

A FAMILY OF q -CONGRUENCES MODULO THE SQUARE OF A CYCLOTOMIC POLYNOMIAL

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ABSTRACT. Using Watson's terminating ${}_8\phi_7$ transformation formula, we prove a family of q -congruences modulo the square of a cyclotomic polynomial, which were originally conjectured by the author and Zudilin [J. Math. Anal. Appl. 475 (2019), 1636–646]. As an application, we deduce two supercongruences modulo p^4 (p is an odd prime) and their q -analogues. This also partially confirms a special case of Swisher's (H.3) conjecture.

1. Introduction. In 1997, Van Hamme [19, (H.2)] proved the following supercongruence: for any prime $p \equiv 3 \pmod{4}$,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}, \quad (1.1)$$

where $(a)_n = a(a+1) \cdots (a+n-1)$ is the rising factorial. It is easy to see that (1.1) is also true when the sum is over k from 0 to $p-1$, since $(1/2)_k/k! \equiv 0 \pmod{p}$ for $(p-1)/2 < k \leq p-1$. Nowadays various generalizations of (1.1) can be found in [8, 10, 11, 12, 13, 14, 16, 17]. For example, Liu [12] proved that, for any prime $p \equiv 3 \pmod{4}$ and positive integer m ,

$$\sum_{k=0}^{mp-1} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}. \quad (1.2)$$

The first purpose of this paper is to prove the following q -analogue of (1.2), which was originally conjectured by the author and Zudilin [10, Conjecture 2].

Theorem 1.1. *Let m and n be positive integers with $n \equiv 3 \pmod{4}$. Then*

$$\sum_{k=0}^{mn-1} \frac{(1+q^{4k+1})(q^2; q^4)_k^3}{(1+q)(q^4; q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2}, \quad (1.3)$$

$$\sum_{k=0}^{mn+(n-1)/2} \frac{(1+q^{4k+1})(q^2; q^4)_k^3}{(1+q)(q^4; q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2}. \quad (1.4)$$

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Here and in what follows, the *q-shifted factorial* is defined by $(a; q)_0 = 1$ and $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ for $n \geq 1$, and the *n-th cyclotomic polynomial* $\Phi_n(q)$ is defined as

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ \gcd(n,k)=1}} (q - \zeta^k),$$

where ζ is an *n*-th primitive root of unity. Moreover, the *q-integer* is given by $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$.

The $m = 1$ case of (1.3) was first conjectured by the author and Zudilin [9, Conjecture 4.13] and has already been proved by themselves in a recent paper [11]. For some other recent progress on *q*-congruences, the reader may consult [2, 3, 4, 5, 6, 7, 8, 10, 15].

In 2016, Swisher [18, (H.3) with $r = 2$] conjectured that, for primes $p \equiv 3 \pmod{4}$ and $p > 3$,

$$\sum_{k=0}^{(p^2-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^2 \pmod{p^5}, \tag{1.5}$$

The second purpose of this paper is to prove the following *q*-congruences related to (1.5) modulo p^4 .

Theorem 1.2. *Let $n \equiv 3 \pmod{4}$ be a positive integer. Modulo $\Phi_n(q)^2 \Phi_{n^2}(q)^2$, we have*

$$\sum_{k=0}^{(n^2-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \frac{[n^2]_{q^2} (q^3; q^4)_{(n^2-1)/2}}{(q^5; q^4)_{(n^2-1)/2}} q^{(1-n^2)/2}, \tag{1.6}$$

$$\sum_{k=0}^{n^2-1} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \frac{[n^2]_{q^2} (q^3; q^4)_{(n^2-1)/2}}{(q^5; q^4)_{(n^2-1)/2}} q^{(1-n^2)/2}. \tag{1.7}$$

Let $n = p \equiv 3 \pmod{4}$ be a prime and take $q \rightarrow 1$ in Theorem 1.2. Then $\Phi_p(1) = \Phi_{p^2}(1) = p$, and

$$\lim_{q \rightarrow 1} \frac{(q^3; q^4)_{(p^2-1)/2}}{(q^5; q^4)_{(p^2-1)/2}} = \prod_{k=1}^{(p^2-1)/2} \frac{4k-1}{4k+1} = \frac{(\frac{3}{4})_{(p^2-1)/2}}{(\frac{5}{4})_{(p^2-1)/2}}.$$

Therefore, we obtain the following conclusion.

Corollary 1. *Let $p \equiv 3 \pmod{4}$ be a prime. Then*

$$\sum_{k=0}^{(p^2-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^2 \frac{(\frac{3}{4})_{(p^2-1)/2}}{(\frac{5}{4})_{(p^2-1)/2}} \pmod{p^4}, \tag{1.8}$$

$$\sum_{k=0}^{p^2-1} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p^2 \frac{(\frac{3}{4})_{(p^2-1)/2}}{(\frac{5}{4})_{(p^2-1)/2}} \pmod{p^4}. \tag{1.9}$$

Comparing (1.5) and (1.8), we would like to propose the following conjecture, which was recently confirmed by Wang and Pan [20].

Conjecture 1. Let $p \equiv 3 \pmod{4}$ be a prime and r a positive integer. Then

$$\prod_{k=1}^{(p^{2r}-1)/2} \frac{4k-1}{4k+1} \equiv 1 \pmod{p^2}. \tag{1.10}$$

Note that the $r = 1$ case is equivalent to say that (1.5) is true modulo p^4 .

2. Proof of Theorem 1.1. We need to use Watson’s terminating ${}_8\phi_7$ transformation formula (see [1, Appendix (III.18)] and [1, Section 2]):

$$\begin{aligned} & {}_8\phi_7 \left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q, \frac{a^2q^{n+2}}{bcde} \right] \\ &= \frac{(aq; q)_n (aq/de; q)_n}{(aq/d; q)_n (aq/e; q)_n} {}_4\phi_3 \left[\begin{matrix} aq/bc, & d, & e, & q^{-n} \\ aq/b, & aq/c, & deq^{-n}/a \end{matrix} ; q, q \right], \end{aligned} \tag{2.1}$$

where the basic hypergeometric ${}_{r+1}\phi_r$ series with $r + 1$ upper parameters a_1, \dots, a_{r+1} , r lower parameters b_1, \dots, b_r , base q and argument z is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1; q)_k (a_2; q)_k \cdots (a_{r+1}; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_r; q)_k} z^k.$$

The left-hand side of (1.4) with $m \geq 0$ can be written as the following terminating ${}_8\phi_7$ series:

$${}_8\phi_7 \left[\begin{matrix} q^2, & q^5, & -q^5, & q^2, & q, & q^2, & q^{4+(4m+2)n}, & q^{2-(4m+2)n} \\ & q, & -q, & q^4, & q^5, & q^4, & q^{2-(4m+2)n}, & q^{4+(4m+2)n} \end{matrix} ; q^4, q \right]. \tag{2.2}$$

By Watson’s transformation formula (2.1) with $q \mapsto q^4$, $a = b = d = q^2$, $c = q$, $e = q^{4+(4m+2)n}$, and $n \mapsto mn + (n - 1)/2$, we see that (2.2) is equal to

$$\begin{aligned} & \frac{(q^6; q^4)_{mn+(n-1)/2} (q^{-(4m+2)n}; q^4)_{mn+(n-1)/2}}{(q^4; q^4)_{mn+(n-1)/2} (q^{2-(4m+2)n}; q^4)_{mn+(n-1)/2}} \\ & \times {}_4\phi_3 \left[\begin{matrix} q^3, & q^2, & q^{4+(4m+2)n} \\ & q^4, & q^5, & q^6 \end{matrix} ; q^4, q^4 \right]. \end{aligned} \tag{2.3}$$

It is not difficult to see that there are exactly $m + 1$ factors of the form $1 - q^{an}$ (a is an integer) among the $mn + (n - 1)/2$ factors of $(q^6; q^4)_{mn+(n-1)/2}$. So are $(q^{-(4m+2)n}; q^4)_{mn+(n-1)/2}$. But there are only m factors of the form $1 - q^{an}$ (a is an integer) in each of $(q^4; q^4)_{mn+(n-1)/2}$ and $(q^{2-(4m+2)n}; q^4)_{mn+(n-1)/2}$. Since $\Phi_n(q)$ is a factor of $1 - q^N$ if and only if n divides N , we conclude that the fraction before the ${}_4\phi_3$ series is congruent to 0 modulo $\Phi_n(q)^2$. Moreover, for any integer x , let $f_n(x)$ be the least non-negative integer k such that $(q^x; q^4)_k \equiv 0$ modulo $\Phi_n(q)$. Since $n \equiv 3 \pmod{4}$, we have $f_n(2) = (n + 1)/2$, $f_n(3) = (n + 1)/4$, $f_n(4) = n$, $f_n(5) = (3n - 1)/4$, and $f_n(6) = (n - 1)/2$. It follows that the denominator of the reduced form of the k -th summand

$$\frac{(q^3; q^4)_k (q^2; q^4)_k (q^{4+(4m+2)n}; q^4)_k (q^{2-(4m+2)n}; q^4)_k}{(q^4; q^4)_k^2 (q^5; q^4)_k (q^6; q^4)_k} q^{4k}$$

in the ${}_4\phi_3$ summation is always relatively prime to $\Phi_n(q)$ for any non-negative integer k . This proves that (2.3) (i.e. (2.2)) is congruent to 0 modulo $\Phi_n(q)^2$, thus establishing (1.4) for $m \geq 0$.

It is easy to see that $(q^2; q^4)_k^3 / (q^4; q^4)_k^3$ is congruent to 0 modulo $\Phi_n(q)^3$ for $mn + (n - 1)/2 < k \leq (m + 1)n - 1$. Therefore, the q -congruence (1.3) with $m \mapsto m + 1$ follows from (1.4).

3. Proof of Theorem 1.2. The author and Zudilin [11, Theorem 1.1] proved that, for any positive odd integer n ,

$$\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \frac{[n]_{q^2}(q^3; q^4)_{(n-1)/2}}{(q^5; q^4)_{(n-1)/2}} q^{(1-n)/2} \pmod{\Phi_n(q)^2}, \tag{3.1}$$

which is also true when the sum on the left-hand side of (3.1) is over k from 0 to $n - 1$. Replacing n by n^2 in (3.1) and its equivalent form, we see that the q -congruences (1.6) and (1.7) hold modulo $\Phi_{n^2}(q)^2$.

It is easy to see that, for $n \equiv 3 \pmod{4}$,

$$\frac{[n^2]_{q^2}(q^3; q^4)_{(n^2-1)/2}}{(q^5; q^4)_{(n^2-1)/2}} q^{(1-n^2)/2} \equiv 0 \pmod{\Phi_n(q)^2}$$

because $[n^2]_{q^2} = (1 - q^{n^2}) / (1 - q^2)$ is divisible by $\Phi_n(q)$, and $(q^3; q^4)_{(n^2-1)/2}$ contains $(n + 1)/2$ factors of the form $1 - q^{an}$ (a is an integer), while $(q^5; q^4)_{(n^2-1)/2}$ only has $(n - 1)/2$ such factors. Meanwhile, by Theorem 1.1, the left-hand sides of (1.6) and (1.7) are both congruent to 0 modulo $\Phi_n(q)^2$ since $(n^2 - 1)/2 = (n - 1)n/2 + (n - 1)/2$. This proves that the q -congruences (1.6) and (1.7) also hold modulo $\Phi_n(q)^2$. Since the polynomials $\Phi_n(q)$ and $\Phi_{n^2}(q)$ are relatively prime, we finish the proof of the theorem.

4. Discussion. Swisher’s (H.3) conjecture also indicates that, for positive integer r and primes $p \equiv 3 \pmod{4}$ with $p > 3$, we have

$$\sum_{k=0}^{(p^{2r}-1)/2} \frac{\binom{1}{2}_k^3}{k!^3} \equiv p^{2r} \pmod{p^{2r+3}}. \tag{4.1}$$

Motivated by (4.1), we shall give the following generalization of Theorem 1.2.

Theorem 4.1. *Let n and r be positive integers with $n \equiv 3 \pmod{4}$. Then, modulo $\Phi_{n^{2r}}(q)^2 \prod_{j=1}^r \Phi_{n^{2j-1}}(q)^2$, we have*

$$\sum_{k=0}^{(n^{2r}-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \frac{[n^{2r}]_{q^2}(q^3; q^4)_{(n^{2r}-1)/2}}{(q^5; q^4)_{(n^{2r}-1)/2}} q^{(1-n^{2r})/2}, \tag{4.2}$$

$$\sum_{k=0}^{n^{2r}-1} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \frac{[n^{2r}]_{q^2}(q^3; q^4)_{(n^{2r}-1)/2}}{(q^5; q^4)_{(n^{2r}-1)/2}} q^{(1-n^{2r})/2}. \tag{4.3}$$

Proof. Replacing n by n^{2r} in (3.1) and its equivalent form, we see that (4.2) and (4.3) are true modulo $\Phi_{n^{2r}}(q)^2$. Similarly as before, we can show that

$$\frac{[n^{2r}]_{q^2}(q^3; q^4)_{(n^{2r}-1)/2}}{(q^5; q^4)_{(n^{2r}-1)/2}} q^{(1-n^{2r})/2} \equiv 0 \pmod{\prod_{j=1}^r \Phi_{n^{2j-1}}(q)^2}.$$

Further, by Theorem 1.1, we can easily deduce that the left-hand sides of (4.2) and (4.3) are also congruent to 0 modulo $\prod_{j=1}^r \Phi_{n^{2j-1}}(q)^2$. \square

Letting $n = p \equiv 3 \pmod{4}$ be a prime and taking $q \rightarrow 1$ in Theorem 4.1, we are led to the following result.

Corollary 2. *Let $p \equiv 3 \pmod{4}$ be a prime and let $r \geq 1$. Then*

$$\sum_{k=0}^{(p^{2r}-1)/2} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \equiv p^{2r} \frac{\left(\frac{3}{4}\right)_{(p^{2r}-1)/2}}{\left(\frac{5}{4}\right)_{(p^{2r}-1)/2}} \pmod{p^{2r+2}}, \tag{4.4}$$

$$\sum_{k=0}^{p^{2r}-1} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \equiv p^{2r} \frac{\left(\frac{3}{4}\right)_{(p^{2r}-1)/2}}{\left(\frac{5}{4}\right)_{(p^{2r}-1)/2}} \pmod{p^{2r+2}}. \tag{4.5}$$

In light of (1.10), the supercongruence (4.4) implies that (4.1) holds modulo p^{2r+2} for any odd prime p .

It is known that q -analogues of supercongruences are usually not unique. See, for example, [2]. The author and Zudilin [10, Conjecture 1] also gave another q -analogue of (1.2), which still remains open.

Conjecture 2 (Guo and Zudilin). *Let m and n be positive integers with $n \equiv 3 \pmod{4}$. Then*

$$\sum_{k=0}^{mn-1} \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv 0 \pmod{\Phi_n(q)^2}, \tag{4.6}$$

$$\sum_{k=0}^{mn+(n-1)/2} \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv 0 \pmod{\Phi_n(q)^2}.$$

The author and Zudilin [10, Theorem 2] themselves have proved (4.6) for the $m = 1$ case. Motivated by Conjecture 2, we would like to give the following new conjectural q -analogues of (1.8) and (1.9).

Conjecture 3. *Let $n \equiv 3 \pmod{4}$ be a positive integer. Modulo $\Phi_n(q)^2 \Phi_{n^2}(q)^2$, we have*

$$\sum_{k=0}^{(n^2-1)/2} \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv \frac{[n^2](q^3; q^4)_{(n^2-1)/2}}{(q^5; q^4)_{(n^2-1)/2}},$$

$$\sum_{k=0}^{n^2-1} \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv \frac{[n^2](q^3; q^4)_{(n^2-1)/2}}{(q^5; q^4)_{(n^2-1)/2}}.$$

There are similar such new q -analogues of (4.4) and (4.5). We omit them here and leave space for the reader’s imagination.

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