order to apply Theorem 4.1, we need to prove that \mathcal{S} generated by the delay lattice system (3) is asymptotically compact.

Lemma 4.3. *Suppose that Assumptions 1–8 hold. Then the semigroup \mathcal{S} is asymptotically compact.*

Proof. We consider $\xi^n := \mathbf{u}_{t_n}^n = \mathcal{S}(t_n, \psi^n)$, where $\psi^n \in B$, a bounded set in $\mathcal{C}([-h, 0], \ell_\rho^2)$. From (14) there is a $C > 0$ such that

$$\|\mathbf{u}_{t_n}^n(s)\| \leq C, \quad \forall s \in [-h, 0], \forall n \in \mathbb{N}.$$

For fixed $s \in [-h, 0]$ we can find a subsequence (which we still denote by \mathbf{u}^n) such that

$$\mathbf{u}^n(t_n + s) \rightharpoonup \zeta(s) \text{ in } \ell_\rho^2.$$

In fact, the weak convergence here is strong, which follows from Lemma 4.2. Indeed, there exists $N_1 > 0$, when $n \geq N_1$, we have $t_n > T$ (where T is the constant in Lemma 4.2). Moreover, for any $\mu > 0$ there exist $K_2(\mu)$ and $N_2(\mu)$ such that

$$\sum_{|i| > K_2} \rho_i |u_i^n(t_n + s)|^2 < \mu, \quad \sum_{|i| > K_2} \rho_i |\zeta_i(s)|^2 < \mu, \quad \sum_{|i| \leq K_2} \rho_i |u_i^n(t_n + s) - \zeta_i(s)|^2 < \mu$$

if $n \geq \max\{N_1, N_2(\mu)\}$. Hence

$$\begin{aligned} \|\mathbf{u}^n(t_n + s) - \zeta(s)\|_\rho^2 &\leq \sum_{|i| \leq K_2} \rho_i |u_i^n(t_n + s) - \zeta_i(s)|^2 + \sum_{|i| > K_2} \rho_i |u_i^n(t_n + s) - \zeta_i(s)|^2 \\ &\leq \sum_{|i| \leq K_2} \rho_i |u_i^n(t_n + s) - \zeta_i(s)|^2 + 2 \sum_{|i| > K_2} \rho_i |u_i^n(t_n + s)|^2 \\ &\quad + 2 \sum_{|i| > K_2} \rho_i |\zeta_i(s)|^2 \\ &< 5\mu. \end{aligned}$$

Thus, $\{\mathbf{u}^n(t_n + s)\}$ is precompact in ℓ_ρ^2 for any $s \in [-h, 0]$. Since G_τ is a bounded map, Proposition 1 and the integral representation of solutions imply that

$$\|\mathbf{u}^n(t_n + s) - \mathbf{u}^n(t_n + t)\|_\rho \leq \int_s^t \|G_\tau(\mathbf{u}_{t_n + \tau}^n)\|_\rho d\tau \leq K(t - s) \text{ if } -h \leq s < t \leq 0.$$

Then, the Ascoli-Arzelà theorem implies that ξ^n is relatively compact in $\mathcal{C}([-h, 0], \ell_\rho^2)$. \square

Remark 2. If Assumption 8 guaranteeing uniqueness of solutions does not hold, then the lattice model (1) generates a set-valued semi-dynamical system, which can be shown to have a global attractor using essentially the same Lemmas as above.

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